Some Remarks on the Blow-up Rate for the 3D Magnetic Zakharov System

Zaihui Gan[1,2] * **Yansheng Ma**[3,4] † **Ting Zhong**[2] ‡

1 Center for Applied Mathematics, Tianjin University, Tianjin, 300072, China

2 College of Mathematics and Software Science, Sichuan Normal University, Chengdu, 610068, China

3 School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, China

4 Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China

Abstract: We continue our study [8] on the Cauchy problem for the three-dimensional magnetic Zakharov system

$$\begin{cases} i\mathbf{E}_t + \Delta\mathbf{E} - n\mathbf{E} + i(\mathbf{E}\wedge\mathbf{B}(\mathbf{E})) = 0, \\ n_t = -\nabla\cdot\mathbf{V}, \\ \mathbf{V}_t = -\nabla n - \nabla|\mathbf{E}|^2, \\ \Delta\mathbf{B} - i\eta\nabla\times\nabla\times(\mathbf{E}\wedge\overline{\mathbf{E}}) + \delta\mathbf{B} = 0, \end{cases} \tag{ZSM}$$

with initial data

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \quad n(0,x) = n_0(x), \quad \mathbf{V}(0,x) = \mathbf{V}_0(x). \tag{ZSM-1}$$

Let $(\mathbf{E}, n, \mathbf{V}) \in \mathcal{C}\left([0,T), H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)\right)$ be a blow-up solution to the Cauchy problem (ZSM)-(ZSM-1), and let $T<\infty$ be its blow-up time. Then $(\mathbf{E}, n)$ satisfies the space-time integral estimate

$$\int_0^T \left[\left(\int_{\mathbb{R}^3}|n(t,x)|^q dx\right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^3}|\mathbf{E}(t,x)|^{2q}dx\right)^{\frac{1}{q}}\right]^\gamma dt = +\infty, \tag{E-1}$$

where $\gamma > \frac{1}{\varepsilon}$, $\varepsilon \in \left(0, \frac{1}{4}\right]$ and $q = \frac{3}{2(1-\varepsilon)} \in \left(\frac{3}{2}, 2\right]$. The estimate (E-1) implies that, for $a<1$,

$$\sup_{t\in[0,T)}\left[(T-t)^{a\varepsilon}\left(|n(t)|_{L^q(\mathbb{R}^3)} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}\right)\right] = +\infty.$$

In particular, if $(\mathbf{E}, n, \mathbf{V})$ is a radially symmetric solution to (ZSM)-(ZSM-1), then

$$\sup_{t\in[0,T)}\left[(T-t)^{a\varepsilon}\left(|n(t)|_{L^q(\mathbb{D})} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{D})}\right)\right] = +\infty,$$

where $a \in (0, \frac{1}{3})$ and $\mathbb{D} = \left\{x\in\mathbb{R}^3 : |x| < 1\right\}$.

Key Words: Magnetic Zakharov system; Blow-up; Space-time integral estimate
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*E-mail: ganzaihui2008cn@tju.edu.cn, ganzaihui2008cn@gmail.com
†E-mail: mays538@nenu.edu.cn
‡E-mail: zhongting89@sina.cn

1 Introduction

The main purpose of this work is to study the blow-up rate for a blow-up solution of the magnetic Zakharov system. We establish space-time integral estimates on the blow-up solution. Specifically, in the present paper we continue our study [8] on the 3D Magnetic Zakharov system

$$\begin{cases} i\mathbf{E}_t + \Delta\mathbf{E} - n\mathbf{E} + i(\mathbf{E}\wedge\mathbf{B}(\mathbf{E})) = 0, \\ n_t = -\nabla\cdot\mathbf{V}, \\ \mathbf{V}_t = -\nabla n - \nabla|\mathbf{E}|^2, \\ \Delta\mathbf{B} - i\eta\nabla\times\nabla\times(\mathbf{E}\wedge\overline{\mathbf{E}}) + \delta\mathbf{B} = 0, \end{cases} \tag{1.1}$$

with initial data

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \quad n(0,x) = n_0(x), \quad \mathbf{V}(0,x) = \mathbf{V}_0(x), \tag{1.2}$$

where $(\mathbf{E}, n, \mathbf{V}) : (t,x) \in [0,T) \times \mathbb{R}^3 \to \mathbb{C}^3 \times \mathbb{R} \times \mathbb{R}^3$, η and δ are two real constants with $\eta > 0$, $\delta \leq 0$, $\wedge$ denotes the exterior product of vector-valued functions, and $\overline{\mathbf{E}}$ the complex conjugate of $\mathbf{E}$ (see [14]). The Zakharov system (1.1) describes the spontaneous generation of a magnetic field in a cold plasma. $\mathbf{E}$ represents the slowly varying complex amplitude of the high-frequency electric field, $\mathbf{B}$ the self-generated magnetic field, and n the fluctuation of the electron density from its equilibrium [7, 13, 14, 22, 23].

Using Fourier transform, we can solve the fourth equation in (1.1) and can obtain that if $\mathbf{E} \in H^1(\mathbb{R}^3)$, then $\mathbf{B}(\mathbf{E}) \in L^2(\mathbb{R}^3)$ and

$$\mathbf{B}(\mathbf{E}) = \mathcal{F}^{-1}\left[\frac{i\eta}{|\xi|^2 - \delta}\left(\xi\wedge\left(\xi\wedge\mathcal{F}\left(\mathbf{E}\wedge\overline{\mathbf{E}}\right)\right)\right)\right], \tag{$B-1$}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ mean the Fourier transform and the Fourier inverse transform, respectively (see [14, 17, 18, 19]).

For the Zakharov system (1.1), on one hand, it is a system with a nonlocal operator. On the other hand, it is a Hamiltonian system, and for $(\mathbf{E}_0, n_0, \mathbf{V}_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, we have two conservation laws:

• L^2-norm:

$$\|\mathbf{E}\|^2_{L^2(\mathbb{R}^3)} = \|\mathbf{E}_0\|^2_{L^2(\mathbb{R}^3)}, \tag{1.3}$$

• Energy:

$$\mathcal{H}(t) = \mathcal{H}(0), \tag{1.4}$$

where

$$\mathcal{H}(t) = \int_{\mathbb{R}^3}\left(|\nabla\mathbf{E}|^2 + \frac{1}{2}|n|^2 + \frac{1}{2}|\mathbf{V}|^2 + n|\mathbf{E}|^2\right)dx + \frac{\eta}{2}\int_{\mathbb{R}^3}\frac{1}{|\xi|^2-\delta}\left[\left|\xi\cdot\mathcal{F}(\mathbf{E}\wedge\overline{\mathbf{E}})\right|^2 - |\xi|^2\left|\mathcal{F}(\mathbf{E}\wedge\overline{\mathbf{E}})\right|^2\right]d\xi. \tag{1.5}$$

We first recall some known results about the system (1.1) without any magnetic field effect which provides the fluid modeling of the interaction between Langmuir and ion-acoustic waves [22]:

$$\begin{cases} i\mathbf{E}_t + \Delta\mathbf{E} - n\mathbf{E} = 0, \\ n_{tt} - \Delta n = \Delta|\mathbf{E}|^2. \end{cases} \tag{$Z-S$}$$

Sulem proved in [21] the global existence of a weak solution for certain small initial data in two and three dimensions, supposing in particular, $n_1 \in \dot{H}^{-1}(\mathbb{R}^d)$ ($\dot{H}^{-1}(\mathbb{R}^d)$ denotes the homogeneous Sobolev space, $\dot{H}^{-1}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : \ \frac{1}{|\xi|}\hat{u}(\xi) \in L^2(\mathbb{R}^d)\}$). With the same assumptions, they also established local existence and uniqueness of a smooth solution $(\mathbf{E}, n)$ with

$$(\mathbf{E}, n) \in L^\infty\left(0, T; H^m(\mathbb{R}^d)\right) \times L^\infty\left(0, T; H^{m-1}(\mathbb{R}^d)\right) \text{ for } m \geq 3.$$

In [11, 12], Glangetas and Merle studied the existence of self-similar blow-up solutions, concentration properties of blow-up solutions and instability of periodic solutions for the Zakharov system (Z-S) under the Hamiltonian case in $\mathbb{R}^2$. In [16], Merle established the blow-up results of virial type for (Z-S) in two and three space dimensions. The solution was shown to be globally well-defined in one space dimension [21], and in two space dimensions for small initial data [1]. In [20], the authors proved the local well-posedness (in time) of the Cauchy problem for (Z-S) if initial data satisfied $(\mathbf{E}_0, n_0, V_0) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. Later, Bourgain-Colliander [5] and Ginibre-Tsutsumi-Velo [10] obtained some results on the local well-posedness through using a method introduced by Bourgain in [2, 3, 4] for nonlinear dispersive equations. In [10], the authors also proved that the Cauchy problem for (Z-S) is locally (in time) wellposed for initial data $(\mathbf{E}_0, n_0, n_1) \in H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$ provided that $l \geq 0$ and $2k - (l + 1) \geq 0$, and the solution satisfied $(\mathbf{E}, n, n_t) \in \mathcal{C}\left([0, T); H^k(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)\right)$. For the system (Z-S) in the Hamiltonian case, Masselin in [15] provided an integral estimate on space and time for n.

