

The Bailey transform and Hecke-Rogers identities for the universal mock theta functions

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Abstract. Recently, Garvan obtained two-variable Hecke-Rogers identities for three universal mock theta functions $g_2(z; q)$, $g_3(z; q)$, $K(z; q)$ by using basic hypergeometric functions, and he proposed a problem of finding direct proofs of these identities by using Bailey pair technology. In this paper, we give proofs of Garvan's identities by applying Bailey's transform with the conjugate Bailey pair of Warnaar and three Bailey pairs deduced from two special cases of ${}_6\psi_6$ given by Slater. In particular, we obtain a compact form of two-variable Hecke-Rogers identity related to $g_3(z; q)$, which implies the corresponding identity given by Garvan. We also extend these two-variable Hecke-Rogers identities into infinite families.

Keywords: universal mock theta function, Hecke-Rogers identity, conjugate Bailey pair, Bailey pair, Bailey transform

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1 Introduction

This paper is concerned with two-variable Hecke-Rogers identities for the universal mock theta functions due to Garvan [10]. Recall that universal mock theta functions are defined

by Gordon and McIntosh [11] and Hickerson [13] as

$$g_2(z; q) = \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{\frac{1}{2}n(n+1)}}{(z; q)_{n+1} (z^{-1}q; q)_{n+1}},$$

$$g_3(z; q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(z; q)_{n+1} (z^{-1}q; q)_{n+1}},$$

$$K(z; q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q; q^2)_n q^{n^2}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n}.$$

Here and throughout the paper, we adopt the standard notation on q -series [1, 9].

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n),$$

$$(a; q)_n = \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}},$$

$$(a_1, a_2, \dots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n,$$

and

$$(a_1, a_2, \dots, a_k; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} \cdots (a_k; q)_{\infty}.$$

Gordon and McIntosh [11] and Hickerson [13] noticed that all of Ramanujan's mock theta functions can be written in terms of $g_2(z; q)$ and $g_3(z; q)$, that is why these two functions are called universal mock theta functions. The function $K(z; q)$ first appears in Ramanujan's lost notebook [2] and is related by modular transformation to the universal mock theta function $g_2(z; q)$ in [16]. For a summary of mock theta functions, see Gordon and McIntosh [11] or Hickerson and Mortenson [14].

In [10], Garvan first obtained four Hecke-Rogers identities for these three universal mock theta functions, which can be viewed as two-variable generalizations of Hecke-Rogers identities. These identities can also be used to establish Hecke-Rogers-type double sum representation for each of Ramanujan's mock theta functions.

More specifically, Garvan [10] considered the Dyson rank function $R(z; q)$ defined by

$$R(z; q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq; q)_n (z^{-1}q; q)_n},$$

which has the following relation with $g_3(z; q)$.

$$g_3(z; q) = -\frac{1}{z} + \frac{1}{z(1-z)} R(z; q).$$

He obtained the following Hecke-Rogers identity for $R(z; q)$ [10, Eq. (1.15)].

$$\begin{aligned}
& (zq, z^{-1}q, q; q)_\infty R(z; q) \\
&= (zq, z^{-1}q, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq; q)_n (z^{-1}q; q)_n} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/2]} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \right. \\
&\quad \left. + \sum_{j=1}^{[n/2]} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \right). \tag{1.1}
\end{aligned}$$

Putting $z = 1$ in (1.1) and noting that $R(1; q) = 1/(q; q)_\infty$, we can recover the following celebrated Hecke-Rogers identity [12, Eq. (7); 18, p. 323]

$$(q; q)_\infty^2 = \sum_{n=0}^{\infty} \sum_{j=-[n/2]}^{[n/2]} (-1)^{n+j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)}. \tag{1.2}$$

Garvan [10] also considered the overpartition rank function $H(z; q)$ defined by

$$H(z; q) = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{\frac{1}{2}n(n+1)}}{(zq; q)_n (z^{-1}q; q)_n}, \tag{1.3}$$

which has the following relation with $g_2(z; q)$ [16].

$$(1+z)H(z; q) = (1-z) + 2z(1-z)g_2(z; q). \tag{1.4}$$

The following Hecke-Rogers identity for $H(z; q)$ is given by Garvan [10, Eqs. (1.16)–(1.17)].

