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Finding the Extreme Z-Eigenvalues of Tensors via a Sequential SDPs Method

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SUMMARY

In this paper, we first introduce the tensor conic linear programming (TCLP for short) which is a generalization of the space tensor conic linear programming introduced by Qi and Ye [1]. Then, an approximation method, by using a sequence of semidefinite programming problems, is proposed to solve the TCLP. In particular, we reformulate the extreme Z-eigenvalue problem as a special TCLP. It gives a numerical algorithm to compute the extreme Z-eigenvalue of an even order tensor with dimension larger than three, which is discussed in the literature. Numerical experiments show the efficiency of the the proposed method. Copyright © 2013 John Wiley & Sons, Ltd.

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KEY WORDS: Tensor conic linear programming; Z-eigenvalue; semidefinite programming

1. INTRODUCTION

Tensors appear in physics, mechanics, statistics and so on. When a basis for the tensor space is chosen, there is a one to one correspondence between tensors and multi-way arrays. Recently, higher order three-dimensional tensors have been extensively investigated and used in many applications (see [1] and references therein), especially in the diffusion magnetic resonance imaging [2–9]. Meanwhile, higher order tensors with dimension larger than three are frequently involved in some numerical computations of engineering science [10–15], which are worth investigating. However, there lacks a unified model and efficient numerical schemes for such problems.

Very recently, Qi and Ye [1] proposed a mathematical model, named *space tensor conic linear programming* (STLP for short), to give a unification for various optimization problems with higher order three-dimensional tensors appeared in the literature. Based on the analysis on the space tensor cone, Qi and Ye established fundamental properties of the STLP, including the duality theory.

In this paper, we first give a unified model for the tensor programming involving higher order higher-dimensional tensors, which is denoted by the TCLP for short. Then, an approximation method, by using a sequence of semidefinite programming problems, is proposed to solve the TCLP. We call this method as *the sequential SDPs method*, and abbreviate it as *T-SSM* for convenience. The prefix letter "T" represents tensors. We show that the eigenvalue problem of even order symmetric tensors [12, 13] is strongly related to the TCLP. Actually, the extreme Z-eigenvalue problem for even order symmetric tensors is a special case of the TCLP. We note that there are many problems which are just reformulations of the extreme Z-eigenvalues of a tensor [14, 16], such as the best rank

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one approximation problem. Since TCLP provides a unification for the computations involving the extreme Z-eigenvalues of even order symmetric tensors, its numerical scheme is applicable and useful. We use T-SSM to solve the extreme Z-eigenvalue problem for even order symmetric tensors in this paper. It gives a method for computing the extreme Z-eigenvalues of even order symmetric tensors with dimensions larger than three, which improves the literature [8, 14]. The numerical experiments demonstrate that the proposed method is efficient.

The rest of this paper is organized as follows. Some preliminaries and the TCLP model are presented in the next section. In Section 3, we describe the sequential SDPs method for the TCLP. Then, in Section 4, we discuss the problem of the extreme Z-eigenvalues of even order symmetric tensors and report some preliminary numerical results by the T-SSM. Some final remarks are given in the last section.

2. PRELIMINARIES

In this section, we present the model of the TCLP and some duality results for it. For the convenience of the subsequent analysis, we give some notation first.

Notation A real m-th order n-dimensional (m is a positive even number throughout this paper) tensor D consists of n^m real entries: $d_{i_1,\ldots,i_m} \in \Re$, where $i_j \in \{1,\ldots,n\}$ for $j \in \{1,\ldots,m\}$. The tensor D is called symmetric if its entries are invariant under any permutation of its indices. Let $\mathcal{T}(m)$ denote the set of all m-th order n-dimensional real symmetric tensors. For any $C = (c_{i_1,\ldots,i_m}), D = (d_{i_1,\ldots,i_m}) \in \mathcal{T}(m)$, their inner product is defined as $C \bullet D = \sum_{i_1,\ldots,i_m=1}^n c_{i_1,\ldots,i_m} d_{i_1,\ldots,i_m}$, and the Hilbert-Schmidt norm induced by the inner product is defined by $\|D\| = \sqrt{D \bullet D}$. Given a vector $x = (x_1,\ldots,x_n)^\top \in \Re^n$, a tensor $D \in \mathcal{T}(m)$ defines a homogeneous polynomial through:

$$d(x) = \sum_{i_1,\ldots,i_m=1}^n d_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}. \tag{1}$$

We denote the degree of a polynomial f by deg(f). A polynomial is homogenous of degree k if each of its monomials has degree k. Then, d defined above is homogenous and has deg$(d) = m$. We use $\mathbf{0}$ to denote a tensor with its all entries being zeros. It defines a polynomial 0, which can be viewed as a polynomial of arbitrary degree. The above d is said to be positive semidefinite, if $d(x) \geq 0$ holds for all $x \in \Re^n$, and d is said to be positive definite, if $d(x) > 0$ holds for all $x \in \Re^n \backslash \{0\}$. Obviously, for nonzero tensors, m being an even integer is a necessity for positive semidefiniteness. Define

$$\mathcal{S}(m) := \{D \in \mathcal{T}(m) \mid d(x) \geq 0, \;\; \forall x \in \Re^n\}. \tag{2}$$

Then, $\mathcal{S}(m)$ is a pointed closed convex cone with nonempty interior, which can be proved in a similar way as those given in [1, 8]. The partial order induced by $\mathcal{S}(m)$ is denoted by $\succeq$, i.e., $C \succeq D$ means $C - D \in \mathcal{S}(m)$ for any $C, D \in \mathcal{T}(m)$. It is easy to see that $X \succeq \mathbf{0}$ if and only if the homogeneous polynomial defined by the tensor X is positive semidefinite. In this case, we say that the tensor X is positive semidefinite.

With the above notation, the TCLP is defined as:

$$\begin{array}{ll} \min & A_0 \bullet X \\ \text{s.t.} & X \in \mathcal{F} := \{X \mid A_i \bullet X = b_i, \; \forall i \in \{1,2,\ldots,p\}, \quad X \succeq \mathbf{0}\}, \end{array} \tag{3}$$

where p is a positive integer, $A_0, A_1, \ldots, A_p \in \mathcal{T}(m)$, and $b_1, \ldots, b_p \in \Re$. It is easy to see that, when $m = 3$, the TCLP reduces to the STLP introduced in [1].

