Sobolev–Kantorovich inequalities under $CD(0,\infty)$ condition

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Abstract

We refine and generalize several interpolation inequalities bounding the L^p norm of a probability density with respect to the reference measure μ by its Sobolev norm and the Kantorovich distance to μ on a smooth weighted Riemannian manifold satisfying $CD(0,\infty)$ condition.

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1. Introduction

In the last decade there has been an increasing interest in functional inequalities relating Sobolev norms with certain other norms and quantities such as entropy and Kantorovich distances, e.g. see [5], [27], [29]. In this paper we discuss some interpolation inequalities which can be viewed as analogs of the classic Hardy–Landau–Littlewood inequality

$$\|f'\|_{L^1}^2 \le C\|f\|_{L^1}\|f''\|_{L^1}$$

as well as the celebrated Otto–Villani HWI inequality

$$\int f\log f\,d\mu \le \biggl(\int \frac{|\nabla f|^2}{f}\,d\mu\biggr)^{1/2} W_2(\mu, f\mu).$$

It was noticed in [11], [12] that the Hardy–Landau–Littlewood inequality can be written as

$$\|f\|_{L^1}^2 \le C\|\nabla f\|_{L^1}\,\|f\|_K,$$

where f belongs to the usual Sobolev class $W^{1,1}$ and has zero integral and $\|f\|_K$ denotes the Kantorovich norm of the signed measure $f\,dx$ with zero value on the whole space. Recall that the Kantorovich norm of a signed measure μ on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) = 0$ integrating Lipschitz functions is defined by

$$\|\mu\|_K = \sup\biggl\{\int f\,d\mu\colon f\in C_b^\infty(\mathbb{R}^n),\ |\nabla f|\le 1\biggr\}.$$

In this form this inequality admits natural multidimensional extensions such as the bound

$$\|\mu\|^2 \le C\|D\mu\|\,\|\mu\|_K$$

established in [11, Theorem 1] for signed Borel measures μ on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) = 0$ possessing a density of class BV, where $\|\mu\|$ is the total variation of μ and $\|D\mu\|$ is the total variation of the vector measure $D\mu$ that is the distributional derivative of μ. Next, in our note [12] a dimension-free version of this bound employing probability reference measures was established, in particular, for the standard Gaussian measure γ_d on $\mathbb{R}^d$ we proved that

$$\|f\|_{L^1(\gamma_d)}^2 \le 2\|\nabla f\|_{L^1(\gamma_d)}\|f\gamma_d\|_K$$

for all smooth functions $f\in L^1(\gamma_d)$ with zero integral against γ_d.

There are also extensions closely related to the inequalities arising in the study of some evolution equations (e.g., the Cahn–Hilliard model) established by Cinti, Kohn, and Otto [14], [18]. In particular, it was proved in [14, Proposition 1.3] that for any periodic smooth probability density f on $[0,1]^n$ one has

$$\|(f-C)_+\|_r^\theta \le C\|\nabla f\|_1 W_2(dx, f\,dx),$$

where

$$r = \frac{3n+2}{3n},\ \ \theta = \frac{3n+2}{2n}$$

and $W_2(dx, f\,dx)$ denotes the Kantorovich distance of order 2 between the probability measures dx and $f\,dx$ on $[0,1]^n$. This result was generalized by Ledoux [19] in the setting of non-negatively curved weighted manifolds. Let (M, g) be a complete connected n-dimensional Riemannian manifold with the Riemannian volume dx and let μ be a probability measure on M with a smooth density with respect to dx. The term "smooth function" will mean below a function of class $C_b^\infty(M)$. The symbol $\|\cdot\|_q$ will refer to the norm in $L^q(\mu)$. In the "finite-dimensional" case Ledoux established the following theorem.

Theorem 1.1. (Theorem 1.1 from [19]) *Suppose that μ satisfies the curvature-dimension condition $CD(0, N)$ for some $N \geq 1$. Given $p, q \geq 1$, there is a constant $C > 0$ depending only on p, q, N such that for any probability measure $\nu = f \cdot \mu$ with a smooth density f one has*

$$\|(f - C)_+\|_r^\theta \leq C\|\nabla f\|_q W_p(\mu, \nu),$$

where

$$r = \frac{1 + \frac{1}{p} + \frac{1}{N}}{\frac{1}{p} + \frac{1}{q}}, \quad \theta = r\Big(\frac{1}{p} + \frac{1}{q}\Big) = 1 + \frac{1}{p} + \frac{1}{N}.$$

In the "infinite-dimensional" case $CD(0, \infty)$ the main result from [19] is summarized in the next theorem.

Theorem 1.2. (Theorem 4.1 from [19]) *Suppose that μ satisfies the curvature-dimension condition $CD(0, \infty)$. Given $1 < q \leq 2$, there is a constant $C > 0$ depending only on q such that for any probability measure $\nu = f \cdot \mu$ with a smooth density f one has*

$$\|(f - C)_+\|_r^{3/2} \leq C\|\nabla f\|_q W_2(\mu, \nu), \quad r = \frac{3q}{q + 2}.$$

Of course, these bounds extend to densities f from the Sobolev class $W^{q,1}(\mu)$ defined as the completion of $C_0^\infty(M)$ with respect to the Sobolev norm

$$\|f\|_{q,1} := \|f\|_{L^q(\mu)} + \big\| |\nabla f| \big\|_{L^q(\mu)},$$

associated with μ. Using uniformly Lipschitz bump functions on M (cf. [2, Chapter 2]), it is readily verified that $W^{q,1}(\mu)$ coincides with the class of locally Sobolev functions on M with finite norm $\|f\|_{q,1}$.

For the Kantorovich distance of order 1 (Kantorovich norm) it was proved in [12] that on a smooth weighted Riemannian manifold (M, g, μ) satisfying the curvature-dimension condition $CD(\kappa, \infty)$ with $\kappa \geq 0$ for any smooth function f with zero integral the following inequality holds:

$$\|f\|_1 \leq \inf_{\tau > 0}\left[\|f\|_K \sqrt{\frac{\kappa}{e^{2\kappa\tau} - 1}} + \|\nabla f\|_1 \int_{[0,\tau]} \sqrt{\frac{\kappa}{e^{2\kappa t} - 1}}\, dt\right],$$

where for $\kappa = 0, t > 0$ we set

$$\frac{\kappa}{e^{2\kappa t} - 1} := \frac{1}{2t}.$$

This bound implies that under the condition $CD(0,\infty)$ for any two smooth probability densities f_1, f_2 the following inequality holds:

$$\|f_1 - f_2\|_1^2 \le 2\|\nabla f_1 - \nabla f_2\| \cdot \|f_1 \cdot \mu - f_2 \cdot \mu\|_K.$$

In the present paper Theorem 1.2 (Theorem 4.1 from [19]) is improved in several directions. We show that a stronger inequality with an extra logarithmic factor on the left-hand side holds true for all $q > 1$. Our approach refines the arguments from [14], [19] and combines them with some ideas from our short notes [12], [13], where a preliminary version of this result was announced without a proof and in smaller generality. The main result is contained in the next theorem.

Theorem 1.3. *Let (M, g, μ) be a complete connected Riemannian manifold with a probability measure μ with a smooth density satisfying the condition $CD(0,\infty)$. For every $q > 1$, there is a constant $C > 0$ depending only on q such that for any probability measure $\nu = f \cdot \mu$ with a smooth density f (or $f \in W^{q,1}(\mu)$) one has*

$$\left\|(f - C)_+\bigl(1 + \log^\alpha(1 + (f - C)_+)\bigr)\right\|_r^{3/2} \le C\|\nabla f\|_q W_2(\mu, \nu),$$

where

$$r = \frac{3q}{q+2},\ \ \alpha = \frac{1}{3}.$$

The assertion does not hold if $\log^\alpha(1 + \cdot)$ *is replaced by any increasing positive function* $\Phi\colon [0,\infty) \to [0,\infty)$ *with*

$$\lim_{u\to\infty} \Phi(u) \log^{-2/3}(1 + u) = \infty.$$

Remark 1.4. It would be interesting to investigate the sharpness of this inequality, i.e., determine the optimal value of α. Theorem 1.3 essentially states that the optimal value of α belongs to the interval $[1/3, 2/3]$.

The paper is organized as follows. In Section 2 we introduce our framework and some basic tools, such as the pseudo-Poincaré inequality, the Kantorovich duality, and the infinite-dimensional Harnack inequality. In Section 3 we prove Theorem 1.3. In Section 4 the case of the Kantorovich distance W_1 of order 1 is discussed. In Section 5 we present some extensions to the case of negatively curved weighted Riemannian manifolds.

