Distribution Dependent SDEs for Navier-Stokes Type Equations *

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Abstract

To characterize Navier-Stokes type equations where the Laplacian is extended to a singular second order differential operator, we propose a class of SDEs depending on the distribution in future. The well-posedness and regularity estimates are derived for these SDEs.

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1 Introduction

Let $d \in \mathbb{N}$. Consider the following incompressible Navier-Stokes equation on $E := \mathbb{R}^d$ or $\mathbb{R}^d/\mathbb{Z}^d$:

$$\partial_t u_t = \kappa \Delta u_t - (u_t \cdot \nabla) u_t - \nabla \wp_t, \quad t \in [0, T] \tag{1.1}$$

with $\nabla \cdot u_t := \sum_{i=1}^d \partial_i u_t^i = 0$, where $T > 0$ is a fixed time,

$$u := (u^1, \cdots, u^d) : [0, T] \times E \to \mathbb{R}^d, \quad \wp : [0, T] \times E \to \mathbb{R},$$

and $u_t \cdot \nabla := \sum_{i=1}^d u_t^i \partial_i$. This equation describes viscous incompressible fluids, where u is the velocity field of a fluid flow, $\wp$ is the pressure, and $\kappa > 0$ is the viscosity constant.

Besides existing probabilistic characterizations on Navier-Stokes equations, see [1] and references therein, in this paper we propose a new type stochastic differential equation (SDE)

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depending on distributions in the future, such that the solution of (1.1) is explicitly given by the initial datum u_0 and the pressure $\wp$. By proving the well-posedness of the SDE, we derive the well-posedness of (1.1) in $\mathscr{C}_b^n(n \geq 2)$ with given pressure (which is however a part of solution in Navier-Stokes equations), see [3] for an analytic characterization on the pressure to ensure $\nabla \cdot u_t = 0$.

Indeed, we will prove a more general result for the following Navier-Stokes type equation on $E := \mathbb{R}^d$ or $E := \mathbb{R}^d/\mathbb{Z}^d$:

$$\partial_t u_t = L_t u_t - (u_t \cdot \nabla) u_t + V_t, \quad t \in [0,T], \tag{1.2}$$

where

$$L_t := \mathrm{tr}\{a_t \nabla^2\} + b_t \cdot \nabla$$

and

$$V,\ b : [0,T] \times E \to \mathbb{R}^d, \quad a : [0,T] \times E \to \mathbb{R}^{d\otimes d}$$

are measurable, and $a_t(x)$ is positive definite for $(t,x) \in [0,T] \times E$.

To characterize (1.2), we consider the following SDE on $\mathbb{R}^d$ where differentials are in $s \in [t,T]$:

$$\begin{aligned} \mathrm{d}X_{t,s}^x &= \sqrt{2a_{T-s}}(X_{t,s}^x)\mathrm{d}W_s \\ &\quad + \bigg\{ b_{T-s}(X_{t,s}^x) - \bigg[\mathbb{E}u_0(X_{s,T}^y) + \mathbb{E}\int_s^T V_{T-r}(X_{s,r}^y)\mathrm{d}r \bigg]_{y=X_{t,s}^x} \bigg\} \mathrm{d}s, \\ &\quad t \in [0,T], s \in [t,T], X_{t,t}^x = x \in \mathbb{R}^d, \end{aligned} \tag{1.3}$$

where $(W_s)_{s\in[0,T]}$ is a d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_s\}_{s\in[0,T]}, \mathbb{P})$. When $E = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, by extending a function f from domain E to domain $\mathbb{R}^d$ as

$$f(x+k) = f(x), \quad x \in [0,1)^d, k \in \mathbb{Z}^d, \tag{1.4}$$

we also have the SDE (1.3) for the case $E = \mathbb{T}^d$.

Regarding s as the present time, the SDE (1.3) depends on the distribution of $(X_{s,r})_{r\in[s,T]}$ coming from the future. So, this is a future distribution dependent equation, but is essentially different from McKean-Vlasov SDEs which depend on the distribution at present rather than future. We will use $X := (X_{t,s}^x)_{0\le t\le s\le T, x\in E}$ to formulate the solution to (1.2).

Let $D_T := \{(t,s) : 0 \le t \le s \le T\}$. We define the solution X of (1.3) as follows.

Definition 1.1. A family $X := (X_{t,s}^x)_{(t,s,x)\in D_T\times\mathbb{R}^d}$ of random variables on $\mathbb{R}^d$ is called a solution of (1.3), if $X_{t,s}^x$ is $\mathscr{F}_s$-measurable for all $x \in \mathbb{R}^d$ and $0 \le t \le s \le T$, $\mathbb{P}$-a.s. continuous in (t,s,x),

$$\mathbb{E}\int_t^T \bigg\{ \|a_{T-s}(X_{t,s}^x)\| + \bigg| b_{T-s}(X_{t,s}^x) - \bigg[\mathbb{E}u_0(X_{s,T}^y) + \mathbb{E}\int_s^T V_{T-r}(X_{s,r}^y)\mathrm{d}r \bigg]_{y=X_{t,s}^x} \bigg| \bigg\} \mathrm{d}s < \infty$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, and $\mathbb{P}$-a.s.

$$X_{t,s}^x = x + \int_t^s \sqrt{2a_{T-r}}(X_{t,r}^x)\mathrm{d}W_r$$

$$+\int_t^s\bigg\{b_{T-r}(X_{t,r}^x)-\bigg[\mathbb{E}u_0(X_{r,T}^y)+\mathbb{E}\int_r^T V_{T-r}(X_{r,\theta}^y)\mathrm{d}\theta\bigg]_{y=X_{t,r}^x}\bigg\}\mathrm{d}r,\quad (t,s,x)\in D_T\times\mathbb{R}^d.$$

We will allow the operator L_t to be singular, where the drift contains a locally integrable term introduced in [4] for singular SDEs. For any $p,q>1$ and $0\le t<s$, we write $f\in\tilde L_q^p(t,s)$ if $f=(f_r(x))_{(r,x)\in[t,s]\times\mathbb{R}^d}$ is a measurable function on $[t,s]\times\mathbb{R}^d$ such that

$$\|f\|_{\tilde L_q^p(t,s)}:=\sup_{z\in\mathbb{R}^d}\bigg(\int_t^s\|f_r1_{B(z,1)}\|_{L^p}^q\mathrm{d}r\bigg)^{\frac{1}{q}}<\infty,$$

where $B(z,1)$ is the unit ball at z, and $\|\cdot\|_{L^p}$ is the L^p-norm for the Lebesgue measure. We denote $f\in\tilde H_q^{2,p}(t,s)$ if $|f|+|\nabla f|+\|\nabla^2 f\|\in\tilde L_q^p(t,s)$. When $(t,s)=(0,T)$ we simply denote

$$\tilde L_q^p=\tilde L_q^p(0,T),\quad \tilde H_q^{2,p}=\tilde H_q^{2,p}(0,T).$$

We will take (p,q) from the following class:

$$\mathscr{K}:=\Big\{(p,q):p,q>2,\ \frac{d}{p}+\frac{2}{q}<1\Big\}.$$

We now make the following assumption on the operator L_t.