The dual programming problem of (3) is:

$$\begin{array}{ll} \max & b^\top y \\ \text{s.t.} & (y,S) \in \mathcal{D} := \left\{(y,S) \mid \sum_{i=1}^p y_i A_i + S = A_0, \; y \in \Re^p, \; S \in \mathcal{S}^*(m)\right\}, \end{array} \tag{4}$$

where $\mathcal{S}^*(m)$ is the dual cone of $\mathcal{S}(m)$. Since $S(m)$ is a closed, convex, pointed cone with nonempty interior, so is $\mathcal{S}^*(m)$. We use $int\mathcal{S}(m)$ (respectively, $int\mathcal{S}^*(m)$) to denote the interior of $\mathcal{S}(m)$

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(respectively, $\mathcal{S}^*(m)$). If the objective function of (3) is bounded below over its feasible set, then we say that (3) is bounded below, and if there exists $X \in int\mathcal{S}(m)$ such that X is feasible to (3), then we say that (3) is strictly feasible. Similar concepts are defined for (4).

From the theory of conic optimization problems [17], there is no difficulty to obtain the following results, whose proofs are hence omitted.

Theorem 2.1
Let the optimization problems be defined as in (3) and (4). Denote

$$a^* := \inf_{X \in \mathcal{F}} A_0 \bullet X \quad \text{and} \quad b^* := \sup_{(y,S) \in \mathcal{D}} b^\top y.$$

Suppose that $X \in \mathcal{F} \neq \emptyset$ and $(y, S) \in \mathcal{D} \neq \emptyset$. Then,

- (weak duality) $b^\top y \leq A_0 \bullet X$.
- (strong duality) Suppose that (3) is bounded below and strictly feasible (respectively, (4) is bounded above and strictly feasible), then $a^* = b^*$ and (4) (respectively, (3)) is solvable.
- (complementarity slackness condition) If $a^* = b^*$, then X is optimal for (3) and (y, S) is optimal for (4) if and only if the complementarity slackness condition holds, i.e., $X \bullet S = 0$.
- (optimality condition) If $b^\top y = A_0 \bullet X$, then X is optimal for (3) and (y, S) is optimal for (4).

We state the following assumption, and assume it holds throughout this paper.

Assumption 2.1
Suppose that both optimization problems (3) and (4) are strictly feasible.

Under the above assumption, both the feasible sets of (3) and (4) have nonempty relative interiors. The following is a well known result in convex analysis [18].

Lemma 2.1
If A is a convex set with nonempty relative interior, then for any convex set B with $\mathrm{rl}(A) \subseteq B \subseteq \mathrm{cl}(A)$, we have $\mathrm{rl}(A) = \mathrm{rl}(B) = \mathrm{rl}(\mathrm{cl}(B))$ and $\mathrm{cl}(B) = \mathrm{cl}(A)$. Here $\mathrm{cl}(\cdot)$ and $\mathrm{rl}(\cdot)$ denotes the closure hull and the relative interior of a set respectively.

Although the TCLP is a convex optimization problem, it is hard to solve. Up to now, there exists no algorithm to solve the TCLP. It is of great interest and importance for giving a numerical scheme to solve such a tensorial optimization problem due to the many applications in statistic, medical imaging, engineering science, and so on. In the next section, we provide a framework based on semidefinite programming to solve the TCLP. Such a framework is an approximation solution method for the TCLP (3) in the sense that, theoretically, it can find a feasible solution to (3) with the distance between its objective value and the optimal value of (3) being within a given precision.

The Z-eigenvalue problem, especially the extreme Z-eigenvalue problem, is somewhat the corner stone in medical imaging and control optimization. This problem has also direct applications in the best rank one approximation of higher order tensors [16]. We show in Section 4 that the extreme Z-eigenvalue problem of even order tensors is a special TCLP (3). Thus, the proposed method serves as a useful tool for solving it.

3. DESCRIPTION OF THE T-SSM

In this section, we give the detailed description of the sequential SDPs method (the T-SSM) to solve the TCLP.

By a similar proof as the one in [1, Theorem 1], we have

$$\mathrm{int}\mathcal{S}(m) = \{D \in \mathcal{T}(m) \mid d(x) > 0, \ \ \forall x \in \Re^n \setminus \{0\}\}. \tag{5}$$

Let $\Re[x]$ be the polynomial ring of multivariate polynomials in the variable x with coefficients in the field $\Re$ of real numbers. A polynomial $p \in \Re[x]$ is called SOS (short for sum of squares), if $p(x) = \sum_{i=1}^t p_i^2(x)$ for some polynomials $p_i \in \Re[x]$ and some integer t. Denote by $g(x) := \sum_{i=1}^n x_i^2$.

The following result is a direct consequence of Reznick's theorem [19, Corollary 3.18].

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Theorem 3.2
Let int$\mathcal{S}(m)$ be defined as (5) and $d(x)$ as (1) for a tensor D. If $D \in$ int$\mathcal{S}(m)$, then for some sufficiently large integer $r \geq 0$, $g(x)^r d(x)$ is SOS.

With Theorem 3.2, define

$$K(m) := \{D \in \mathcal{T}(m) \mid g(x)^r d(x) \text{ is SOS for some } r \geq 0\}. \tag{6}$$

We have the following result.

Theorem 3.3
Let $K(m)$ be defined as (6). The set $K(m)$ is a convex cone, and int$\mathcal{S}(m) \subseteq K(m) \subseteq \mathcal{S}(m)$.

Proof. The statement that $K(m)$ is a cone is trivial. In the following, we show that it is also convex. If both tensors D_1 and D_2 are in $K(m)$, then both $g(x)^r d_1(x)$ and $g(x)^s d_2(x)$ are sums of squares for some integers r and s respectively. Since $g(x)$ is a sum of squares and the product of two SOS polynomials is again SOS, we see that $g(x)^{r+s}(d_1(x) + d_2(x))$ is SOS. Consequently, by (1), we see that $D_1 + D_2 \in K(m)$.