2. Framework and geometric tools

Throughout this paper, we assume that (M, g) is a smooth complete connected Riemannian manifold and $V \in C^2(M)$ such that $\mu(dx) := e^{-V(x)}dx$ is a probability measure, where dx stands for the Riemannian volume measure, such that $L := \Delta - \nabla V \cdot \nabla$ generates a diffusion semigroup $\{P_t\}_{t\geq 0}$ on $L^2(\mu)$, which means that $\{P_t\}_{t\geq 0}$ is a symmetric strongly continuous operator semigroup on $L^2(\mu)$ and its generator coincides with L on smooth compactly supported functions, $P_t f \geq 0$ if $f \geq 0$ and $P_t 1 = 1$. Consequently, μ is P_t-invariant, i.e.,

$$\int P_t f \, d\mu = \int f \, d\mu \quad \forall\, t \geq 0,\ f \in L^2(\mu).$$

This triple (M, g, μ) will be called below a smooth weighted Riemannian manifold.

The curvature-dimension condition $CD(K, N)$ for the operator L, where $K \in \mathbb{R}$, $N \geq 1$, is described by the Bochner-type inequality

$$\frac{1}{2} L|\nabla f|^2 - \nabla f \cdot \nabla L f \geq K|\nabla f|^2 + \frac{1}{N}(Lf)^2$$

for all smooth functions $f \in C_0^\infty(M)$. The standard references on this topic are [4] and [5]. For example, by the classical Bochner formula the Laplace operator on an n-dimensional Riemannian manifold with Ricci curvature bounded from below by K satisfies the curvature-dimension condition $CD(K, N)$ for all $N \geq n$. However, many important diffusion operators are intrinsically of infinite dimension, for example, for $M = \mathbb{R}^n$ the standard Ornstein–Uhlenbeck operator $L = \Delta - x \cdot \nabla$ satisfies the condition $CD(1, \infty)$, but does not satisfy $CD(K, N)$ with a finite number N. We recall several results from [3] and [20]. Let us define the Riesz transform $\mathcal{R}_\varrho$ by the formula

$$\mathcal{R}_\varrho := \nabla(\varrho - L)^{-1/2},\ \ \varrho > 0.$$

Proposition 2.1. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(-\varrho, \infty)$, $\varrho > 0$. Then, for each $p > 1$, there exists a constant $C_p > 0$ depending only on p such that for all $f \in C_0^\infty$ one has*

$$\|\mathcal{R}_\varrho f\|_p \leq C_p \|f\|_p.$$

Proof. This is a well-known result, first proved by Bakry in [3], for a self-contained exposition and an analytical approach to this fundamental estimate we refer the reader to the more recent work [15], see also [20, Theorem 1.4]. □

The formulation of Theorem 1.4 in [20] also includes the case $\varrho = 0$, but we would like to notice that one has to be careful with the definition of $\mathcal{R}_0$, since the range of $\sqrt{-L}$ on C_0^∞ is not dense in $L^2(\mu)$.

Proposition 2.2. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(-\varrho, \infty)$ with some $\varrho \geq 0$. For every $p > 1$, there exists a constant $C_p > 0$ depending only on p such that for all $f \in C_0^\infty$ one has*

$$C_p^{-1} \|\nabla f\|_p \leq \|\sqrt{\varrho - L} f\|_p \leq \sqrt{\varrho} \|f\|_p + C_q \|\nabla f\|_p,$$

where $1/p + 1/q = 1$ and C_p, C_q are the constants provided by Proposition 2.1.

Proof. For $\varrho > 0$ this is the statement of Theorem 5.5 from [20] and in fact the proof of these inequalities was presented in [20] only in this case and the constants C_p, C_q do not depend on ϱ. The operator $-L$ is essentially self-adjoint on C_0^∞ (see, e.g., [5, Corollary 3.2.2]) and non-negative. Let $\{E_\lambda\}_{\lambda \geq 0}$ be the projection-valued measure such that

$$-L = \int_{[0,\infty)} \lambda \, dE_\lambda.$$

Let us fix $f \in C_0^\infty$. Since $f \in D(L)$

$$\int_{[0,\infty)} \lambda^2 \, d\langle E_\lambda f, E_\lambda f \rangle < \infty.$$

Then by the dominated convergence theorem

$$\lim_{\varrho \to 0+} \|\sqrt{-L} f - \sqrt{\varrho - L} f\|_2^2 = \lim_{\varrho \to 0+} \int_{[0,\infty)} (\sqrt{\lambda} - \sqrt{\varrho + \lambda})^2 \, d\langle E_\lambda f, E_\lambda f \rangle = 0,$$

or, equivalently, $\lim_{\varrho \to 0+} \sqrt{\varrho - L} f = \sqrt{-L} f$ in $L^2(\mu)$. Now the case $\varrho = 0$ easily follows by passing to the limit as $\varrho \to 0$. □

The next theorem strengthens Proposition 2.2 from [19], established in the case $q \in [1, 2]$. We thank the anonymous referee for pointing out that for $q > 1$ it also follows from [1, Proposition 2.3]. However, we include a short proof, because in Section 5 we refer to this proof with some modification in order to cover the negative curvature case.

Theorem 2.3. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(0, \infty)$. For every*

$q \geq 1$, there exists a positive constant C depending only on q such that for any smooth function f on M and any $t > 0$ one has

$$\|f - P_t f\|_q \leq C\sqrt{t}\|\nabla f\|_q.$$

Proof. For any bounded measurable function h on M we have the following reverse Poincaré inequality (see [5]):

$$2s|\nabla P_s h|^2 \leq P_s(h^2) - (P_s h)^2.$$

Then by Jensen's inequality for all $q' \in [2, \infty]$ we obtain

$$\|\nabla P_s h\|_{q'} \leq \frac{1}{\sqrt{2s}}\|h\|_{q'}. \tag{2.1}$$

Now one can observe that

$$\int_M h(f - P_t f)\, d\mu = -\int_0^t \int_M hLP_s f\, d\mu\, ds = \int_0^t \int_M \nabla P_s h \cdot \nabla f\, d\mu\, ds$$

and by duality it is easy to see that for any $q \in [1, 2]$ one has

$$\|f - P_t f\|_q \leq \int_0^t \frac{1}{\sqrt{2s}}\|\nabla f\|_q\, ds = \sqrt{2t}\|\nabla f\|_q.$$

Now let us consider $q > 2$. For $f \in C_0^\infty$ and $t > 0$ we have (in in $L^2(\mu)$)

$$P_t f - f = \int_0^\infty K(s,t) P_s \sqrt{-L} f\, ds,$$

$$K(s,t) := \frac{1}{\sqrt{\pi}}\left(\frac{\chi_{s>t}}{(s-t)^{1/2}} - \frac{\chi_{s>0}}{s^{1/2}}\right).$$

One can easily check that for all $t > 0$

$$\int_0^\infty |K(s,t)|\, ds = \frac{4}{\sqrt{\pi}}\sqrt{t}.$$

Taking into account Proposition 2.2, we obtain

$$\begin{aligned}\|f - P_t f\|_q &\leq \int_0^\infty |K(s,t)|\, \|P_s \sqrt{-L} f\|_q\, ds \\ &\leq \int_0^\infty |K(s,t)|\, \|\sqrt{-L} f\|_q\, ds \leq C(q)\sqrt{t}\|\nabla f\|_q,\end{aligned}$$

which completes the proof. □

Remark 2.4. For the standard Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ given by the Mehler formula

$$T_t f(x) := \int_{\mathbb{R}^d} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\, \gamma(dy)$$

the estimate from Theorem 2.3 can be established directly with an explicit constant:

$$\|T_t f - f\|_q \le K_q c_t \|\nabla f\|_q,$$

$$K_q^q := \int_{\mathbb{R}} |x|^q \, \gamma(dx),\ c_t := \int_0^t \frac{e^{-s}}{\sqrt{1-e^{-2s}}}\, ds = \arccos(e^{-t}).$$

Indeed, for $f \in C_0^\infty(\mathbb{R}^d)$ one has

$$f(e^{-t}x + \sqrt{1-e^{-2t}}y) - f(x) = \int_0^1 \frac{d}{d\tau} f(e^{-t\tau}x + \sqrt{1-e^{-2t\tau}}y)\, d\tau$$
$$= t\int_0^1 \nabla f(e^{-t\tau}x + \sqrt{1-e^{-2t\tau}}y) \cdot \Bigl(-e^{-t\tau}x + \frac{e^{-2t\tau}}{\sqrt{1-e^{-2t\tau}}} y\Bigr)\, d\tau,$$

$$\|T_t f - f\|_q^q \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |f(e^{-t}x + \sqrt{1-e^{-2t}}y) - f(x)|^q \gamma(dx)\gamma(dy)$$
$$\le c_t^{q-1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \int_{[0,1]} \frac{te^{-t\tau}}{\sqrt{1-e^{-2t\tau}}}$$
$$\times \Bigl|\nabla f(e^{-t\tau}x + \sqrt{1-e^{-2t\tau}}y) \cdot (-\sqrt{1-e^{-2t\tau}}x + e^{-t\tau}y)\Bigr|^q \, d\tau\gamma(dx)\gamma(dy)$$
$$= c_t^{q-1} \int_0^1 \frac{te^{-t\tau}}{\sqrt{1-e^{-2t\tau}}} \int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla f(x)\cdot y|^q \gamma(dy)\gamma(dx)\, d\tau$$
$$= K_q^q c_t^q \int_{\mathbb{R}^d} |\nabla f(x)|^q \gamma(dx),$$

as announced.

