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A note on nowhere-zero 3-flow and Z_3-connectivity

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Abstract

There are many major open problems in integer flow theory, such as Tutte's 3-flow conjecture that every 4-edge-connected graph admits a nowhere-zero 3-flow, Jaeger et al.'s conjecture that every 5-edge-connected graph is Z_3-connected and Kochol's conjecture that every bridgeless graph with at most three 3-edge-cuts admits a nowhere-zero 3-flow (an equivalent version of 3-flow conjecture). Thomassen proved that every 8-edge-connected graph is Z_3-connected and therefore admits a nowhere-zero 3-flow. Furthermore, Lov*á*sz, Thomassen, Wu and Zhang improved Thomassen's result to 6-edge-connected graphs. In this paper, we prove that: (1) Every 4-edge-connected graph with at most seven 5-edge-cuts admits a nowhere-zero 3-flow. (2) Every bridgeless graph containing no 5-edge-cuts but at most three 3-edge-cuts admits a nowhere-zero 3-flow. (3) Every 5-edge-connected graph with at most five 5-edge-cuts is Z_3-connected. Our main theorems are partial results to Tutte's 3-flow conjecture, Kochol's conjecture and Jaeger et al.'s conjecture, respectively.

Keywords: Integer flow, nowhere-zero 3-flow, Z_3-connected, modulo 3-orientation, edge-cuts.

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1 Introduction

All graphs considered in this paper are loopless, but allowed to have multiple edges. A graph G is called *k-edge-connected*, if $G-S$ is connected for each edge set S with $|S| < k$. Let X, Y be two disjoint subsets of $V(G)$. Let $\partial_G(X, Y)$ be the set of edges of G with one end in X and the other in Y. In particular, if $Y = \overline{X}$, we simply write $\partial_G(X)$ for $\partial_G(X, Y)$, which is the *edge-cut* of G associated with X. The edge set $C = \partial_G(X)$ is called a *k-edge-cut* if $|\partial_G(X)| = k$. If X is nontrivial, we use G/X to denote the graph obtained from G by replacing X by a single vertex x that is incident with all the edges in $\partial_G(X)$.

Let D be an orientation of $E(G)$. The *out-cut* of D associated with X, denoted by $\partial^+_D(X)$, is the set of arcs of D whose tails lie in X. Analogously, the *in-cut* of D associated with X, denoted by $\partial^-_D(X)$, is the set of arcs of D whose heads lie in X. We refer to $|\partial^+_D(X)|$ and $|\partial^-_D(X)|$ as the out-degree and in-degree of X, and denote these quantities by $d^+_D(X)$ and $d^-_D(X)$, respectively.

Definition 1.1. (1) An orientation D of $E(G)$ is called a *modulo 3-orientation* if

$$d^+_D(v) - d^-_D(v) \equiv 0 \pmod 3$$

for every vertex $v \in V(G)$.

(2) A pair (D, f) is called a *nowhere-zero 3-flow* of G if D is an orientation of $E(G)$ and f is a function from $E(G)$ to $\{\pm 1, \pm 2\}$, such that

$$\sum_{e \in \partial^+_D(v)} f(e) = \sum_{e \in \partial^-_D(v)} f(e)$$

for every vertex $v \in V(G)$.

The 3-flow conjecture, proposed by Tutte as a dual version of Grötzsch's 3-color theorem for planar graphs, may be one of the most major open problems in integer flow theory.

Conjecture 1.2 (3-Flow conjecture, Tutte [9])**.** *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Kochol proved that Tutte's 3-flow conjecture is equivalent to the following two conjectures.

Conjecture 1.3 (Kochol [4])**.** *Every 5-edge-connected graph admits a nowhere-zero 3-flow.*

Conjecture 1.4 (Kochol [5])**.** *Every bridgeless graph with at most three 3-edge-cuts admits a nowhere-zero 3-flow.*

A weakened version of Conjecture 1.2, the so-called weak 3-flow conjecture, was proposed by Jaeger.