$$\begin{aligned}
& (1+z)(zq, z^{-1}q, q; q)_\infty H(z; q) \\
&= (1+z)(zq, z^{-1}q, q; q)_\infty \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{\frac{1}{2}n(n+1)}}{(zq; q)_n (z^{-1}q; q)_n} \\
&= \sum_{n=0}^{\infty} \sum_{|m| \leq [n/2]} (-1)^{n+m} (z^{n-2|m|+1} + z^{2|m|-n}) q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} \tag{1.5}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{|m| \leq [n/3]} (-1)^n (z^{n-4|m|+1} + z^{4|m|-n}) q^{\frac{1}{2}(n^2-8m^2)+\frac{1}{2}n}. \tag{1.6}$$

Letting $z = 1$ in (1.5) and (1.6), and using the fact that $H(1; q) = (q^2; q^2)_\infty / (q; q)_\infty^2$, we get the following two Hecke-Rogers type identities given by Kac and Peterson [3, Eq.

(3.16); 14, final equation].

$$(q; q)_\infty (q^2; q^2)_\infty = \sum_{n=0}^{\infty} \sum_{|m| \leq [n/2]} (-1)^{n+m} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}, \quad (1.7)$$

$$(q; q)_\infty (q^2; q^2)_\infty = \sum_{n=0}^{\infty} \sum_{|m| \leq [n/3]} (-1)^n q^{\frac{1}{2}(n^2-8m^2)+\frac{1}{2}n}. \quad (1.8)$$

For $K(z; q)$, Garvan [10, Eq. (1.18)] established the following identity.

$$\begin{aligned} & (zq^2, z^{-1}q^2, q^2; q^2)_\infty K(z; q) \\ &= (zq^2, z^{-1}q^2, q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q; q^2)_n}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (-1)^n z^{m-n} q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n-m)} + \sum_{m=1}^n (-1)^n z^{n-m+1} q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n+m)} \right). \end{aligned} \quad (1.9)$$

Setting $z = 1$ in (1.9), and noting that $K(1; q) = (q; q)_\infty / (q^2; q^2)_\infty^2$, we obtain the following Hecke-Rogers type identity due to Bressoud [7, Eq. (3.8)].

$$(q; q)_\infty (q^2; q^2)_\infty = \sum_{n=0}^{\infty} \sum_{m=-n}^n (-1)^n q^{\frac{1}{2}(2n^2-m^2)+\frac{1}{2}(2n+m)}. \quad (1.10)$$

In [10], Garvan showed that (1.5) and (1.6) are equivalent with the aid of a transform of Milne [17]. He then proved (1.1), (1.5) and (1.9) are valid by showing the coefficients of z^k on both sides of these identities are equal. At the end of his paper, Garvan proposed a problem of finding direct proofs of these three identities by the method of Bailey pairs. In this paper, we will show these identities hold by applying Bailey's transform with a conjugate Bailey pair of Warnaar [22] and three Bailey pairs derived from three identities of Slater [20]. In particular, we obtain a more compact form of Hecke-Rogers identity for $R(z; q)$.

Theorem 1.1. *We have*

$$(zq, z^{-1}q, q; q)_{\infty} R(z; q) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{n-3j+1} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \right) \quad (1.11)$$

$$= \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} + \sum_{j=0}^{\infty} q^{3j^2+j} + \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} - \sum_{j=1}^{\infty} q^{3j^2-j}. \quad (1.12)$$

Note that Garvan's identity (1.1) can immediately follow from (1.11). To be specific, if we replace z by z^{-1} in (1.11), we get

$$(zq, z^{-1}q, q; q)_{\infty} R(z^{-1}; q) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{-n+3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{-n+3j-1} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \right). \quad (1.13)$$

Combining (1.11) and (1.13), and noting $R(z; q) = R(z^{-1}; q)$, we get Garvan's identity (1.1).

Using the Bailey machinery, we obtain the following Hecke-Rogers identity for $g_2(z; q)$.

Theorem 1.2. *We have*

$$(1-z)(zq, z^{-1}q, q; q)_{\infty} g_2(z; q) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^{m+n} z^{n-2m} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} + \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^{m+n} z^{2m-n-1} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} \right). \quad (1.14)$$

When substitute (1.14) into the relation (1.4), and employ Jacobi's triple product identity, we derive Garvan's identity (1.5).

More generally, we generalized these two-variable Hecke-Rogers identities to infinite families. The next theorem embeds (1.12) into an infinite family.

