Constructions of Lagrangian Cobordisms

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1 Introduction

A contact manifold is an odd-dimensional manifold Y^{2n+1} together with a maximally non-integrable hyperplane distribution ξ. In a contact manifold, *Legendrian submanifolds* play a central role. These are the maximal integral submanifolds of

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$\xi : \Lambda^n$ such that $T_p\Lambda \subset \xi$, for all $p \in \Lambda$. In general, Legendrian submanifolds are plentiful and easy to construct. In this article we will restrict our attention to the contact manifold $\mathbb{R}^3$ with its standard contact structure $\xi = \ker \alpha$, where $\alpha = dz - y\,dx$. In this setting, every smooth knot or link has an infinite number of non-equivalent Legendrian representatives. More background on Legendrian knots is given in Sect. 2.

The even-dimensional siblings of contact manifolds are symplectic manifolds. These are even-dimensional manifolds M^{2n} equipped with a closed, non-degenerate 2-form ω. In symplectic manifolds, *Lagrangian submanifolds* play a central role. Lagrangian submanifolds are the maximal dimensional submanifolds where ω vanishes on the tangent spaces: L^n such that $\omega|_L = 0$. When the symplectic manifold is exact, $\omega = d\lambda$, it is important to understand the more restrictive subset of *exact* Lagrangians: these are submanifolds where $\lambda|_L$ is an exact 1-form. Geometrically, L exact means that for any closed curve $\gamma \subset L$, $\int_\gamma \lambda = 0$. In this article, we will restrict our attention to a symplectic manifold that is symplectomorphic to $\mathbb{R}^4$ with its standard symplectic structure $\omega_0 = \sum dx_i \wedge dy_i$. In contrast to Legendrians, Lagrangians are scarce. For example, in $\mathbb{R}^4$ with its standard symplectic structure, the torus is the only closed surface that will admit a Lagrangian embedding into $\mathbb{R}^4$. A famous theorem of Gromov [37] states that there are *no* closed, exact Lagrangian submanifolds of $\mathbb{R}^4$.

There has been a great deal of recent interest in a certain class of non-closed, exact Lagrangian submanifolds, known as *Lagrangian cobordisms*. These Lagrangian submanifolds live in the symplectization of a contact manifold and have cylindrical ends over Legendrians. In this article, we will focus on exact, orientable Lagrangian cobordisms from the Legendrian Λ_- to the Legendrian Λ_+ that live in the symplectization of $\mathbb{R}^3$; this symplectization is $\mathbb{R} \times \mathbb{R}^3$ equipped with the exact symplectic form $\omega = d(e^t\alpha)$, where t is the coordinate on $\mathbb{R}$ and $\alpha = dz - y\,dx$ is the standard contact form on $\mathbb{R}^3$. See Fig. 5 for a schematic picture of a Lagrangian cobordism and Definition 1 for a formal definition. Such Lagrangian cobordisms were first introduced in Symplectic Field Theory (SFT) [25]: in relative SFT, we get a category whose objects are Legendrians and whose morphisms are Lagrangian cobordisms. *Lagrangian fillings* occur when $\Lambda_- = \emptyset$ and are key objects in the Fukaya category, which is an important invariant of symplectic four-manifolds. A *Lagrangian cap* occurs when $\Lambda_+ = \emptyset$.

A basic question tied to understanding the general existence and behavior of Lagrangian submanifolds is to understand the existence of Lagrangian cobordisms: *Given two Legendrians $\Lambda_\pm$, when does there exist a Lagrangian cobordism from Λ_- to Λ_+?* There are known to be a number of obstructions to this relation on Legendrian submanifolds coming from both classical and non-classical invariants of the Legendrians $\Lambda_\pm$. Some of these obstructions are described in Sect. 2.3. To complement the obstructions, there are some known constructions. For example, it is well known [7, 23, 26] that there exists a Lagrangian cobordism between Legendrians $\Lambda_\pm$ that differ by Legendrian isotopy. In addition, by Ekholm et al. [23], Chantraine [8], it is known that there exists a Lagrangian cobordism from Λ_- to Λ_+ if Λ_- can be obtained from Λ_+ by a "pinch" move or if $\Lambda_+ = \Lambda_- \cup U$, where

$$(A) \qquad\qquad (B)$$
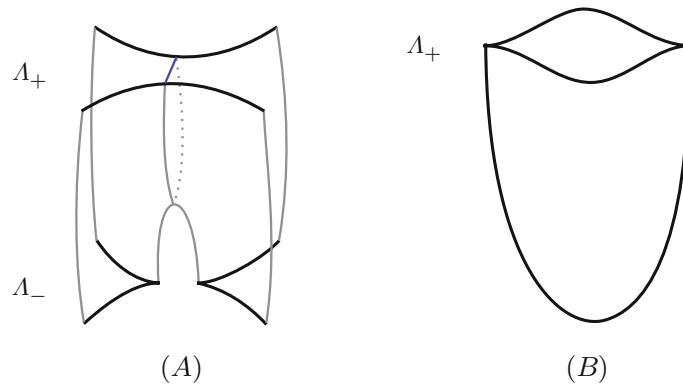
Fig. 1 (**a**) The pinch move on Λ_+ produces a Lagrangian saddle; (**b**) Λ_+ obtained by introducing an unknotted component to Λ_- corresponds to the Lagrangian cobordism having a local min

U denotes a Legendrian unknot with maximal Thurston-Bennequin number of -1 that is contained in the complement of a ball containing Λ_-. Topologically, between these slices, the cobordism changes by a saddle move (1-handle) and the addition of a local minimum (0-handle); see Fig. 1. It is important to notice that there is *not* an elementary move corresponding to a local maximum (2-handle) move. By stacking these individual cobordisms obtained from isotopy, saddles, and minimums, one obtains what is commonly referred to as a *decomposable* Lagrangian cobordism. Through these moves, it is easy to construct Lagrangian cobordisms and fillings; see an example in Fig. 7.

Towards understanding the existence of Lagrangians, it is natural to ask: *Does there exist a Lagrangian cobordism from Λ_- to Λ_+ if and only if there exists a decomposable Lagrangian cobordism from Λ_- to Λ_+?* We know the answer to this question is "No": by studying the "movies" of the not necessarily Legendrian slices of a Lagrangian, Sauvaget, Murphy, and Lin [42, 59] have shown that there exists a genus two Lagrangian cap of the Legendrian unknot with Thurston-Bennequin number equal to -3 and rotation number 0. The *Lagrangian diagram moves* used by Lin [42] to construct a Lagrangian cap are described in Sect. 3.3. The necessity of a local maximum when $\Lambda_+ \neq \emptyset$ is not currently understood.

To formulate some precise motivating questions, we will use *ribbon cobordism* to denote a $2n$-dimensional manifold that can be built from k-handles with $k \leq n$. This idea of restricting the handle index is well known in symplectic topology: Eliashberg [15, 51] has shown that any $2n$-dimensional Stein manifold admits a handle decomposition with handles of dimension at most n, and thus any $2n$-dimensional Stein cobordism between closed, $(2n - 1)$-dimensional contact manifolds must be ribbon. Working in the relative setting with submanifolds and using the handle decomposition from the "height" function given by the $\mathbb{R}$ coordinate on $\mathbb{R} \times \mathbb{R}^3$, we see that all decomposable 2-dimensional Lagrangian cobordisms between 1-dimensional Legendrian submanifolds are ribbon cobordisms. We are led to the following natural questions.

Motivating Questions 1 Suppose $\Lambda_+ \neq \emptyset$ and there exists a connected Lagrangian cobordism L from Λ_- to Λ_+. Then:

1. Does there exist a decomposable Lagrangian cobordism from Λ_- to Λ_+?
2. Does there exist a ribbon Lagrangian cobordism from Λ_- to Λ_+?
3. Is L Lagrangian isotopic to a ribbon and/or decomposable Lagrangian cobordism?

There are some results known about Motivating Question 3 for the special case of the simplest Legendrian unknot. If U denotes the Legendrian unknot with Thurston-Bennequin number -1, it is known that every (exact) Lagrangian filling is orientable [55], and there is a unique (exact, orientable) Lagrangian filling of U up to compactly supported Hamiltonian isotopy [27]. Moreover, any Lagrangian cobordism from U to U is Lagrangian isotopic, via a compactly supported Hamiltonian isotopy, to one in a countable collection given by the trace of a Legendrian isotopy induced by a rotation [11].

Motivating Questions (1) and (2) are closely related and have deep ties to important questions in topology. Observe that a "yes" answer to (1) implies a "yes" to (2): if the existence of a Lagrangian cobordism implies the existence of a decomposable Lagrangian cobordism, then we also know the existence of a ribbon cobordism. Also note that when Λ_+ is topologically a slice knot and $\Lambda_- = \emptyset$, (2) is a symplectic version of the topological Slice-Ribbon conjecture: is every Lagrangian slice disk a ribbon disk? Cornwell, Ng, and Sivek conjecture that the answer to Motivating Question (1) and (3) is "No": using the theory of satellites, we know that there is a Lagrangian concordance between $\Lambda_\pm$ shown in Fig. 2, and in [17, Conjecture 3.3] it is conjectured that the concordance between the pair is not decomposable.