Recall that the Kantorovich distance $W_p(\mu,\nu)$ of order $p \ge 1$ between two probability measures μ and ν with finite moments of order p is defined by the formula

$$W_p(\mu,\nu)^p = \inf_{\sigma\in\Pi(\mu,\nu)} \int_{M\times M} d(x,y)^p \, \sigma(dx,dy),$$

where inf is taken over all measures σ from the set $\Pi(\mu,\nu)$ of Borel probability measures on $M \times M$ having projections μ and ν onto the first and second factors, respectively; see [10], [9] or [24] (the case $p = 1$ was considered in [16], [17]). Now let us introduce the Hopf–Lax infimum-convolutions $(Q_s)_{s>0}$ defined by the formula

$$Q_s\varphi(x) := \inf_{y\in M} \Bigl[\varphi(y) + d^p(x,y)/s\Bigr],\ x \in M,\ s > 0.$$

The dual description of the Kantorovich metric of order p is given by the equality

$$W_p^p(\mu,\nu)=\sup_{\varphi}\Bigl(\int_M Q_1\varphi\,d\nu-\int_M\varphi\,d\mu\Bigr),$$

where the supremum is taken over all bounded continuous functions φ, see [24]. Alternatively, one can use the sup-convolutions defined by

$$\widehat{Q}_s\varphi(x):=\sup_{y\in M}\Bigl[\varphi(y)-d^p(x,y)/s\Bigr],\ x\in M,\ s>0,$$

$$W_p^p(\mu,\nu)=\sup_{\varphi}\Bigl(\int_M\varphi\,d\mu-\int_M\widehat{Q}_1\varphi\,d\nu\Bigr).$$

A crucial role in our considerations in the next section will be played by the "infinite-dimensional" Harnack inequality that states that, under the curvature-dimension condition $CD(0,\infty)$, for any non-negative square-integrable function g on M and all $t>0$, $x,y\in M$ one has one has

$$[P_tg(y)]^2\le P_t(g^2)(x)e^{d^2(x,y)/2t}. \tag{2.2}$$

For more information on Harnack inequalities of this type, see [25], [27], [28], and [6].

3. Main results

We start with establishing a weak-type bound. The next theorem strengthens Proposition 4.2 from [19] (see also [13]). Set

$$\kappa(s)=\frac{s\log s}{s\log s+1-s},\quad s>1.$$

It is easy to see that the function κ is decreasing in $s\in(1,\infty)$ with

$$\lim_{s\to1+}\kappa(s)=\infty,\ \lim_{s\to\infty}\kappa(s)=1.$$

Theorem 3.1. *Let (M,g,μ) be a smooth weighted Riemannian manifold satisfying the condition $CD(0,\infty)$. For every $q\ge1$, there exists a positive constant C depending only on q such that for any probability measure $\nu=f\cdot\mu$ with a smooth density f (or $f\in W^{q,1}(\mu)$) and any $s>1$ one has*

$$\sup_{u\ge s}\Bigl[u^{3/2}\log^{1/2}u\Bigr]\mu(f\ge 2u)^{3/(2r)}\le C\kappa^{1/2}(s)\|\nabla f\|_qW_2(\mu,\nu),$$

where $r=\frac{3q}{q+2}$. On the other hand, the assertion does not hold if $\log^{1/2}$ is rep laced by an increasing positive function $\Phi\colon[0,\infty)\to[0,\infty)$ with $\lim_{u\to\infty}\Phi(u)\log^{-1}(u)=\infty$.

Proof. (a) The proof of the first assertion is inspired by the approach of Ledoux [19]. For any $t > 0$ let us represent f as

$$f = (f - P_t f) + P_t f.$$

Then for $u > 0$ we have

$$\mu(f \geq 2u) \leq \mu\big(|f - P_t f| \geq u\big) + \mu(P_t f \geq u).$$

Taking into account Theorem 2.3 we obtain the estimate

$$\mu(f \geq 2u) \leq \frac{C_q t^{q/2}}{u^q} \|\nabla f\|_q^q + \mu(P_t f \geq u).$$

The next step is to apply the classical entropy inequality

$$\int_M h P_t f \, d\mu \leq \int_M P_t f \log P_t f \, d\mu + \log \int_M e^h \, d\mu.$$

Now let us take

$$h := \mathbb{1}_F \log P_t f,$$

where

$$F = \{P_t f \geq u\}.$$

Then

$$\int_M \mathbb{1}_F P_t f \log P_t f \, d\mu \leq \int_M P_t f \log P_t f \, d\mu + \log \int_M \big[1 + \mathbb{1}_F (P_t f - 1)\big] \, d\mu,$$

$$\int_F P_t f \log P_t f \, d\mu \leq \int_M P_t f \log P_t f \, d\mu + \log \bigg[\int_M 1 \, d\mu + \int_F (P_t f - 1) \, d\mu\bigg].$$

Since for every $x \geq 0$ one has $\log(1 + x) \leq x$, and μ is a probability measure, we obtain

$$\int_F P_t f \log P_t f \, d\mu \leq \int_M P_t f \log P_t f \, d\mu + \int_F (P_t f - 1) \, d\mu.$$

Note that due to [7, Lemma 4.2] or [6, Lemma 1.11] we have

$$\int_M P_t f \log P_t f \, d\mu \leq \frac{1}{4t} W_2^2(\nu, \mu).$$

This implies the inequality

$$\int_F \big(1 + P_t f[\log P_t f - 1]\big) \, d\mu \leq \int_M P_t f \log P_t f \, d\mu \leq \frac{1}{4t} W_2^2(\nu, \mu) \; \forall \, t > 0.$$

Combining this estimate with the definition and monotonicity of the function κ we obtain

$$(u \log u)\mu(F) \leq \frac{\kappa(u)}{4t} W_2^2(\nu, \mu) \leq \frac{\kappa(s)}{4t} W_2^2(\nu, \mu), \; 1 < s \leq u.$$

Consequently,

$$\mu(f \geq 2u) \leq \frac{C_q t^{q/2}}{u^q} \|\nabla f\|_q^q + \frac{\kappa(s)}{4tu\log u} W_2^2(\nu,\mu).$$

Optimizing in $t > 0$ we obtain the desired inequality.