Next, we prove the chain of inclusions. The inclusion int$\mathcal{S}(m) \subseteq K(m)$ follows from Theorem 3.2 immediately. Hence, it needs only to prove the other one. Let $D \in K(m)$, then $g(x)^r d(x)$ is SOS for some r by the definition of $K(m)$ given in (6). Consequently, $g(x)^r d(x) \geq 0$ for all $x \in \Re^n$. Since $g(x) > 0$ for any nonzero $x \in \Re^n$, we have that $d(x) \geq 0$ for any nonzero $x \in \Re^n$. Hence, D is positive semidefinite and $D \in \mathcal{S}(m)$ by (2). The proof is complete. □

Theorem 3.4
$K(m)^*$=$\mathcal{S}(m)^*$.

Proof. From Theorem 3.3 and the result in [18, Page 121] which is known as bi-polar theorem, we have that $K(m)^{**}$=cl$(K(m))$=$\mathcal{S}(m)$=$\mathcal{S}(m)^{**}$. Again by the same result in [18, Page 121], we have $K(m)^*$=$K(m)^{***}$=$\mathcal{S}(m)^{***}$=$\mathcal{S}(m)^*$. The proof is complete. □

From Theorem 3.3, we obtain that cl$(K(m)) = \mathcal{S}(m)$. Thus, the following optimization problem is an approximation of the optimization problem (3):

$$\begin{array}{ll} \inf & A_0 \bullet X \\ \text{s.t.} & A_i \bullet X = b_i, \ \forall i = 1, 2, \dots, p, \\ & X \in K(m). \end{array} \tag{7}$$

Hence, instead of (3), we can solve optimization problem (7) by some numerical algorithms to find a feasible solution such that the distance between its objective value and the optimal value of (7), hence (3), being within a given precision. Moreover, we can choose this feasible solution as an approximation solution of (3).

In the following, we discuss the promised sequential SDPs method for solving the optimization problem (7) and prove the convergent result. Let $\mathcal{N}$ denote the set of all nonnegative integers, and $\mathcal{S}^h_+$ denote the set of all $h \times h$ real positive semidefinite symmetric matrices. For an $s \in \mathcal{N}$, the total number of monomials with its degree being s in the variables x_i $(i \in \{1, \dots, n\})$ is denoted by $t(s)$; it is C^s_{n+s-1}. The corresponding monomials vector is denoted by $v_s(x)$, which is ordered lexicographically. Then, $v_s(x)$ can be written as

$$v_s(x) := \left(x_1^s, x_1^{s-1}x_2, \dots, x_2^s, \dots, x_n^s\right)^\top. \tag{8}$$

Consequently, the following result is immediate.

Theorem 3.5
A homogenous polynomial p of positive degree $2s \in \mathcal{N}$ is SOS if and only if $p(x) = v_s(x)^\top H v_s(x)$, where $v_s(x)$ is given by (8) and $H \in \mathcal{S}^{t(s)}_+$.

In Theorem 3.5, the matrix H is given by

$$H := \begin{pmatrix} h_{2s,0,\dots,0} & h_{2s-1,1,\dots,0} & h_{2s-1,0,\dots,1} & \cdots & h_{s,0,\dots,0,s} \\ h_{2s-1,1,\dots,0} & h_{2s-1,2,\dots,0} & h_{2s-2,1,1,\dots,0} & \cdots & h_{s-1,1,\dots,s} \\ \cdots & & & & \\ h_{s,0,\dots,s} & h_{s-1,1,\dots,s} & h_{s-1,0,\dots,s+1} & \cdots & h_{0,0,\dots,0,2s} \end{pmatrix}.$$

The subscript of h corresponds to the multidegree of the monomial in $v_{2s}(x)$. For example, $h_{2s-1,1,\ldots,0}$ corresponds to the monomial $x_1^{2s-1}x_2$. When $n=3$, we have

$$H=\begin{pmatrix} h_{4,0,0} & h_{3,1,0} & h_{3,0,1} & h_{2,2,0} & h_{2,1,1} & h_{2,0,2}\\ h_{3,1,0} & h_{2,2,0} & h_{2,1,1} & h_{1,3,0} & h_{1,2,1} & h_{1,1,2}\\ h_{3,0,1} & h_{2,1,1} & h_{2,0,2} & h_{1,2,1} & h_{1,1,2} & h_{1,0,3}\\ h_{2,2,0} & h_{1,3,0} & h_{1,2,1} & h_{0,4,0} & h_{0,3,1} & h_{0,2,2}\\ h_{2,1,1} & h_{1,2,1} & h_{1,1,2} & h_{0,3,1} & h_{0,2,2} & h_{0,1,3}\\ h_{2,0,2} & h_{1,1,2} & h_{1,0,3} & h_{0,2,2} & h_{0,1,3} & h_{0,0,4} \end{pmatrix}.$$

We see that some monomials correspond to multiple entries of the matrix H.

Theorem 3.5 is fundamental, with which we can parameterize the cone $K(m)$; and hence, get the sequential SDPs. To this end, we introduce operators $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{M}$ first.