We now return to mention some known results on the Cauchy problem (1.1)-(1.2). From results of Laurey [14], the global existence of a weak solution, the local existence and uniqueness of a smooth solution for the system were achieved for space dimension 2 and 3. Concerning the singularity of solutions to the Cauchy problem (1.1)-(1.2), Gan-Guo-Han-Zhang [8] established the following blow-up result.

Proposition 1.1. Let $\eta > 0$ and $\delta \leq 0$. Assume that for all time, the solutions $(\mathbf{E}, n, \mathbf{V})(t)$ of the Cauchy problem (1.1)-(1.2) are radially symmetric and $\mathcal{H}(0) < 0$. Then we have the following alternatives:

i) $(\mathbf{E}, n, \mathbf{V})(t)$ blows up in finite time.

ii) $(\mathbf{E}, n, \mathbf{V})(t)$ blows up in infinite time in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. That is, $(\mathbf{E}, n, \mathbf{V})(t)$ is defined for all t, and

$$\lim_{t \to T} |(\mathbf{E}, n, \mathbf{V})(t)|_{H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} = +\infty. \qquad (B - U)$$

□

In addition, Gan-Guo-Huang in [9] constructed a kind of blow-up solutions, established the existence of blow-up solutions through considering an elliptic system, showed the nonlinear instability of periodic solutions, studied the concentration properties of blow-up solutions and obtained the global existence of weak solutions.

We now mention some notations which will be imposed in the present paper.

Notations. Let $\mathbb{D} = \{x \in \mathbb{R}^3 : |x| < 1\}$. For an $\varepsilon \in (0, \frac{1}{4}]$ fixed, let $q = \frac{3}{2(1-\varepsilon)} \in (\frac{3}{2}, 2]$ and $p \in [4, 6)$ defined by the relation $\frac{2}{p} + \frac{1}{q} = 1$. In particular, if $\varepsilon = \frac{1}{4}$, then $q = 2$ and $p = 4$. Throughout this paper, C denotes any positive constant which depends only on ε, and $|(\mathbf{E}_0, n_0, \mathbf{V}_0)|_{H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)}$.

In this paper we shall be interested in the blow-up rate of the blow-up solutions to the Cauchy problem for the system (1.1), and establish the integral estimate on space and time for n and $\mathbf{E}$.

The following one theorem and two corollaries are the main results of the paper.

Theorem 1.1. For $\eta > 0$ and $\delta \leq 0$, let $(\mathbf{E}, n, \mathbf{V}) \in \mathcal{C}\left([0,T); H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\right)$ be a blow-up solution to the Cauchy problem (1.1)-(1.2). We assume that $(\mathbf{E}, n, \mathbf{V})$ blows up at time T $(T < \infty)$. If $\gamma > \frac{1}{\varepsilon}$, then

$$\int_0^T \left[\left(\int_{\mathbb{R}^3} |n(t,x)|^q dx \right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^3} |\mathbf{E}(t,x)|^{2q} dx \right)^{\frac{1}{q}} \right]^{\gamma} dt = +\infty. \tag{1.6}$$

Theorem 1.1 implies the conclusion below.

Corollary 1.1. For $\eta > 0$ and $\delta \leq 0$, let $(\mathbf{E}, n, \mathbf{V}) \in \mathcal{C}\left([0,T); H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\right)$ be a blow-up solution to the Cauchy problem (1.1)-(1.2), and that $T < \infty$ be its blow-up time. Then for $a < 1$, there holds that

$$\sup_{t \in [0,T)} \left[(T-t)^{a\varepsilon} \left(|n(t)|_{L^q(\mathbb{R}^3)} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)} \right) \right] = +\infty. \tag{1.7}$$

In particular, if $(\mathbf{E}, n, \mathbf{V})$ is a radially symmetric solution to the Cauchy problem (1.1)-(1.2), we then achieve the following estimates of n in $L^q(\mathbb{D})$ and $\mathbf{E}$ in $L^{2q}(\mathbb{D})$.

Corollary 1.2. For $\eta > 0$ and $\delta \leq 0$, let $(\mathbf{E}, n, \mathbf{V}) \in \mathcal{C}\left([0,T); H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\right)$ be a radially symmetric solution of the Cauchy problem (1.1)-(1.2), and that $T < \infty$ be its blow-up time. Then for $a \in (0, \frac{1}{3})$, there holds that

$$\sup_{t \in [0,T)} \left[(T-t)^{a\varepsilon} \left(|n(t)|_{L^q(\mathbb{D})} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{D})} \right) \right] = +\infty. \tag{1.8}$$

It is difficult to construct a kind of blow-up solutions for the system (1.1) in three space dimensions although the authors in [8] have established an existence result of the blow-up solutions for the Cauchy problem (1.1)-(1.2) . Fortunately, for some special forms of $\mathbf{E}$, n and $\mathbf{B}$ in the system (1.1), we can construct a kind of blow-up solutions to the system (1.1). Let's here give a brief remark.

Remark 1.1. We consider the Zakharov system (1.1) in the cylindrical coordinate (r, θ, z). We can construct the following blow-up solutions:

$$\begin{cases} \mathbf{E} = (E_1, 0, iE_1), \quad E_1(t,x) = \dfrac{\omega/\sqrt{2}}{T-t} P\left(\dfrac{\omega r}{T-t}, \dfrac{\omega z}{T-t}\right) e^{i(T-t)^{-1}}, \\[2ex] n(t,x) = \dfrac{\omega^2}{(T-t)^2} N\left(\dfrac{\omega r}{T-t}, \dfrac{\omega z}{T-t}\right), \\[2ex] \mathbf{B} = (0, \widetilde{B}, 0), \quad \widetilde{B}(t,x) = \dfrac{\omega^2}{(T-t)^2} M\left(\dfrac{\omega r}{T-t}, \dfrac{\omega z}{T-t}\right), \end{cases} \tag{1.9}$$

where

$$P(r,z) = P\left(\sqrt{x_1^2 + x_2^2}, z\right), N(r,z) = N\left(\sqrt{x_1^2 + x_2^2}, z\right), M(r,z) = M\left(\sqrt{x_1^2 + x_2^2}, z\right)$$

are real radially symmetric functions on $\mathbb{R}^2 \times \mathbb{R}$ in the cylindrical coordinate and $\omega > 0$. In addition, (P, N, M) satisfies the following system:

$$\begin{cases} \Delta P - P - NP + MP = 0, \\ \lambda^2 (r^2 N_{rr} + rzN_{rz} + zrN_{zr} + z^2 N_{zz} \\ \qquad +6(rN_r + zN_z) + 6N) - \Delta N = \Delta(|P|^2), \\ \Delta M + \delta(T-t)^2 \lambda^2 M = \eta \Delta(|P|^2), \end{cases} \tag{1.10}$$

where $\lambda = \dfrac{1}{\omega}$. Furthermore, let

$$M = \eta \mathcal{F}^{-1} \left(\frac{|\xi|^2}{|\xi|^2 - \delta(T-t)^2 \lambda^2} \mathcal{F}(P^2) \right),$$

(1.10) then reduces to

$$\begin{cases} \Delta P - P\eta \mathcal{F}^{-1} \left(\dfrac{|\xi|^2}{|\xi|^2 - \delta(T-t)^2 \lambda^2} \mathcal{F}(P^2) \right) + P = NP, \\ \lambda^2 (r^2 N_{rr} + rzN_{rz} + zrN_{zr} + z^2 N_{zz} \\ \qquad +6(rN_r + zN_z) + 6N) - \Delta N = \Delta(|P|^2). \end{cases} \tag{1.11}$$

For $\forall T > 0$, $0 \leq t < T$ fixed, the existence of solutions to the system (1.11) in $H_r^1(\mathbb{R}^3) \times L_r^2(\mathbb{R}^3)$ may be obtained by referring papers [11, 12]. We will discuss this topic in our future work. Here, we assume that there exists a solution for the system (1.11) to establish the main results of the present paper. In particular, if $(P, N) \in H_r^1(\mathbb{R}^3) \times L_r^2(\mathbb{R}^3)$ is a solution to (1.11), then $(\mathbf{E}, n)$ defined in (1.9) is a blow-up solution to the Cauchy problem (1.1)-(1.2). When $\delta = 0$, (1.11) reduces to

$$\begin{cases} \Delta P - \eta P^3 + P = NP, \\ \lambda^2 (r^2 N_{rr} + rzN_{rz} + zrN_{zr} + z^2 N_{zz} \\ \qquad +6(rN_r + zN_z) + 6N) - \Delta N = \Delta(|P|^2). \end{cases} \tag{1.12}$$

If $(P, N) \in H_r^1(\mathbb{R}^3) \times L_r^2(\mathbb{R}^3)$ is a solution to (1.12), then $(\mathbf{E}, n)$ defined in (1.9) is a self-similar blow-up solution to the Cauchy problem (1.1)-(1.2) with $\delta = 0$. It is worth mentioning that in two space dimensions, we have constructed a special kind of blow-up solutions to the system (1.1) (see [9]) .