Theorem 1.3. For $k \geq 2$, we have

$$\begin{aligned}
& (zq, z^{-1}q, q; q)_\infty \sum_{n=0}^{\infty} \frac{(q; q)_{2n} q^n}{(zq; q)_n (z^{-1}q; q)_n} \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{k-2}^2 + 2N_{k-1}^2 + N_1 + \dots + N_{k-2}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q; q)_{2n_{k-1}}} \\
&= \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2) + \frac{1}{2}(n-j) + 3(k-1)j(3j+1)} \\
&+ \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2) + \frac{1}{2}(n+j) + 3(k-1)j(3j-1)} \\
&+ \sum_{j=0}^{\infty} q^{(3k-2)j(3j+1)} - \sum_{j=1}^{\infty} q^{(3k-2)j(3j-1)}, \tag{1.15}
\end{aligned}$$

where $N_j = n_j + n_{j+1} + \dots + n_{k-1}$.

We extends (1.14) into an infinite family.

Theorem 1.4. For $k \geq 2$, we have

$$\begin{aligned}
& (zq, z^{-1}q, q; q)_\infty \sum_{n=0}^{\infty} \frac{(q^2; q)_{2n} q^n}{(zq; q)_n (z^{-1}q; q)_{n+1}} \\
& \quad \times \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{k-2}^2 + \frac{3}{2}N_{k-1}^2 + 2(N_1 + \dots + N_{k-2}) + \frac{3}{2}N_{k-1}} (-q; q)_{n_{k-1}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q^2; q)_{2n_{k-1}}} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{[n/2]} (-1)^{m+n} z^{n-2m} q^{\frac{1}{2}(n^2-2m^2) + \frac{1}{2}n + 4(k-1)(m^2+m)} \right. \\
& \quad \left. + \sum_{m=1}^{[n/2]} (-1)^{m+n} z^{2m-n-1} q^{\frac{1}{2}(n^2-2m^2) + \frac{1}{2}n + 4(k-1)(m^2-m)} \right), \tag{1.16}
\end{aligned}$$

where $N_j = n_j + n_{j+1} + \dots + n_{k-1}$.

Finally, we generalize (1.9) to an infinite family.

Theorem 1.5. For $k \geq 2$, we have

$$\begin{aligned}
& (zq^2, z^{-1}q^2, q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{(q^2; q^2)_{2n} q^{2n}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \\
& \quad \times \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{(-1)^{n_{k-1}} q^{2N_1^2 + \dots + 2N_{k-2}^2 + 3N_{k-1}^2 + 2N_1 + \dots + 2N_{k-2}} (q; q^2)_{n_{k-1}}}{(q^2; q^2)_{n-N_1} (q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{k-2}} (q^2; q^2)_{2n_{k-1}}} \\
& = \sum_{n=0}^{\infty} \left(\sum_{m=0}^n (-1)^n z^{m-n} q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n-m) + 2(k-1)(m^2+m)} \right. \\
& \quad \left. + \sum_{m=1}^n (-1)^n z^{n-m+1} q^{\frac{1}{2}(2n^2-m^2) + \frac{1}{2}(2n+m) + 2(k-1)(m^2-m)} \right), \tag{1.17}
\end{aligned}$$

where $N_j = n_j + n_{j+1} + \dots + n_{k-1}$.

2 Conjugate Bailey pairs and Bailey pairs

In this section, we present a conjugate Bailey pair and three Bailey pairs which are needed in the proof of Hecke-Rogers identities for the universal mock theta functions. Recall that a pair of sequences (δ_n, γ_n) is called a conjugate Bailey pair relative to a if they satisfy

$$\gamma_n = \sum_{r=n}^{\infty} \frac{\delta_r}{(q; q)_{r-n} (aq; q)_{r+n}}. \tag{2.1}$$

A pair of sequences (α_n, β_n) is called a Bailey pair relative to a if they satisfy

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}. \tag{2.2}$$

From the definitions of Bailey pairs and conjugate Bailey pairs, one could easily obtain the following transform first presented by Bailey in [6, p. 1] and known as Bailey's transform [21, Section 2.4, pp. 58–74].