(b) On the other hand, let $\Phi\colon [0,\infty) \to [0,\infty)$ be increasing such that

$$\lim_{u\to\infty} \Phi(u)\log^{-1} u = \infty. \tag{3.1}$$

For any constant $C > 0$, we intend to disprove the inequality

$$\sup_{u\geq s} \Big[u^{3/2}\Phi(u)\Big] \mu(f \geq 2u)^{3/(2r)} \leq C\kappa^{1/2}(s)\|\nabla f\|_q W_2(\mu,\nu), \tag{3.2}$$

where $\nu = f\cdot\mu$, under the condition $CD(0,\infty)$. To this end, we take $M = \mathbb{R}$ and

$$V(x) = c + \int_0^{|x|} ds \int_0^s h(r)dr, \quad x \in \mathbb{R},$$

where $h \in C^\infty([0,\infty))$ is nonnegative such that $h|_{[0,1/2]} = 1, h|_{[1,\infty)} = 0$, and $c \in \mathbb{R}$ is such that $\mu(dx) := \exp^{-V(x)}\,dx$ is a probability measure. Then $V \in C^\infty(\mathbb{R})$ with $V'' \geq 0$ such that the condition $CD(0,\infty)$ holds. Moreover, there exists a constant c_0 such that

$$V(x) = |x| + c_0 \quad \text{wnenever } |x| \geq 1. \tag{3.3}$$

For all $k \geq 1$, we take

$$f_k(x) = \delta_k\{(x-k)^+ \wedge 1\}, \quad \delta_k = \left(\int_{\mathbb{R}} \{(x-k)^+ \wedge 1\} e^{-V(x)}\,dx\right)^{-1}.$$

Let $\nu_k = f_k \cdot \mu$ and take $u = \frac{1}{2}e^k$. Then there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
&2c_1 u = c_1 e^k \leq \delta_k \leq c_2 e^k = 2c_2 u,\\
&\mu(f_k \geq 2u) \geq c_1 e^{-k} = c_1(2u)^{-1},\\
&\|\nabla f\|_q \leq c_2 e^{(1-1/q)k} = c_2(2u)^{1-1/q},\\
&W_2(\mu,\mu_k) \leq c_2 k = c_2\log(2u), \quad u = \frac{1}{2}e^k, k \geq 1.
\end{aligned}$$

Thus, (3.2) with $f = f_k$ implies the bound

$$\begin{aligned}
2^{-3/2}(2u)^{1-1/q}\Phi(u) &= u^{3/2}\Phi(u)(2u)^{-(q+2)/(2q)}\\
&\leq C(2u)^{1-1/q}\log(2u), \quad u = \frac{1}{2}e^k, k \geq 1.
\end{aligned}$$

Therefore, $\liminf_{u\to\infty} \Phi(u)\log^{-1} u < \infty$ contrary to (3.1). $\square$

Remark 3.2. In the Gaussian case, for any probability measure of the form $\nu = f\cdot\gamma$ with $f \in W^{q,1}(\gamma),\ q \geq 1$ one has

$$\gamma(f \geq 2u) \leq \inf_{t>0}\Big[\frac{K_q^q \arccos^q(e^{-t})}{u^q}\|\nabla f\|_q^q + \frac{\kappa(s)}{2(e^{2t}-1)u\log u}W_2^2(\nu,\gamma)\Big]$$

for all $u \geq s > 1$. Indeed, this follows from Remark 2.4 along the lines of the proof of Theorem 3.1 where one needs to take into account that in the Gaussian case we have the bound

$$\int_{\mathbb{R}^d} T_t f \log T_t f\, d\gamma \leq \frac{W_2^2(\nu,\gamma)}{2(e^{2t}-1)},\ t > 0.$$

The next lemma is a reinforcement of the remark made in [19] regarding Claim B in the proof of Proposition 1.3 from [14]. This simple observation will be used in the proof of Theorem 1.3.

Lemma 3.3. *For all $a > 1$ and $\beta \geq 0$, any collection $\{F_k\}_{k=1}^\infty$ of subsets of M and any non-empty set $I \subset \mathbb{Z}_+$ one has*

$$\sum_{k\in I}(1+k)^\beta a^k \mathbb{1}_{F_k}(x) \leq \frac{a}{a-1}\sup_{k\in I}\big\{(1+k)^\beta a^k \mathbb{1}_{F_k}(x)\big\},\ x \in M.$$

Proof. It suffices to prove this inequality for finite $I \subset \mathbb{Z}_+$ such that $x \in F_k$ for some $k \in I$. In this case let us set $k_x := \sup I_x$, where $I_x := \{k \in I\colon x \in F_k\}$. Then for any $a > 1$ and $\alpha \geq 0$

$$\begin{aligned}\sum_{k\in I}(1+k)^\beta a^k \mathbb{1}_{F_k}(x) &= \sum_{k\in I_x}(1+k)^\beta a^k \leq (1+k_x)^\beta \sum_{i=0}^{k_x} a^k \\ &\leq (1+k_x)^\beta \cdot \frac{a^{k_x+1}-1}{a-1} \leq (1+k_x)^\beta\cdot\frac{a^{k_x+1}}{a-1} \\ &\leq \frac{a}{a-1}\sup_{k\in I}\big\{(1+k)^\beta a^k \mathbb{1}_{F_k}(x)\big\},\end{aligned}$$

as announced. □

Lemma 3.4. *Let $a > 1, \beta \geq 0$ and $p \geq 1$. Then for any collection $\{F\}_{k=1}^\infty$ of Borel subsets of M, every finite non-empty set $I \subset \mathbb{Z}_+$ and $\varepsilon > 0$ one has*

$$\int \varphi f\, d\mu \leq \frac{1}{\varepsilon}W_p^p(\mu, f\mu) + \frac{a}{a-1}\int \psi_\varepsilon\, d\mu,$$

where

$$\varphi(x) = \sum_{k\in I}(1+k)^\beta a^k \mathbb{1}_{F_k}(x),$$

$$\psi_\varepsilon(x) = \sum_{k\in I}\sup_{y\in M_{k,\varepsilon}(x)}(1+k)^\beta a^k \mathbb{1}_{F_k}(y),$$

$$M_{k,\varepsilon}(x) = \Big\{ y \in M\colon\ d(x,y)^p \le \frac{a}{a-1}(1+k)^\beta a^k \varepsilon \Big\}.$$

Proof. By the Kantorovich duality for every $\varepsilon > 0$ we have

$$\int \varphi f \, d\mu \le \frac{1}{\varepsilon} W_p^p(\mu, f\mu) + \int \widehat{Q}_\varepsilon \varphi \, d\mu, \tag{3.4}$$

$$\widehat{Q}_\varepsilon \varphi(x) = \sup_{y\in M} \Big[\varphi(y) - \frac{1}{\varepsilon} d(x,y)^p \Big].$$

Applying Lemma 3.3 and letting $\Lambda := \frac{a}{a-1}$ we obtain

$$\begin{aligned}
\widehat{Q}_\varepsilon \varphi(x) &= \sup_{y\in M} \Big[\sum_{k\in I} (1+k)^\alpha a^k \mathbb{1}_{F_k}(y) - \frac{1}{\varepsilon} d(x,y)^p \Big] \\
&\le \sup_{y\in M} \Big[\Lambda \sup_{k\in I} (1+k)^\alpha a^k \mathbb{1}_{F_k}(y) - \frac{1}{\varepsilon} d(x,y)^p \Big] \\
&= \Lambda \sup_{y\in M} \sup_{k\in I} \Big[(1+k)^\alpha a^k \mathbb{1}_{F_k}(y) - \frac{1}{\Lambda\varepsilon} d(x,y)^p \Big] \\
&= \Lambda \sup_{k\in I} \sup_{y\in M} \Big[(1+k)^\alpha a^k \mathbb{1}_{F_k}(y) - \frac{1}{\Lambda\varepsilon} d(x,y)^p \Big] \\
&\le \Lambda \sup_{k\in I} \sup_{y\in M_{k,\varepsilon}(x)} (1+k)^\alpha a^k \mathbb{1}_{F_k}(y) \le \Lambda \sum_{k\in I} \sup_{y\in M_{k,\varepsilon}(x)} (1+k)^\alpha a^k \mathbb{1}_{F_k}(y),
\end{aligned}$$

which together with (3.4) completes the proof. □

Now we are ready to prove Theorem 1.3. We use Ledoux's strategy from [19] with some modifications. For the reader's convenience all the details are presented. The key difference with the proof of Ledoux concerns the application of Harnack's inequality (2.2), where the parameters are "balanced" in such a way that the exponential term is no longer bounded by a constant.