We will give the formal derivation of the blow-up solution (1.9) to the Cauchy problem (1.1)-(1.2) in Appendix A.

2 Preliminaries

In order to prove Theorem 1.1, Corollary 1.1 and Corollary 1.2, we give some key ingredients.

Lemma 2.1. For $\eta > 0$ and $\delta \leq 0$, let $\varepsilon \in (0, \frac{1}{4}]$ be fixed, $q = \frac{3}{2(1-\varepsilon)} \in (\frac{3}{2}, 2]$ and $\frac{2}{p} + \frac{1}{q} = 1$. Then the following estimates hold.

(i) If $\phi \in H^1(\mathbb{R}^3)$ is radially symmetric, then $\phi \in L^\infty(\mathbb{R}^3 \setminus \mathbb{D})$ and

$$|\phi|^2_{L^\infty(\mathbb{R}^3 \setminus \mathbb{D})} \leq C|\nabla\phi|_{L^2(\mathbb{R}^3)}|\phi|_{L^2(\mathbb{R}^3)}. \tag{2.1}$$

(ii) If Ω is a domain in $\mathbb{R}^3$ with sufficiently smooth boundary and $\phi \in H^1(\Omega)$, then $\phi \in L^p(\Omega)$, and

$$|\phi|_{L^p(\Omega)} \leq C|\nabla\phi|^{1-\varepsilon}_{L^2(\Omega)}|\phi|^{\varepsilon}_{L^2(\Omega)}. \tag{2.2}$$

In particular $\phi \in L^4(\Omega)$, and

$$|\phi|_{L^4(\Omega)} \leq C|\nabla\phi|^{\frac{3}{4}}_{L^2(\Omega)}|\phi|^{\frac{1}{4}}_{L^2(\Omega)}. \tag{2.3}$$

(iii) Let $U(t) = e^{it\Delta}$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There then exists a constant $C > 0$ such that for any $\phi \in L^{p'}(\mathbb{R}^3)$, and for any $t > 0$,

$$|U(t)\phi|_{L^p(\mathbb{R}^3)} \leq \frac{C}{t^{1-\varepsilon}}|\phi|_{L^{p'}(\mathbb{R}^3)}. \tag{2.4}$$

Proof. (i) From [6], it follows that

$$\sup_{x\in\mathbb{R}^3} |x||\phi(x)| \leq C|\phi|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}|\nabla\phi|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \tag{2.5}$$

for $\phi \in H^1(\mathbb{R}^3)$. Note that $\mathbb{D} = \{x \in \mathbb{R}^3 : |x| < 1\}$ and (2.5), for $\phi \in L^\infty(\mathbb{R}^3 \setminus \mathbb{D})$, we achieve

$$|\phi|_{L^\infty(\mathbb{R}^3\setminus\mathbb{D})} < \sup_{x\in\mathbb{R}^3\setminus\mathbb{D}} |x||\phi(x)| \leq \sup_{x\in\mathbb{R}^3} |x||\phi(x)| \leq C|\phi|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}|\nabla\phi|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}.$$

With an eye on exploiting (2.5), we obtain (2.1).

(ii) With the aid of Gagliardo-Nirenberg inequality, we conclude (2.2) and (2.3).

(iii) Note that $p' = \frac{p}{p-1}$, $q = \frac{3}{2(1-\varepsilon)}$ and $\frac{2}{p} + \frac{1}{q} = 1$, (2.4) follows immediately from the standard semigroup theory arguments. □

One can of course obtain the limit estimates of a blow-up solution for the Cauchy problem (1.1)-(1.2).

Lemma 2.2. For $\eta > 0$, $\delta \leq 0$, let $(\mathbf{E}, n, \mathbf{V})$ be a blow-up solution to the Cauchy problem (1.1)-(1.2) in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, and let T ($T < \infty$) be the blow-up time. Then

$$\lim_{t\to T} |\mathbf{E}(t)|_{L^4(\mathbb{R}^3)} = +\infty, \tag{2.6}$$

$$\lim_{t\to T} |\nabla\mathbf{E}(t)|_{L^2(\mathbb{R}^3)} = +\infty. \tag{2.7}$$

Proof. We first show (2.6). Suppose for contradiction that there is a sequence $\{t_k\}$ with $t_k < T$ satisfying $\lim\limits_{k\to\infty} t_k = T$ such that

$$\lim_{t\to T} |\mathbf{E}(t_k)|_{L^4(\mathbb{R}^3)} \leq C \tag{2.8}$$

for any k and for a constant $C>0$. From (1.4), (1.5) and Hölder's inequality we have

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\mathbb{R}^3} n^2(t_k,x)dx\\
&\quad\le \mathcal{H}(0)+\left|\int_{\mathbb{R}^3} n(t_k,x)|\mathbf{E}(t_k,x)|^2dx\right|\\
&\qquad+\tfrac{\eta}{2}\int_{\mathbb{R}^3}\tfrac{1}{|\xi|^2-\delta}\left[|\xi|^2\left|\mathcal{F}(\mathbf{E}\wedge\overline{\mathbf{E}})\right|^2-\left|\xi\cdot\mathcal{F}(\mathbf{E}\wedge\overline{\mathbf{E}})\right|^2\right]d\xi \qquad (2.9)\\
&\quad\le \mathcal{H}(0)+|n(t_k)|_{L^2(\mathbb{R}^3)}|\mathbf{E}(t_k)|^2_{L^4(\mathbb{R}^3)}+\tfrac{\eta}{2}\int_{\mathbb{R}^3}|(\mathbf{E}\wedge\overline{\mathbf{E}})(t_k)|^2dx\\
&\quad\le \mathcal{H}(0)+|n(t_k)|_{L^2(\mathbb{R}^3)}|\mathbf{E}(t_k)|^2_{L^4(\mathbb{R}^3)}+\tfrac{\eta}{2}|\mathbf{E}(t_k)|^4_{L^4(\mathbb{R}^3)}.
\end{aligned}
$$

Hence $|n(t_k)|_{L^2(\mathbb{R}^3)}$ is bounded.

On the other hand, combining (2.9) with (1.4) and (1.5) yields that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\nabla\mathbf{E}(t_k,x)|^2dx+\tfrac{1}{2}\int_{\mathbb{R}^3}(n^2(t_k,x)+|\mathbf{V}(t_k,x)|^2)dx\\
&\quad\le \mathcal{H}(0)+|n(t_k)|_{L^2(\mathbb{R}^3)}|\mathbf{E}(t_k)|^2_{L^4(\mathbb{R}^3)}+\tfrac{\eta}{2}|\mathbf{E}(t_k)|^4_{L^4(\mathbb{R}^3)} \qquad (2.10)\\
&\quad\le C.
\end{aligned}
$$

But this together with (2.9) contradicts the blow-up assumption. Thus (2.6) holds true. (2.7) follows easily from (2.3) and (2.6). □

The following estimates play an important role to the proofs of Theorem 1.1, Corollary 1.1 and Corollary 1.2.