Fig. 2 There is a Lagrangian concordance between these Legendrian knots that is conjectured to be non-decomposable. Here Λ_- is a Legendrian trefoil and Λ_+ is a Legendrian Whitehead double of $m(9_{46})$
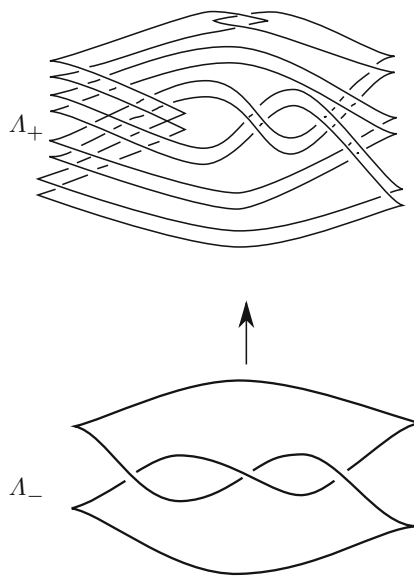
Very recently, Roberta Guadagni has discovered additional combinatorial moves that can be used to construct a "movie," meaning a sequence of slice pictures, of a Lagrangian cobordism; Fig. 9 illustrates one of these tangle moves. With one of Guadagni's moves, it is possible to construct a movie of a Lagrangian cobordism between the Legendrians pictured in Fig. 2; see Fig. 10. Guadagni's moves are "geometric": they are developed through proofs similar to those used in the satellite procedure, and thus the handle attachments involved in the cobordism are not obvious. In particular, at this point it is not known if Guadagni's tangle moves are independent from the decomposable moves.

This survey article is organized as follows. In Sect. 2, we provide some background on Legendrians and Lagrangians, formally define Lagrangian cobordisms, and summarize known obstructions to the existence of Lagrangian cobordisms. In Sect. 3, we describe three "combinatorial" ways to construct Lagrangian cobordisms, and in Sect. 4, we describe more abstract "geometric" ways to construct Lagrangian concordances and cobordisms through satellites. Then in Sect. 5, we describe some potential pathways—through the theory of rulings, Heegaard-Floer homology, and contact surgery—to potentially show the existence of Legendrians that are Lagrangian cobordant but are not related by a decomposable Lagrangian cobordism.

2 Background

2.1 Legendrian Knots and Links

In this section, we give a very brief introduction to Legendrian submanifolds in $\mathbb{R}^3$ and their invariants. More details can be found, for example, in the survey paper [28].

In $\mathbb{R}^3$, the **standard contact structure** ξ is a 2-dimensional plane field given by the kernel of the 1-form $\alpha = dz - y dx$. In $(\mathbb{R}^3, \xi = \ker \alpha)$, a **Legendrian knot** is a knot in $\mathbb{R}^3$ that is tangent to ξ everywhere. A useful way to visualize a Legendrian knot is to project it from $\mathbb{R}^3$ to $\mathbb{R}^2$. There are two useful projections: the **Lagrangian projection**

$$\pi_L : \quad \mathbb{R}^3 \quad \to \quad \mathbb{R}^2$$
$$(x, y, z) \mapsto (x, y),$$

as well as the **front projection**

$$\pi_F : \quad \mathbb{R}^3 \quad \to \quad \mathbb{R}^2$$
$$(x, y, z) \mapsto (x, z).$$

An example of a Legendrian trefoil is shown in Fig. 3.
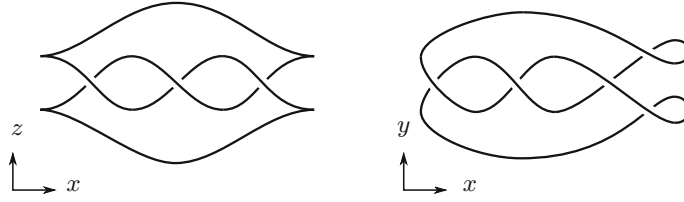
Fig. 3 The front projection (left) and the Lagrangian projection (right) of a Legendrian trefoil
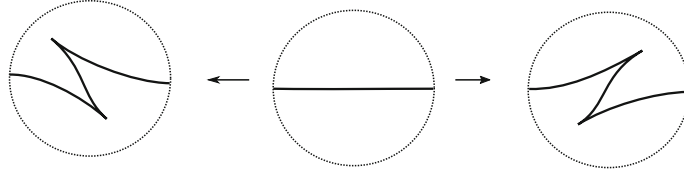
Fig. 4 Two ways to stabilize a Legendrian knot in the front projection

Legendrian submanifolds are equivalent if they can be connected by a 1-parameter family of Legendrian submanifolds. In fact, for each topological knot type there are infinitely many different Legendrian knots. Indeed, we can stabilize a Legendrian knot (as shown in Fig. 4) to get another Legendrian knot of the same topological knot type. We can see that these are not Legendrian equivalent using Legendrian invariants.

Two useful classical invariants of Legendrian knots Λ are the Thurston-Bennequin number $tb(\Lambda)$ and the rotation number $r(\Lambda)$. They can be computed easily from front projections. Given the front projection of a Legendrian knot or link Λ, the **Thurston-Bennequin number** is

$$tb(\Lambda) = \mathrm{writhe}(\pi_F(\Lambda)) - \#(\text{right cusps}),$$

where the writhe is the number of crossings counted with sign. Once the Legendrian knot is equipped with an orientation, the **rotation number** is

$$r(\Lambda) = \frac{1}{2}\Big(\#(\text{down cusps}) - \#(\text{up cusps})\Big).$$

One can use these two invariants to see that stabilizations change the Legendrian knot type.

In future sections, we will not assume that our Legendrians $\Lambda_\pm$ come equipped with an orientation. In our Motivating Questions described in Sect. 1, our Lagrangian cobordisms are always orientable, so the existence of a Lagrangian cobordism from Λ_- to Λ_+ will induce orientations on $\Lambda_\pm$.

There are many powerful non-classical invariants that can be assigned to a Legendrian knot. Although this will not be a focus of this paper, we will give a brief description of some of these invariants. One important invariant stems from *normal rulings*, defined independently by Chekanov and Pushkar [54] and Fuchs

[30]. A count of normal rulings leads to *ruling polynomials* [54]; more details will be discussed in Sect. 5.1. Through the closely related theory of generating families, one can also associate invariant polynomials that record the dimensions of *generating family homology groups* [32, 40, 58, 61]. In addition, through the theory of pseudo-holomorphic curves, one can associate to a Legendrian Λ a *differential graded algebra (DGA)*, $\mathcal{A}(\Lambda)$ [13, 24]. An *augmentation* is a DGA map from $\mathcal{A}(\Lambda)$ to a field. The count of augmentations is closely related to the count of ruling polynomials [30, 47, 50]. Augmentations can be used to construct finite-dimensional *linearized contact homology groups* [13], which are often known to be isomorphic to the generating family homology groups [32]. In addition, there are invariants for Legendrian knots coming from *Heegaard Floer Homology* [43, 52].

2.2 Lagrangian Cobordisms

Lagrangian cobordisms between Legendrian submanifolds always have "cylindrical ends" over the Legendrians, but other conditions vary: sometimes it is specified that the Lagrangian is exact, is embedded (or immersed), is orientable, or has a fixed Maslov class. In this paper, a Lagrangian cobordism is always exact, embedded, and orientable.

Definition 1 Let $\Lambda_\pm$ be two Legendrian knots or links in $(\mathbb{R}^3, \xi = \ker \alpha)$. A **Lagrangian cobordism** L **from** Λ_- **to** Λ_+ is an embedded, orientable Lagrangian surface in the symplectization $(\mathbb{R} \times \mathbb{R}^3, d(e^t \alpha))$ such that for some $N > 0$,

1. $L \cap ([-N, N] \times \mathbb{R}^3)$ is compact,
2. $L \cap ((N, \infty) \times \mathbb{R}^3) = (N, \infty) \times \Lambda_+$,
3. $L \cap ((-\infty, -N) \times \mathbb{R}^3) = (-\infty, -N) \times \Lambda_-$, and
4. there exists a function $f : L \to \mathbb{R}$ and constant numbers $\mathfrak{c}_\pm$ such that $e^t \alpha|_{TL} = df$, where $f|_{(-\infty, -N) \times \Lambda_-} = \mathfrak{c}_-$, and $f|_{(N, \infty) \times \Lambda_+} = \mathfrak{c}_+$.