(H) Let $b_t=b_t^{(0)}+b_t^{(1)}$, and when $E=\mathbb{T}^d$ we extend $a_t,b_t^{(0)}$ and $b_t^{(1)}$ to $\mathbb{R}^d$ as in (1.4).

(1) a is positive definite with

$$\|a\|_\infty+\|a^{-1}\|_\infty:=\sup_{(t,x)\in[0,T]\times E}\|a_t(x)\|+\sup_{(t,x)\in[0,T]\times E}\|a_t(x)^{-1}\|<\infty,$$

$$\lim_{\varepsilon\to0}\sup_{|x-y|\le\varepsilon,t\in[0,T]}\|a_t(x)-a_t(y)\|=0.$$

(2) There exist $l\in\mathbb{N}$, $\{(p_i,q_i)\}_{0\le i\le l}\subset\mathscr{K}$ and $0\le f_i\in\tilde L_{q_i}^{p_i}, 0\le i\le l$, such that

$$|b^{(0)}|\le f_0,\qquad \|\nabla a\|\le\sum_{i=1}^l f_i.$$

(3) $\|b^{(1)}(0)\|_\infty:=\sup_{(t,x)\in[0,T]}|b^{(1)}(0)|<\infty$, and

$$\|\nabla b^{(1)}\|_\infty:=\sup_{t\in[0,T]}\sup_{x\neq y}\frac{|b_t^{(1)}(x)-b_t^{(1)}(y)|}{|x-y|}<\infty.\tag{1.5}$$

Under this assumption, we will prove the well-posedness of (1.3) and solve (1.2) in the class

$$\mathscr{U}(p_0,q_0):=\Big\{u:[0,T]\times E\to\mathbb{R}^d;\ \|u\|_\infty+\|\nabla u\|_\infty+\|\nabla^2u\|_{\tilde L_{q_0}^{p_0}}<\infty\Big\}.$$

Recall that $W^{1,\infty}(E;\mathbb{R}^d)$ is the space of all weakly differentiable functions $f:E\to\mathbb{R}^d$ with $\|f\|_\infty+\|\nabla f\|_\infty<\infty$.

Theorem 1.1. *Assume* (H). *Let* $u_0 \in W^{1,\infty}(E;\mathbb{R}^d)$ *and* $\int_0^T \|V_t\|_\infty^2 \mathrm{d}t < \infty$. *Then the following assertions hold.*

(1) *The SDE* (1.3) *has a unique solution* $X := (X_{t,s}^x)_{(t,s,x)\in D_T\times\mathbb{R}^d}$.

(2) *If* u *solves* (1.2) *and* $u \in \mathscr{U}(p_0,q_0)$, *then*

$$u_t(x) = \mathbb{E}\bigg[u_0(X_{T-t,T}^x) + \int_{T-t}^T V_{T-s}(X_{T-t,s}^x)\mathrm{d}s\bigg], \quad (t,x)\in[0,T]\times E. \tag{1.6}$$

Moreover, there exists a constant $c>0$ *such that for any* $i\in\{1,2\}$ *and* $j,j'\in\{0,1\}$,

$$\|\nabla^i u_t\|_\infty \le ct^{-\frac{i-j}{2}}\|\nabla^j u_0\|_\infty + c\int_{T-t}^T (s+t-T)^{-\frac{i-j'}{2}}\|\nabla^{j'}V_{T-s}\|_\infty \mathrm{d}s, \quad t\in(0,T]. \tag{1.7}$$

(3) *If* $b^{(1)}=0$ *and* $u_0, V_t \in \mathscr{C}_b^2$ *with* $\int_0^T \|V_t\|_{\mathscr{C}_b^2}\mathrm{d}t<\infty$, *then* u *given by* (1.6) *solves* (1.2), *and* u *is in the class* $\mathscr{U}(p_0,q_0)$.

In the next two sections, we prove assertions (1) and (2)-(3) of Theorem 1.1 respectively, where in Section 2 the well-posedness is proved for a more general equation than (1.3). Finally, in Section 4 we apply Theorem 1.1 to the equation (1.1).

2 Proof of Theorem 1.1(1)

Let $\mathscr{P}$ be the set of all probability measures on $\mathbb{R}^d$ equipped with the weak topology, let $\mathscr{L}_\xi$ be the distribution of a random variable ξ on $\mathbb{R}^d$. Let

$$\Gamma := C(D_T\times\mathbb{R}^d;\mathscr{P})$$

be the space of continuous maps from $D_T\times\mathbb{R}^d$ to $\mathscr{P}$. For any $\lambda>0$, Γ is a complete space under the metric

$$\rho_\lambda(\gamma^1,\gamma^2) := \sup_{(t,s,x)\in D_T\times\mathbb{R}^d} \mathrm{e}^{-\lambda(T-t)}\|\gamma_{t,s,x}^1-\gamma_{t,s,x}^2\|_{var}, \quad \gamma^1,\gamma^2\in\Gamma,$$

where $\|\cdot\|_{var}$ is the total variation norm defined by

$$\|\mu-\nu\|_{var} := \sup_{|f|\le1}|\mu(f)-\nu(f)|, \quad \mu,\nu\in\mathscr{P}$$

for $\mu(f):=\int_{\mathbb{R}^d} f\mathrm{d}\mu$. Note that the convergence in $\|\cdot\|_{var}$ is stronger than the weak convergence.