Lemma 2.3. For $\eta>0$ and $\delta\le 0$, there exists a positive constant C such that for any $t\in[0,T)$,

$$
|\mathbf{E}(t)|_{L^p(\mathbb{R}^3)}\le C\left[1+\int_0^t\frac{1}{(t-s)^{1-\varepsilon}}\left(|n(s)|_{L^q(\mathbb{R}^3)}+|\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)}\right)|\mathbf{E}(s)|_{L^p(\mathbb{R}^3)}ds\right]. \qquad (2.11)
$$

In particular,

$$
|\mathbf{E}(t)|_{L^4(\mathbb{R}^3)}\le C\left[1+\int_0^t\frac{1}{(t-s)^{3/4}}\left(|n(s)|_{L^2(\mathbb{R}^3)}+|\mathbf{E}(s)|^2_{L^4(\mathbb{R}^3)}\right)|\mathbf{E}(s)|_{L^4(\mathbb{R}^3)}ds\right]. \qquad (2.12)
$$

Proof. Consider the Cauchy problem

$$
\begin{cases}
i\mathbf{E}_t+\Delta\mathbf{E}-n\mathbf{E}+i(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))=0,\\
\mathbf{B}(\mathbf{E})=\mathcal{F}^{-1}\left[\frac{i\eta}{|\xi|^2-\delta}\left(\xi\wedge\left(\xi\wedge\mathcal{F}\left(\mathbf{E}\wedge\overline{\mathbf{E}}\right)\right)\right)\right],\\
\mathbf{E}(0,x)=\mathbf{E}_0(x),
\end{cases}
$$

it is easy to write down its formal solution as follows by integrating the corresponding equations and then applying the corresponding initial value condition.

$$
\mathbf{E}(t)=U(t)\mathbf{E}_0-i\int_0^t U(t-s)\left[n(s)\mathbf{E}(s)-i(\mathbf{E}(s)\wedge\mathbf{B}(\mathbf{E}(s)))\right]ds. \qquad (2.13)
$$

Here, $U(t)=e^{it\Delta}$ is the unitary semigroup generated by the free Schrödinger equation $i\mathbf{E}_t+\Delta\mathbf{E}=0$ in the Hilbert space $H^k(\mathbb{R}^3)$ $(k\in\mathbb{R})$. Applying the Minkowski inequality, we then have

$$\begin{aligned}|\mathbf{E}(t)|_{L^p(\mathbb{R}^3)} \leq\ & |U(t)\mathbf{E}_0|_{L^p(\mathbb{R}^3)}\\ & + \int_0^t |U(t-s)n(s)\mathbf{E}(s)|_{L^p(\mathbb{R}^3)}ds\\ & + \int_0^t |U(t-s)(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))|_{L^p(\mathbb{R}^3)}ds.\end{aligned} \tag{2.14}$$

On the other hand,

$$|U(t-s)n(s)\mathbf{E}(s)|_{L^p(\mathbb{R}^3)} \leq \frac{C}{(t-s)^{1-\varepsilon}}|n(s)\mathbf{E}(s)|_{L^{p'}(\mathbb{R}^3)}, \tag{2.15}$$

$$|U(t-s)(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))|_{L^p(\mathbb{R}^3)} \leq \frac{C}{(t-s)^{1-\varepsilon}}|(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))|_{L^{p'}(\mathbb{R}^3)}, \tag{2.16}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{2}{p}+\frac{1}{q}=1$, $\frac{1}{p}+\frac{1}{q}=1-\frac{1}{p}=\frac{1}{p'}$. By Hölder's inequality, the right-hand side of (2.15) and (2.16) can be bounded as

$$|n(s)\mathbf{E}(s)|_{L^{p'}(\mathbb{R}^3)} \leq |n(s)|_{L^q(\mathbb{R}^3)}|\mathbf{E}(s)|_{L^p(\mathbb{R}^3)}, \tag{2.17}$$

$$\begin{aligned}&|(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))|_{L^{p'}(\mathbb{R}^3)}\\ &\quad= \left|\mathbf{E}\wedge\mathcal{F}^{-1}\left[\tfrac{i\eta}{|\xi|^2-\delta}\left(\xi\wedge\left(\xi\wedge\mathcal{F}\left(\mathbf{E}\wedge\overline{\mathbf{E}}\right)\right)\right)\right]\right|_{L^{p'}(\mathbb{R}^3)}\\ &\quad\leq C|\mathbf{E}(s)|_{L^p(\mathbb{R}^3)}|\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)}.\end{aligned} \tag{2.18}$$

Combining (2.15) with (2.16), (2.17) and (2.18) yields that

$$|U(t-s)n(s)\mathbf{E}(s)|_{L^p(\mathbb{R}^3)} \leq \frac{C}{(t-s)^{1-\varepsilon}}|n(s)|_{L^q(\mathbb{R}^3)}|\mathbf{E}(s)|_{L^p(\mathbb{R}^3)}, \tag{2.19}$$

$$|U(t-s)(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))|_{L^p(\mathbb{R}^3)} \leq \frac{C}{(t-s)^{1-\varepsilon}}|\mathbf{E}(s)|_{L^p(\mathbb{R}^3)}|\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)}. \tag{2.20}$$

With the aid of (2.2) we have

$$\begin{aligned}|U(t)\mathbf{E}_0|_{L^p(\mathbb{R}^3)} &\leq C|U(t)\mathbf{E}_0|^{\varepsilon}_{L^2(\mathbb{R}^3)}|\nabla(U(t)\mathbf{E}_0)|^{1-\varepsilon}_{L^2(\mathbb{R}^3)}\\ &\leq C|\mathbf{E}_0|^{\varepsilon}_{L^2(\mathbb{R}^3)}|\nabla\mathbf{E}_0|^{1-\varepsilon}_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{2.21}$$

(2.14)-(2.21) then yield the desired inequality (2.11). In particular, we obtain (2.12) by choosing $\varepsilon=\frac{1}{4}$. □

3 Proofs of the Main Results

We are now in the position to prove Theorem 1.1, Corollary 1.1 and Corollary 1.2.

3.1 Proof of Theorem 1.1.

Suppose for contradiction that there exists $\gamma > \frac{1}{\varepsilon}$ such that

$$\int_0^T \left[\left(\int_{\mathbb{R}^3} |n(t,x)|^q dx \right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}^3} |\mathbf{E}(t,x)|^{2q} dx \right)^{\frac{1}{q}} \right]^\gamma dt < +\infty. \tag{3.1}$$

Applying Lemma 2.3 we have

$$|\mathbf{E}(t)|_{L^p(\mathbb{R}^3)} \leq C \left(1 + \int_0^t \frac{1}{(t-s)^{1-\varepsilon}} (|n(s)|_{L^q(\mathbb{R}^3)} + |\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)}) |\mathbf{E}(s)|_{L^p(\mathbb{R}^3)} ds \right), \tag{3.2}$$

where $1 - \varepsilon + \frac{1}{\gamma} < 1$ and $|n(s)|_{L^q(\mathbb{R}^3)} + |\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)} \in L^\gamma(0,T)$. By Gronwall lemma, there exists a constant $C > 0$ such that

$$\forall t \in [0,T), \quad |\mathbf{E}(t)|_{L^p(\mathbb{R}^3)} \leq C. \tag{3.3}$$

Indeed, from (3.2) Gronwall lemma implies that

$$\begin{aligned} |\mathbf{E}(t)|_{L^p(\mathbb{R}^3)} &\leq C \exp \left[\int_0^t \frac{1}{(t-s)^{1-\varepsilon}} \left(|n(s)|_{L^q(\mathbb{R}^3)} + |\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)} \right) ds \right] \\ &\leq C \exp \left\{ \left[\int_0^t \left(|n(s)|_{L^q(\mathbb{R}^3)} + |\mathbf{E}(s)|^2_{L^{2q}(\mathbb{R}^3)} \right)^\gamma ds \right]^{\frac{1}{\gamma}} \right. \\ &\qquad \left. \cdot \left[\int_0^t \left(\frac{1}{(t-s)^{1-\varepsilon}} \right)^{\frac{\gamma}{\gamma-1}} ds \right]^{\frac{\gamma-1}{\gamma}} \right\}, \end{aligned}$$

which requires that $1 - \frac{\gamma(1-\varepsilon)}{\gamma-1} > 0$ if the right hand side of the above inequality is bounded. That is, $\gamma > \frac{1}{\varepsilon}$.

We now need to divide the remaining proof into two cases:

Case 1: $\varepsilon = \frac{1}{4}$;

Case 2: $\varepsilon \in (0, \frac{1}{4})$.

Let us first consider **Case 1:** $\varepsilon = \frac{1}{4}$. In this case, noting that $q = \frac{3}{2(1-\varepsilon)}$ and $\frac{2}{p} + \frac{1}{q} = 1$, we have $p = 4$ and $q = 2$. (3.3) then contradicts Lemma 2.2. Hence for any $\gamma > 4$, one always has

$$\int_0^T \left[\left(\int_{\mathbb{R}^3} |n(t,x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^3} |\mathbf{E}(t,x)|^4 dx \right)^{\frac{1}{2}} \right]^\gamma dt = +\infty. \tag{3.4}$$

We now consider **Case 2:** $\varepsilon \in (0, \frac{1}{4})$. In this case, from (1.4) and (1.5) we have

$$\begin{aligned} &\tfrac{1}{2} \int_{\mathbb{R}^3} |n(t,x)|^2 dx \\ &\quad \leq \mathcal{H}(0) + \left| \int_{\mathbb{R}^3} n(t,x) |\mathbf{E}(t,x)|^2 dx \right| \\ &\qquad + \tfrac{\eta}{2} \int_{\mathbb{R}^3} \tfrac{1}{|\xi|^{2-\delta}} \left[|\xi|^2 \left| \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}}) \right|^2 - \left| \xi \cdot \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}}) \right|^2 \right] d\xi \\ &\quad \leq \mathcal{H}(0) + |n(t)|_{L^q(\mathbb{R}^3)} |\mathbf{E}(t)|^2_{L^p(\mathbb{R}^3)} + C |\mathbf{E}(t)|^4_{L^4(\mathbb{R}^3)}, \end{aligned} \tag{3.5}$$

where $\frac{2}{p}+\frac{1}{q}=1$, $\eta>0$ and $\delta\le 0$.