Conjecture 1.5 (Weak 3-flow conjecture, Jaeger [2])**.** *There is a natural number h such that every h-edge-connected graph admits a nowhere-zero 3-flow.*

Lai and Zhang [6] and Alon et al. [1] gave partial results on Conjectures 1.2 and 1.5.

Theorem 1.6 (Lai and Zhang [6]). *Every $4\lceil \log_2 n_0 \rceil$-edge-connected graph with at most n_0 odd-degree vertices admits a nowhere-zero 3-flow.*

Theorem 1.7 (Alon, Linial and Meshulam [1]). *Every $2\lceil \log_2 n \rceil$-edge-connected graph with n vertices admits a nowhere-zero 3-flow.*

Recently, Thomassen [8] confirmed weak 3-flow conjecture. He proved

Theorem 1.8 (Thomassen [8]). *Every 8-edge-connected graph is Z_3-connected and therefore admits a nowhere-zero 3-flow.*

Thomassen's method was further refined by Lovász, Thomassen, Wu and Zhang [7] to obtain the following theorem.

Theorem 1.9 (Lovász, Thomassen, Wu and Zhang [7]). *Every 6-edge-connected graph is Z_3-connected and therefore admits a nowhere-zero 3-flow.*

For more results on Tutte's 3-flow conjecture, we refer the reader to the introduction part of [7] and the book written by Zhang [11].

In this paper, we will give the following conjecture which is equivalent to Tutte's 3-flow conjecture.

Conjecture 1.10. *Every 5-edge-connected graph with minimum degree at least 6 has a nowhere-zero 3-flow.*

To prove the equivalence of Conjectures 1.2 and 1.10, the following lemma is needed.

Lemma 1.11 (Tutte [10]). *Let $F(G, k)$ be the number of nowhere-zero k-flows of G. Then $F(G, k) = F(G/e, k) - F(G \setminus e, k)$ if e is not a loop of G.*

Proposition 1.12. *Conjectures 1.2 and 1.10 are equivalent.*

Proof. It is obvious that Conjecture 1.2 implies Conjecture 1.3, and Conjecture 1.3 implies Conjecture 1.10. Now we prove that Conjecture 1.10 can imply Conjecture 1.3. Let G be a 5-edge-connected graph. Let G' be the graph obtained from G by gluing $|V(G)|$ disjoint copies of K_7, such that for each such copy H_i, $|V(H_i) \cap V(G)| = 1\,(i = 1, 2, \cdots, |V(G)|)$. Then G' is 5-edge-connected and its minimum degree is at least 6, and thus has a nowhere-zero 3-flow. By Lemma 1.11, G has a nowhere-zero 3-flow. Therefore Conjecture 1.10 implies Conjecture 1.3. Note that Conjecture 1.2 is equivalent to Conjecture 1.3. This completes the proof. □

Our first main result is the following theorem.

Theorem 1.13. *Let G be a bridgeless graph and let $P = \{C = \partial_G(X) : |C| = 3,\ X \subset V(G)\}$ and $Q = \{C = \partial_G(X) : |C| = 5,\ X \subset V(G)\}$. If $2|P| + |Q| \leq 7$, then G has a modulo 3-orientation (and therefore has a nowhere-zero 3-flow).*

As corollaries of Theorem 1.13, we obtain Theorems 1.14 and 1.15.

Theorem 1.14. *Every 4-edge-connected graph with at most seven 5-edge-cuts admits a nowhere-zero 3-flow.*

Theorem 1.15. *Every bridgeless graph containing no 5-edge-cuts but at most three 3-edge-cuts admits a nowhere-zero 3-flow.*

Remark. *The number of 3-edge-cuts in Theorem 1.15 can not be improved from three to four, since K_4 or any graph contractable to K_4 has no nowhere-zero 3-flow.*

Theorems 1.14 and 1.15 partially confirm Conjectures 1.2 and 1.4, respectively.