We consider the following more general equation than (1.3):

$$\begin{aligned}\mathrm{d}X_{t,s}^x &= \Big\{b_{T-s}^{(1)}(X_{t,s}^x) + Z_s(X_{t,s}^x,\mathscr{L}_X)\Big\}\mathrm{d}s + \sqrt{2a_{T-s}}(X_{t,s}^x)\mathrm{d}W_s,\\ &t\in[0,T], s\in[t,T], X_{t,t}^x = x\in\mathbb{R}^d,\end{aligned} \tag{2.1}$$

where $\mathscr{L}_X \in \Gamma$ is defined by $\{\mathscr{L}_X\}_{t,s,x} := \mathscr{L}_{X_{t,s}^x}$, and

$$Z : [0,T] \times \mathbb{R}^d \times \Gamma \to \mathbb{R}^d$$

is measurable.

It is easy to see that (2.1) covers (1.3) for

$$\begin{aligned} Z_t(x,\gamma) &:= b_{T-t}^{(0)}(x) - \int_{\mathbb{R}^d} u_0(y)\gamma_{t,T,x}(\mathrm{d}y) - \int_t^T \mathrm{d}s \int_{\mathbb{R}^d} V_{T-s}(y)\gamma_{t,s,x}(\mathrm{d}y), \\ &(t,x,\gamma) \in [0,T] \times \mathbb{R}^d \times \Gamma. \end{aligned} \tag{2.2}$$

The solution of (2.1) is defined as in Definition 1.1 using $b_{T-s}^{(1)}(X_{t,s}^x) + Z_s(X_{t,s}^x, \mathscr{L}_X)$ replacing

$$b_{T-s}(X_{t,s}^x) - \bigg[\mathbb{E}u_0(X_{s,T}^y) + \mathbb{E}\int_s^T V_{T-r}(X_{s,r}^y)\mathrm{d}r\bigg]_{y=X_{t,s}^x}.$$

We make the following assumption.

(A) $b^{(1)}$ and a satisfy (H), and there exists $(p_0,q_0) \in \mathscr{K}$ and $f_0 \in \tilde{L}_{q_0}^{p_0}$ such that

$$|Z_t(x,\gamma)| \le f_0(t,x), \quad (t,x,\gamma) \in [0,T] \times \mathbb{R}^d \times \Gamma.$$

Moreover, there exists $0 \le g \in L^2([0,T])$ such that

$$\sup_{x\in\mathbb{R}^d} |Z_t(x,\gamma^1) - Z_t(x,\gamma^2)| \le g_t \sup_{(s,x)\in[t,T]\times\mathbb{R}^d} \|\gamma_{t,s,x}^1 - \gamma_{t,s,x}^2\|_{var}, \quad t \in [0,T], \gamma^1,\gamma^2 \in \Gamma.$$

When $\|u_0\|_\infty + \int_0^T \|V_t\|_\infty^2 \mathrm{d}t < \infty$, ($H$) implies ($A$) for Z given by (2.2). So, Theorem 1.1(1) follows from the following result, which also includes regularity estimates on the solution.

Theorem 2.1. *Assume (A). Then the following assertions hold.*

(1) (2.1) *has a unique solution, and the solution has the flow property*

$$X_{t,r}^x = X_{s,r}^{X_{t,s}^x}, \quad 0 \le t \le s \le r \le T, \ x \in \mathbb{R}^d. \tag{2.3}$$

(2) *For any $j \ge 1$,*

$$\nabla_v X_{t,s}^x := \lim_{\varepsilon\downarrow 0} \frac{X_{t,s}^{x+\varepsilon v} - X_{t,s}^x}{\varepsilon}, \quad s \in [t,T]$$

exists in $L^j(\Omega \to C([t,T];\mathbb{R}^d),\mathbb{P})$, and there exists a constant $c(j) > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\bigg[\sup_{s\in[t,T]} |\nabla_v X_{t,s}^x|^j\bigg] \le c(j)|v|^j, \quad v \in \mathbb{R}^d. \tag{2.4}$$

(3) *For any $0 \le t < s \le T$, $v \in \mathbb{R}^d$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$,*

$$\nabla_v\big\{\mathbb{E}f(X_{t,s}^\cdot)\big\}(x) = \frac{1}{s-t}\mathbb{E}\bigg[f(X_{t,s}^x)\int_t^s \Big\langle\big(\sqrt{2a_{T-r}}\big)^{-1}(X_{t,r}^x)\nabla_v X_{t,r}^x, \ \mathrm{d}W_r\Big\rangle\bigg]. \tag{2.5}$$

Proof. (a) We first explain the idea of proof using fixed point theorem on Γ. For any $\gamma \in \Gamma$, we consider the following classical SDE

$$\begin{aligned} \mathrm{d}X_{t,s}^{\gamma,x} &= \Big\{b_{T-s}^{(1)}(X_{t,s}^{\gamma,x}) + Z_s(X_{t,s}^{\gamma,x},\gamma)\Big\}\mathrm{d}s + \sqrt{2a_{T-s}}(X_{t,s}^{\gamma,x})\mathrm{d}W_s,\\ & t\in[0,T], s\in[t,T], X_{t,t}^{\gamma,x} = x\in\mathbb{R}^d. \end{aligned} \tag{2.6}$$

By [2, Theorem 2.1] for $[t,T]$ replacing $[0,T]$, see also [4] for $b^{(1)} = 0$, this SDE is well-posed, such that for any $j \geq 1$ and $v \in \mathbb{R}^d$, the directional derivative

$$\nabla_v X_{t,s}^{\gamma,x} := \lim_{\varepsilon\downarrow 0} \frac{X_{t,s}^{\gamma,x+\varepsilon v} - X_{t,s}^{\gamma,x}}{\varepsilon}, \quad s\in[t,T]$$

exists in $L^j(\Omega \to C([t,T];\mathbb{R}^d),\mathbb{P})$, and there exists a constant $c(j) > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}\Big[\sup_{s\in[t,T]} |\nabla_v X_{t,s}^{\gamma,x}|^j\Big] \leq c(j)|v|^j, \quad v\in\mathbb{R}^d, \tag{2.7}$$

and for any $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$\nabla_v\big\{\mathbb{E}f(X_{t,s}^{\gamma,\cdot})\big\}(x) = \frac{1}{s-t}\mathbb{E}\Big[f(X_{t,s}^{\gamma,x})\int_t^s \Big\langle \big(\sqrt{2a_{T-r}}\big)^{-1}(X_{t,r}^{\gamma,x})\nabla_v X_{t,r}^{\gamma,x},\ \mathrm{d}W_r\Big\rangle\Big]. \tag{2.8}$$