Since

$$
\begin{aligned}
|\mathbf{E}(t)|^4_{L^4(\mathbb{R}^3)} &\le |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}|\mathbf{E}(t)|^2_{L^p(\mathbb{R}^3)}\\
&\le \left(|\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}+|n(t)|_{L^q(\mathbb{R}^3)}\right)|\mathbf{E}(t)|^2_{L^p(\mathbb{R}^3)}, \qquad (3.6)
\end{aligned}
$$

using (3.3), a direct computation allows one to achieve

$$
\begin{aligned}
&\tfrac{1}{2}\left(|n(t)|_{L^2(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^4(\mathbb{R}^3)}\right)^2\\
&\quad\le |n(t)|^2_{L^2(\mathbb{R}^3)}+|\mathbf{E}(t)|^4_{L^4(\mathbb{R}^3)} \qquad (3.7)\\
&\quad\le C\left(|n(t)|_{L^q(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}\right).
\end{aligned}
$$

We then have

$$
\begin{aligned}
&\int_0^T\left(|n(t)|_{L^2(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^4(\mathbb{R}^3)}\right)^{2\gamma}dt\\
&\quad\le C\int_0^T\left(|n(t)|_{L^q(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}\right)^{\gamma}dt<+\infty. \qquad (3.8)
\end{aligned}
$$

In view of $2\gamma>\frac{2}{\varepsilon}\ge 8$, (3.8) contradicts the estimate (3.4), which completes the proof of the space-time estimate (1.6). □

3.2 Proof of Corollary 1.1.

We show Corollary 1.1 by contradiction.

Assume that there is a constant C such that for $a<1$,

$$
(T-t)^{a\varepsilon}\left(|n(t)|_{L^q(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}\right)\le C. \qquad (3.9)
$$

A simple computation yields that for $\gamma>\frac{1}{\varepsilon}$,

$$
\int_0^T\left(|n(t)|_{L^q(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}\right)^{\gamma}dt\le C\int_0^T(T-t)^{-a\varepsilon\gamma}dt. \qquad (3.10)
$$

That is, for $a<1$, $\varepsilon\in(0,\frac{1}{4}]$ fixed and for $\gamma>\frac{1}{\varepsilon}$, there holds the following inequality:

$$
\int_0^T\left(|n(t)|_{L^q(\mathbb{R}^3)}+|\mathbf{E}(t)|^2_{L^{2q}(\mathbb{R}^3)}\right)^{\gamma}dt\le CT^{-a\varepsilon\gamma+1}\le C,
$$

which contradicts (1.6) in Theorem 1.1. This shows (1.7), which completes the proof of Corollary 1.1. □

3.3 Proof of Corollary 1.2.

We prove Corollary 1.2 by contradiction here. For $a \in (0, \frac{1}{3})$, assume that $(\mathbf{E}, n, \mathbf{V})$ is a radially symmetric blow-up solution to the Cauchy problem (1.1)-(1.2) such that $(\mathbf{E}, n, \mathbf{V}) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, and

$$|n(t)|_{L^q(\mathbb{D})} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{D})} \leq \frac{C}{(T-t)^{a\varepsilon}} \tag{3.11}$$

for all $t \in [0, T)$. We first claim the following estimate:

$$|n(t)|_{L^2(\mathbb{R}^3)} + |\mathbf{E}(t)|^2_{L^4(\mathbb{R}^3)} \leq \frac{C}{(T-t)^{\frac{3}{4}a}}. \tag{3.12}$$

Indeed, combining (1.4) with (1.5) yields that

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^3} |\nabla \mathbf{E}(t,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |n(t,x)|^2 dx \\
&\quad \leq \mathcal{H}(0) + \left| \textstyle\int_{\mathbb{D}} n(t,x) |\mathbf{E}(t,x)|^2 dx \right| \\
&\quad + \left| \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} n(t,x) |\mathbf{E}(t,x)|^2 dx \right| \\
&\quad + \tfrac{\eta}{2} \textstyle\int_{\mathbb{R}^3} \frac{|\xi|^2}{|\xi|^2 - \delta} \left| \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}}) \right|^2 d\xi.
\end{aligned} \tag{3.13}$$

Note that $\frac{2}{p} + \frac{1}{q} = 1$, from Hölder's inequality, (2.2) and (3.11) we have

$$\begin{aligned}
&\left| \textstyle\int_{\mathbb{D}} n(t,x) |\mathbf{E}(t,x)|^2 dx \right| \\
&\quad \leq \left(\textstyle\int_{\mathbb{D}} |n(t,x)|^q dx \right)^{\frac{1}{q}} \left(\textstyle\int_{\mathbb{D}} |\mathbf{E}(t,x)|^p dx \right)^{\frac{2}{p}} \\
&\quad \leq \tfrac{C}{(T-t)^{a\varepsilon}} |\nabla \mathbf{E}(t,x)|^{2(1-\varepsilon)}_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{3.14}$$

On the other hand, from (1.3) and (2.1) we conclude

$$\begin{aligned}
&\left| \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} n(t,x) |\mathbf{E}(t,x)|^2 dx \right| \\
&\quad \leq \tfrac{1}{2} \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |n(t,x)|^2 dx + \tfrac{1}{2} \int_{\mathbb{R}^3 \setminus \mathbb{D}} |\mathbf{E}(t,x)|^4 dx \\
&\quad \leq \tfrac{1}{2} \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |n(t,x)|^2 dx + \tfrac{1}{2} |\mathbf{E}(t)|^2_{L^\infty(\mathbb{R}^3) \setminus \mathbb{D})} |\mathbf{E}(t)|^2_{L^2(\mathbb{R}^3)} \\
&\quad \leq \tfrac{1}{2} \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |n(t,x)|^2 dx + C |\nabla \mathbf{E}(t)|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\tfrac{\eta}{2} \textstyle\int_{\mathbb{R}^3} \frac{|\xi|^2}{|\xi|^2 - \delta} \left| \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}}) \right|^2 d\xi \\
&\quad \leq C \textstyle\int_{\mathbb{R}^3} |\mathbf{E}(t,x)|^4 dx \\
&\quad \leq C \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |\mathbf{E}(t,x)|^4 dx + C \int_{\mathbb{D}} |\mathbf{E}(t,x)|^4 dx \\
&\quad \leq C |\nabla \mathbf{E}(t)|_{L^2(\mathbb{R}^3)} + \tfrac{C}{(T-t)^{2a\varepsilon}}.
\end{aligned} \tag{3.16}$$

Combining (3.13), (3.14), (3.15) and (3.16) together yields that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla \mathbf{E}(t,x)|^2 dx + \tfrac{1}{2}\int_{\mathbb{R}^3} |n(t,x)|^2 dx \\
&\le \mathcal{H}(0) + \tfrac{C}{(T-t)^{a\varepsilon}} |\nabla \mathbf{E}(t)|_{L^2(\mathbb{R}^3)}^{2(1-\varepsilon)} \\
&\quad + \tfrac{1}{2}\int_{\mathbb{R}^3\setminus\mathbb{D}} |n(t,x)|^2 dx \\
&\quad + C|\nabla \mathbf{E}(t)|_{L^2(\mathbb{R}^3)} + \tfrac{C}{(T-t)^{2a\varepsilon}}.
\end{aligned} \tag{3.17}$$

A computation shows that

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3} |\nabla \mathbf{E}(t,x)|^2 dx\right)^{\varepsilon} \\
&\le \mathcal{H}(0)|\nabla \mathbf{E}|_{L^2(\mathbb{R}^3)}^{2\varepsilon-2} + \tfrac{C}{(T-t)^{a\varepsilon}} \\
&\quad + |\nabla \mathbf{E}|_{L^2(\mathbb{R}^3)}^{2\varepsilon-1} + \tfrac{C}{(T-t)^{2a\varepsilon}} |\nabla \mathbf{E}|_{L^2(\mathbb{R}^3)}^{2(\varepsilon-1)}.
\end{aligned} \tag{3.18}$$

According to Lemma 2.2, for $\varepsilon \in (0, \frac{1}{4}]$ we have

$$\lim_{t\to T} |\nabla \mathbf{E}|_{L^2(\mathbb{R}^3)}^{2\varepsilon-1} = 0, \quad \lim_{t\to T} |\nabla \mathbf{E}|_{L^2(\mathbb{R}^3)}^{2(\varepsilon-1)} = 0,$$

which together with (3.18) implies that

$$\left(\int_{\mathbb{R}^3} |\nabla \mathbf{E}(t,x)|^2 dx\right)^{\frac{1}{2}} \le \frac{C}{(T-t)^{\frac{a}{2}}}. \tag{3.19}$$