Proof of Theorem 1.3. (a) We first prove the claimed inequality. For $k \in \mathbb{Z}_+$ and $t_k > 0$ let us set

$$A_k = \big\{ 2^k \le f < 2^{k+1} \big\},\ f_k = \min((f - 2^k)_+, 2^k),$$

$$F_k = \big\{ P_{t_k} f_k \ge 2^{k-1} \big\},\ r = \frac{3q}{q+2},\ \alpha = \frac{1}{3}.$$

For $k_0 \geq 1$ one has

$$\int_M (f-2^{k_0+1})_+^r \big(1+\log^{r\alpha}(1+(f-2^{k_0+1})_+)\big)\,d\mu$$
$$= \sum_{k=k_0+1}^{\infty} \int_{A_k} (f-2^{k_0+1})_+^r \big(1+\log^{r\alpha}(1+(f-2^{k_0+1})_+)\big)\,d\mu$$
$$\leq \sum_{k=k_0+1}^{\infty} 2^{r(k+1)}(1+\log^{r\alpha}(1+2^{k+1}))\mu(A_k)$$
$$\leq C\sum_{k=k_0}^{\infty}(1+k)^{r\alpha}2^{rk}\mu(f_k \geq 2^k), \quad (3.5)$$

where we have used the inclusion $A_k \subseteq \{f_{k-1} \geq 2^{k-1}\}$. The constant C depends only on q and k_0. Now let us observe that for any sufficiently smooth non-negative function g (which need not be a probability density) and all $u > 0$ we have

$$\mu(g \geq 2u) \leq \mu(g \geq 2u, P_t g \leq u) + \mu(g \geq 2u, P_t g \geq u)$$
$$\leq \mu\big(|g - P_t g| \geq u\big) + \frac{1}{2u}\int_M \mathbb{1}_F g\,d\mu$$
$$\leq \frac{Ct^{q/2}}{u^q}\|\nabla g\|_q^q + \frac{1}{2u}\int_M \mathbb{1}_F g\,d\mu, \quad (3.6)$$

where $F := \{P_t g \geq u\}$ and the constant C depends only on q. Applying (3.6) to $g = f_k$ and $u = 2^{k-1}$, $t = t_k$ we obtain the bound

$$\mu(f_k \geq 2^k) \leq \frac{Ct_k^{q/2}}{2^{qk}}\int_{A_k}|\nabla f|^q\,d\mu + \frac{1}{2^k}\int_M \mathbb{1}_{F_k} f_k\,d\mu$$
$$\leq \frac{Ct_k^{q/2}}{2^{qk}}\int_{A_k}|\nabla f|^q\,d\mu + \frac{1}{2^k}\int_M \mathbb{1}_{F_k} f\,d\mu,$$

where the inequality $f_k \leq f$ has been used. For any $k_1 \geq k_0$ one has

$$S_I := \sum_{k\in I}(1+k)^{r\alpha}2^{rk}\mu(f_k \geq 2^k)$$
$$\leq C\sum_{k\in I}(1+k)^{r\alpha}2^{(r-q)k}t_k^{q/2}\int_{A_k}|\nabla f|^q\,d\mu + \int_M \varphi f\,d\mu, \quad (3.7)$$

where

$$I := \{k_0,\dots,k_1\},\ \ \varphi := \sum_{k\in I}(1+k)^{r\alpha}2^{(r-1)k}\mathbb{1}_{F_k}.$$

Let ε be a fixed positive number. Now we can apply Lemma 3.4 to $a = 2^{r-1}$, $\beta = \alpha r$, $p = 2$ and obtain the inequality

$$\int \varphi f \, d\mu \leq \frac{1}{\varepsilon} W_2^2(f\mu, \mu) + \frac{a}{a-1} \int_M \psi_\varepsilon \, d\mu, \tag{3.8}$$

where

$$\psi_\varepsilon(x) = \sum_{k \in I} \sup_{y \in M_{k,\varepsilon}(x)} (1+k)^{r\alpha} 2^{(r-1)k} \mathbb{1}_{F_k}(y),$$

$$M_{k,\varepsilon}(x) = \Big\{ y \in M \colon d(x,y)^2 \leq \frac{a}{a-1}(1+k)^{r\alpha} 2^{(r-1)k} \varepsilon \Big\},$$

Set

$$\eta := \frac{1}{\log 2} \cdot \frac{a}{a-1}.$$

Let us take t_k such that

$$(1+k)^{r\alpha} 2^{(r-q)k} t_k^{q/2} = \eta^{q/2} \varepsilon^{q/2},$$

that is,

$$t_k = \frac{1}{\log 2} \frac{a}{a-1} (1+k)^{-2r\alpha/q} 2^{2(1-r/q)k} \varepsilon.$$

By the definition of the set F_k we have the pointwise estimate

$$\mathbb{1}_{F_k}(y) \leq 2^{-2k+2} \big(P_{t_k} f_k(y) \big)^2, \ y \in M.$$

Harnack's inequality (2.2) for the function f_k and $y \in M_{k,\varepsilon}(x)$ yields

$$\begin{aligned} \mathbb{1}_{F_k}(y) &\leq 2^{-2k+2} \big(P_{t_k} f_k(y) \big)^2 \\ &\leq 2^{-2k+2} P_{t_k} f_k^2(x) \exp\Big(\frac{1}{2t_k} \frac{a}{a-1} (1+k)^{r\alpha} 2^{(r-1)k} \varepsilon \Big). \end{aligned}$$

Due to our choice of t_k we have

$$r\alpha + 2r\alpha/q = \alpha(r + 2r/q) = \frac{1}{3}\Big(\frac{3q}{q+2} + \frac{6}{q+2} \Big) = 1,$$

$$(r-1)k - 2(1-r/q)k = (r + 2r/q - 3)k = \Big(\frac{3q}{q+2} + \frac{6}{q+2} - 3 \Big) k = 0.$$

This leads to the equality

$$\exp\Big(\frac{1}{2t_k} \frac{a}{a-1} (1+k)^{r\alpha} 2^{(r-1)k} \varepsilon \Big) = 2^{(1+k)/2}.$$

Then

$$\sup_{y \in M_{k,\varepsilon}(x)} \mathbb{1}_{F_k}(y) \leq 2^{-2k+2} P_{t_k} f_k^2(x) 2^{(1+k)/2} = 2^{-3k/2+5/2} P_{t_k} f_k^2(x).$$

Since

$$\int_M P_{t_k} f_k^2 \, d\mu = \int_M f_k^2 \, d\mu \leq 2^{2k} \mu(f \geq 2^k) = 2^{2k} \mu(f_{k-1} \geq 2^{k-1}),$$

we have

$$\int \sup_{M_{k,\varepsilon}(x)} (1+k)^{r\alpha} 2^{(r-1)k} \mathbb{1}_{F_k}\, d\mu$$
$$\le (1+k)^{r\alpha} 2^{(r-1)k} 2^{k/2+5/2} \mu(f_{k-1} \ge 2^{k-1})$$
$$= (1+k)^{r\alpha} 2^{rk} 2^{-k/2+5/2} \mu(f_{k-1} \ge 2^{k-1}).$$

Consequently,

$$\int \varphi f\, d\mu \le \frac{1}{\varepsilon} W_2^2(f\mu,\mu) + \frac{a}{a-1} \int_M \psi_\varepsilon\, d\mu$$
$$= \frac{1}{\varepsilon} W_2^2(f\mu,\mu) + \frac{a}{a-1} \sum_{k\in I} \int_M \sup_{M_{k,\varepsilon}(x)} (1+k)^{r\alpha} 2^{(r-1)k} \mathbb{1}_{F_k}\, d\mu$$
$$\le \frac{1}{\varepsilon} W_2^2(f\mu,\mu) + \frac{a}{a-1} \sum_{k\in I} (1+k)^{r\alpha} 2^{rk} 2^{-k/2+5/2} \mu(f_{k-1} \ge 2^{k-1}).$$

Combining this with (3.7) and taking into account that

$$(1+k)^{r\alpha} 2^{(r-q)k} t_k^{q/2} = \eta^{q/2} \varepsilon^{q/2},$$

we finally have the following bound for S_I:

$$S_I = \sum_{k\in I} (1+k)^{r\alpha} 2^{rk} \mu(f_k \ge 2^k)$$
$$\le C\eta^{q/2}\varepsilon^{q/2} \sum_{k\in I} \int_{A_k} |\nabla f|^q\, d\mu + \frac{1}{\varepsilon} W_2^2(f\mu,\mu)$$
$$+ \frac{a}{a-1} \sum_{k\in I} (1+k)^{r\alpha} 2^{rk} 2^{-k/2+5/2} \mu(f_{k-1} \ge 2^{k-1})$$
$$\le C\eta^{q/2}\varepsilon^{q/2} \int |\nabla f|^q\, d\mu + \frac{1}{\varepsilon} W_2^2(f\mu,\mu)$$
$$+ k_0^{r\alpha} 2^{r(k_0-1)-k_0+3} \mu(f \ge 2^{k_0})$$
$$+ C\frac{a}{a-1} \sum_{k\in I} (1+k)^{r\alpha} 2^{rk} 2^{-k/2+5/2} \mu(f_k \ge 2^k). \quad (3.9)$$

where the constant C depends only on q, since

$$(1+1/k)^{r\alpha} \le 2^{r\alpha},\ k \ge 1.$$

We can assume that k_0 is sufficiently large and for all $k \ge k_0$

$$C\frac{a}{a-1} 2^{-k/2+5/2} \le \frac{1}{2}.$$

Then the last term on the right-hand side of (3.9) can be replaced with $S_I/2$:

$$S_I \le C'\varepsilon^{q/2} \int |\nabla f|^q \, d\mu + \frac{1}{\varepsilon} W_2^2(f\mu, \mu) + C'\mu(f \ge 2^{k_0}) + \frac{1}{2} S_I,$$

or, equivalently,

$$\frac{1}{2} S_I \le C'\varepsilon^{q/2} \int |\nabla f|^q \, d\mu + \frac{1}{\varepsilon} W_2^2(f\mu, \mu) + C'\mu(f \ge 2^{k_0}),$$

the constant C' depends only on q and k_0. Optimizing in ε and using the weak-type bound (3.1) we obtain the inequality

$$S_I^{3/2r} \le C(q, k_0) \|\nabla f\|_q W_2(\mu, \nu).$$

Passing to the limit $k_1 \to \infty$ and using (3.5) we derive the claimed inequality.