By the pathwise uniqueness of (2.6), the solution satisfies the flow property

$$X_{t,r}^{\gamma,x} = X_{s,r}^{\gamma,X_{t,s}^{\gamma,x}}, \quad 0\leq t\leq s\leq r\leq T,\ x\in\mathbb{R}^d. \tag{2.9}$$

Moreover,

$$\Phi(\gamma)_{t,s,x} := \mathscr{L}_{X_{t,s}^{\gamma,x}}, \quad (t,s,x)\in D_T\times\mathbb{R}^d$$

defines a map $\Phi : \Gamma \to \Gamma$. If Φ has a unique fixed point $\bar\gamma \in \Gamma$, then (2.6) with $\gamma = \bar\gamma$ reduces to (2.1), the well-posedness of (2.6) implies that of (2.1), and the unique solution is given by

$$X_{t,s}^x = X_{t,s}^{\bar\gamma,x}.$$

Then (2.3), (2.4) and (2.5) follow from (2.9), (2.7) and (2.8) for $\gamma = \bar\gamma$ respectively. Therefore, it remains to prove that Φ has a unique fixed point.

(b) By the fixed point theorem, we only need to find constants $\lambda > 0$ and $\delta \in (0,1)$ such that

$$\rho_\lambda(\Phi(\gamma^1),\Phi(\gamma^2)) \leq \delta\rho_\lambda(\gamma^1,\gamma^2), \quad \gamma^1,\gamma^2\in\Gamma. \tag{2.10}$$

Below, we prove this estimate using Girsanov's theorem.

For $i = 1, 2$, consider the SDE

$$\begin{aligned} \mathrm{d}X_{t,s}^{i,x} &= \Big\{b_{T-s}^{(1)}(X_{t,s}^{i,x}) + Z_s(X_{t,s}^{i,x},\gamma^i)\Big\}\mathrm{d}s + \sqrt{2a_{T-s}}(X_{t,s}^{i,x})\mathrm{d}W_s,\\ & t\in[0,T], s\in[t,T], X_{t,t}^{i,x} = x\in\mathbb{R}^d. \end{aligned}$$

By the definition of Φ, we have

$$\Phi(\gamma^i)_{t,s,x} = \mathscr{L}_{X^{i,x}_{t,s}}, \quad i=1,2,\ (t,s,x)\in D_T\times\mathbb{R}^d. \tag{2.11}$$

Let

$$\xi_s := \big(\sqrt{2a_{T-s}}(X^{1,x}_{t,s})\big)^{-1}\big\{Z_s(X^{1,x}_{t,s},\gamma^1) - Z_s(X^{1,x}_{t,s},\gamma^2)\big\},\quad s\in[t,T].$$

By (A), there exists a constant $K>0$ such that

$$|\xi_s| \le K g_s \sup_{(r,x)\in[s,T]\times\mathbb{R}^d} \|\gamma^1_{s,r,x} - \gamma^2_{s,r,x}\|_{var}. \tag{2.12}$$

By Girsanov theorem,

$$\tilde W_s := W_s - \int_t^s \xi_r \mathrm{d}r,\quad s\in[t,T]$$

is a Brownian motion under the weighted probability $\mathrm{d}\mathbb{Q}_t := R_t\mathrm{d}\mathbb{P}$, where

$$R_t := \mathrm{e}^{\int_t^T\langle \xi_s,\mathrm{d}W_s\rangle - \frac12\int_t^T|\xi_s|^2\mathrm{d}s}.$$

With this new Brownian motion, the SDE for X^1 becomes

$$\mathrm{d}X^{1,x}_{t,s} = \Big\{b^{(1)}_{T-s}(X^{1,x}_{t,s}) + Z_s(X^{1,x}_{t,s},\gamma^2)\Big\}\mathrm{d}s + \sqrt{2a_{T-s}}(X^{1,x}_{t,s})\mathrm{d}\tilde W_s,\quad s\in[t,T].$$

By the (weak) uniqueness for the SDE with $i=2$, we derive

$$\mathscr{L}_{X^{1,x}_{t,s}|\mathbb{Q}_t} = \mathscr{L}_{X^{2,x}_{t,s}} = \Phi(\gamma^2)_{t,s,x},$$

where $\mathscr{L}_{X^{1,x}_{t,s}|\mathbb{Q}_t}$ is the distribution of $X^{1,x}_{t,s}$ under $\mathbb{Q}_t$. Combining this with (2.11), we get

$$\|\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}\|_{var} = \sup_{|f|\le1}\big|\mathbb{E}[f(X^{1,x}_{t,s}) - f(X^{1,x}_{t,s})R_t]\big| \le \mathbb{E}|R_t-1|. \tag{2.13}$$

By Pinsker's inequality and the definition of R_t, we obtain

$$(\mathbb{E}|R_t-1|)^2 \le 2\mathbb{E}[R_t\log R_t] = 2\mathbb{E}_{\mathbb{Q}_t}[\log R_t] = 2\mathbb{E}_{\mathbb{Q}_t}\int_t^T|\xi_s|^2\mathrm{d}s, \tag{2.14}$$

where $\mathbb{E}_{\mathbb{Q}_t}$ is the expectation under the probability $\mathbb{Q}_t$. Combining (2.13) and (2.14) with (2.12), and using the definition of ρ_λ, we arrive at

$$\begin{aligned}\|\Phi(\gamma^1)_{t,s,x} - \Phi(\gamma^2)_{t,s,x}\|_{var} &\le \bigg(2K^2\int_t^T g_s^2 \sup_{(r,y)\in[s,T]\times\mathbb{R}^d}\|\gamma^1_{s,r,y}-\gamma^2_{s,r,y}\|^2_{var}\mathrm{d}s\bigg)^{\frac12}\\ &\le \rho_\lambda(\gamma^1,\gamma^2)\bigg(2K^2\int_t^T g_s^2\mathrm{e}^{2\lambda(T-s)}\mathrm{d}s\bigg)^{\frac12},\quad (t,x)\in[0,T]\times\mathbb{R}^d.\end{aligned}$$

Therefore

$$\rho_\lambda(\Phi(\gamma^1),\Phi(\gamma^2)) \le \varepsilon_\lambda\rho_\lambda(\gamma^1,\gamma^2),$$

where

$$\varepsilon_\lambda := \sup_{t\in[0,T]} \left(2K^2 \int_t^T g_s^2 \mathrm{e}^{-2\lambda(s-t)} \mathrm{d}s\right)^{\frac{1}{2}} \downarrow 0 \text{ as } \lambda \uparrow \infty.$$

By taking large enough $\lambda > 0$, we prove (2.10) for some $\delta < 1$.