Definition 1.16. (1) A mapping $\beta_G : V(G) \mapsto Z_k$ is called a Z_k*-boundary* of G if

$$\sum_{v \in V(G)} \beta_G(v) \equiv 0 \pmod{k}$$

(2) A graph G is called Z_k*-connected*, if for every Z_k-boundary β_G, there is an orientation D_{β_G} and a function f_{β_G}: $E(G) \mapsto Z_k - \{0\}$, such that

$$\sum_{e \in \partial^+_{D_{\beta_G}}(v)} f_{\beta_G}(e) - \sum_{e \in \partial^-_{D_{\beta_G}}(v)} f_{\beta_G}(e) \equiv \beta_G(v) \pmod{k}$$

for every vertex $v \in V(G)$.

Jaeger, Linial, Payan and Tarsi [3] conjectured that

Conjecture 1.17 (Jaeger, Linial, Payan and Tarsi [3])**.** *Every 5-edge-connected graph is Z_3-connected.*

By applying a similar argument as in the proof of Theorem 1.13, we could obtain the second main result, which is a partial result to Conjecture 1.17.

Theorem 1.18. *Every 5-edge-connected graph with at most five 5-edge-cuts is Z_3-connected.*

In the next section, some necessary preliminaries will be given. In Sections 3 and 4, proofs of Theorems 1.13 and 1.18 will be given, respectively.

2 Preliminaries

In this section, we will give additional but necessary notations and definitions, and then give some useful lemmas.

Definition 2.1. Let β_G be a Z_3-boundary of G. An orientation D of G is called a β_G*-orientation* if

$$d^+_D(v) - d^-_D(v) \equiv \beta_G(v) \pmod{3}$$

for every vertex $v \in V(G)$.

Let G be a graph and A be a vertex subset of G. The *degree* of A, denoted by $d_G(A)$, is the number of edges with precisely one end in A. Moreover if $A = \{x\}$, we simply write $d_G(x)$.

Let G be a graph and β_G be a Z_3-boundary of G. Define a mapping $\tau_G : V(G) \mapsto \{0, \pm1, \pm2, \pm3\}$ such that, for each vertex $x \in V(G)$,

$$\tau_G(x) \equiv \begin{cases} \beta_G(x) & \pmod{3} \\ d_G(x) & \pmod{2}. \end{cases}$$

Now, the mapping τ_G can be further extended to any nonempty vertex subset A as follows:

$$\tau_G(A) \equiv \begin{cases} \beta_G(A) & \pmod 3 \\ d_G(A) & \pmod 2. \end{cases}$$

where $\beta_G(A) \equiv \sum_{x\in A} \beta_G(x) \in \{0,1,2\} \pmod 3$.

Proposition 2.2. *Let G be a graph and A be a vertex subset of G.*
(1) If $d_G(A) \le 5$, then $d_G(A) \le 4 + |\tau_G(A)|$.
(2) If $d_G(A) \ge 6$, then $d_G(A) \ge 4 + |\tau_G(A)|$.

Proposition 2.2 follows from the fact that $|\tau_G(A)| \le 3$ and $d_G(A) - |\tau_G(A)|$ is even.

Lemma 2.3 (Tutte [9])**.** *Let G be a graph.*
(1) G has a nowhere-zero 3-flow if and only if G has a modulo 3-orientation.
(2) G has a nowhere-zero 3-flow if and only if G has a β_G-orientation with $\beta_G = 0$.

The following lemma is Theorem 3.1 in [7] by Lovász et al. This lemma will play the main role in our proofs.