(b) For the second assertion, we let Φ be an increasing function such that

$$\lim_{u \to \infty} \Phi(u) \log^{-2/3}(1+u) = \infty. \tag{3.10}$$

For any constant $C > 0$, we intend to disprove the inequality

$$\big\|(f-C)_+ \big(1+\Phi((f-C)_+)\big)\big\|_r^{3/2} \le C \|\nabla f\|_q W_2(\mu, \nu), \quad \nu = f \cdot \mu. \tag{3.11}$$

To this end, let $M = \mathbb{R}$ and let V, μ, $\nu_k := f_k \cdot \mu$ be taken as in Step (b) of the proof of Theorem 3.1. By (3.3), we have $\mu = e^V \, dx$ and $\nu_k = f_k \cdot \mu$, hence we can find constants $c_2 > c_1 > 0$ such that for all $k \ge 1$ one has

$$c_1 e^k \le \delta_k \le c_2 e^k, \quad W_2(\mu, \nu_k) \le c_2 k, \quad \|\nabla f_k\|_q \le c_2 e^{(1-1/q)k},$$

$$\big\|(f-C)_+ \big(1 + F((f-C)_+)\big)\big\|_r^{3/2} \ge c_1 e^{(1-1/q)k} \{F(e^k)\}^{3/2}.$$

Therefore, inequality (3.11) implies that

$$\limsup_{k \to \infty} k^{-1} \{\Phi(e^k)\}^{3/2} < \infty,$$

which contradicts (3.10). □

4. The Kantorovich distance W_1

In this section we show that Theorem 1.1 from [19] admits a generalization to the case $N = \infty$, although, unlike Theorem 1.3 above, where the case $p = 2$ was considered, the inequality does not include any extra logarithmic factors. It might be possible that this result can be further improved, we leave this question for future research. It would be also interesting to find a unified proof of these inequalities covering the full scale of the Kantorovich metrics W_p with $p \ge 1$. Recall that the dual representation of the Kantorovich distance W_1 is given by the formula

$$W_1(\mu,\nu) = \sup_{\varphi} \int_M \varphi \, d(\mu - \nu), \tag{4.1}$$

where the supremum is taken over all bounded 1-Lipschitz functions φ, see, e.g., [9] or [24]. The next theorem is a generalization of Proposition 4.3 from [19].

Theorem 4.1. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the condition $CD(0, \infty)$. For each $q \geq 1$, there exists a positive constant C depending only on q such that for any probability measure $\nu = f \cdot \mu$ with a smooth density f (or $f \in W^{q,1}(\mu)$) one has*

$$\sup_{u>0} \Big[u^2 \cdot \mu\big(|f-1| \geq 2u\big)^{2/r} \Big] \leq C \|\nabla f\|_q W_1(\mu,\nu), \ r = \frac{2q}{q+1}.$$

Proof. For each $t > 0$ let us represent $f - 1$ as

$$f - 1 = f - P_t f + P_t(f-1).$$

Then for $u > 0$ we have

$$\begin{aligned} \mu\big(|f-1| \geq 2u\big) &\leq \mu\big(|f - P_t f| \geq u\big) + \mu\big(|P_t(f-1)| \geq u\big) \\ &\leq \mu\big(|f - P_t f| \geq u\big) + \frac{1}{u} \int_M (\mathbb{1}_{F_+} - \mathbb{1}_{F_-}) P_t(f-1) \, d\mu \\ &\leq \frac{C t^{q/2}}{u^q} \|\nabla f\|_q^q + \frac{1}{u} \int_M (\mathbb{1}_{F_+} - \mathbb{1}_{F_-}) P_t(f-1) \, d\mu \\ &= \frac{C t^{q/2}}{u^q} \|\nabla f\|_q^q + \frac{1}{u} \int_M P_t(\mathbb{1}_{F_+} - \mathbb{1}_{F_-}) \, d(\nu - \mu), \end{aligned}$$

where

$$F_+ := \big\{ P_t(f-1) \geq u \big\}, \ F_- := \big\{ P_t(f-1) \leq -u \big\}$$

and C depends only on q. By the gradient estimate (2.1) we have

$$|\nabla P_t(\mathbb{1}_{F_+} - \mathbb{1}_{F_-})| \leq \frac{1}{\sqrt{2t}},$$

hence the Kantorovich duality (4.1) yields the bound

$$\int_M P_t(\mathbb{1}_{F_+} - \mathbb{1}_{F_-}) \, d(\nu - \mu) \leq \frac{1}{\sqrt{2t}} \frac{1}{u} W_1(\mu,\nu).$$

Finally, we have

$$\mu\big(|f-1| \geq 2u\big) \leq \frac{C t^{q/2}}{u^q} \|\nabla f\|_q^q + \frac{1}{\sqrt{2t}} \frac{1}{u} W_1(\mu,\nu),$$

so optimizing in $t > 0$ we arrive at the desired inequality. □

From the weak-type bounds provided by Theorem 4.1 we deduce the corresponding strong ones.

Theorem 4.2. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the condition $CD(0,\infty)$. For each $q > 1$, there exists a positive constant C depending only on q such that for any probability measure $\nu = f \cdot \mu$ with a smooth density f (or $f \in W^{q,1}(\mu)$) one has*

$$\big\|(f-C)_+\big\|_r^2 \le C\|\nabla f\|_q W_1(\mu,\nu),\ r = \frac{2q}{q+1}.$$

Proof. Following the lines of the proof of Theorem 1.3, we introduce

$$A_k = \big\{2^k \le f < 2^{k+1}\big\},\ f_k = \min((f-2^k)_+, 2^k),$$

$$F_k = \big\{P_{t_k} f_k \ge 2^{k-1}\big\},\ r = \frac{2q}{q+1},\ a := 2^{r-1}$$

and establish the inequality

$$\int_M \big(f - 2^{k_0+1}\big)_+^r \, d\mu \le C \sum_{k=k_0}^{\infty} 2^{rk} \mu(f_k \ge 2^k),$$

where C depends only on q and k_0. Next we bound $\mu(f_k \ge 2^k)$ using the estimate

$$\mu(f_k \ge 2^k) \le \frac{C t_k^{q/2}}{2^{qk}} \int_{A_k} |\nabla f|^q \, d\mu + \frac{1}{2^k} \int_M \mathbb{1}_{F_k} f \, d\mu.$$

Then for

$$I := \{k_0, \dots, k_1\},\ \varphi := \sum_{k\in I} 2^{(r-1)k} \mathbb{1}_{F_k}.$$

we obtain the chain of inequalities

$$\begin{aligned} S_I &:= \sum_{k\in I} 2^{rk} \mu(f_k \ge 2^k) \\ &\le C \sum_{k\in I} 2^{(r-q)k} t_k^{q/2} \int_{A_k} |\nabla f|^q \, d\mu + \int_M \varphi f \, d\mu \\ &\le C \sum_{k\in I} 2^{(r-q)k} t_k^{q/2} \int_{A_k} |\nabla f|^q \, d\mu + \frac{1}{\varepsilon} W_1(\mu,\nu) \\ &\qquad + \frac{a}{a-1} \sum_{k\in I} \int \sup_{M_{k,\varepsilon}(x)} 2^{(r-1)k} \mathbb{1}_{F_k} \, d\mu, \end{aligned}$$

where

$$M_{k,\varepsilon}(x) := \Big\{ y \in M \colon d(x,y) \le \frac{a}{a-1} 2^{(r-1)k} \varepsilon \Big\}.$$