A **Lagrangian filling of** Λ_+ is a Lagrangian cobordism with $\Lambda_- = \emptyset$; a **Lagrangian cap of** Λ_- is a Lagrangian cobordism with $\Lambda_+ = \emptyset$. A **Lagrangian concordance** occurs when $\Lambda_\pm$ are knots and L has genus 0.

Figure 5 is a schematic representation of a Lagrangian cobordism.

Remark 1 In condition (4) of Definition 1, the fact that $\Lambda_\pm$ are Legendrian will guarantee that $f_\pm$ will be locally constant. Using this, it follows that any genus zero Lagrangian surface that is cylindrical over Legendrian knots will be exact. When $\Lambda_\pm$ have multiple components, one needs to check that the constant does not vary: this condition guarantees the exactness of the Lagrangian cobordism obtained by "gluing" together Lagrangian cobordisms [10].

Fig. 5 A Lagrangian cobordism from Λ_- to Λ_+
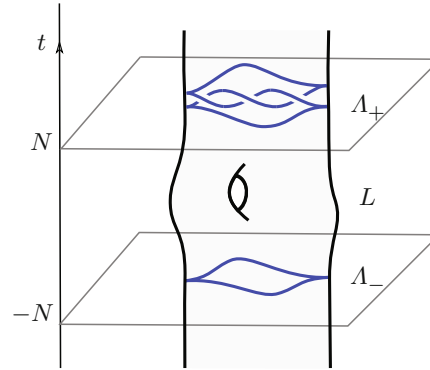
Remark 2 In contrast to topological cobordisms, Lagrangian cobordisms form a non-symmetric relationship on Legendrian knots [9]. In this article we will always denote the direction of increasing $\mathbb{R}_t$ coordinate by an arrow.

2.3   Obstructions to Lagrangian Cobordisms

The focus of this paper is on constructing Lagrangian cobordisms between two given Legendrians $\Lambda_\pm$. In the smooth world, any two knots are related by a smooth cobordism, but in this more restrictive Lagrangian world, there are a number of obstructions that are important to keep in mind when trying to explicitly construct Lagrangian cobordisms. Here we mention a few that come from classical and non-classical invariants of the Legendrians $\Lambda_\pm$.

Obstructions

1. If there exists a Lagrangian cobordism of genus g between Λ_- and Λ_+, then there must exist a smooth cobordism of genus g between the smooth knot types of Λ_- and Λ_+. Thus any obstruction of a smooth genus g cobordism between Λ_- and Λ_+ would obstruct a Lagrangian genus g cobordism.
2. Since there are no closed, exact Lagrangian surfaces [37], if there exists a Lagrangian cap (respectively, filling) for Λ, then there cannot exist a Lagrangian filling (respectively, cap) of Λ.
3. As shown in [7], if there exists a Lagrangian cobordism L from Λ_- to Λ_+, then

$$r(\Lambda_-) = r(\Lambda_+) \quad \text{and} \quad tb(\Lambda_+) - tb(\Lambda_-) = -\chi(L).$$

   In particular, if a Legendrian knot Λ admits a Lagrangian filling or cap, then $r(\Lambda) = 0$. Also, combining this equality on tb and the slice-Bennequin inequality [56], we see that, when Λ is a single component knot, if there exists a Lagrangian cap L of Λ, then $tb(\Lambda) \le -1$ and $g(L) \ge 1$.

4. If there exists a Maslov 0 ([21]) Lagrangian cobordism Σ from Λ_- to Λ_+, and Λ_- has an augmentation, then

   (a) $\#Aug(\Lambda_+; \mathbb{F}_2) \geq \#Aug(\Lambda_-; \mathbb{F}_2)$, where $\mathbb{F}_2$ is the finite field of two elements, and $\#Aug(\Lambda; \mathbb{F}_2)$ denotes the number of augmentations of Λ to $\mathbb{F}_2$ up to DGA homotopy [6, 53], and
   (b) the ruling polynomials $R_{\Lambda_\pm}(z)$ (see Sect. 5.1 for definitions) satisfy

   $$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq q^{-\chi(\Sigma)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2}),$$

   for any q that is a power of a prime number [53].

5. If Λ admits a Maslov 0 Lagrangian filling L, and if ϵ_L denotes the augmentation of Λ induced by L, then $LCH^k_{\epsilon_L}(\Lambda) \cong H_{n-k}(L)$, which is known as the Ekholm-Seidel isomorphism [20], and whose proof was completed by Dimitroglou Rizell in [19]. More generally, if there is a cobordism from Λ_- to Λ_+, and if Λ_- admits an augmentation, then [12] provides several long exact sequences relating the homology of the cobordism and the Legendrian contact (co)homologies of its Legendrian ends. A version of this isomorphism and these long exact sequences using generating families are given in [58].
6. If Λ admits an augmentation, Λ does not admit a Lagrangian cap, as the augmentation implies the non-acyclicity of the DGA $\mathcal{A}(\Lambda)$ [22, Theorem 5.5], and from [18, Corollary 1.9] if a Legendrian admits a Lagrangian cap then its DGA $\mathcal{A}(\Lambda)$ (with $\mathbb{F}_2$ coefficients) is acyclic.

There are additional obstructions, obtained through Heegaard Floer Theory, that can be used to obstruct Lagrangian concordances and cobordisms [1, 3, 35]. Some of these will be discussed more in Sect. 5.3.

Remark 3 Observe that the Obstructions 4 and 6 assume that the bottom Λ_- has an augmentation, and stabilized knots will never have an augmentation. It would be nice to have more obstructions when Λ_- is a stabilized knot. This might be possible using the theory of "satellites" described in Sect. 4.1: it is possible for the satellite of a stabilized Legendrian to admit an augmentation. See Sect. 4.3 for more discussions in this direction.

3 Combinatorial Constructions of Lagrangian Cobordisms

A convenient way of visualizing topological cobordisms is through "movies": a sequence of pictures that represent slices of the Lagrangian. In this section, we describe three known combinatorial ways to construct Lagrangian cobordisms through such an approach.

3.1 Decomposable Moves

It is well known that if Λ_- and Λ_+ are Legendrian isotopic, then there exists a Lagrangian cobordism from Λ_- to Λ_+; see, for example, [7, 23, 26]. Isotopy, together with two types of handle moves, form the basis for decomposable Lagrangian cobordisms.

Theorem 1 ([5, 23]) *If the front diagrams of two Legendrian links Λ_- and Λ_+ are related by any of the following moves, there is a Lagrangian cobordism L from Λ_- to Λ_+.*

Isotopy   *There is a Legendrian isotopy between Λ_- and Λ_+; see Fig. 6a–c for Reidemeister Moves I–III.*

1-**handle**   *The front diagram of Λ_- can be obtained from the front diagram of Λ_+ by "pinching" two oppositely-oriented strands; see Fig. 6d. We will also refer to this move as a "Pinch Move."*

0-**handle**   *The front diagram of Λ_- can be obtained from the front diagram of Λ_+ by deleting a component of Λ_+ that is the front diagram of a standard Legendrian unknot U with maximal Thurston-Bennequin number of -1 as long as there exist disjoint disks D_U, $D_{U^c} \subset \mathbb{R}^2_{xz}$ containing the xz-projection of U and the other components of Λ_+, respectively. Such an "unknot filling" can be seen in Fig. 6e.*

Definition 2 A Lagrangian cobordism L from Λ_- to Λ_+ is called **elementary** if it arises from isotopy, a single 0-handle, or a single 1-handle. A Lagrangian cobordism L from Λ_- to Λ_+ is **decomposable** if it is obtained by stacking elementary Lagrangian cobordisms.

(a) $\qquad\qquad$ (b) $\qquad\qquad$ (c) $\qquad\qquad$ (d) $\qquad\qquad$ (e)
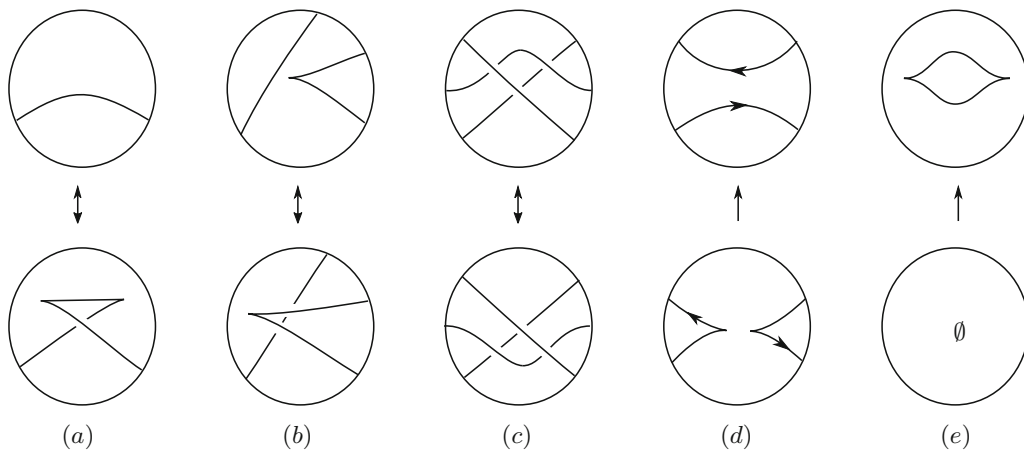
Fig. 6 Decomposable moves in terms of front projections. Arrows indicate the direction of increasing $\mathbb{R}_t$ coordinate in the symplectization. The move in (**b**) only shows the Reidemeister II move in the left cusp case, but there is an analogous move for the right cusp case

Observe that there is *not* an elementary move corresponding to a 2-handle (maximum). Also note that the elementary 1-handle (saddle) move can be used to connect two components or to split one component into two.