- $\mathcal{V}$: For any positive $s\in\mathcal{N}$, define an operator $\mathcal{V}:\Re^{t(s)\times t(s)}\to\Re^{t(2s)}$ such that $[\mathcal{V}(H)]_i$ is the coefficient of the i-th monomial in the vector $v_{2s}(x)$ of the polynomial $v_s(x)^\top Hv_s(x)$, for any $H\in\Re^{t(s)\times t(s)}$.
- $\mathcal{W}$: For any positive $s\in\mathcal{N}$, define an operator $\mathcal{W}$ such that it maps a homogenous polynomial p of degree s to a vector in $\Re^{t(s)}$ satisfying $p(x)=v_s(x)^\top\mathcal{W}(p)$.
- $\mathcal{M}$: For any positive $s\in\mathcal{N}$, we defined an operator $\mathcal{M}:\Re^{t(2s)}\to\mathcal{T}(2s)$. To this end, some notation is necessary. For the $t(2s)$ independent elements of a tensor $D\in\mathcal{T}(2s)$, we order them in a vector $w_{2s}(D)$ use the lexicographic order of the indices of the elements of D. Actually, the monomials associated to this order are in the same order of the monomials of $d(x)$ in the vector $v_{2s}(x)$. We call the i-th element in the vector $w_{2s}(D)$ the i-th independent element of D. Define operator $\mathcal{M}:\Re^{t(2s)}\to\mathcal{T}(2s)$, such that for any $y\in\Re^{t(2s)}$, $\mathcal{M}$ sends $y_i/u(i)$ to the i-th independent element of the tensor $\mathcal{M}(y)\in\mathcal{T}(2s)$, where $u(i)$ is the total number of the i-th independent element of $\mathcal{M}(y)$ among its n^{2s} elements.

Obviously, all the operators $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{M}$ are linear, and they are dependent on the integer s. However, for the convenience of the subsequent discussion, we omit the parameter s and the value of s will be clear from the content. It is also easy to see that operators $\mathcal{W}$ and $\mathcal{M}$ are invertible.

With the operators defined above and Theorem 3.5, the set $K(m)$ can be written as

$$K(m):=\left\{D\in\mathcal{T}(m)\ \middle|\ \mathcal{W}[g(x)^s d(x)]=\mathcal{V}(Q),\ s\in\mathcal{N},\ \ Q\in\mathcal{S}_+^{t(s+m/2)}\right\}. \tag{9}$$

By using the description of $K(m)$ given in (9) and the optimization problem (7), an approximation optimization problem of the TCLP can be given by

$$\begin{array}{ll} \inf & A_0\bullet X\\ \text{s.t.} & A_i\bullet X=b_i,\ \forall i=1,2,\ldots,p,\\ & \mathcal{V}(Q)=\mathcal{W}[g(x)^s d(x)],\\ & \mathcal{M}\circ\mathcal{W}(d(x))=X,\\ & s\in\mathcal{N},\ Q\in\mathcal{S}_+^{t(s+m/2)}. \end{array} \tag{10}$$

In this problem, the variables are the tensor X, the integer s and the positive semidefinite matrix Q. There are altogether $p+t(m+2s)$ linear constraints, since we can embed the linear constraints $\mathcal{M}\circ\mathcal{W}(d(x))=X$ into the constraints $\mathcal{V}(Q)=\mathcal{W}[g(x)^s d(x)]$ by parameterizing $d(x)$ with X.

The optimization problem (10) is still hard to solve, since it involves the positive semidefinite matrix variable Q with indeterminate size $t(s+m/2)$. However, for every fixed $s\in\mathcal{N}$, the optimization problem (10) becomes

$$SDP(s)\qquad \begin{array}{ll} \inf & A_0\bullet X\\ \text{s.t.} & A_i\bullet X=b_i,\ \forall i=1,2,\ldots,p,\\ & \mathcal{V}(Q)=\mathcal{W}[g(x)^s d(x)],\\ & \mathcal{M}\circ\mathcal{W}(d(x))=X,\\ & Q\in\mathcal{S}_+^{t(s+m/2)}. \end{array} \tag{11}$$

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In the constraints of the problem (11), $\mathcal{V}(Q)$ is linear in the variable Q, $\mathcal{W}[g(x)^s d(x)]$ is linear in the coefficients of $d(x)$ as s is fixed, and $\mathcal{M} \circ \mathcal{W}(d(x))$ is linear in the coefficients of $d(x)$ as $\mathcal{M} \circ \mathcal{W}$ is linear. Consequently, the problem (11) is a linear semidefinite programming problem (SDP). Hence, it can be solved efficiently in polynomial time [20].

For every integer $s \in \mathcal{N}$, we define

$$K(m)_s := \left\{ D \in \mathcal{T}(m) \;\middle|\; \mathcal{W}[g(x)^s d(x)] = \mathcal{V}(Q),\ Q \in \mathcal{S}_+^{t(s+\text{[illegible]}/2)} \right\}. \tag{12}$$

Consequently, for every $s \in \mathcal{N}$, we get an SDP which is an approximation to the original TCLP as $K(m)_s$ is an approximation of $K(m)$ and hence $S(m)$. Thus, we get a sequential SDPs as $\{SDP(s),\ \ s \in \mathcal{N}\}$. The SDP(s) is called the s-th order relaxation of the TCLP. It is hence expected that for solve this sequential SDPs along $s \to \infty$, the optimal values of the SDPs converge to the original optimal value of the TCLP. Note that by Theorem 2.1 and Assumption 2.1, the TCLP has finite optimal value and is solvable. In this following, we will prove that the optimal values of the SDPs do converge to the optimal value of the TCLP. Then, this numerical scheme works, at least theoretically. It is called the sequential SDPs method for the TCLP. We abbreviate it as T-SSM.

We first show that the approximations of $K(m)_s$ ($s \in \mathcal{N}$) to $K(m)$ form an ascending chain, and converge to $K(m)$.

Theorem 3.6
Let $K(m)_s$ be defined by (12), then we have

(i) $K(m)_s \subseteq K(m)_{s+1}$ for every $s \in \mathcal{N}$, and
(ii) $\lim_{s\to\infty} K(m)_s = \bigcup_{s=0}^{\infty} K(m)_s = K(m)$.

Proof. We show that $K(m)_s \subseteq K(m)_{s+1}$ and the other results follow from the definitions of $K(m)_s$ and $K(m)$ immediately.

Let $s \in \mathcal{N}$ be fixed. Suppose that $D \in K(m)_s$. Then, $g(x)^s d(x)$ is SOS by Theorem 3.5 and (12). This, together with the fact that $g(x)$ is SOS, implies that $g(x)^{s+1} d(x)$ is SOS. Consequently, by Theorem 3.5 again, $D \in K(m)_{s+1}$. □

It is interesting to investigate whether this ascending chain stops finitely or not.