Let us set

$$\eta := \frac{1}{\log 2} \cdot \frac{a}{a-1}$$

and take t_k such that

$$2^{(r-q)k} t_k^{q/2} = \eta^{q/2} \varepsilon^{q/2},$$

that is,

$$t_k = \frac{1}{\log 2} \frac{a}{a-1} 2^{2(1-r/q)k} \varepsilon.$$

Then

$$\mathbb{1}_{F_k}(y) \le 2^{-2k+2} \big(P_{t_k} f_k(y)\big)^2, \; y \in M,$$

and by Harnack's inequality applied to the function f_k and $y \in M_{k,\varepsilon}(x)$ we have

$$\mathbb{1}_{F_k}(y) \le 2^{-2k+2} \big(P_{t_k} f_k(y)\big)^2 \le 2^{-2k+2} P_{t_k} f_k^2(x) \exp\Big(\frac{1}{2t_k} \frac{a}{a-1} 2^{(r-1)k} \varepsilon\Big).$$

The definition of t_k yields the equality

$$(r-1)k - 2(1-r/q)k = (r + 2r/q - 3)k = \Big(\frac{3q}{q+2} + \frac{6}{q+2} - 3\Big)k = 0.$$

This leads to the equality

$$\exp\Big(\frac{1}{2t_k} \frac{a}{a-1} 2^{(r-1)k} \varepsilon\Big) = 2^{1/2}.$$

Then

$$\sup_{y \in M_{k,\varepsilon}(x)} \mathbb{1}_{F_k}(y) \le 2^{-2k+2} P_{t_k} f_k^2(x) 2^{1/2} = 2^{-2k+5/2} P_{t_k} f_k^2(x).$$

Recalling that

$$\int_M P_{t_k} f_k^2 \, d\mu = \int_M f_k^2 \, d\mu \le 2^{2k} \mu(f \ge 2^k) = 2^{2k} \mu(f_{k-1} \ge 2^{k-1}),$$

similarly to the proof of Theorem 1.3 we obtain the "recursive" inequality for all sufficiently large k_0:

$$S_I \le C \eta^{q/2} \varepsilon^{q/2} \int |\nabla f|^q \, d\mu + \frac{1}{\varepsilon} W_1(\mu, \nu) + 2^{r(k_0-1)-k_0+3} \mu\big(f \ge 2^{k_0}\big) + \frac{1}{2} S_I.$$

Taking into account Theorem 4.1 it is easy to complete the proof. $\square$

Now let us consider the case $q = 1$.

Theorem 4.3. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the condition $CD(0, \infty)$. Then for each probability measure $\nu = f \cdot \mu$ with a smooth density f (or $f \in W^{1,1}(\mu)$) one has*

$$\big\|f - 1\big\|_1^2 \le 2\|\nabla f\|_1 W_1(\mu, \nu).$$

Proof. This inequality is a particular case of the results from [12]. We present a proof here for the reader's convenience. For a smooth function $g \in C_0^\infty$ with $\|g\|_\infty \le 1$ we have

$$\begin{aligned}\left|\int_M (f-1)g\,d\mu\right| &= \left|\int_M (f-1)P_t g\,d\mu - \int_M (f-1)\int_{[0,t]} \frac{d}{ds}P_s g\,d\mu\right| \\ &\le W_1(\nu,\mu)\|\nabla P_t g\|_\infty + \int_0^t \int |\nabla f|\,|\nabla P_s g|\,d\mu\,ds \\ &\le W_1(\nu,\mu)\|\nabla P_t g\|_\infty + \|\nabla f\|_1 \int_{[0,t]} \|\nabla P_s g\|_\infty\,ds \\ \le W_1(\nu,\mu)\frac{1}{\sqrt{2t}} + \|\nabla f\|_1 &\int_{[0,t]} \frac{1}{\sqrt{2s}}\,ds = W_1(\nu,\mu)\frac{1}{\sqrt{2t}} + \sqrt{2t}\|\nabla f\|_1.\end{aligned}$$

Optimizing in $t > 0$ we obtain the desired inequality. □

Of course, this theorem covers the case of the standard Gaussian measure γ_d on $\mathbb{R}^d$ with the usual metric. As already mentioned in our note [12], in this case the following two-sided inequality holds for all functions f from the Gaussian Sobolev class $W^{1,1}(\gamma_d)$ having zero integral against γ_d:

$$\frac{\|f\|_{L^1(\gamma_d)}^2}{2\|\nabla f\|_{L^1(\gamma_d)}} \le \|f \cdot \gamma_d\|_K \le \|\nabla f\|_{L^1(\gamma_d)}.$$

In [12, Proposition 1 and Proposition 2] we constructed two examples showing that the bound from this theorem can fail with any constant if μ does not satisfy the indicated condition (in one of these examples μ is a measure on the real line with the usual metric and in the other example M is a two-dimensional complete connected Riemannian submanifold in $\mathbb{R}^d$ and μ is its Riemannian volume).

The obtained inequalities involving the Kantorovich distance W_1 of order 1 can be combined with the estimate provided by Theorem 1.1 from [19].

Proposition 4.4. *Let (M, g, μ) be a smooth Riemannian manifold satisfying the curvature-dimension condition $CD(0, N)$. Then, given $p, q \ge 1$, there exists a constant $C > 0$ depending only on p, q, N such* *that for each probability measure $\nu = f \cdot \mu$ with a smooth density f (or $f \in W^{1,1}(\mu)$) one has*

$$\|f-1\|_r^\theta \le C\Big(\|\nabla f\|_q W_p(\mu,\nu) + \big[\|\nabla f\|_1 W_1(\mu,\nu)\big]^{\theta/(2r)}\Big),$$

where

$$r = \frac{1+\frac{1}{p}+\frac{1}{N}}{\frac{1}{p}+\frac{1}{q}},\ \theta = 1+\frac{1}{p}+\frac{1}{N}.$$

In particular, for $p = q = 1$ this becomes

$$\|f-1\|_{1+\frac{1}{2N}}^{2+\frac{1}{N}} \le C\|\nabla f\|_1 W_1(\mu,\nu). \tag{4.2}$$

Proof. By [19, Theorem 1.1] there exists a constant $C > 1$ such that

$$\|(f-C)_+\|_r^\theta \le C\|\nabla f\|_q W_p(\mu,\nu) \tag{4.3}$$

and by Theorem 4.3 we have

$$\|f-1\|_1 \le \sqrt{2}\|\nabla f\|_1 W_1(\mu,\nu). \tag{4.4}$$

Since f is non-negative, we have the following trivial bound:

$$|f-1|^r \le 2^{r-1}\big[(f-C)_+\big]^r + (2C)^{r-1}|f-1|.$$

Integrating this inequality over M and applying (4.3) and (4.4) we get the desired bound. □

Remark 4.5. The inequality (4.2) is sharp in the sense that, whenever $C > 0$ and $\Phi\colon [0,\infty) \to [0,\infty)$ is an increasing function with

$$\lim_{u\to\infty} \Phi(u)u^{-2-\frac{1}{N}} = \infty, \tag{4.5}$$

the following inequality does not hold:

$$\Phi(\|f-1\|_{1+\frac{1}{2N}}) \le C\|\nabla f\|_1 W_1(\mu,\nu). \tag{4.6}$$

Indeed, let $M = (\mathbb{S}^1)^N$, where $\mathbb{S}^1$ is the unit circle, which is equivalent to $[0,2\pi)$ with the periodic boundary. For every $n \ge 1$ we set $h_n(s) = \min\{ns, (2-ns)^+\}, s \in [0,2\pi)$ and

$$f_n(x) = \prod_{i=1}^N h_n(x_i), \quad x = (x_1,\cdots,x_N) \in [0,2\pi)^N.$$

Let $\nu_n = f_n \cdot dx$, $\mu = \frac{1}{2\pi}\,dx$, We have $W_1(\nu_n,\mu) \le 2\pi$, and there exist constants $c_2 \ge c_1 > 0$ such that

$$\|f_n-1\|_{1+\frac{1}{2N}} \ge c_1 n^{\frac{N}{2N+1}}, \quad \|\nabla f_n\|_1 \le c_2 n, \quad n \ge 1.$$

Thus, (4.6) implies that $\liminf_{u\to\infty} \Phi(u)u^{-2+\frac{1}{N}} < \infty$, which contradicts (4.5).