Decomposable cobordisms are particularly convenient as they are easy to describe in a combinatorial fashion, through a list of embedded Legendrian curves,

$$\Lambda_- = \Lambda_0 \to \Lambda_1 \to \cdots \to \Lambda_n = \Lambda_+,$$

where the front projection of the Legendrian Λ_{i+1} is related to that of Λ_i by isotopy or one of the 0-handle or 1-handle moves.

Example 1 One can construct a Lagrangian filling of a positive Legendrian trefoil with maximal Thurston-Bennequin number using the series of moves shown in Fig. 7: a 0-handle, followed by three Reidemeister I moves, followed by two 1-handles (or pinch moves). This gives a genus 1 (orientable, exact) Lagrangian filling of this Legendrian trefoil. Since we are assuming that Lagrangian fillings and caps are always exact, this implies that this trefoil cannot admit a Lagrangian cap; see Sect. 2.3 Obstruction 2.

Example 2 Using elementary moves, one can also construct a Lagrangian concordance from the unknot with $tb = -1$ to a Legendrian representative of the knot $m(9_{46})$, as shown on Fig. 8.
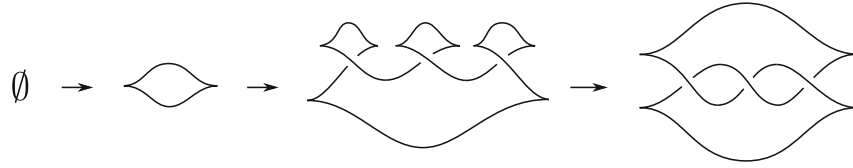
**Fig. 7** A decomposable Lagrangian filling of a Legendrian trefoil
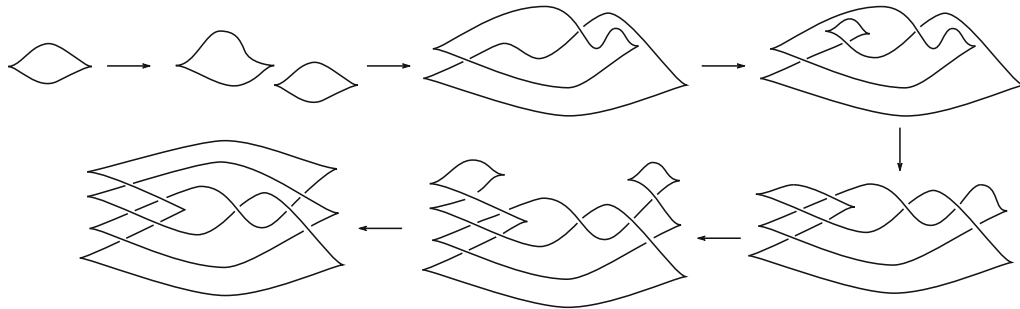
Fig. 8 A decomposable Lagrangian cobordism from a Legendrian unknot to a Legendrian $m(9_{46})$

3.2 Guadagni Moves

Very recently, Roberta Guadagni has discovered a new "tangle" move; see Fig. 9. This is not a local move: there are some global requirements. In particular, this move cannot be applied if all components of the tangle are contained in the same component of Λ_-: the component of Λ_- containing the blue strand must be different than the components containing the other strands of the tangle.

Example 3 With Guadagni's tangle move, it is possible to construct a Lagrangian cobordism between the Legendrians pictured in Fig. 2; see Fig. 10. However, at this point it is not known if Guadagni's tangle move is independent of the decomposable moves.

Fig. 9 Under some global conditions, there exists a Lagrangian cobordism between these tangles
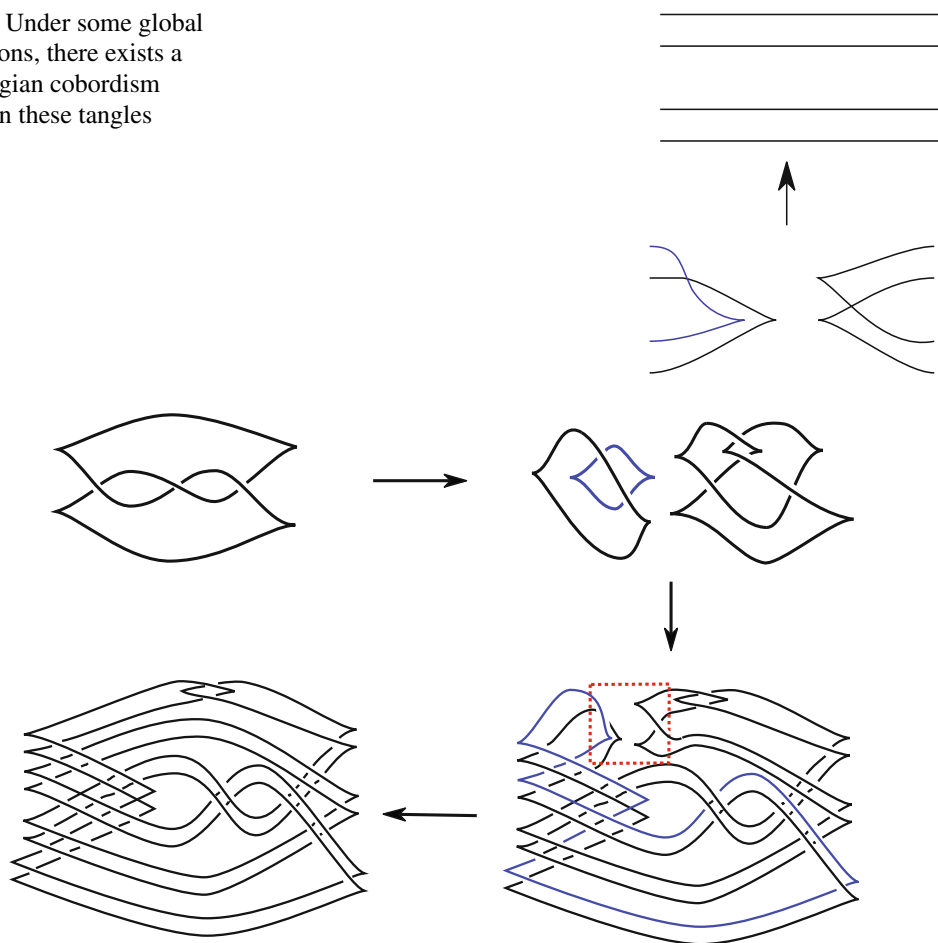
Fig. 10 A movie, using a Guadagni move, of an (orientable, exact) Lagrangian cobordism from the trefoil to the Whitehead double of $m(9_{46})$ in Fig. 2
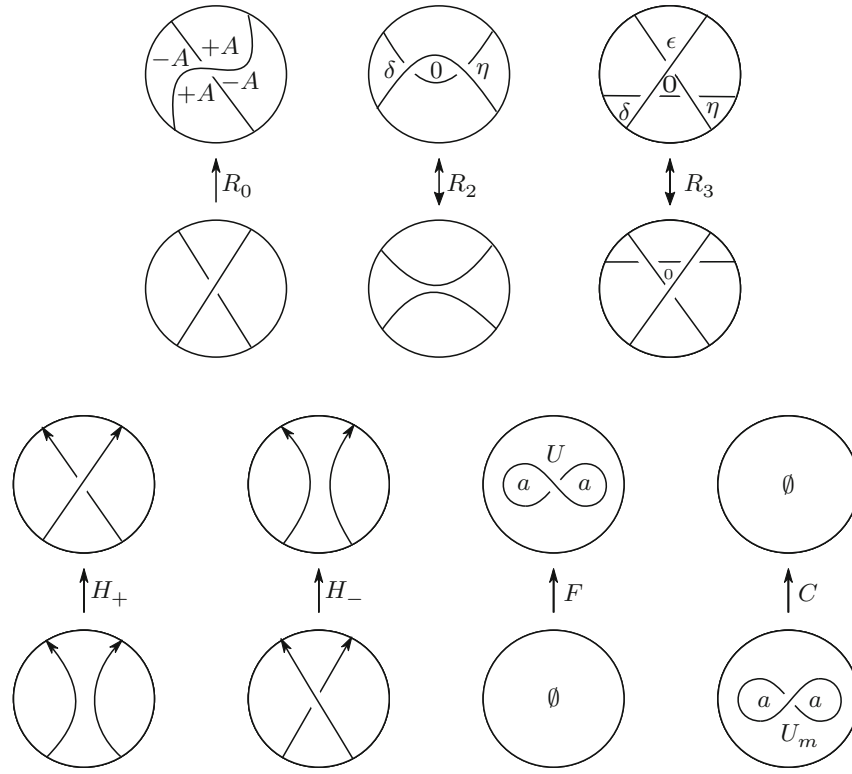
Fig. 11 The Lagrangian diagram moves. The labels in the R_0 move represent the change of area through the move, while other labels $0, \epsilon, \delta, \eta, a$ indicate the area of the corresponding regions; here 0 represents a positive area that is smaller than either the area ϵ, the area δ or the area η