Lemma 2.4 (Lovász, Thomassen, Wu and Zhang [7])**.** *Let G be a graph, β_G be a Z_3-boundary of G, and let $z_0 \in V(G)$ and D_{z_0} be a pre-orientation of $E(z_0)$ of all edges incident with z_0. Assume that*
(i) $|V(G)| \ge 3$.
(ii) $d_G(z_0) \le 4 + |\tau_G(z_0)|$ and $d^+_{D_{z_0}}(z_0) - d^-_{D_{z_0}}(z_0) \equiv \beta_G(z_0) \pmod 3$, and
(iii) $d_G(A) \ge 4 + |\tau_G(A)|$ for each nonempty vertex subset A not containing z_0 with $|V(G) \setminus A| > 1$.
Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to an orientation D of the entire graph G, that is, for every vertex x of G,

$$d^+_D(x) - d^-_D(x) \equiv \beta_G(x) \pmod 3.$$

3 Proof of Theorem 1.13

If not, suppose that G is a counterexample, such that $|V(G)| + |E(G)|$ is as small as possible. Let $P' = \{x \in V(G) : d_G(x) = 3\}$ and $Q' = \{x \in V(G) : d_G(x) = 5\}$.

Claim 3.1. $|V(G)| \ge 3$.

Proof. If $|V(G)| = 1$, then G has a nowhere-zero 3-flow, a contradiction. If $|V(G)| = 2$, let $V(G) = \{x, y\}$, then all the edges of G are all between x and y. Since G is bridgeless, $|E(G)| \ge 2$. Let a be the integer in $\{0,1,2\}$ such that $a \equiv |E(G)| - a \pmod 3$. Orient a edges from x to y and the remaining $|E(G)| - a$ edges from y to x. Clearly, the resulting orientation is a modulo 3-orientation of G, a contradiction. Therefore $|V(G)| \ge 3$. □

Claim 3.2. *G is 3-edge-connected, and G has no nontrivial 3-edge-cuts.*

Proof. If G has a vertex x of degree 2, then suppose that $xx_1, xx_2 \in E(G)$. By the minimality of G, $(G - \{xx_1, xx_2\}) \cup \{x_1x_2\}$ has a nowhere-zero 3-flow f'. However, f' can be extended to a nowhere-zero 3-flow f of G, a contradiction. If G has a nontrivial k-edge-cut ($k = 2, 3$), then contract one side and find a mod 3-orientation by the minimality of G. Merge such two mod 3-orientations and we will get one for G, a contradiction. □

Claim 3.3. *For any $U \subset V(G)$, if $d_G(U) \leq 5$ and $|U| \geq 2$, then $U \cap (P' \cup Q') \neq \emptyset$.*

Proof. If not, choose U to be a minimal one such that: for any $A \subset U$ with $2 \leq |A| < |U|$, we have $d_G(A) \geq 6$.

By the minimality of G, G/U has a modulo 3-orientation D' which is a partial modulo 3-orientation of G, such that $d^+_{D'}(x) \equiv d^-_{D'}(x) \pmod 3$ for each $x \in V(G) \setminus U$.

Let G' be a graph obtained from G by contracting $V(G) \setminus U$ as z_0 and let $\beta_{G'} = 0$.

(i) Since $V(G') = U + z_0$, $|V(G')| = |U| + 1 \geq 3$.

(ii) Since $d_{G'}(z_0) = d_G(U) \leq 5$, by Proposition 2.2 (1), $d_{G'}(z_0) \leq 4 + |\tau_{G'}(z_0)|$.

(iii) By the assumption and minimality of U, we have that for any $A \subset U$, $d_G(A) \neq 5$ and $d_G(A) \neq 3$. If $d_G(A) = 4$, then $d_{G'}(A) = d_G(A) = 4$ and $\tau_{G'}(A) = \beta_{G'}(A) = \beta_G(A) = 0$. Thus $d_{G'}(A) = 4 = 4 + |\tau_{G'}(A)|$. If $d_G(A) \geq 6$, then by Proposition 2.2 (2), $d_{G'}(A) = d_G(A) \geq 4 + |\tau_{G'}(A)|$.

By Lemma 2.4, we could see that the pre-orientation of $E'(z_0)$ of all edges incident with z_0 can be extended to a $\beta_{G'}$-orientation of G'. Then G has a modulo 3-orientation, which is a contradiction. □

Let G'_1 be a graph obtained from G by adding a new vertex z_0 and $2|P'| + |Q'|$ edges between z_0 and $P' \cup Q'$, such that:

(i) For each vertex $v \in P'$, we add two arcs with the same direction between it and z_0; and

(ii) For each vertex $v \in Q'$, we add one arc between it and z_0.