Theorem 3.7
Let $K(m)_s$ be defined by (12), then we have

(i) $K(m)_s^* \supset K(m)_{s+1}^*$ for every $s \in \mathcal{N}$, and
(ii) $\lim_{s\to\infty} K(m)_s^* = \cap_{s=0}^{\infty} K(m)_s^* = K(m)^*$.

Proof. The results follow from the definitions of $K(m)_s^*$ and $K(m)^*$, and Theorem 3.6 immediately. □

Theorem 3.6 indicates that the optimal value of the SDP(s) given in (11) tends to the one of (7), hence the original TCLP, as s tends to ∞. We show the detailed proof in the following theorem.

Theorem 3.8
Suppose that Assumption 2.1 holds. Denote the optimal value of the SDP(s) (11) by $p^{(s)}$. Then, we have $p^{(s)} \to p^*$ as $s \to \infty$.

Proof. Note that the problem SDP(s) (11) is just the problem (10) with the cone $K(m)$ in (10) being replaced by the cone $K(m)_s$. Since Assumption 2.1 holds, there exists an $X \in \mathrm{int}\mathcal{S}(m)$ which is feasible to (3), and also to (10). In fact, by Theorem 3.3, we have $X \in K(m)$. Thus, by Theorem 3.6, it follows that there exists a positive integer s_0 such that $X \in K(m)_{s_0}$. Furthermore, it is easy to see from Theorem 3.6 that $\infty > p^{(s_0)} \geq p^{(s)}$ when $s \geq s_0$.

From Theorem 3.6, we have $K(m)_s \subseteq K(m)$ for all $s \in \mathcal{N}$. Then, $K(m) \subseteq \mathcal{S}(m)$, together with the fact that both (3) and (4) are strictly feasible (hence solvable), implies that $p^{(s)} \geq p^* > -\infty$. By Theorem 3.6, we know that $K(m)_s \to K(m)$ as $s \to \infty$. Hence, $p^{(s)} \to p^*$ as $s \to \infty$. Actually, (3) is solvable by Assumption 2.1 and Theorem 2.1. Since the relative interior of the feasible set of (3) is nonempty, there exists a sequence of points $\{X^{(\text{[illegible]})}\}$ in the relative interior of the feasible set of (3), which are also in the relative interior of the feasible set of (10) by Lemma 1, converges to an optimal solution of (3). Denote the corresponding objective value of $X^{(k)}$ by p_k. Since $p_k \to p^*$, for any $\varepsilon > 0$, there exists a $k(\varepsilon)$ such that for any $k \geq k(\varepsilon)$, $p_k - p^* \leq \varepsilon$. For $X^{(k(\varepsilon))} \in K(m)$, there exists an $s(\varepsilon)$ such that $X^{(k(\varepsilon))} \in K(m)_{s(\varepsilon)}$ by Theorem 3.6. Hence, $p^{(s(\varepsilon))} - p^* \leq \varepsilon$. Consequently, we have that $p^{(s)} \to p^*$ as $s \to \infty$.

The proof is complete. □

Therefore, by Theorem 3.8 we can, theoretically, solve a sequence of SDPs given as (11) by increasing s to obtain an approximation solution of the original TCLP up to a priori fixed precision. For every problem SDP(s), we can find a feasible $X^{(s)} \in K(m)_s \subset K(m) \subset \mathcal{S}(m)$ whose objective value is sufficiently close to $p^{(s)}$ by solving the SDP(s) (11). Since $p^{(s)} \to p^*$, for a sufficiently large s, $p^{(s)}$ is sufficiently close to p^*, and hence $X^{(s)} \in K(m)_s \subset K(m) \subset \mathcal{S}(m)$ severs as an approximation solution to (3). Note that such an $X^{(s)}$ is feasible to (3) as well. While, as the size of the resulting SDPs increases drastically, it is impossible in practical computation to increase s arbitrarily large due to the present ability to solve SDP. Moreover, we do not know in advance for which s, $p^{(s)}$ would be within the given accuracy region of p^*. Consequently, Theorem 3.8 only serves as a theoretical result.

4. FINDING THE EXTREME Z-EIGENVALUES

In this section, we consider how to find the extreme Z-eigenvalues of an even order symmetric tensor. Such a problem is crucial in many applications [14, 16]. We transform the extreme Z-eigenvalue problem into a special TCLP, and then use the T-SSM to solve it. This section partitions into two subsections. The concept of the Z-eigenvalues of tensors and the reformulation of the extreme Z-eigenvalue problem are given in Subsection 4.1, and the numerical computation is given in the other subsection.

4.1. *The extreme Z-eigenvalues*

In this subsection, we reformulate the extreme Z-eigenvalues of an even order symmetric tensor into a TCLP. We first recall the concept of the Z-eigenvalues of tensors. For extensive discussions on eigenvalues of tensors, please refer to [12–14, 16, 21–26] and references therein. For an m-th order n-dimensional symmetric tensor D and a vector $x \in \Re^n$, we denote by Dx^{m-1} a vector in $\Re^n$ with its i-th coordinate being $\sum_{i_2,\dots,i_m=1}^{n} d_{ii_2\dots i_m} x_{i_2} \cdots x_{i_m}$, and Dx^m the inner product of the vectors x and Dx^{m-1}. Given the tensor D, a Z-eigenvalue pair $(\lambda, x) \in \Re \times \Re^n$ means a solution to the following system

$$\begin{cases} Dx^{m-1} = \lambda x, \\ x^T x = 1. \end{cases}$$

Obviously, $\lambda = Dx^m$ for a Z-eigenvalue pair (λ, x) of D. In many applications, it is crucial to compute the largest or the smallest Z-eigenvalues of a given tensor D.

Note that the largest and the smallest Z-eigenvalues of D are the optimal values of the following optimization problems:

$$\begin{array}{ll} \max & Dx^m \\ s.t. & x^T x = 1, \end{array} \tag{13}$$

and

$$\begin{array}{ll} \min & Dx^m \\ s.t. & x^T x = 1, \end{array} \tag{14}$$

respectively.