3.3 Lagrangian Diagram Moves

As shown in Sect. 3.1, decomposable cobordisms are constructed from 0-handles and some 1-handles (saddles) but no 2-handles (caps). Based on the work of Sauvaget [59], Lin [42] constructs a genus two cap of a twice stabilized unknot, and thus gives the first explicit example of a non-decomposable Lagrangian cobordism. The construction describes time-slices of a Lagrangian cobordism through a list of moves on "decorated Lagrangian diagrams."

   A **decorated Lagrangian diagram** is a curve in the xy-plane with the compact regions decorated by a positive number, which is the **area** of the region. Figure 11 shows some examples: in the illustration of the F move, U is a Lagrangian projection of the Legendrian unknot with maximal Thurston-Bennequin number; in the illustration of the C move, U_m is a decorated Lagrangian diagram, but is *not* the Lagrangian projection of a Legendrian knot.

Theorem 2 ([42]) *Let $\Lambda_{\pm}$ be Legendrian links and $D_{\pm}$ be their corresponding decorated Lagrangian projections. If one can create a sequence of decorated Lagrangian diagrams*

$$D_- = D_0 \to D_1 \to \cdots \to D_n = D_+$$

such that each diagram D_{i+1} can be obtained from D_i by the following combinatorial moves, then there is a compact Lagrangian submanifold in $\mathbb{R} \times \mathbb{R}^3$ with boundary $\Lambda_- \cup \Lambda_+$, where $\Lambda_\pm \subset \{\pm N\} \times \mathbb{R}^3$, for some $N > 0$.

1. *R_0: a planar isotopy that changes areas by the amount $\pm A$, for $A > 0$. This operation can only be done in the direction specified.*
2. *R_2: a Reidemeister II move. One can either introduce or eliminate two crossings assuming some area conditions are satisfied: it is possible to introduce or remove two crossings as long as the area of the inner region, denoted by 0 in the diagram, is less than either the area δ or the area η. One can also do this move with the lower strand passing under the upper strand.*
3. *R_3: a Reidemeister III move. One can perform a Reidemeister III move as long as the area of the inner region, denoted by 0 in the diagram, is less than either the area ϵ, the area δ or the area η. The fixed center crossing can be reversed. Additionally, the moving strand can also occur as an overstrand.*
4. *H_+: a handle attachment that creates a positive crossing in the diagram.*
5. *H_-: a handle attachment that removes a negative crossing in the diagram.*
6. *F: a filling that creates the diagram U, which is the Lagrangian projection of an unknot with maximal Thurston-Bennequin number.*
7. *C: a cap that eliminates the diagram U_m, which is the topological mirror of U.*

*These moves are called **Lagrangian diagram moves**. Moreover, the constructed Lagrangian will be exact if, in addition,*

(E1) Each move results in a diagram with all components having a total signed area equal to 0. The signed area of a region is determined by the sum of the signed heights of its Reeb chords.

(E2) If a handle attachment merges two components of a link, the components being merged must be vertically split, meaning that the images of the xy-projections of these components are contained in disjoint disks.

Remark 4

1. For condition (E2), the H_- can never be applied to merge components, and H_+ can only be applied if the components being merged are vertically split.
2. A main distinction between the Lagrangian diagram moves and the decomposable moves is that each diagram D_i in the middle of the sequence is not necessarily the Lagrangian projection of a Legendrian link. They are just the xy-projection of some time t_i-slice of the cobordism. Thus the Lagrangian diagram moves are more flexible than the decomposable moves. However, keeping track of the areas is an added complication.

Example 4 Figure 12 illustrates the construction of a Lagrangian torus using the Lagrangian diagram moves. This torus fails to be exact since condition (E1) is violated. Figure 13 gives another construction of a Lagrangian torus. This time, all components have signed area 0, but now condition (E2) is violated.

Fig. 12 A (non-exact)
Lagrangian torus constructed
using the Lagrangian diagram
moves. The middle figure
violates (E1)
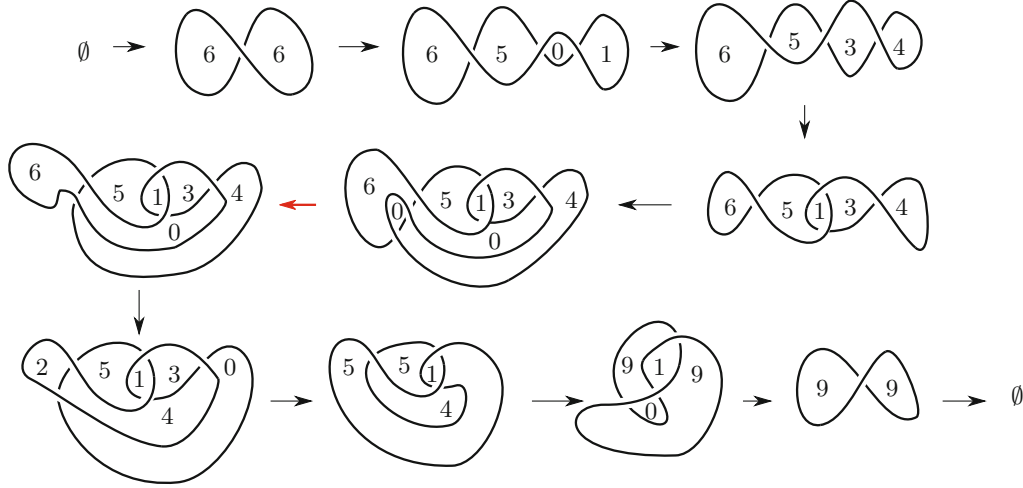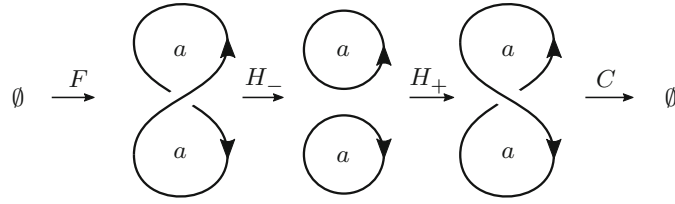
Fig. 13 A (non-exact) Lagrangian torus constructed using the Lagrangian diagram moves. These figures satisfy (E1) but (E2) is violated in the step labelled by a red arrow

4   Geometrical Constructions of Lagrangian Cobordisms

An important general way to know of the existence of Lagrangian cobordisms without using the constructions described in Sect. 3 comes through the satellite operation. In this section, we review the satellite construction and then state results from [17, 38] about the existence of a Lagrangian concordance/cobordism from Λ_- to Λ_+ implying the existence of a Lagrangian concordance/cobordism between corresponding satellites.

4.1   The Legendrian Satellite Construction

We begin by reviewing the construction of a Legendrian satellite; see also [48, Appendix] and [17, Sect. 2.2]. To construct a Legendrian satellite, begin by identifying the open solid torus $S^1 \times \mathbb{R}^2$ with the 1-jet space of the circle, $J^1 S^1 \cong$

Fig. 14 An example of a
Legendrian satellite
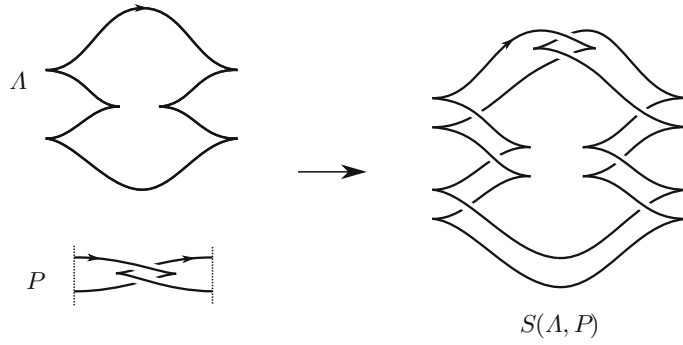
$T^*S^1 \times \mathbb{R}$, equipped with the contact form $\alpha = dz - ydx$, where x, y are the coordinates in T^*S^1 and z is the coordinate in $\mathbb{R}$. Similar to the situation for $\mathbb{R}^3 \cong J^1\mathbb{R}$, we can recover a Legendrian knot in J^1S^1 from its front projection in $S^1_x \times \mathbb{R}_z$, which is typically drawn by representing S^1 as an interval with its endpoints identified.