Combining (3.19) with (2.3) gives

$$|\mathbf{E}(t)|_{L^4(\mathbb{R}^3)} \le \frac{C}{(T-t)^{\frac{3a}{8}}}. \tag{3.20}$$

On the other hand, from (1.4), (1.5) and (3.20) one has

$$\begin{aligned}
&\tfrac{1}{4}\left(|n(t)|_{L^2(\mathbb{R}^3)} + |\mathbf{E}(\mathbf{t})|_{L^4(\mathbb{R}^3)}^2\right)^2 \\
&\le \tfrac{1}{2}\left(|n(t)|_{L^2(\mathbb{R}^3)}^2 + |\mathbf{E}(t)|_{L^4(\mathbb{R}^3)}^4\right) \\
&\le \mathcal{H}(0) + C\left(|n(t)|_{L^2(\mathbb{R}^3)}| + |\mathbf{E}(t)|_{L^4(\mathbb{R}^3)}^2\right)|\mathbf{E}(t)|_{L^4(\mathbb{R}^3)}^2 \\
&\le \mathcal{H}(0) + \left(|n(t)|_{L^2(\mathbb{R}^3)} + |\mathbf{E}(\mathbf{t})|_{L^4(\mathbb{R}^3)}^2\right)\tfrac{C}{(T-t)^{\frac{3a}{4}}}
\end{aligned} \tag{3.21}$$

which concludes that

$$|n(t)|_{L^2(\mathbb{R}^3)} + |\mathbf{E}(\mathbf{t})|_{L^4(\mathbb{R}^3)}^2 \le \frac{C}{(T-t)^{\frac{3a}{4}}}. \tag{3.22}$$

That is, the estimate (3.12) holds. Therefore, for $\gamma \in (4, \frac{4}{3a})$ with $a < \frac{1}{3}$ we have

$$\int_0^T \left(|n(t)|_{L^2(\mathbb{R}^3)} + |\mathbf{E}(t)|^2_{L^4(\mathbb{R}^3)} \right)^\gamma dt < +\infty.$$

This contradicts Theorem 1.1, and hence we have proven that

$$|n(t)|_{L^q(\mathbb{D})} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{D})} > \frac{C}{(T-t)^{a\varepsilon}}. \tag{3.23}$$

We are now in the position to prove that

$$|n(t)|_{L^q(\mathbb{D})} + |\mathbf{E}(t)|^2_{L^{2q}(\mathbb{D})} \to +\infty \text{ as } t \to T \tag{3.24}$$

by contradiction. Assume that there exists $t_k \to T$ such that

$$|n(t_k)|_{L^q(\mathbb{D})} + |\mathbf{E}(t_k)|^2_{L^{2q}(\mathbb{D})} \le C. \tag{3.25}$$

From (1.4) and (1.5) we have

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^3} |\nabla \mathbf{E}(t_k,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} |n(t_k,x)|^2 dx \\ &\le \mathcal{H}(0) + \left| \textstyle\int_{\mathbb{D}} n(t_k,x) |\mathbf{E}(t_k,x)|^2 dx \right| \\ &+ \left| \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} n(t_k,x) |\mathbf{E}(t_k,x)|^2 dx \right| \\ &+ \tfrac{\eta}{2} \textstyle\int_{\mathbb{R}^3} \frac{|\xi|^2}{|\xi|^2 - \delta} \left| \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}}) \right|^2 d\xi. \end{aligned} \tag{3.26}$$

Applying Lemma 2.1, Hölder's inequality and the relation $\frac{2}{p} + \frac{1}{q} = 1$, for $\eta > 0$ and $\delta \le 0$, we conclude the following three estimates:

$$\begin{aligned} \left| \textstyle\int_{\mathbb{D}} n(t_k,x) |\mathbf{E}(t_k,x)|^2 dx \right| &\le |n(t_k)|_{L^q(\mathbb{D})} |\mathbf{E}(t_k)|^2_{L^p(\mathbb{D})} \\ &\le C |\nabla \mathbf{E}(t_k)|^{2(1-\varepsilon)}_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\left| \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} n(t_k,x) |\mathbf{E}(t_k,x)|^2 dx \right| \\ &\le \tfrac{1}{2} \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |n(t_k,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3 \setminus \mathbb{D}} |\mathbf{E}(t_k,x)|^4 dx \\ &\le \tfrac{1}{2} \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |n(t_k,x)|^2 dx + \frac{1}{2} |\mathbf{E}(t_k)|^2_{L^\infty(\mathbb{R}^3 \setminus \mathbb{D})} |\mathbf{E}(t_k)|^2_{L^2(\mathbb{R}^3)} \\ &\le \tfrac{1}{2} \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |n(t_k,x)|^2 dx + C |\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}, \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\tfrac{\eta}{2} \textstyle\int_{\mathbb{R}^3} \frac{|\xi|^2}{|\xi|^2 - \delta} \left| \mathcal{F}(\mathbf{E} \wedge \overline{\mathbf{E}}) \right|^2 d\xi \\ &\le \textstyle\int_{\mathbb{R}^3} |\mathbf{E}(t_k,x)|^4 dx \\ &\le C \textstyle\int_{\mathbb{R}^3 \setminus \mathbb{D}} |\mathbf{E}(t_k,x)|^4 dx + C \int_{\mathbb{D}} |\mathbf{E}(t_k,x)|^4 dx \\ &\le C |\mathbf{E}(t_k)|^2_{L^\infty(\mathbb{R}^3 \setminus \mathbb{D})} |\mathbf{E}(t_k)|^2_{L^2(\mathbb{R}^3)} + C \\ &\le |\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)} + C. \end{aligned} \tag{3.29}$$

Combining (3.26) with (3.27), (3.28) and (3.29) yields that

$$\int_{\mathbb{R}^3} |\nabla \mathbf{E}(t_k, x)|^2 dx \leq \mathcal{H}(0) + C|\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2-2\varepsilon} + C|\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)} + C. \tag{3.30}$$

That is,

$$\begin{aligned} |\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2\varepsilon} &\leq \mathcal{H}(0)|\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2\varepsilon-2} + C \\ &\qquad + C|\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2\varepsilon-1} + C|\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2\varepsilon-2}. \end{aligned} \tag{3.31}$$

By Lemma 2.2, for $\varepsilon \in (0, \frac{1}{4})$ we have

$$\lim_{t_k \to T} |\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2\varepsilon-2} = 0, \quad \lim_{t_k \to T} |\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^{2\varepsilon-1} = 0,$$

which together with (3.31) implies that

$$|\nabla \mathbf{E}(t_k)|_{L^2(\mathbb{R}^3)}^2 \leq C. \tag{3.32}$$

(3.32) contradicts Lemma 2.2.

This completes the proof of Corollary 1.2. □

Appendix A

We consider the cauchy problem of the 3D magnetic Zakharov system

$$\begin{aligned} & i\mathbf{E}_t + \Delta \mathbf{E} - n\mathbf{E} + i(\mathbf{E} \wedge \mathbf{B}) = 0, \\ & n_{tt} - \Delta n = \Delta |\mathbf{E}|^2, \\ & \Delta \mathbf{B} - i\eta \nabla \times (\nabla \times (\mathbf{E} \wedge \bar{\mathbf{E}})) + \delta \mathbf{B} = 0, \end{aligned} \tag{$A-1$}$$

with initial condition

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \quad n(0, x) = n_0(x), \quad n_t(0, x) = n_1(x). \tag{$A-2$}$$

Here, we give the formal derivation of the blow-up solution (1.9) for (A-1)-(A-2) in the cylindrical coordinate (r, θ, z) on $(t, T]$. Let

$$\begin{cases} \mathbf{E} = (E_1, 0, iE_1), \quad E_1(t, x) = \dfrac{a}{(T-t)^{\alpha}} P\left(\dfrac{\omega r}{(T-t)^{\beta}}, \dfrac{\omega z}{(T-t)^{\beta}}\right) e^{i(T-t)^{\sigma}}, \\ n(t, x) = \dfrac{b}{(T-t)^{2\alpha}} N\left(\dfrac{\omega r}{(T-t)^{\beta}}, \dfrac{\omega z}{(T-t)^{\beta}}\right), \\ \mathbf{B} = (0, \widetilde{B}, 0), \quad \widetilde{B}(t, x) = \dfrac{c}{(T-t)^{2\alpha}} M\left(\dfrac{\omega r}{(T-t)^{\beta}}, \dfrac{\omega z}{(T-t)^{\beta}}\right), \end{cases} \tag{$A-3$}$$

where $\omega > 0$ is a constant, $P(r, z) = P(\sqrt{x_1^2 + x_2^2}, z)$, $N(r, z) = N(\sqrt{x_1^2 + x_2^2}, z)$, $M(r, z) = M(\sqrt{x_1^2 + x_2^2}, z)$ are real radially symmetric functions on $\mathbb{R}^2 \times \mathbb{R}$ in the cylindrical coordinate, $\alpha, \beta, a, b, c, \sigma$ are parameters to be determined later. Supposing that the Zakharov system (A-1) has the above formal solution (A-3), we first make some formal computations in order to derive an elliptical system, whose existence will be considered in our forthcoming work.