In the following, we will focus on optimization problem (14), since (13) can be easily handled if we know a method to solve (14). We first notice that (14) is equivalent to

$$\begin{array}{ll} \max & \gamma \\ s.t. & Dx^m \geq \gamma, \quad \forall x \in \{x \in \Re^n \mid x^T x = 1\}. \end{array} \tag{15}$$

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Define an m-th order n-dimensional tensor E as $E = I \otimes I \otimes \cdots \otimes I$ (for $m/2$ times), where I is the $n \times n$ identity matrix. So, $Dx^m \geq \gamma, \forall x \in \{x \in \Re^n \mid x^\top x = 1\}$ is the same as $D - \gamma E \in \mathcal{S}(m)$, which is the same to say $D - \gamma E \succeq \mathbf{0}$. Hence, (15) is a special TCLP given by

$$\begin{array}{ll} \max & \gamma \\ s.t. & D - \gamma E \succeq \mathbf{0}, \end{array}$$

which is further equivalent to

$$\begin{array}{ll} \min & \gamma \\ s.t. & D + \gamma E \succeq \mathbf{0}. \end{array} \tag{16}$$

Since the optimization problem (16) is a special TCLP, it can be solved by using the T-SSM. For any $s \in \mathcal{N}$, by replacing the constraint $D + \gamma E \succeq \mathbf{0}$ with $D + \gamma E \succeq \mathbf{0} \in K(m)_s$, we get the s-th order relaxation problem of (16). Let us write out it explicitly as:

$$SDP(s) \qquad \begin{array}{ll} \inf & \gamma \\ \text{s.t.} & \mathcal{V}(Q) - \gamma \mathcal{W}(g(x)^{s+m/2}) = \mathcal{W}[g(x)^s d(x)], \\ & Q \in \mathcal{S}_+^{t(s+m/2)}. \end{array} \tag{17}$$

Note that for any given tensor $D \in \mathcal{T}(m)$, the vector $\mathcal{W}[g(x)^s d(x)]$ is a constant vector. Here are the main parameters of the SDP(s) above.

- It has $t(m+2s)$ linear constraints.
- The positive semidefinite matrix variable is of dimension $t(m/2+s)$.

Recall that for n dimensional tensors, $t(k) = C_{n+k-1}^{k}$ for $k \in \mathcal{N}$. Hence, the size of the SDP(s) increases drastically when (m, n, s) increases. In the tables of the next subsection, we show the number of the linear constraints (**Lin**) and the dimension of the matrix variable (**Dim**) for every computed case.

4.2. *Numerical computation*

In this subsection, we present some preliminary numerical results for solving (16) by using the T-SSM. We just follow the discussions in the above subsection to develop the code. We implement the optimization problem (17) in Matlab on our PC. The PC is with CPU of 2.4 GHz and RAM of 2.0 GB. We use SDPT3 [27] to solve the resulting conic linear programming problem which has both free variable and positive semidefinite variable.

Firstly, we test three examples of tensors to show that the T-SSM can work very well.

Example 4.1
The first example is a 6*-th order 3-dimensional tensor. The corresponding tensor is made up of the coefficients of the following polynomial (Stengle's form) taken from [28]:*

$$h_{Ste}(x) = x_1^3 x_3^3 + (x_2^2 x_3 - x_1^3 - x_1 x_3^2)^2.$$

It is well known that h_{Ste} is a positive semidefinite polynomial but not SOS. We denote the corresponding tensor as T_{Ste}. It is easy to see that the Z-eigenvalue system of T_{Ste} is

$$\left\{ \begin{array}{rcl} \frac{1}{6}[3x_1^2 x_3^3 + 2(x_2^2 x_3 - x_1^3 - x_1 x_3^2)(-3x_1^2 - x_3^2)] & = & \lambda x_1, \\ \frac{1}{6}[2(x_2^2 x_3 - x_1^3 - x_1 x_3^2)(2x_2)] & = & \lambda x_2, \\ \frac{1}{6}[3x_1^3 x_3^2 + 2(x_2^2 x_3 - x_1^3 - x_1 x_3^2)(x_2^2 - 2x_1 x_3)] & = & \lambda x_3, \end{array} \right. \tag{18}$$

where $x_1^2 + x_2^2 + x_3^2 = 1$. From (18), we see that $\lambda = 0$ is a Z-eigenvalue of T_{Ste} with the corresponding Z-eigenvector being $(0,1,0)^\top$. Furthermore, $\lambda = 0$ is the smallest Z-eigenvalue of T_{Ste} since $h_{Ste}(x) \geq 0$. We use the T-SSM to find approximations for the smallest Z-eigenvalue of T_{Ste} with $s \in \{0,1,2,3\}$. The computed results are listed in Table I. In the table, **it** *means the iteration number of the SDP solver,* **cpu** *means the total time in seconds spent for both setting up the problem and solving it,* **opt** *means the approximation value computed, and* **vol** *means the norm of the violation of the constraints of the approximation solution. From the table, we see that the method can find a good approximation solution, even for the zero order relaxation.*

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Table I. Computation results for Example 4.1

m	n	s	Lin	Dim	it	cpu	opt	vol
6	3	0	28	10	15	1.203	-1.7466e-005	1.5047e-011
6	3	1	45	15	19	1.563	-1.3928e-009	1.4234e-010
6	3	2	66	21	20	2.000	-6.3799e-010	1.7391e-010
6	3	3	91	28	23	3.453	-2.5448e-010	3.5135e-010

Example 4.2
In this example, fourth order four-dimensional positive semidefinite symmetric tensors are randomly generated. To generate a positive semidefinite symmetric tensor, we first randomly generate a vector x. Then form a fourth order rank one tensor $x \otimes \cdots \otimes x$. By definition, this tensor is positive semidefinite. We take a sum of 15 such randomly generated rank one tensors to form a positive semidefinite symmetric tensor. We test the T-SSM with such tensors. In the following, we display the computation results for one of the simulations as an example. The coefficients of the polynomial $d(x)$ in the order of the monomials vector $v_4(x)$ are put in the following vector

$$\begin{gathered}(0.6795, 0.5696, 0.7268, -0.4051, 2.4625, 1.4716, 2.1854, 2.0333, -1.1188, 2.0347,\\ 0.6176, 3.9182, -0.7433, 2.8720, 2.4059, -1.1220, 0.2469, -0.5837, 0.9269, -0.6628,\\ 1.2701, 0.7049, 1.3213, 2.5168, -0.4932, 1.9971, 0.2712, 0.6999, -0.9938, 0.9199,\\ 0.7409, -0.3241, 1.6088, -1.0541, 0.5114).\end{gathered}$$

The computed approximation values of the s-th order relaxations for $s \in \{0, 1, 2, 3, 4\}$ are the same 0.1706.