□

For later use we present the following consequence of Theorem 2.1.

Corollary 2.2. *Assume (A) and let*

$$P_{t,s}f(x) := \mathbb{E}[f(X_{t,s}^x)], \quad (t,s,x) \in D_T \times \mathbb{R}^d, f \in \mathscr{B}_b(\mathbb{R}^d).$$

Then there exists a constant $c > 0$ such that for any function f,

$$\|\nabla P_{t,s} f\|_\infty \le c \min\left\{(s-t)^{-\frac{1}{2}}\|f\|_\infty,\ \|\nabla f\|_\infty\right\},$$
$$\|\nabla^2 P_{t,s} f\|_\infty \le c(s-t)^{-\frac{1}{2}}\|\nabla f\|_\infty, \quad 0 \le t < t \le T.$$

Proof. By (2.5) we have

$$\|\nabla P_{t,s} f\|_\infty \le c(t-s)^{-\frac{1}{2}}\|f\|_\infty$$

for some constant $c > 0$. Next, by chain rule and (2.4),

$$|\nabla P_{t,s} f(x)| = \left|\mathbb{E}[\langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x\rangle]\right| \le c\|\nabla f\|_\infty, \quad (t,s,x) \in D_T \times \mathbb{R}^d$$

holds for some constant $c > 0$. Moreover,

$$\nabla P_{t,s} f(x) = \mathbb{E}[\langle \nabla f(X_{t,s}^x), \nabla X_{t,s}^x\rangle] = \mathbb{E}[g(X_{t,s}^x)],$$

where $g(X_{t,s}^x) := \left\langle \nabla f(X_{t,s}^x), \mathbb{E}(\nabla X_{t,s}^x | X_{t,s}^x)\right\rangle$. Combining this with (2.5) and (2.4), we find a constant $c > 0$ such that

$$\begin{aligned}
&\|\nabla^2 P_{t,s} f(x)\| \le \|\nabla \mathbb{E}[g(X_{t,s}^x)]\| \\
&\le \frac{1}{s-t}\mathbb{E}\left[\left|g(X_{t,s}^x)\right| \cdot \left|\int_s^t \left\langle \left(\sqrt{2a_{T-r}}\right)^{-1}(X_{t,r}^x)\nabla_v X_{t,r}^x,\ \mathrm{d}W_r\right\rangle\right|\right] \\
&\le \frac{1}{t-s}\left(\mathbb{E}|g(X_{t,s}^x)|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\int_t^s \|a^{-1}\|_\infty \|\nabla X_{t,r}^x\|^2 \mathrm{d}r\right)^{\frac{1}{2}} \le c\|\nabla f\|_\infty.
\end{aligned}$$

Then the proof is finished. □

3 Proofs of Theorem 1.1(2)-(3)

We will need the following lemma implied by [5, Theorem 2.1, Theorem 3.1, Lemma 3.3], see also [4] and references within for the case $b^{(1)} = 0$.

Lemma 3.1. *Assume (A)(1), (A)(3) and $\|b^{(0)}\|_{\tilde L_{q_0}^{p_0}} < \infty$ for some $(p_0, q_0) \in \mathscr{K}$. Let $\sigma_t = \sqrt{2a_t}$. Then the following assertions hold.*

(1) *For any $p,q>1$, $\lambda\ge 0$, $0\le t_0<t_1\le T$ and $f\in \tilde L_q^p(t_0,t_1)$, the PDE*

$$(\partial_t+L_t)u_t=\lambda u_t+f_t,\quad t\in[t_0,t_1], u_{t_1}=0, \tag{3.1}$$

has a unique solution in $\tilde H_q^{2,p}(t_0,t_1)$. If $(2p,2q)\in\mathscr{K}$, then there exist a constant $c>0$ such that for any $0\le t_0<t_1\le T$ and $f\in \tilde L_q^p(t_0,t_1)$, the solution satisfies

$$\|u\|_\infty+\|\nabla u\|_\infty+\|(\partial_t+\nabla_{b^{(1)}})u\|_{\tilde L_q^p(t_0,t_1)}+\|\nabla^2 u\|_{\tilde L_q^p(t_0,t_1)}\le c\|f\|_{\tilde L_q^p(t_0,t_1)}.$$

(2) *Let $(X_t)_{t\in[0,T]}$ be a continuous adapted process on $\mathbb{R}^d$ satisfying*

$$X_t=X_0+\int_0^t b_s(X_s)\mathrm{d}s+\int_0^t\sigma_s(X_s)\mathrm{d}W_s,\quad t\in[0,T]. \tag{3.2}$$

For any $p,q>1$ with $(2p,2q)\in\mathscr{K}$, there exists a constant $c>0$ such that for any X_t satisfying (3.2)*,*

$$\mathbb{E}\bigg(\int_t^s|f_r(X_r)|\mathrm{d}r\bigg|\mathscr{F}_t\bigg)\le c\|f\|_{\tilde L_q^p(t,s)},\quad (t,s)\in D_T, f\in\tilde L_q^p(t,s).$$

(3) *Let $p,q>1$ with $\frac{d}{p}+\frac{2}{q}<1$. For any $u\in\tilde H_q^{2,p}$ with $\|(\partial_t+b^{(1)})u\|_{\tilde L_q^p}<\infty$, $\{u_t(X_t)\}_{t\in[0,T]}$ is a semimartingale satisfying*

$$\mathrm{d}u_t(X_t)=L_tu_t(X_t)\mathrm{d}t+\big\langle\nabla u_t(X_t),\sigma_t(X_t)\mathrm{d}W_t\big\rangle,\quad t\in[0,T].$$

In the following we consider $E=\mathbb{R}^d$ and $\mathbb{T}^d$ respectively.