Given an oriented Legendrian **companion** knot $\Lambda \subset \mathbb{R}^3$ and a oriented Legendrian **pattern** knot $P \subset J^1(S^1)$, the Legendrian neighborhood theorem says that Λ has a standard neighborhood $N(\Lambda)$ such that there is a contactomorphism $\kappa : J^1(S^1) \to N(\Lambda)$. The **Legendrian satellite**, $S(\Lambda, P)$, is then the image $\kappa(P)$. The front projection of $S(\Lambda, P)$ is as shown in Fig. 14. In particular, suppose that the front projection of the pattern P intersects the vertical line at the boundary of the S^1 interval n times. We then make an n-copy of Λ by using n-disjoint copies of Λ that all differ by small translations in the z-direction. Take a point on the front projection of Λ that is oriented from left to right, cut the front of the n-copy open along the n-copy at that point, and insert the front diagram of P. The orientation on the satellite $S(\Lambda, P)$ is induced by the orientation on P.

Remark 5 The satellite operation often makes Legendrian knots "nicer"; for example, in Fig. 14, the companion Λ is stabilized and does not admit an augmentation or a normal ruling. However, the satellite $S(\Lambda, P)$ does admit a normal ruling and an augmentation.

## 4.2 Lagrangian Cobordisms for Satellites

In [17, Theorem 2.4], Cornwell, Ng, and Sivek, show that Lagrangian *concordance* is preserved by the Legendrian satellite operation.

Theorem 3 ([17]) *Suppose $P \subset J^1S^1$ is a Legendrian knot. If there exists a Lagrangian concordance L from a Legendrian knot Λ_- to a Legendrian knot Λ_+, then there exists a Lagrangian concordance L_P from $S(\Lambda_-, P)$ to $S(\Lambda_+, P)$.*

In particular, as shown in Fig. 8, there is a Lagrangian concordance from Λ_-, which is the Legendrian unknot with $tb = -1$, to Λ_+, which is the Legendrian

Fig. 15 For an n-stranded tangle, repeating this basic tangle n times produces a full twist
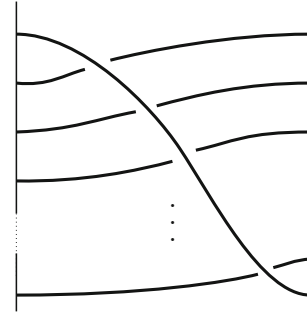
$m(9_{46})$ with maximal $tb = -1$. Using the Legendrian "clasp" tangle P as shown in Fig. 14—which produces the Legendrian Whitehead double—we can conclude that there exists a Lagrangian concordance from $S(\Lambda_-, P)$ to $S(\Lambda_+, P)$. In fact, $S(\Lambda_-, P)$ is the positive trefoil with $tb = 1$. Thus Theorem 3 implies that there exists a Lagrangian concordance between the Legendrian knots in Fig. 2.

Conjecture 1 ([17, Conjecture 3.3]) The Lagrangian concordance from $S(\Lambda_-, P)$ to $S(\Lambda_+, P)$ built through the satellite construction is not decomposable.

Theorem 3 has been extended to higher genus cobordisms by Guadagni, Sabloff, and Yacavone in [38]. To state their theorem, we need to first introduce the notion of "twisting" and then closing a tangle $T \subset J^1[0, 1]$. Given a Legendrian tangle $T \subset J^1[0, 1]$, ΔT is the tangle obtained by adding the tangle T and the full twist tangle Δ, which is illustrated in Fig. 15; the tangle $\Delta^t T$ can be thought of as T followed by t full twists. Given a Legendrian tangle $T \subset J^1[0, 1]$, $\overline{T} \subset J^1(S^1)$ will denote the associated closure to a Legendrian link.

Theorem 4 ([38]) *Suppose $T \subset J^1[0, 1]$ is a Legendrian tangle whose closure $\overline{T} \subset J^1(S^1)$ is a Legendrian knot. If there exists a Lagrangian cobordism L from Λ_- to Λ_+ of genus $g(L)$, then there exists a Lagrangian cobordism L_T from $S(\Lambda_-, \overline{\Delta^{2g(L)+1}T})$ to $S(\Lambda_+, \overline{\Delta T})$.*

In fact, Theorem 4 can be generalized to use the closure of different tangles T_- and T_+ that are Lagrangian cobordant; for details, see [38].

Remark 6 It is natural to wonder if, along the lines of Conjecture 1, this higher genus satellite procedure can create additional *candidates* for Legendrians that can be connected by a Lagrangian cobordism but not by a decomposable Lagrangian cobordism. In [38, Theorem 1.5], it is shown that if the cobordism L from Λ_- to Λ_+ is decomposable and the handles in the decomposition satisfy conditions known as "Property A", then the corresponding satellites $S(\Lambda_-, \Delta^{2g(L)+1}P)$ and $S(\Lambda_+, \Delta P)$ will also be connected by a decomposable Lagrangian cobordism. In particular, if there exists a decomposable cobordism L that does *not* satisfy Property A and is not isotopic to a cobordism that satisfies Property A, then the satellite construction would lead to a higher genus candidate that generalizes Conjecture 1.

4.3   Obstructions to Cobordisms Through Satellites

In Sect. 2.3, some known obstructions to the existence of a Lagrangian cobordism were mentioned. As mentioned in Remark 3, a number of these obstructions require Λ_- to admit an augmentation, and thus in particular Λ_- must be non-stabilized. However, as mentioned in Remark 5, it is possible for the satellite of a Legendrian Λ to admit an augmentation even if Λ does not. So the contrapositive of Theorem 3 provides a potential strategy for further obstructions to the existence of a Lagrangian cobordism when Λ_- does not admit an augmentation. For example, motivated by Obstruction 4 in Sect. 2.3, one can ask: *Can a count of augmentations give an obstruction to the existence of a Lagrangian concordance from $S(\Lambda_-, P)$ to $S(\Lambda_+, P)$ and thereby obstruct the existence of a Lagrangian concordance from Λ_- to Λ_+?* In fact, this augmentation count will not likely provide a further obstruction: a simple computation shows that when Λ is stabilized enough, the number of augmentations of $S(\Lambda, P)$ only depends on the Legendrian pattern P. If trying to pursue this path to obtain further obstructions to Lagrangian cobordisms, it is useful to keep in mind the following result of Ng that shows the DGA of the satellite of a Legendrian Λ might only remember the underlying knot type of Λ.

Theorem 5 ([49]) *Suppose Λ_1 and Λ_2 are stabilized Legendrian knots that are of the same topological knot type and have the same Thurston-Bennequin and rotation numbers. For a Legendrian pattern P whose front intersects a vertical line by two points, the DGAs of $S(\Lambda_1, P)$ and $S(\Lambda_2, P)$ are equivalent.*

## 5   Candidates for Non-decomposable Lagrangian Cobordisms

Now that we have developed some ways to construct a Lagrangian cobordism through combinatorial moves and satellites, we state some theorems that show *if* a Lagrangian cobordism does exist, then it cannot be decomposable: this addresses Motivating Question 1. While we discuss these theorems, it is useful to keep in mind the known obstructions to Lagrangian cobordisms that were mentioned in Sect. 2.3.

5.1   Candidates for Non-decomposable Lagrangian Cobordisms from Normal Rulings

One simple way to show that two Legendrians $\Lambda_\pm$ cannot be connected by a decomposable Lagrangian cobordism comes from a count of "combinatorial" rulings. Roughly, a **normal ruling** of a Legendrian Λ is a "decomposition" of the front projection into pairs of paths from left cusps to right cusps such that
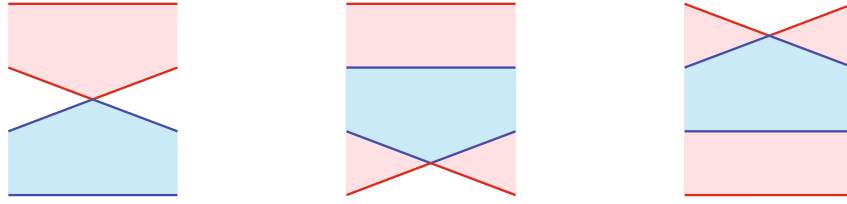
Fig. 16 Normal rulings near a switch
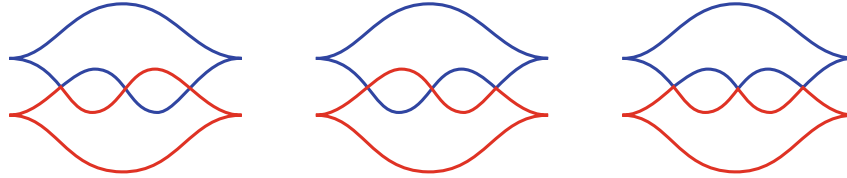
Fig. 17 All normal rulings of this max tb positive Legendrian trefoil

1. each pair of paths starts from a common left cusp and ends at a common right cusp, has no further intersections, and bounds a topological disk whose boundary is smooth everywhere other than at the cusps and certain crossings called **switches**, and
2. near a switch, the pair of paths must be arranged as in one of the diagrams in Fig. 16; observe that near the switch, vertical slices of the associated disks are either disjoint or the slices of one are contained in the slices of the other.