5. Extensions to the negative curvature case

In this section we briefly discuss some extensions for negatively curved weighted Riemannian manifolds. We assume that (M, g, μ) satisfies the curvature-dimension condition $CD(-\varrho, \infty)$, $\varrho > 0$ and that additionally the logarithmic Sobolev inequality holds:

$$\int_M f^2 \log f^2 \, d\mu \le \frac{2}{\lambda} \int_M |\nabla f|^2 \, d\mu \tag{5.1}$$

for all $f \in C^1(M)$ with

$$\int_M f^2 \, d\mu = 1.$$

For example, according to [26], under the curvature-dimension condition $CD(-\varrho, \infty)$ the finiteness of the integral

$$\int_M \exp\bigl(\varepsilon d^2(x_0, x)\bigr) \, d\mu$$

for some $x_0 \in M$ and $\varepsilon > \varrho/2$ ensures the validity of the log-Sobolev inequality 5.1. The main idea of the considerations below is that even though in this case the curvature bound alone does not guarantee the required semigroup estimates, nevertheless they can be established under some additional assumptions about (M, g, μ).

Proposition 5.1. *Let (M, g, μ) be a smooth weighted Riemannian satisfying the curvature-dimension condition $CD(-\varrho, \infty)$ with some $\varrho \ge 0$. Assume also that the log-Sobolev inequality (5.1) holds for some $\lambda > 0$. Then for each $p \in [2, \infty)$ there exists $C > 0$ depending only on ϱ, λ, p such that*

$$\|\nabla P_t h\|_p \le \frac{C}{\sqrt{t}} \|h\|_p, \ t > 0, \ h \in L^p(M, \mu).$$

Proof. Using the standard approximation arguments one can see that it is sufficient to establish this inequality just for $h \in C_b(M)$. Applying [25, Corollary 4.2] we obtain the inequality

$$|\nabla P_t h| \le \frac{C}{\sqrt{t \wedge 1}} \bigl(P_t |h|^p\bigr)^{\frac{1}{p}}.$$

This implies the bound

$$\|\nabla P_t h\|_p \le \frac{C}{\sqrt{t \wedge 1}} \|h\|_p. \tag{5.2}$$

Now one can observe that the log-Sobolev inequality (5.1) ensures that the generator of the semigroup $\{P_t\}_{t \ge 0}$ has a spectral gap larger or equal to λ, in particular, for any $\varphi \in L^2(\mu)$ with zero integral against μ one has

$$\|P_t\varphi\|_2 \le e^{-\lambda t}\|\varphi\|_2.$$

Consequently, for $h \in C_b(M)$ we have

$$\|\nabla P_t h\|_2 \le \frac{Ce^{-\lambda t}}{\sqrt{t\wedge 1}}\|h\|_2. \tag{5.3}$$

Combining estimates (5.2), (5.3) and applying the standard interpolation theorem we obtain the desired inequality. □

Theorem 5.2. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(-\varrho, \infty)$ with some $\varrho \ge 0$. Assume also that the log-Sobolev inequality* (5.1) *holds for some $\lambda > 0$. Then, for each $q \in (1, 2]$, there exists $C > 0$ depending only on ϱ, λ, p such that for all $f \in W^{q,1}(\mu)$ one has*

$$\|f - P_t f\|_q \le C\sqrt{t}\|\nabla f\|_q,\ t > 0.$$

Proof. This follows from Proposition 5.1 along the lines of the proof of Theorem 2.3. □

Theorem 5.3. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(-\varrho, \infty)$ with some $\varrho \ge 0$. Assume also that the log-Sobolev inequality* (5.1) *holds for some $\lambda > 0$. Then there exists $C > 0$ depending only on ϱ, λ, p such that for every probability density f one has*

$$\int_M P_t f \log P_t f\, d\mu \le \frac{C}{t} W_2^2(\mu, f\cdot\mu).$$

Proof. According to [28] the curvature condition $CD(-\varrho, \infty)$ implies the log-Harnack inequality

$$P_t \log g(x) \le \log P_t g(y) + \frac{\varrho\cdot d^2(x,y)}{2(1-e^{-2\varrho t})}.$$

Applying this inequality to $g := P_t f$ and integrating with respect to the optimal coupling of the measures $f\cdot\mu$ and μ (see, e.g., [22, Corollary 1.2]) we get the bound

$$\int_M P_t f \log P_t f\, d\mu \le \frac{\varrho}{2(1-e^{-2\varrho t})} W_2^2(\mu, f\cdot\mu).$$

Consequently, for all $t \in (0, 1)$ we have

$$\int_M P_t f \log P_t f\, d\mu \le \frac{C}{t} W_2^2(\mu, f\cdot\mu).$$

Next, it is known that the log-Sobolev inequality (5.1) implies the following bound for any probability density g with respect to μ:

$$\int_M P_t g \log P_t g \, d\mu \le e^{-\lambda t} \int_M g \log g \, d\mu.$$

Applying this inequality to $g = P_1 f$, we obtain the estimate

$$\int_M P_t f \log P_t f \, d\mu \le e^{-\lambda(t-1)} \int_M P_1 f \log P_1 f \, d\mu$$
$$\le C e^{-\lambda t} W_2^2(\mu, f\cdot\mu), \ t \ge 1.$$

Now it is easy to complete the proof. □

Theorem 5.4. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(-\varrho, \infty)$ with some $\varrho \ge 0$. Assume also that the log-Sobolev inequality (5.1) holds for some $\lambda > 0$. Then for each $q \in (1, 2]$ there exists $C > 0$ depending only on ϱ, λ, q such that for every smooth probability density f (or $f \in W^{q,1}(\mu)$) and every $s > 1$ one has*

$$\sup_{u\ge s} \Big[u^{3/2} \log^{1/2} u\Big] \mu(f \ge 2u)^{3/(2r)} \le C\kappa^{1/2}(s) \|\nabla f\|_q W_2(\mu, \nu),$$

where

$$r = \frac{3q}{q+2}, \ \kappa(s) := \frac{s\log s}{s \log s + 1 - s}, \ s > 1.$$

Proof. This follows from Theorem 5.2 and Theorem 5.3 along the lines of the proof of Theorem 3.1. □

Let us conclude this section with a generalization of Theorem 4.3. For a function $f \in L^1(\mu)$ let

$$\|f - \mu(f)\|_K = \sup_{g\in C^\infty(M), \ \|\nabla g\|_\infty \le 1} \int_M f g \, d\mu.$$

When $\nu = f \cdot \mu$ is a probability measure, we have $W_1(\mu, \nu) = \|f - 1\|_K$.

Theorem 5.5. *Let (M, g, μ) be a smooth weighted Riemannian manifold satisfying the curvature-dimension condition $CD(-\varrho, \infty)$, $\varrho \ge 0$. Assume also that the semigroup $\{P_t\}_{t\ge 0}$ satisfies the inequality*

$$\|P_t g\|_\infty \le c e^{-\lambda t} \|g\|_\infty, \ \int_M g \, d\mu = 0, \ t \ge 0. \tag{5.4}$$

with some $c, \lambda > 0$. Then there exists $C > 0$ depending only on ϱ, c, λ such that for every integrable smooth function f with zero integral against μ one has

$$\|f\|_1^2 \le C \|\nabla f\|_1 \|f\|_K.$$

Proof. First, let us remind that for any $h \in C_b(M)$ we have the pointwise inequality (see [25, Corollary 4.2])

$$|\nabla P_t h| \le \frac{C}{\sqrt{t \wedge 1}} \big(P_t |h|^2\big)^{1/p},$$

consequently,

$$\|\nabla P_t h\|_\infty \le \frac{C}{\sqrt{t \wedge 1}} \|h\|_\infty.$$

Next, using our additional assumption about the semigroup $\{P_t\}_{t \ge 0}$ it readily seen that for $t \ge 1$ one has

$$\|\nabla P_t h\|_\infty = \|\nabla P_1 P_{t-1} h\|_\infty \le C' e^{-\lambda t} \|h\|_\infty.$$

Combining these two bounds we obtain

$$\|\nabla P_t h\|_\infty \le \frac{C}{\sqrt{t}} \|h\|_\infty$$

and, consequently,

$$\|f - P_t f\|_1 \le C\sqrt{t} \|\nabla f\|_1.$$

Now it is easy to complete the proof similarly to Theorem 4.3, see also our short note [12]. □

Remark 5.6. According to [26], the log-Sobolev inequality and the strong ergodicity (inequality (5.4)) are incomparable, but both follow from the ultraboundedness: $\|P_t\|_{1\to\infty} < \infty$ for $t > 0$. See also [21] for more details.

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