Theorem 2.1 (The Bailey transform). *If (α_n, β_n) is a Bailey pair relative to a and (γ_n, δ_n) is a conjugate Bailey pair relative to a . Then we have*

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n.$$

We first state a conjugate Bailey pair which is needed in the proof of Garvan's three identities. The proof of this conjugate Bailey pair is mainly based on the following generalization of Jacobi's triple product identity given by Warnaar [22, Theorem 1.5]. Andrews

and Warnaar provided an alternative proof of this identity in [5].

$$1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2}} (a^r + b^r) = (a, b, q; q)_{\infty} \sum_{r=0}^{\infty} \frac{(ab/q; q)_{2r} q^r}{(q, a, b, ab; q)_r}. \quad (2.3)$$

Setting $a = z$ and $b = z^{-1}q$ in (2.3), we obtain Jacobi's triple product [9, p.15]

$$1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2}} (z^r + z^{-r} q^r) = (z, z^{-1}q, q; q)_{\infty}. \quad (2.4)$$

Lemma 2.2. *The following pair of sequences (δ_n, γ_n) forms a conjugate Bailey pair relative to ab , where*

$$\delta_n = (aq, b, q; q)_{\infty} \frac{(ab; q)_{2n} q^n}{(aq; q)_n (b; q)_n},$$

and

$$\gamma_n = \frac{(1-ab)q^n}{1-abq^{2n}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2}} ((aq^{n+1})^r + (bq^n)^r) \right).$$

Proof. By the definition of conjugate Bailey pair, we see that

$$\begin{aligned} \gamma_n &= \sum_{r=n}^{\infty} \frac{\delta_r}{(q; q)_{r-n} (abq; q)_{r+n}} \\ &= (aq, b, q; q)_{\infty} \sum_{r=n}^{\infty} \frac{(ab; q)_{2r} q^r}{(q; q)_{r-n} (abq; q)_{r+n} (aq; q)_r (b; q)_r} \\ &= (aq, b, q; q)_{\infty} \sum_{r=0}^{\infty} \frac{(ab; q)_{2n+2r} q^{r+n}}{(q; q)_r (abq; q)_{r+2n} (aq; q)_{r+n} (b; q)_{r+n}} \\ &= (aq, b, q; q)_{\infty} \frac{(ab; q)_{2n} q^n}{(abq; q)_{2n} (aq; q)_n (b; q)_n} \sum_{r=0}^{\infty} \frac{(abq^{2n}; q)_{2r} q^r}{(q, abq^{2n+1}, aq^{n+1}, bq^n; q)_r} \\ &= \frac{(1-ab)q^n}{1-abq^{2n}} (aq^{n+1}, bq^n, q; q)_{\infty} \sum_{r=0}^{\infty} \frac{(abq^{2n}; q)_{2r} q^r}{(q, abq^{2n+1}, aq^{n+1}, bq^n; q)_r}. \end{aligned}$$

Using (2.3) with a replaced by aq^{n+1} and b by bq^n in the summation of the above identity, we derive that

$$\begin{aligned} \gamma_n &= \frac{(1-ab)q^n}{1-abq^{2n}} (aq^{n+1}, bq^n, q; q)_{\infty} \frac{1}{(aq^{n+1}, bq^n, q; q)_{\infty}} \\ &\quad \times \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2}} ((aq^{n+1})^r + (bq^n)^r) \right) \\ &= \frac{(1-ab)q^n}{1-abq^{2n}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2}} ((aq^{n+1})^r + (bq^n)^r) \right), \end{aligned}$$

as desired. This completes the proof. ■

To derive (1.12), we also need the following Bailey pair given by Warnaar [22, p. 375], which can be established by an identity of Slater [20, Eq. (3.4)].

Lemma 2.3. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to q , where*

$$\alpha_{3n} = q^{(3n-2)n} \frac{1 - q^{6n+1}}{1 - q}, \quad \alpha_{3n+1} = 0, \quad \alpha_{3n+2} = -q^{(3n+2)n} \frac{1 - q^{6n+5}}{1 - q},$$

$$\beta_n = \frac{q^{n(n-1)}}{(q; q)_{2n}}.$$

The following Bailey pair is necessary in the proof of (1.14). The proof of this Bailey pair is mainly based on a special case of ${}_6\psi_6$ given by Slater [20, Eq. (4.2)]:

$$\sum_{r=-[n/2]}^{[n/2]} \frac{(1 - aq^{4r})(q^{-n}; q)_{2r}(d; q^2)_r(e; q^2)_r}{(1 - a)(aq^{n+1}; q)_{2r}(aq^2/d; q^2)_r(aq^2/e; q^2)_r} \left(\frac{a^2 q^{2n+1}}{de} \right)^r \quad (2.5)$$

$$= \frac{(q^2/a, aq/d, aq/e, aq^2/de; q^2)_\infty}{(q, q^2/d, q^2/e, a^2 q/de; q^2)_\infty} \cdot \frac{(q; q)_n (aq; q)_n (a^2 q/de; q^2)_n}{(aq; q^2)_n (aq/d; q)_n (aq/e; q)_n}.$$