Formal definitions of normal rulings can be found in, for example, [54] and [30].

As an illustration, all normal rulings of a particular Legendrian trefoil are shown in Fig. 17.

For each normal ruling R, let $s(R)$ and $d(R)$ be the number of switches and number of disks, respectively. By Pushkar and Chekanov [54], the **ruling polynomial**

$$R_\Lambda(z) = \sum_R z^{s(R)-d(R)},$$

where the sum is over all the normal rulings, is an invariant of Λ under Legendrian isotopy. Normal rulings and augmentations are closely related even though they are defined in very different ways [30, 31, 50, 57].

We have the following obstruction to decomposable cobordisms in terms of normal rulings.

Theorem 6 *If Λ_- has m normal rulings and Λ_+ has n normal rulings with $m > n$, then there is no decomposable Lagrangian cobordism from Λ_- to Λ_+.*

Proof One can compare the number of normal rulings of the two ends for the decomposable moves, as shown in Fig. 18. Thus any normal ruling of Λ_- induces a normal ruling of Λ_+. Different normal rulings of Λ_- induce different normal rulings

Fig. 18 Comparison of
normal rulings for
decomposable moves
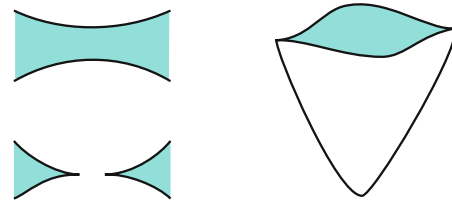
of Λ_+. Therefore the number of normal rulings of Λ_+ is bigger than or equal to the number of normal rulings of Λ_-.   □

Here is a strategy to show the existence of Legendrians that can be connected by a Lagrangian cobordism but not by one that is decomposable.

Strategy 1 *Choose Legendrians $\Lambda_\pm$ such that:*

1. *Λ_+ has fewer graded normal rulings than Λ_-, and*
2. *it is possible to construct, via a combination of the combinatorial constructions from Sect. 3 or the satellite construction from Sect. 4, a Lagrangian cobordism from Λ_- to Λ_+.*

Remark 7 If $\Lambda_\pm$ admit normal rulings, they will admit augmentations [31, 57]. From Sect. 2.3 Obstruction 4(b), we then know that if there is a Lagrangian cobordism from Λ_- to Λ_+, their ruling polynomials satisfy

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq q^{-\chi(\Sigma)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2}),$$

for any q that is a power of a prime number. Satisfying condition (1) in Strategy 1 means that the polynomial on the right side of the inequality has fewer terms than the polynomial on the left side of the inequality. If following this approach, it may be helpful to start by first finding a pair of positive integer coefficient polynomials that satisfy this inequality and condition (1) at the same time. One can start with checking the ruling polynomials of small crossing number Legendrian knots in [14].

## 5.2 Candidates for Non-decomposable Lagrangian Concordances from Topology

Observe that any decomposable Lagrangian concordance will be a smooth ribbon concordance. Thus it is potentially possible to use known obstructions to ribbon concordances to find examples of smooth knots whose Legendrian representatives cannot be connected by a decomposable Lagrangian concordance: constructing a Lagrangian concordance between very stabilized Legendrian representatives of these knot types, via the combinatorial techniques of Sect. 3 or geometric techniques of Sect. 4, will give an example of an exact Lagrangian concordance between knots that cannot be connected by a decomposable Lagrangian concordance.

For example, it is known [36, 41, 62] that the only knot that admits a ribbon concordance to the unknot is the unknot itself. This has as a corollary the following obstruction to a decomposable Lagrangian concordance.

Theorem 7 ([17, Theorem 3.2]) *If Λ_- is topologically non-trivial and Λ_+ is topologically an unknot, then there is no decomposable Lagrangian concordance from Λ_- to Λ_+.*

Example 5 To illustrate this theorem, here is a possible low crossing number Legendrian knot to examine as Λ_-. Consider the topological knot 6_1 which is slice and ribbon. Its maximum tb Legendrian representative Λ_{6_1} (see Fig. 19) has $tb = -5$ and $r = 0$. The DGA of this Legendrian $\mathcal{A}(\Lambda_{6_1})$ admits an augmentation, and thus Λ_{6_1} does not admit a Lagrangian cap; see Obstruction 6. in Sect. 2.3. Since we are trying to construct a Legendrian Λ_- that could be Lagrangian concordant to a stabilized unknot, which might have a Lagrangian cap, we will add some stabilizations that will prevent augmentations and thereby allow the possibility of a Lagrangian cap. If we now add a positive and a negative stabilization to Λ_{6_1}, we get a knot $\Lambda_{6_1}^{\pm}$ with $tb = -7$ and $r = 0$, which has no augmentation and is still topologically the knot 6_1. If, by a sequence of moves in Sect. 3, one can construct a concordance from $\Lambda_{6_1}^{\pm}$ to the $tb = -7$ stabilized unknot, then by Theorem 7 this Lagrangian concordance will not be decomposable; see Fig. 20. In fact, one can stabilize Λ_{6_1} *as many times as we wish* resulting in $tb(\Lambda_-) = t$ and $r(\Lambda_-) = r$ and try, using the combinatorial constructions of Sect. 3, to construct a Lagrangian concordance to Λ_+, where Λ_+ is a Legendrian unknot with $tb(\Lambda_+) = t$ and $r(\Lambda_+) = r$. *If possible*, such a construction would prove the existence of a non-decomposable Lagrangian concordance.

Fig. 19 Front diagram of Λ_{6_1}
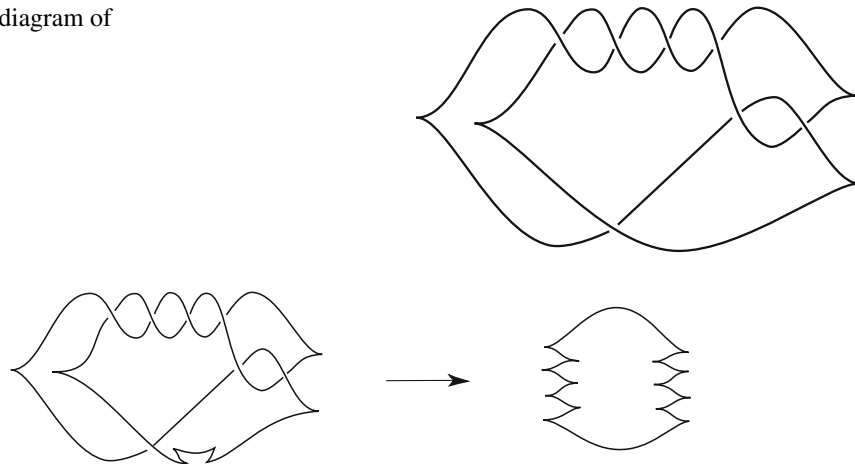
Fig. 20 Any Lagrangian concordance from the doubly stabilized Λ_{6_1} to the $tb = -7$, $r = 0$ Legendrian unknot would necessarily be non-decomposable

There are additional results from topology that give obstructions to the existence of ribbon concordances. For example, as shown by Gilmer [34] and generalized by Friedl and Powell [29], if K_- is ribbon concordant to K_+, then the Alexander polynomial of K_- divides the Alexander polynomial of K_+. We can invoke these results in a strategy to show the existence of non-decomposable Lagrangian concordances.

Strategy 2

1. *Use results from smooth topology to find examples of smooth knots $K_\pm$ such that K_- is not ribbon concordant to K_+.*
2. *For any pair of Legendrian representatives $\Lambda_\pm$ of the knot type $K_\pm$, even highly stabilized, use a combination of the combinatorial moves described in Sect. 3 to construct a Lagrangian concordance from Λ_- to Λ_+.*

The example with the knot 6_1 given above is a concrete example to try to apply this strategy with $K_- = 6_1$ and K_+ being an unknot. A possible example when K_+ is non-trivial is the following.