Set

$$A = \frac{a}{(T-t)^{\alpha}}, \quad B = P\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right), \quad C = e^{i(T-t)^{\sigma}}.$$

A computation shows that

$$A_t = \frac{\partial}{\partial t}\left(\frac{a}{(T-t)^{\alpha}}\right) = \frac{a\alpha}{(T-t)^{\alpha+1}},$$

$$B_t = \frac{\partial}{\partial t} P\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right) = \frac{\beta}{T-t}(rP_r + zP_z),$$

$$C_t = \frac{\partial}{\partial t}\left(e^{i(T-t)^{\sigma}}\right) = e^{i(T-t)^{\sigma}} \cdot i \cdot \sigma \cdot (T-t)^{\sigma-1}.$$

It follows from (A-3) that

$$\begin{aligned} E_{1t}(t,x) =& A_t BC + AB_t C + ABC_t \\ =& \frac{a\alpha}{(T-t)^{\alpha+1}} \cdot P \cdot e^{i(T-t)^{\sigma}} + \frac{a}{(T-t)^{\alpha}} \cdot \frac{\beta}{T-t}(rP_r + zP_z) \cdot e^{i(T-t)^{\sigma}} \\ & + \frac{a}{(T-t)^{\alpha}} \cdot P \cdot e^{i(T-t)^{\sigma}} \cdot i \cdot \sigma \cdot (T-t)^{\sigma-1} \\ =& \frac{e^{i(T-t)^{\sigma}}}{(T-t)^{\alpha+1-\sigma}}\Big((T-t)^{-\sigma} a(\alpha P + \beta r P_r + \beta z P_z) - ia\sigma P\Big). \end{aligned}$$

That is,

$$\begin{aligned} iE_{1t} &= \frac{e^{i(T-t)^{\sigma}}}{i(T-t)^{\alpha+1-\sigma}} \cdot a \cdot \Big(i(T-t)^{-\sigma}(\alpha P + \beta r P_r + \beta z P_z) + \sigma P\Big), \\ i\mathbf{E}_t &= i(E_{1t}, 0, iE_{1t}) = (iE_{1t}, 0, -E_{1t}). \end{aligned} \qquad (A-4)$$

Extra calculation yields that

$$\begin{cases} \Delta E_1 &= \Delta\left(\frac{a}{(T-t)^{\alpha}} P\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right) e^{i(T-t)^{\sigma}}\right) \\ &= \frac{a}{(T-t)^{\alpha}} e^{i(T-t)^{\sigma}} \Delta P\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right) \\ &= \frac{a\omega^2}{(T-t)^{\alpha+2\beta}} e^{i(T-t)^{\sigma}} \Delta P \\ &= \frac{a\omega^2}{(T-t)^{\alpha+2\beta}} e^{i(T-t)^{\sigma}} \Big(P_{rr} + \frac{1}{r}P_r + P_{zz}\Big), \\ \Delta \mathbf{E} &= (\Delta E_1, 0, i\Delta E_1). \end{cases} \qquad (A-5)$$

The coupled terms $n\mathbf{E}$ and $\mathbf{E}\wedge\mathbf{B}$ can also be written as

$$\begin{cases} nE_1 = \dfrac{ab}{(T-t)^{\alpha+2\alpha}} P\cdot N\cdot e^{i(T-t)^{\sigma}}, \\ \\ n\mathbf{E} = (nE_1, 0, i\sigma E_1), \\ \\ \mathbf{E}\wedge\mathbf{B} = det\begin{vmatrix} i & j & k \\ E_1 & 0 & iE_1 \\ 0 & \widetilde{B} & 0 \end{vmatrix} = (-iE_1\widetilde{B}, 0, E_1\widetilde{B}). \end{cases} \qquad (A-6)$$

Next we calculate some terms about n, where

$$n = \frac{b}{(T-t)^{2\alpha}} N\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right). \qquad (A-7)$$

Let

$$D = \frac{b}{(T-t)^{2\alpha}}, \quad G = N\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right).$$

A simple computation yields

$$D_t = \frac{\partial}{\partial t}\left(\frac{b}{(T-t)^{2\alpha}}\right) = \frac{2\alpha\cdot b}{(T-t)^{2\alpha+1}},$$

$$D_{tt} = \frac{\partial}{\partial t}\left(\frac{2\alpha\cdot b}{(T-t)^{2\alpha+1}}\right) = \frac{2\alpha(2\alpha+1)b}{(T-t)^{2(\alpha+1)}},$$

$$G_t = \frac{\partial}{\partial t}\left(N\left(\frac{\omega r}{(T-t)^{\beta}}, \frac{\omega z}{(T-t)^{\beta}}\right)\right) = \frac{\beta}{T-t}\Big(rN_r + zN_z\Big),$$

$$\begin{aligned} G_{tt} &= \frac{\partial}{\partial t}\left(\frac{\beta}{T-t}\Big(rN_r + aN_z\Big)\right) \\ &= \frac{1}{(T-t)^2}\Big(\beta^2(r^2N_{rr} + rzN_{rz} + zrN_{zr} + z^2N_{zz}) + \beta(\beta+1)(rN_r + zN_z)\Big). \end{aligned}$$

The detail of G_{tt} is a little complex and will be given at the end of the appendix. We further have

$$\begin{aligned} n_{tt} &= D_{tt}G + 2D_tG_t + DG_{tt} \\ &= \frac{2\alpha(2\alpha+1)b}{(T-t)^{2(\alpha+1)}}N + 2\cdot\frac{2\alpha b}{(T-t)^{2\alpha+1}}\cdot\frac{\beta}{T-t}\beta(\beta+1)(rN_r + zN_z)\Big) \\ &= \frac{b}{(T-t)^{2(\alpha+1)}}\Big(2\alpha(2\alpha+1)N + \beta(4\alpha+\beta+1)(rN_r + zN_z) \\ &\qquad + \beta^2(r^2N_{rr} + rzN_{rz} + zrN_{zr} + z^2N_{zz})\Big). \end{aligned} \qquad (A-8)$$

$$
\begin{aligned}
\Delta n &= \Delta\left(\frac{b}{(T-t)^{2\alpha}}N\left(\frac{\omega r}{(T-t)^{\beta}},\frac{\omega z}{(T-t)^{\beta}}\right)\right)\\
&= \frac{b\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta N\\
&= \frac{b\omega^2}{(T-t)^{2(\alpha+\beta)}}\Big(N_{rr}+\frac{1}{r}N_r+N_{zz}\Big).
\end{aligned}
\qquad (A-9)
$$

On the other hand, a computation claims

$$
|\mathbf{E}|^2=\mathbf{E}\cdot\overline{\mathbf{E}}=(E_1,0,iE_1)\cdot(\overline{E}_1,0,-i\overline{E}_1)=2|E_1|^2=2\frac{a^2}{(T-t)^{2\alpha}}|P|^2, \qquad (A-10)
$$

$$
\begin{aligned}
\Delta(|\mathbf{E}|^2) &= \Delta(2|E_1|^2)\\
&= 2\Delta\left(\frac{a^2}{(T-t)^{2\alpha}}\left|P\left(\frac{\omega r}{(T-t)^{\beta}},\frac{\omega z}{(T-t)^{\beta}}\right)\right|^2\right)\\
&= \frac{2a^2\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta(|P|^2).
\end{aligned}
\qquad (A-11)
$$

We finally calculate the terms on magnetic field $\mathbf{B}$, where

$$
\mathbf{B}=(0,\widetilde{B},0),\quad \widetilde{B}(t,x)=\frac{c}{(T-t)^{2\alpha}}M\left(\frac{\omega r}{(T-t)^{\beta}},\frac{\omega z}{(T-t)^{\beta}}\right).
$$

We claim

$$
\begin{aligned}
\Delta\widetilde{B} &= \Delta\left(\frac{c}{(T-t)^{2\alpha}}M\left(\frac{\omega r}{(T-t)^{\beta}},\frac{\omega z}{(T-t)^{\beta}}\right)\right)\\
&= \frac{c\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta M\\
&= \frac{c\omega^2}{(T-t)^{2(\alpha+\beta)}}\Big(M_{rr}+\frac{1}{r}M_r+M_{zz}\Big),\\
\Delta\mathbf{B} &= (0,\Delta\widetilde{B},0).
\end{aligned}
\qquad (A-12)
$$