Example 4.3
This example is similar to Example 4.2, except for fourth order five-dimensional tensors. The taken example is as follows. The coefficients of the polynomial $d(x)$ in the order of monomials vector $v_4(x)$ are put in the following vector

$$\begin{gathered}(2.3525, -0.7663, -2.0847, 1.8713, 2.4123, 1.0810, 0.0019, 1.8585, 0.3500, 2.8223,\\ -1.2591, -1.1789, 4.1955, 4.5213, 4.3148, -0.2060, -0.3665, -0.4879, -0.5095, 0.0551,\\ 0.2353, 0.1936, -1.2513, -0.5995, 0.0117, -1.5205, 1.4961, -0.7163, -1.2764, 0.0210,\\ -0.7210, 1.2239, 0.3303, 2.9108, 1.2443, 0.3552, 0.8959, 0.5733, -0.8267, 1.4233,\\ 0.3718, -2.4382, 0.8745, -0.4801, 1.8166, 0.7590, 0.4976, -2.3369, 0.4780, -0.9982,\\ 2.1721, 0.4117, 0.0751, 1.0824, -0.5833, 0.6672, -0.6814, -0.2190, 1.2729, -1.2632,\\ 2.2729, -0.4839, -0.7655, -2.1708, -1.9507, 0.9113, 0.6238, 3.8270, 2.7994, 2.2072).\end{gathered}$$

The computed approximation values of the s-th order relaxations for $s \in \{0, 1, 2, 3\}$ are the same 0.0508.

Secondly, we present in the following three systems of preliminary numerical results for some randomly generated symmetric tensors.

(I) We compare the accuracy of the T-SSM with the roots finding method proposed in [14] by randomly generated 4-th order 3-dimensional symmetric tensors. The reason why we use 3-dimensional tensors is that the roots finding method can only be applied to 3-dimensional tensors. We use $s = 0$ in the numerical computation, since Hilbert's result say that $K(4)_0 = K(4)$ in this case [28]. The numerical results are listed in Table II. In the table, **r** means that the corresponding tensor is equal to a sum of r positive semidefinite rank one tensors generated similarly to Example 4.2; while $r = 0$ means that every entry of the corresponding tensor is randomly generated, hence it is indefinite in general. We list ten simulations for $r = 0$, and four for each other values of r. **it**, **opt** and **vol** are the same as those in Example 4.1. **cpu1** is the the total cpu time of the T-SSM in seconds spent for both setting up the problem and

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solving it; and **cpu2** for the roots finding method. Finally, **Root** is the smallest Z-eigenvalue computed through the roots finding method.

(II) We test the numerical behaviors of the T-SSM for randomly generated positive semidefinite symmetric tensors. The positive semidefinite symmetric tensors are generated similarly to those in Example 4.2. The results are listed in Table III. The parameters $(\mathbf{m}, \mathbf{n}, \mathbf{r}, \mathbf{s})$ of the tested cases are clear from the table. For each case, we simulate ten times to get the average number of iterations (**it**), the average cpu time spent for both setting up the problem and solving it (**cpu**), the average approximation value computed by the T-SSM (**opt**) and the average violation of the constraints (**vol**).

(III) We test the T-SSM for positive semidefinite symmetric tensors with the smallest Z-eigenvalues being zeros. The tensors are generated similarly to those in Example 4.2 with the summation of the rank one tensors being the number of its dimension minus one. Consequently, the generated tensors are positive semidefinite and with the smallest Z-eigenvalues being zeros. We call that the proposed method can successfully solve the extreme Z-eigenvalue problem if the approximation value computed has absolute value less than 10^{-8}. In this case, we put the smallest relaxation order **s** into the corresponding cross of $\mathbf{m}$ and $\mathbf{n}$ in Table IV. If the problem setting is out of the memory of our PC, we put a "-" in the corresponding cross.

Table II. Comparisons of roots finding method (Lin=15 and Dim=6)

r	it	cpu1	opt	vol	cpu2	Root
0	11	1.094	-8.4506e-001	4.3178e-013	1.125	-8.4506e-001
0	11	0.969	-7.2252e-001	1.3786e-011	1.078	-7.2252e-001
0	11	1.078	-4.5716e-001	4.4230e-014	1.094	-4.5716e-001
0	12	1.016	-3.8369e-001	4.9058e-014	1.125	-3.8369e-001
0	11	0.938	-7.4017e-001	3.4039e-013	1.172	-7.4017e-001
0	12	1.109	-6.3362e-001	1.3135e-011	1.109	-6.3362e-001
0	11	1.063	-5.8289e-001	4.4416e-011	1.109	-5.8289e-001
0	12	1.125	-2.5095e-001	5.3866e-011	1.156	-2.5095e-001
0	11	1.109	-2.5795e-001	8.6319e-015	1.109	-2.5795e-001
0	12	1.188	-1.5701e-001	2.0914e-014	2.938	-1.5701e-001
1	19	1.156	-5.1666e-009	2.1909e-010	1.125	-2.3823e-022
1	16	1.063	-3.8401e-009	2.6203e-011	1.094	-1.3764e-021
1	17	1.156	-2.8225e-009	5.2696e-011	1.125	-4.3368e-019
1	17	1.219	-3.0147e-009	3.6141e-011	1.094	-8.6736e-018
5	11	1.141	1.2166e-002	1.8395e-014	1.094	1.2166e-002
5	15	1.281	6.2471e-002	1.2187e-012	1.172	6.2471e-002
5	11	1.141	8.0927e-002	9.7049e-015	1.156	8.0927e-002
5	11	1.188	8.8725e-002	3.2066e-013	1.109	8.8725e-002
10	11	1.078	3.9850e-001	2.0244e-014	1.094	3.9850e-001
10	12	1.234	2.2917e-001	8.8872e-015	1.141	2.2917e-001
10	13	1.281	3.1400e-001	4.0493e-015	1.109	3.1400e-001
10	12	1.172	8.1200e-002	3.4809e-014	1.172	8.1200e-002
15	12	1.281	2.5545e-001	9.9169e-012	1.203	2.5545e-001
15	11	1.266	1.3178e-001	1.5459e-014	1.156	1.3178e-001
15	11	1.234	2.8711e-001	2.2521e-014	1.094	2.8711e-001
15	11	1.234	2.1930e-001	8.2069e-014	1.125	2.1930e-001