3.1 $E=\mathbb{R}^d$

Proof of Theorem 1.1(2). Let $u\in\mathscr{U}(p_0,q_0)$ solve (1.2). Then

$$u\in\tilde H_{q_0}^{2,p_0},\quad \|(\partial_t+b^{(1)}\cdot\nabla)u\|_{\tilde L_{q_0}^{p_0}}<\infty \tag{3.3}$$

as required by Lemma 3.1(3). It remains to prove (1.6), which together with Corollary 2.2 implies (1.7).

Let

$$\begin{aligned}&\mathscr{L}_t:=\mathrm{tr}\{a_{T-t}\nabla^2\}+\tilde b_t\cdot\nabla,\\ &\tilde b_t(x):=b_{T-t}(x)-\mathbb{E}u_0(X_{t,T}^x)-\mathbb{E}\int_t^T V_{T-s}(X_{t,s}^x)\mathrm{d}s,\quad (t,x)\in[0,T]\times\mathbb{R}^d.\end{aligned} \tag{3.4}$$

Since $\|u_0\|_\infty+\int_0^T\|V_t\|_\infty\mathrm{d}t<\infty$, $\|b^{(0)}\|_{\tilde L_{q_0}^{p_0}}<\infty$ implies $\tilde b_t(x):=b_{T-t}^{(1)}(x)+\tilde b_t^{(0)}(x)$ with $\|\tilde b^{(0)}\|_{\tilde L_{q_0}^{p_0}}<\infty$. Then (A) holds for $\tilde b$ replacing b, so that by (3.3) and Lemma 3.1(3), the following Itô's formula holds for $X_{t,s}^x$ solving (1.3):

$$\mathrm{d}u_{T-s}(X_{t,s}^x)=\big(\partial_s+\mathscr{L}_s\big)u_{T-s}(X_{t,s}^x)\mathrm{d}s+\big\{\nabla u_{T-s}(X_{t,s}^x)\big\}^*\sqrt{2a_{T-s}(X_{t,s}^x)}\mathrm{d}W_s,\quad s\in[t,T], \tag{3.5}$$

where $(\nabla u)^*_{ij} := (\partial_j u^i)_{1\le i,j\le d}$. By (1.2) and (3.4), we obtain

$$\begin{aligned}&\big(\partial_s + \mathscr{L}_s\big) u_{T-s}(X^x_{t,s}) + V_{T-s}(X^x_{t,s})\\ &= \bigg\{\bigg[u_{T-s}(y) - \mathbb{E}u_0(X^y_{s,T}) - \mathbb{E}\int_s^T V_{T-r}(X^y_{s,r})\mathrm{d}r\bigg]_{y=X^x_{t,s}} \cdot \nabla\bigg\} u_{T-s}(X^x_{t,s}).\end{aligned}$$

Combining this with the follow property (2.3) and (3.5), we derive

$$\begin{aligned}&\mathbb{E}u_0(X^x_{t,T}) - u_{T-t}(x) = \mathbb{E}\big[u_{T-T}(X^x_{t,T}) - u_{T-t}(X^x_{t,t})\big]\\ &= \mathbb{E}\int_t^T \bigg\{\bigg(u_{T-s}(y) - \mathbb{E}u_0(X^y_{s,T}) - \mathbb{E}\int_s^T V_{T-r}(X^y_{s,r})\mathrm{d}r\bigg)_{y=X^x_{t,s}} \cdot \nabla\bigg\} u_{T-s}(X^x_{t,s})\mathrm{d}s\\ &\quad - \mathbb{E}\int_t^T V_{T-s}(X^x_{t,s})\mathrm{d}s, \quad (t,x)\in[0,T]\times\mathbb{R}^d.\end{aligned}$$

Letting

$$h_t := \sup_{x\in\mathbb{R}^d}\bigg|u_{T-t}(x) - \mathbb{E}u_0(X^x_{t,T}) - \mathbb{E}\int_t^T V_{T-s}(X^x_{t,s})\mathrm{d}s\bigg|, \quad t\in[0,T],$$

we arrive at

$$h_t \le \int_t^T h_s\|\nabla u\|_\infty \mathrm{d}s, \quad t\in[0,T].$$

By Grownwall's inequality we prove $h_t = 0$ for $t\in[0,T]$, hence (1.6) holds. □

Proof of Theorem 1.1(3). (a) Let $P_{t,s}f = \mathbb{E}[f(X^x_{t,s})]$ for $f\in\mathscr{B}_b(\mathbb{R}^d)$, where $X^x_{t,s}$ solves (1.3). For u given by (1.6) we have

$$u_t = P_{T-t,T}u_0 + \int_{T-t}^T P_{T-t,s}V_{T-s}\mathrm{d}s, \quad t\in[0,T]. \tag{3.6}$$

By $\|u_0\|_\infty + \int_0^T \|V_t\|_\infty \mathrm{d}t < \infty$ and (1.7), we find a constant $c>0$ such that

$$\|u\|_\infty + \|\nabla u\|_\infty \le c, \quad \|\nabla^2 u_t\|_\infty \le ct^{-\frac{1}{2}}, \quad t\in(0,T]. \tag{3.7}$$

Moreover, the SDE (1.3) becomes

$$\begin{aligned}\mathrm{d}X^x_{t,s} &= \sqrt{2a_{T-s}}(X^x_{t,s})\mathrm{d}W_s + \big\{b_{T-s} - u_{T-s}\big\}(X^x_{t,s})\mathrm{d}s,\\ &\quad t\in[0,T], s\in[t,T], X^x_{t,t} = x\in\mathbb{R}^d,\end{aligned} \tag{3.8}$$

and the generator in (3.4) reduces to

$$\mathscr{L}_s := \mathrm{tr}\big\{a_{T-s}\nabla^2\big\} + \big\{b_{T-s} - u_{T-s}\big\}\cdot\nabla, \quad s\in[0,T].$$

(b) We prove the Kolmogorov backward equation

$$\partial_t P_{t,s}f = -\mathscr{L}_t P_{t,s}f, \quad f\in\mathscr{C}^2_b, t\in[0,s], s\in(0,T]. \tag{3.9}$$

For any $f \in \mathscr{C}_b^2$, by Itô's formula we have

$$P_{t,s}f(x) = f(x) + \int_t^s P_{t,r}(\mathscr{L}_r f)(x)\mathrm{d}r, \quad (t,s) \in D_T, \tag{3.10}$$

where $\int_t^s P_{t,r}(\mathscr{L}_r f)(x)\mathrm{d}r = \mathbb{E}\int_t^s \mathscr{L}_r f(X_{t,r}^x)\mathrm{d}r$ exists, since Krylov's estimate in Lemma 3.1(2) holds under (A) and $\|u\|_\infty < \infty$.