Due to

$$
\mathbf{E}\wedge\overline{\mathbf{E}}=det\begin{vmatrix} i & j & k\\ E_1 & 0 & iE_1\\ \overline{E}_1 & 0 & -i\overline{E}_1\end{vmatrix}=(0,2i|E_1|^2,0)
$$

and

$$
\Delta(2i|E_1|^2)=2i\Delta(|E_1|^2)=2i\Delta\left(\frac{a^2}{(T-t)^{2\alpha}}|P|^2\right)=2i\frac{a^2\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta(P^2),
$$

one achieves

$$
\nabla\times(\nabla\times(\mathbf{E}\wedge\overline{\mathbf{E}}))=(0,-\Delta(2i|E_1|^2),0)=\left(0,-2i\frac{a^2\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta(P^2),0\right). \qquad (A-13)
$$

Compiling the above computations, we can rewrite the first equation in (A-1) as

$$iE_{1t} + \Delta E_1 - nE_1 + E_1\widetilde{B} = 0. \qquad (A-14)$$

The second equation in the Zakharov system (A-1) is a scalar equation in $\mathbb{R}$

$$n_{tt} - \Delta n = \Delta|\mathbf{E}|^2. \qquad (A-15)$$

However, the third equation in the Zakharov system (A-1) is vector-valued in $\mathbb{C}^3$, and it can be written as

$$\Delta\widetilde{B} + i\eta\Delta(2i|E_1|^2) + \delta\widetilde{B} = 0. \qquad (A-16)$$

Therefore, taking (A-4)-(A-12) into equations (A-14)-(A-16), we derive

$$\begin{cases} \frac{e^{i(T-t)^\sigma}}{(T-t)^{\alpha+1-\sigma}}\left(anP + i(T-t)^{-\sigma}a(\alpha P + \beta rP_r + zP_z)\right) \\ \qquad + \frac{e^{i(T-t)^\sigma}}{(T-t)^{\alpha+2\beta}}a\omega^2\Delta P - \frac{e^{i(T-t)^\sigma}}{(T-t)^{\alpha+2\alpha}}abPN \\ \qquad + \frac{e^{i(T-t)^\sigma}}{(T-t)^{\alpha+2\alpha}}acPM = 0, \\ \frac{b}{(T-t)^{2(\alpha+1)}}\left(2\alpha(2\alpha+1)N + (4\alpha+\beta+1)\beta(rN_r + zN_z)\right. \\ \qquad \left. + \beta^2(r^2N_{rr} + rzN_{rz} + zrN_{zr} + z^2N_{zz})\right) \\ \qquad - \frac{b\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta N \\ \qquad = \frac{2a^2\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta(P^2), \\ \frac{c\omega^2}{(T-t)^{2(\alpha+\beta)}}M + \sigma\frac{c}{(T-t)^{2\alpha}}M = 2\eta\frac{a^2\omega^2}{(T-t)^{2(\alpha+\beta)}}\Delta(|P|^2). \end{cases} \qquad (A-17)$$

Here, $\Delta P = P_{rr} + \frac{1}{r}P_r + P_{zz}$ as the operator Δ was calculated in the cylindrical coordinate. In order to get a related elliptical system, the coefficients in (A-17) need to satisfy the following conditions:

$$\alpha + 1 - \sigma = \alpha + 2\beta = \alpha + 2\alpha$$
$$2(\alpha+1) = 2(\alpha+\beta),$$

which implies that $\alpha = \beta = 1, \sigma = -1$. Let $b = c = 2a^2 = \omega^2$, Equations (A-17) reduce to

$$\begin{cases} \Delta P - P - NP + MP = 0, \\ \lambda^2(r^2N_{rr} + rzN_{rz} + zrN_{zr} + z^2N_{zz} \\ \qquad + 6(rN_r + zN_z) + 6N) - \Delta N = \Delta(|P|^2), \\ \Delta M + \delta(T-t)^2\lambda^2 M = \eta\Delta(|P|^2), \end{cases} \qquad (A-18)$$

where $\lambda = \dfrac{1}{\omega}$ and $\omega > 0$.

Let $M = \eta \mathcal{F}^{-1}\left(\dfrac{|\xi|^2}{|\xi|^2 - \delta(T-t)^2\lambda^2}\mathcal{F}(P^2)\right)$, then (P, N) solves the following system:

$$\left\{\begin{array}{l} \Delta P - P\eta\mathcal{F}^{-1}\left(\dfrac{|\xi|^2}{|\xi|^2 - \sigma(T-t)^2\lambda^2}\mathcal{F}(P^2)\right) + P = NP, \\ \lambda^2(r^2 N_{rr} + rzN_{rz} + zrN_{zr} + z^2 N_{zz} \\ \qquad +6(rN_r + zN_z) + 6N) - \Delta N = \Delta(|P|^2). \end{array}\right. \qquad (A-19)$$

The existence of solutions for (A-19) will be discussed in our next work.

At the end of this appendix, we will calculate G_t and G_{tt}. Firstly, the derivative G_t is

$$\begin{aligned} G_t &= \tfrac{\partial}{\partial t}\left(N\left(\tfrac{\omega r}{(T-t)^\beta}, \tfrac{\omega z}{(T-t)^\beta}\right)\right) \\ &= N_1' \cdot \tfrac{\partial}{\partial t}\left(\tfrac{\omega r}{(T-t)^\beta}\right) + N_2' \cdot \tfrac{\partial}{\partial t}\left(\tfrac{\omega z}{(T-t)^\beta}\right) \\ &= N_1' \cdot \omega r \tfrac{\beta}{(T-t)^{\beta+1}} + N_2' \cdot \omega z \tfrac{\beta}{(T-t)^{\beta+1}}, \end{aligned} \qquad (A-20)$$

where N_1' denotes the partial derivative of $N(\circ, \bullet)$ with respect to its first variable $\circ$, and N_2' the partial derivative of $N(\circ, \bullet)$ with respect to its second variable $\bullet$. In view of

$$N_r = N_1' \cdot \tfrac{\partial}{\partial r}\left(\tfrac{\omega r}{(T-t)^\beta}\right) + N_2' \cdot 0 = N_1' \tfrac{\omega}{(T-t)^\beta}$$

$$N_z = N_1' \cdot 0 + N_2' \cdot \tfrac{\partial}{\partial z}\left(\tfrac{\omega z}{(T-t)^\beta}\right) = N_2' \tfrac{\omega}{(T-t)^\beta},$$

we have

$$G_t = \frac{\beta}{T-t}(rN_r + zN_z).$$

Next, it follows from (A-20) that the second order derivative G_{tt} is

$$\begin{aligned} G_{tt} &= \tfrac{\partial}{\partial t}\left(N_1' \tfrac{\omega r\beta}{(T-t)^{\beta+1}}\right) + \tfrac{\partial}{\partial t}\left(N_2' \tfrac{\omega z\beta}{(T-t)^{\beta+1}}\right) \\ &= \tfrac{\partial N_1'}{\partial t} \cdot \tfrac{\omega r\beta}{(T-t)^{\beta+1}} + N_1' \cdot \tfrac{\partial}{\partial t}\left(\tfrac{\omega r\beta}{(T-t)^{\beta+1}}\right) \\ &\quad + \tfrac{\partial N_2'}{\partial t} \cdot \tfrac{\omega z\beta}{(T-t)^{\beta+1}} + N_2' \cdot \tfrac{\partial}{\partial t}\left(\tfrac{\omega z\beta}{(T-t)^{\beta+1}}\right) \\ &= \left(N_{11}'' \tfrac{\omega r\beta}{(T-t)^{\beta+1}} + N_{12}'' \tfrac{\omega z\beta}{(T-t)^{\beta+1}}\right) \tfrac{\omega r\beta}{(T-t)^{\beta+1}} \\ &\quad + N_1' \cdot \omega r\beta(\beta+1) \tfrac{1}{(T-t)^{\beta+2}} \\ &\quad + \left(N_{21}'' \tfrac{\omega r\beta}{(T-t)^{\beta+1}} + N_{22}'' \tfrac{\omega z\beta}{(T-t)^{\beta+1}}\right) \tfrac{\omega r\beta}{(T-t)^{\beta+1}} \\ &\quad + N_2' \cdot \omega r\beta(\beta+1) \tfrac{1}{(T-t)^{\beta+2}}. \end{aligned}$$

Since

$$N_{rr} = N''_{11}\tfrac{\omega^2}{(T-t)^{2\beta}}, \quad N_{rz} = N''_{12}\tfrac{\omega^2}{(T-t)^{2\beta}},$$

$$N_{zr} = N''_{21}\tfrac{\omega^2}{(T-t)^{2\beta}}, \quad N_{zz} = N''_{22}\tfrac{\omega^2}{(T-t)^{2\beta}},$$

we derive

$$G_{tt} = \frac{1}{(T-t)^2}\left(\beta^2(r^2N_{rr} + rzN_{rz} + zrN_{zr} + z^2N_{zz}) + \beta(\beta+1)(rN_r + zN_z)\right).$$

□

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