If $2|P'| + |Q'| \leq 5$, then all added arcs could be from z_0 to $P' \cup Q'$. Define $\beta_{G'_1}$ as follows:

(1) $\beta_{G'_1}(x) = 0$ if $x \notin (P' \cup Q') + z_0$;

(2) $\beta_{G'_1}(x) = 1$ if $x \in P'$;

(3) $\beta_{G'_1}(x) = 2$ if $x \in Q'$;

(4) $\beta_{G'_1}(z_0) \equiv 2|P'| + |Q'| \pmod 3$ and $\beta_{G'_1}(z_0) \in \{0, 1, 2\}$.

If $2|P'| + |Q'| = 6$ or 7, in this case, if $|P'| \neq 0$, choose one vertex $v \in P'$, such that the two arcs with ends z_0 and v are from v to z_0, the other arcs incident with z_0 are all directed from z_0. If $|P'| = 0$, then two arcs are from Q' to z_0, the others verse. Define $\beta_{G'_1}$ as follows:

(1) $\beta_{G'_1}(x) = 0$ if $x \notin (P' \cup Q') + z_0$;

(2) $\beta_{G'_1}(x) = 2$ if $x \in Q'$ and the arc (z_0, x) exists or $x \in P'$ and the two arcs with ends z_0 and x are from x to z_0;

(3) $\beta_{G'_1}(x) = 1$ if $x \in Q'$ and the arc (x, z_0) exists or $x \in P'$ and the two arcs with ends z_0 and x are from z_0 to x;

(4) $\beta_{G'_1}(z_0) \equiv (2|P'| + |Q'| - 2) - 2 \pmod 3$.

Now $d_{G'_1}(z_0) \leq 4 + |\tau_{G'_1}(z_0)|$ and $|V(G'_1)| = |V(G)| + 1 \geq 4$. We claim that: $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$, for each nonempty vertex subset A not containing z_0 with $|V(G'_1) \setminus A| > 1$.

If $A \cap (P' \cup Q') = \emptyset$, then by Claim 3.3, $d_G(A) = 4$ or $d_G(A) \geq 6$. In each case we could get that $d_{G'_1}(A) = d_G(A) \geq 4 + |\tau_{G'_1}(A)|$.

If $A \cap (P' \cup Q') \neq \emptyset$, then by Claim 3.2, $d_{G'_1}(A) \geq 5$. If $d_{G'_1}(A) = 5$, then $d_G(A) = 3$ or 4 and $|A \cap (P' \cup Q')| = 1$, and it follows that $\beta_{G'_1}(A) = 1$ or 2, and $|\tau_{G'_1}(A)| = 1$. Thus $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$. If $d_{G'_1}(A) \geq 6$, by Proposition 2.2 (2), we have that $d_{G'_1}(A) \geq 4 + |\tau_{G'_1}(A)|$.

Now G_1' satisfies all the conditions of Lemma 2.4. By Lemma 2.4, G_1' has a $\beta_{G_1'}$-orientation extended from the pre-orientation of $E_1'(z_0)$ of all edges incident with z_0, which implies that G has a β_G-orientation with $\beta_G = 0$. By Lemma 2.3, G has a nowhere-zero 3-flow, which is a contradiction. □

4 Proof of Theorem 1.18

Assume not. Suppose that G is a counterexample, such that $|V(G)| + |E(G)|$ is as small as possible. Let $S' = \{x \in V(G) : d_G(x) = 5\}$ and $S = \{C = \partial_G(X) : |C| = 5,\ X \subset V(G)\}$. Let β_G be a Z_3-boundary, such that G has no β_G-orientation.