By (3.10), we obtain the Kolmogorov forward equation

$$\partial_s P_{t,s} f = P_{t,s}(\mathscr{L}_s f), \quad s \in [t,T]. \tag{3.11}$$

On the other hand, $b^{(1)} = 0$ and (A) imply

$$\|\mathscr{L} f\|_{\tilde{L}_{q_0}^{p_0}} \le c_0 \|f\|_{\mathscr{C}_b^2} \tag{3.12}$$

for some constant $c_0 > 0$. By Lemma 3.1(1), for any $s \in (0,T]$, the PDE

$$(\partial_t + \mathscr{L}_t)\tilde{u}_t = -\mathscr{L}_t f, \quad t \in [0,s], \tilde{u}_s = 0 \tag{3.13}$$

has a unique solution $\tilde{u} \in \mathscr{U}(p_0,q_0)$, such that

$$\|\nabla^2 \tilde{u}\|_{\tilde{L}_{q_0}^{p_0}(0,s)} \le c_1 \|\mathscr{L} f\|_{\tilde{L}_{q_0}^{p_0}(0,s)} \tag{3.14}$$

holds for some constant $c_1 > 0$ independent of s. By Itô's formula in Lemma 3.1(3),

$$\mathrm{d}\tilde{u}_t(X_{0,t}^x) = -\mathscr{L}_t f(X_{0,t}^x) + \left\langle \nabla f(X_{0,t}^x), \sqrt{2a_{T-t}}(X_{0,t}^x)\mathrm{d}W_t \right\rangle, \quad t \in [0,s].$$

This and (3.11) imply

$$\begin{aligned} 0 = \tilde{u}_s(x) &= \tilde{u}_t(x) - \int_t^s (P_{t,r}\mathscr{L}_r f)(x)\mathrm{d}r \\ &= \tilde{u}_t(x) - \int_t^s \frac{\mathrm{d}}{\mathrm{d}r}(P_{t,r}f)\mathrm{d}r = \tilde{u}_t(x) - P_{t,s}f(x) + f(x), \quad t \in [0,s]. \end{aligned}$$

Thus,

$$\tilde{u}_t = P_{t,s}f - f, \quad t \in [0,s]. \tag{3.15}$$

Combining this with (3.13) we derive (3.9).

(c) By (3.7) and (3.9), we see that u solves (1.6) with $u \in \mathscr{U}(p_0,q_0)$ provided

$$\|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}} < \infty. \tag{3.16}$$

By (3.12), (3.14) and (3.15), we find a constant $c_2 > 0$ such that

$$\sup_{t\in[0,s]} \|\nabla^2 P_{\cdot,s} f\|_{\tilde{L}_{q_0}^{p_0}(0,s)} \le c_2 \|f\|_{\mathscr{C}_b^2}, \quad s \in (0,T], f \in \mathscr{C}_b^2.$$

Combining this with (3.6), $b^{(1)} = 0$ and $\|u_0\|_{\mathscr{C}_b^2} + \int_0^T \|V_t\|_{\mathscr{C}_b^2}\mathrm{d}t < \infty$, we prove (3.16).

□

3.2 $E = \mathbb{T}^d$

In this case, all functions on E are extended to $\mathbb{R}^d$ as in (1.4), so that the proof for $E = \mathbb{R}^d$ works also for the present setting if we could verify the following periodic property for the solution of (1.3):

$$X_{t,s}^{x+k} = X_{t,s}^x + k, \quad (t,s) \in D_T,\ x \in \mathbb{R}^d,\ k \in \mathbb{Z}^d. \tag{3.17}$$

Let $\tilde{X}_{s,t}^x := X_{t,s}^x + k$. Since the coefficients of (1.3) satisfies (1.4), $\tilde{X}_{t,s}^x$ solves (1.3) with $\tilde{X}_{t,t}^x = x + k$. By the uniqueness of (1.3) ensured by Theorem 1.1(1), we derive (3.17).

4 Application to (1.1)

For any $n \in \mathbb{N}$, let $\mathscr{C}_b^n$ be the class of real functions f on E having derivatives up to order n such that

$$\|f\|_{\mathscr{C}_b^n} := \sum_{i=0}^n \|\nabla^i f\|_\infty < \infty,$$

where $\nabla^0 f := f$. Moreover, for $\alpha \in (0,1)$, we denote $f \in \mathscr{C}_b^{n+\alpha}$ if $f \in \mathscr{C}_b^n$ such that

$$\|f\|_{\mathscr{C}_b^{n+\alpha}} := \|f\|_{\mathscr{C}_b^n} + \sup_{x \neq y} \frac{\|\nabla^n f(x) - \nabla^n f(y)\|}{|x-y|^\alpha} < \infty.$$

Consider the following future distribution dependent SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_{t,s}^x = \left[\mathbb{E}\int_s^T \nabla\wp_{T-r}(X_{s,r}^y)\mathrm{d}r - \mathbb{E}u_0(X_{s,T}^y)\right]_{y=X_{t,s}^x} \mathrm{d}s + \sqrt{2\kappa}\mathrm{d}W_s, \quad X_{t,t}^x = x, s \in [t,T]. \tag{4.1}$$

See Definition 1.1 below for the definition of solution. When $E = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, we extend u_0 and $\wp_t$ to $\mathbb{R}^d$ periodically, i.e. for a function f on $\mathbb{T}^d$, it is extended to $\mathbb{R}^d$ as in (1.4). With this extension, we also have the SDE (4.1) for the case $E = \mathbb{T}^d$.