Example 6 Let K_- be the connect sum of the right- and left-handed trefoils, $K_- = T_r \# T_l$, and let K_+ be the connect sum of the figure-8 knot, F_8, with itself, $K_+ = F_8 \# F_8$. These knots are concordant but there is no ribbon concordance from K_- to K_+, as first shown by Gordon [36]. Choose Legendrian representatives $\Lambda_\pm$ of $K_\pm$ such that $tb(\Lambda_-) = tb(\Lambda_+)$ and $r(\Lambda_-) = r(\Lambda_+)$; note that $\Lambda_\pm$ can be very stabilized. *If* we can construct a Lagrangian concordance from Λ_- to Λ_+, via the combinatorial moves of Sect. 3, then we will have shown the existence of a pair of Legendrians that are (exactly, orientablility) Lagrangian concordant but cannot be connected by a decomposable Lagrangian concordance.

Remark 8 Some known obstructions to ribbon concordance are, in fact, obstructions to generalizations of ribbon concordance, namely **strong homotopy ribbon concordance** and **homotopy ribbon concordance**. A strong homotopy ribbon concordance is one whose complement is ribbon, i.e., can be built with only 1-handles and 2-handles. A homotopy ribbon concordance from K_- to K_+ is a concordance where the induced map on π_1 of the complement of K_- (resp. K_+) injects (resp. surjects) into π_1 of the complement of the concordance. Gordon [36] showed that

$$\text{ribbon concordant} \implies \text{strong homotopy ribbon concordant}$$

$$\implies \text{homotopy ribbon concordant.}$$

There have been a number of recent results obstructing (homotopy or strong homotopy) ribbon concordances from Heegaard-Floer and Khovanov homology [39, 41, 45, 62]; these results play an important role in Strategy 2.

5.3   Candidates for Non-decomposable Lagrangian Cobordisms from GRID Invariants

Some candidates for non-decomposable Lagrangian cobordisms of higher genus come from knot Floer homology. Using the grid formulation of knot Floer homology [52], Ozsváth, Szabó, and Thurston defined Legendrian invariants of a Legendrian link $\Lambda \subset \mathbb{R}^3$, called GRID invariants, which are elements in the *hat* flavor of knot Floer homology of $\Lambda \subset -S^3$:

$$\widehat{\lambda}^+(\Lambda), \widehat{\lambda}^-(\Lambda) \in \widehat{HFK}(-S^3, \Lambda).$$

For more background, see [44, 52].

Baldwin et al. [1] have shown that these GRID invariants can be used to obstruct the existence of decomposable Lagrangian cobordisms.

Theorem 8 ([1, Theorem 1.2]) *Suppose that $\Lambda_\pm$ are Legendrian links in $\mathbb{R}^3$ such that either*

1. $\widehat{\lambda}^+(\Lambda_+) = 0$ and $\widehat{\lambda}^+(\Lambda_-) \neq 0$, or
2. $\widehat{\lambda}^-(\Lambda_+) = 0$ and $\widehat{\lambda}^-(\Lambda_-) \neq 0$.

Then there is no decomposable Lagrangian cobordism from Λ_- to Λ_+.

Remark 9 By Baldwin et al. [4], in the standard contact manifold $\mathbb{R}^3$, the GRID invariants agree with the LOSS invariant [43]. The LOSS invariant is functorial on Lagrangian concordances by Baldwin and Sivek [2, 3]. Thus Theorem 8 would also obstruct the existence of general Lagrangian concordances and not only the decomposable ones. To find non-decomposable cobordisms using obstructions from [1], we should focus on non-zero genus cobordisms.

Using the facts that the GRID invariants are non-zero for the $tb = -1$ Legendrian unknot and that $\widehat{\lambda}^+(\Lambda_+)$ (resp. $\widehat{\lambda}^-(\Lambda_+)$) vanish for positively (negatively) stabilized Legendrian links, Theorem 8 gives the following corollary.

Corollary 1 ([1, Corollaries 1.3, 1.4])

1. If $\Lambda \subset \mathbb{R}^3$ is a Legendrian link such that $\widehat{\lambda}^+(\Lambda) = 0$ or $\widehat{\lambda}^-(\Lambda) = 0$, then there is no decomposable Lagrangian filling of Λ.
2. Suppose $\Lambda_\pm$ are Legendrian links such that either

   (a) $\widehat{\lambda}^+(\Lambda_-) \neq 0$ and Λ_+ is the positive stabilization of a Legendrian link, or
   (b) $\widehat{\lambda}^-(\Lambda_-) \neq 0$ and Λ_+ is the negative stabilization of a Legendrian link.

   Then there is no decomposable Lagrangian cobordism from Λ_- to Λ_+.

This provides another strategy to show the existence of Legendrians $\Lambda_\pm$ that are Lagrangian conbordant but cannot be connected by a decomposable Lagrangian cobordism.

Strategy 3

1. *Find Legendrians $\Lambda_\pm$ satisfying the GRID invariants conditions of Corollary 1 and Theorem 8 such that there are no known obstructions, as described in Sect. 2.3, to the existence of a Lagrangian cobordism from Λ_- to Λ_+.*
2. *Use a combination of the combinatorial moves described in Sect. 3 to construct a Lagrangian cobordism from Λ_- to Λ_+.*

Example 7 Concrete examples mentioned in [1, Sect. 4.1] can be used for Strategy 3. Let Λ_0, Λ_1 be the Legendrian $m(10_{132})$ knots and Legendrian $m(12n_{200})$ knots shown in [46, Figs. 2 and 3]. Modify them with a pattern shown in [1, Fig. 13] to get Λ_0' and Λ_1', which are of knot type $m(12n_{199})$ and $m(14n_{5047})$ (or its mirror), respectively. For $i, j = 0, 1$ we have $tb(\Lambda_i') = tb(\Lambda_i) + 2$ and $r(\Lambda_i') = r(\Lambda_i)$. There is no decomposable Lagrangian cobordism from

1. Λ_0 to Λ_1', or
2. Λ_1 to Λ_0'.

If we can construct, using the combinatorial techniques of Sect. 3, a Lagrangian cobordism (necessarily of genus 1) from Λ_0 to Λ_1' or from Λ_1 to Λ_0', then we will have found a non-decomposable Lagrangian cobordism.

Example 8 In [1, Sect. 4.3], the authors provide an infinite family of pairs of Legendrian knots where there does not exist a decomposable Lagrangian cobordism between them.

Remark 10 In Strategies 2 and 3, we emphasized the construction of Lagrangian cobordisms using the combinatorial techniques of Sect. 3. It would be interesting to know if the geometric constructions of Sect. 4 could also be used to show the existence of a Lagrangian concordance/cobordism from the theory of normal rulings, topology, or grid invariants, that are known to not be decomposable.

5.4 Non-decomposable Candidates Through Surgery

An additional strategy to show the existence of a non-decomposable Lagrangian filling comes from understanding properties of the contact manifold that is obtained from surgery on the Legendrian knot. In particular, Conway, Etnyre, and Tosun [16] have detected a relationship between Lagrangian fillings of a Legendrian and symplectic fillings of the contact manifold obtained by performing a particular type of surgery on the Legendrian.

Theorem 9 ([16, Theorem 1.1]) *There is a Lagrangian disk filling of Λ_+ if and only if the contact $+1$-surgery on $\Lambda_+ \subset \mathbb{R}^3 \subset S^3$ produces a contact manifold that is strongly symplectically fillable. If Λ_+ has a decomposable Lagrangian filling, then the filling can be taken to be Stein.*

In fact, [16] also shows that a filling will be a Stein filling if and only if Λ_+ bounds a *regular* Lagrangian disk: a Lagrangian disk is regular if there is a Liouville vector field that is tangent to the disk. Any decomposable Lagrangian filling is regular.

We now see another strategy to construct a non-decomposable Lagrangian filling.

Strategy 4 *Find a Legendrian Λ such that the $+1$-surgery on Λ produces a contact manifold that is strongly symplectically fillable but does not admit a Stein filling.*

An issue with this approach is a lack of examples: there are very few manifolds which carry strongly fillable but not Stein fillable contact structures. The main examples are the $1/n$ surgeries on the positive and negative trefoils; see works by Ghiggini [33] and Tosun [60]. However it is not obvious whether any of these contact structures are a contact $+1$-surgery on a Legendrian knot in S^3.

6 Conclusion

The desire to understand the flexibility and rigidity of Lagrangian submanifolds has led to a great deal of interesting research in symplectic topology. Similarly, trying to understand constructions of and obstructions to Lagrangian cobordisms has led to many interesting results. At this point, we have few concrete answers to the Motivating Questions stated in our Introduction. In particular, regarding Motivating Question (1), there are presently many candidates for Legendrians $\Lambda_\pm$ that can be connected by a Lagrangian cobordism but not by a decomposable Lagrangian cobordism: by understanding all the obstructions to Lagrangian cobordisms, one can come up with some good candidates. When trying and failing to construct a Lagrangian cobordism between a given pair, one may gain intuition for additional obstructions to Lagrangian cobordisms that are waiting to be discovered.

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