Claim 4.1. *$|V(G)| \geq 3$ and $|S'| \leq |S| \leq 5$.*

Proof. Since G is 5-edge-connected, $|V(G)| \geq 2$. If $|V(G)| = 2$, let $V(G) = \{x, y\}$, then all the edges of G are between x and y, and $|E(G)| \geq 5$. Let D_x be an orientation of x, such that $d^+_{D_x}(x) - d^-_{D_x}(x) \equiv \beta_G(x) \pmod 3$. Since β_G is a Z_3-boundary, $d^+_{D_x}(y) - d^-_{D_x}(y) \equiv \beta_G(y) \pmod 3$. Therefore G has a β_G-orientation, a contradiction. Hence $|V(G)| \geq 3$ and $|S'| \leq |S| \leq 5$. □

Claim 4.2. *Let $U \subset V(G)$ with $|U| \geq 2$. If $d_G(U) = 5$, then $U \cap S' \neq \emptyset$.*

Proof. If not, choose U to be a minimal one such that: for any $A \subset U$ with $2 \leq |A| < |U|$, we have $d_G(A) \neq 5$.

By the minimality of G, G/U has a β_G-orientation D' which is a partial β_G-orientation of G, such that $d^+_{D'}(x) - d^-_{D'}(x) \equiv \beta_G(x) \pmod 3$ for each $x \in V(G) \setminus U$.

Let G' be a graph obtained from G by contracting $V(G) \setminus U$ as z_0, and let $\beta_{G'} = \beta_G$.

(i) Since $V(G') = U + z_0$, $|V(G')| = |U| + 1 \geq 3$.

(ii) Since $d_{G'}(z_0) = d_G(U) = 5$, by Proposition 2.2 (1), we have that $d_{G'}(z_0) \leq 4 + |\tau_{G'}(z_0)|$.

(iii) By the assumption and minimality of U, we have that for any $A \subset U$, $d_G(A) \neq 5$. Therefore $d_G(A) \geq 6$. By Proposition 2.2 (2), $d_{G'}(A) = d_G(A) \geq 4 + |\tau_{G'}(A)|$.

By Lemma 2.4, the pre-orientation of $E'(z_0)$ of all edges incident with z_0 can be extended to a $\beta_{G'}$-orientation of G'. Therefore, G has a β_G-orientation, which is a contradiction. □

Let G_1' be a graph obtained from G by adding a new vertex z_0 and $|S'|$ arcs from z_0 to S', such that each vertex in S' has degree 6 in G_1'.

Define $\beta_{G_1'}$ as follows:

(1) $\beta_{G_1'}(x) = \beta_G(x)$ if $x \notin S' + z_0$;

(2) $\beta_{G_1'}(x) \equiv \beta_G(x) - 1 \pmod 3$ if $x \in S'$;

(3) $\beta_{G_1'}(z_0) \equiv |S'| \pmod 3$ and $\beta_{G_1'}(z_0) \in \{0, 1, 2\}$.

Now $d_{G_1'}(z_0) \leq 4 + |\tau_{G_1'}(z_0)|$ and $|V(G_1')| = |V(G)| + 1 \geq 4$. We claim that $d_{G_1'}(A) \geq 4 + |\tau_{G_1'}(A)|$, for each nonempty vertex subset A not containing z_0 with $|V(G_1') \setminus A| > 1$.

If $A \cap S' = \emptyset$, then by Claim 4.2, $d_{G_1'}(A) = d_G(A) \neq 5$. Thus $d_{G_1'}(A) \geq 6$. By Proposition 2.2 (2), $d_{G_1'}(A) \geq 4 + |\tau_{G_1'}(A)|$.

If $A \cap S' \neq \emptyset$, then $d_{G_1'}(A) \geq d_G(A) + 1 \geq 6$. By Proposition 2.2 (2), we have that $d_{G_1'}(A) \geq 4 + |\tau_{G_1'}(A)|$.

Now G_1' satisfies all the conditions of Lemma 2.4. By Lemma 2.4, G_1' has a $\beta_{G_1'}$-orientation extended from the pre-orientation of $E_1'(z_0)$ of all edges incident with z_0, which implies that G has a β_G-orientation, a contradiction.

The proof is complete. □

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