Theorem 4.1. *If there exists $n \geq 2$ such that $u_0 \in \mathscr{C}_b^n$ and $\wp_t \in \mathscr{C}_b^n$ for a.e. $t \in [0,T]$ with*

$$\int_0^T \big(\|\nabla\wp_t\|_\infty^2 + \|\wp_t\|_{\mathscr{C}_b^n}\big)\mathrm{d}t < \infty.$$

Then (4.1) is well-posed and (1.1) has a unique solution satisfying

$$\sup_{t\in[0,T]} \|u_t\|_{\mathscr{C}_b^n} < \infty, \tag{4.2}$$

and the solution is given by

$$u_t(x) = \mathbb{E}u_0(X_{T-t,T}^x) - \mathbb{E}\int_{T-t}^T \nabla\wp_{T-s}(X_{T-t,s}^x)\mathrm{d}s. \tag{4.3}$$

We only prove for $E=\mathbb{R}^d$ as the case for $E=\mathbb{T}^d$ follows by extending functions from $\mathbb{T}^d$ to $\mathbb{R}^d$ as in (1.4).

Let I_d be the $d\times d$ identity matrix. By Theorem 1.1 with $b=0, a=\kappa I_d$ and $V=-\nabla\wp$, for any $(p_0,q_0)\in\mathscr{K}$, (1.1) has a unique solution in the class $\mathscr{U}(p_0,q_0)$, and by (4.3),

$$
\begin{aligned}
u_t(x) &:= \mathbb{E}u_0(X^x_{T-t,T}) - \mathbb{E}\int_{T-t}^T \nabla\wp_{T-s}(X^x_{T-t,s})\mathrm{d}s \\
&= P_{T-t,T}u_0(x) - \int_{T-t}^T P_{T-t,s}\nabla\wp_{T-s}(x)\mathrm{d}s,\quad (t,x)\in[0,T]\times\mathbb{R}^d.
\end{aligned}
\tag{4.4}
$$

By (3.8) for the present a and b, $X^x_{t,s}$ solves the SDE

$$
\mathrm{d}X^x_{t,s} = \sqrt{2\kappa}\mathrm{d}W_s - u_{T-s}(X^x_{t,s})\mathrm{d}s,\quad X^x_{t,t}=x, t\in[0,T], s\in[t,T],
\tag{4.5}
$$

and the generator is

$$
\mathscr{L}_s := \kappa\Delta - u_{T-s}\cdot\nabla,\quad s\in[0,T].
$$

It remains to prove (4.2). To this end, we present the following lemma.

Lemma 4.2. *Let $P_{t,s}f := \mathbb{E}[f(X^x_{t,s})]$ for the SDE (4.5). Let $m\geq 1$ such that*

$$
\sup_{t\in[0,T]}\|u_t\|_{\mathscr{C}_b^m} + \|f\|_{\mathscr{C}_b^{m+1}} < \infty,
\tag{4.6}
$$

then $\sup_{(t,s)\in D_T}\|P_{t,s}f\|_{\mathscr{C}_b^{m+1}}<\infty$.

Proof. By (4.5) and $\sup_{t\in[0,T]}\|u_t\|_{\mathscr{C}_b^m}<\infty$, we have

$$
\sup_{(t,s,x)\in D_T\times\mathbb{R}^d}\mathbb{E}\big[\|\nabla^i X^x_{t,s}\|\big]<\infty,\quad 1\leq i\leq m.
$$

By chain rule, this implies that for some constant $c_0>0$,

$$
\sup_{(t,s)\in D_T}\|P_{t,s}g\|_{\mathscr{C}_b^m}\leq c_0\|g\|_{\mathscr{C}_b^m},\quad g\in\mathscr{C}_b^m.
\tag{4.7}
$$

Let $P^0_t=\mathrm{e}^{\kappa\Delta t}$. By $\partial_r P^0_{r-t}=P^0_{r-t}\kappa\Delta$ and (3.9), we have

$$
\partial_r P^0_{r-t}P_{r,s}f = P^0_{r-t}\langle\nabla P_{r,s}f, u_{T-r}\rangle,\quad r\in[t,s].
$$

So,

$$
P_{t,s}f = P^0_{s-t}f - \int_t^s P^0_{r-t}\langle\nabla P_{r,s}f, u_{T-r}\rangle\mathrm{d}r.
\tag{4.8}
$$

It is well known that for any $\alpha,\beta\geq 0$ there exists a constant $c_{\alpha,\beta}>0$ such that

$$
\|P^0_t g\|_{\mathscr{C}_b^{\alpha+\beta}}\leq c_{\alpha,\beta}t^{-\frac{\alpha}{2}}\|g\|_{\mathscr{C}_b^\beta},\quad t>0, g\in\mathscr{C}_b^\beta.
\tag{4.9}
$$

This together with (4.8) implies that for some constants $c_1, c_2 > 0$,

$$\|P_{t,s}f\|_{\mathscr{C}_b^{m+\frac{1}{2}}} \le c_1\|f\|_{\mathscr{C}_b^{m+\frac{1}{2}}} + c_1\int_t^s (t+r-s)^{-\frac{3}{4}}\|\langle\nabla P_{r,s}f, u_{T-r}\rangle\|_{\mathscr{C}_b^{m-1}}\mathrm{d}r.$$

Combining this with (4.7) and $\|f\|_{\mathscr{C}_b^m} + \sup_{t\in[0,T]}\|u_t\|_{\mathscr{C}_b^m} < \infty$, we obtain

$$\sup_{(t,s)\in D_T}\|P_{t,s}f\|_{\mathscr{C}_b^{m+\frac{1}{2}}} < \infty.$$

By this together with (4.8) and (4.6), we find a constant $c_2 > 0$ such that

$$\begin{aligned}&\sup_{(t,s)\in D_T}\|P_{t,s}f\|_{\mathscr{C}_b^{m+1}} \le c_2\|f\|_{\mathscr{C}_b^{m+1}}\\&+ c_2\sup_{(t,s)\in D_T}\int_t^s (t+r-s)^{-\frac{3}{4}}\|\langle\nabla P_{r,s}f, u_{T-r}\rangle\|_{\mathscr{C}_b^{m-\frac{1}{2}}}\mathrm{d}r < \infty.\end{aligned}$$

□

We now prove (4.2) as follows. By $u \in \mathscr{U}(p_0, q_0)$, we have

$$\|u\|_\infty + \|\nabla u\|_\infty < \infty.$$

Combining this with (4.4) and Lemma 4.2, we prove (4.2) by inducing in m up to $m = n$.

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