From Tables II-IV, we have the following observations:

- From Table II, we see that the T-SSM can find the smallest Z-eigenvalues with high accuracy for three dimensional tensors. For the rows corresponding to rank one tensors (the optimal

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Table III. Positive semidefinite tensors

m	n	r	s	Lin	Dim	it	cpu	opt	vol
6	3	5	0	28	10	15.3	0.822	2.2712e-003	1.2264e-012
6	3	5	1	45	15	15.8	1.052	1.6621e-003	9.8237e-013
6	3	5	2	66	21	17.2	1.522	1.8533e-003	9.5328e-012
8	3	5	0	45	15	17.5	1.261	2.4100e-004	4.1855e-010
8	3	5	1	66	21	18.4	1.719	3.6409e-005	3.4950e-010
8	3	5	2	91	28	19.5	2.802	8.8981e-004	4.8231e-011
10	3	5	0	66	21	18.1	2.077	1.7700e-004	4.7274e-011
10	3	5	1	91	28	18.8	3.267	5.2268e-005	4.5275e-009
10	3	5	2	120	36	19.9	4.622	7.3900e-006	1.2359e-009
16	3	5	0	153	45	23.2	11.063	2.2106e-007	1.7308e-008
16	3	5	1	190	55	23.3	12.220	4.9569e-009	1.3141e-008
16	3	10	0	153	45	23.9	16.255	3.3441e-005	2.5583e-010
16	3	10	1	190	55	26.1	17.430	1.2871e-004	2.5695e-010
20	3	10	0	231	66	27.7	38.175	4.0633e-007	6.7615e-009
24	3	15	0	325	91	33.1	113.817	5.1880e-007	2.3918e-009
4	4	5	0	35	10	14.7	0.892	7.8790e-003	3.7801e-012
4	4	5	1	84	20	16.1	1.852	5.5990e-003	7.8961e-012
4	4	5	2	165	35	19.2	5.819	5.1730e-003	9.7661e-012
4	6	10	0	126	21	16.4	6.377	7.8242e-003	4.1298e-012
4	6	10	1	462	56	18.8	35.897	1.9803e-002	1.1370e-011
4	8	15	0	330	36	17.9	53.856	1.4943e-002	1.0850e-011
6	4	15	0	84	20	16.6	4.314	2.0749e-002	1.0611e-012
6	5	15	0	210	35	18.0	21.472	1.9167e-002	1.9113e-011
6	6	15	0	462	56	20.8	109.661	5.4306e-003	8.6242e-012

Table IV. The order of relaxation

$m\backslash n$	3	4	5	6	7	8	9	10
4	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	-	-
8	0	0	0	0	-	-	-	-
10	0	0	0	-	-	-	-	-
12	0	0	-	-	-	-	-	-
14	0	0	-	-	-	-	-	-
16	0	0	-	-	-	-	-	-
18	0	-	-	-	-	-	-	-
20	0	-	-	-	-	-	-	-
22	0	-	-	-	-	-	-	-
24	1	-	-	-	-	-	-	-

value would be zero), we see that the **opt** is of magnitude 10^{-9}, while that for **Root** is 10^{-20}. That is because the former is based on SDPT3 which terminates once the duality gap being smaller than 10^{-8}, while the latter is a direct method. However, both are with high accuracy to the true value zero.

- From Table III, we see that the T-SSM can find the smallest Z-eigenvalues of positive semidefinite symmetric tensors in few iterations and cpu time. Note that for the case $(m, n) =$

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$(24, 3)$ in this table, the number of linear constraints of the corresponding SDP is 325. For $(m, n) = (6, 6)$, the number of linear constraints of the corresponding SDP is 462. Hence, it takes a bit more time to solve these problems.

- From Table IV, we see that many cases can be solved with low order relaxation ($s = 0$ mostly). For the failed cases, take $(m, n) = (6, 9)$ as an example. The number of linear constraints of the SDP is 3003 and the dimension of the positive semidefinite matrix variable is 165. Such a problem size for SDP is not small. This problem is out of the reach of our PC.

From the numerical results presented above, we see that the T-SSM performs quite well. Hence, it would serve as a research tool for analyzing tensors and its related problems.

5. FINAL REMARKS

In this paper, we introduced the TCLP which is a generalization of the STLP proposed by Qi and Ye [1]. For the numerical method to the TCLP, we proposed a sequential SDPs method to solve the TCLP. It is abbreviated as T-SSM. In particular, we reformulated the extreme Z-eigenvalue problem for even order symmetric tensors as a special TCLP. Some preliminary numerical results for finding the smallest Z-eigenvalue of an even order symmetric tensor based on the T-SSM were reported. The numerical results showed the potential application for both practical use and theoretical research of the T-SSM in various applications.

There are also some problems need to be further studied: (i) The T-SSM is essentially an SOS relaxation method [29, 30]. It is well-known that such methods cannot handle larger size problems. Then, how to improve the performance of the T-SSM? (ii) The extreme Z-eigenvalue problem for odd order tensors are not included in the framework of the TCLP, since we cannot define positive semidefiniteness for odd order tensors. (iii) Whether it is possible to develop some techniques based on the T-SSM to find out all the Z-eigenvalues of a given tensor or not?

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