Lemma 2.4. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to q^2 , where*

$$\alpha_{2n} = (-1)^n q^{n^2-n} \frac{1 - q^{4n+2}}{1 - q^2}, \quad \alpha_{2n+1} = 0, \quad (2.6)$$

$$\beta_n = \frac{(-q; q)_n q^{\binom{n}{2}}}{(q^2; q)_{2n}}.$$

Proof. In (2.5), letting $a = d = q^2$, $e \rightarrow 0$ and noting that $1/(q^2; q^2)_n = 0$ when $n < 0$, we obtain

$$\sum_{r=0}^{[n/2]} \frac{(1 - q^{4r+2})(q^{-n}; q)_{2r} (-1)^r q^{2nr-r^2}}{(1 - q^2)(q^{n+3}; q)_{2r}} = \frac{(q^3; q)_n q^{\binom{n}{2}}}{(q^3; q^2)_n}.$$

Divide both sides of the above identity by $(q; q)_n (q^3; q)_n$ to get

$$\sum_{r=0}^{[n/2]} \frac{1}{(q; q)_{n-2r} (q^3; q)_{n+2r}} \frac{(1 - q^{4r+2})(-1)^r q^{r^2-r}}{1 - q^2} = \frac{(-q; q)_n q^{\binom{n}{2}}}{(q^2; q)_{2n}},$$

which yields the desired Bailey pair. This completes the proof. ■

To verify (1.9), we require the following Bailey pair.

Lemma 2.5. *The following pair of sequences (α_n, β_n) forms a Bailey pair relative to q , where*

$$\begin{aligned}\alpha_n &= (-1)^n q^{\frac{n^2-3n}{4}} \frac{1-q^{2n+1}}{1-q}, \\ \beta_n &= \frac{(-1)^n q^{\frac{n^2}{2}-n} (q^{\frac{1}{2}}; q)_n}{(q; q)_{2n}}.\end{aligned}\tag{2.7}$$

Proof. In (2.5), set $a = q$, $d = -q^{3/2}$ and $e \rightarrow 0$ to get

$$\sum_{r=-[n/2]}^{[n/2]} \frac{(1-q^{4r+1})(q^{-n}; q)_{2r} q^{2nr-\frac{1}{2}r-r^2}}{(1-q)(q^{n+2}; q)_{2r}} = \frac{(q; q)_n (q^2; q)_n (-1)^n q^{\frac{n^2}{2}-n}}{(q^2; q^2)_n (-q^{\frac{1}{2}}; q)_n}.$$

Dividing both sides of the above identity by $(q; q)_n (q^2; q)_n$, we obtain

$$\sum_{r=-[n/2]}^{[n/2]} \frac{(1-q^{4r+1})q^{r^2-\frac{3}{2}r}}{(q; q)_{n+2r+1} (q; q)_{n-2r}} = \frac{(-1)^n q^{\frac{n^2}{2}-n} (q^{\frac{1}{2}}; q)_n}{(q; q)_{2n}}.$$

Simplifying the left hand side of the above identity yields

$$\begin{aligned}& \sum_{r=-[n/2]}^{[n/2]} \frac{(1-q^{4r+1})q^{r^2-\frac{3}{2}r}}{(q; q)_{n+2r+1} (q; q)_{n-2r}} \\ &= \frac{1}{(q^2; q)_n (q; q)_n} + \sum_{r=1}^{[n/2]} \frac{(1-q^{4r+1})q^{r^2-\frac{3}{2}r}}{(q; q)_{n+2r+1} (q; q)_{n-2r}} + \sum_{r=-[n/2]}^{-1} \frac{(1-q^{4r+1})q^{r^2-\frac{3}{2}r}}{(q; q)_{n+2r+1} (q; q)_{n-2r}} \\ &= \frac{1}{(q^2; q)_n (q; q)_n} + \sum_{r=1}^{[n/2]} \frac{(1-q^{4r+1})q^{r^2-\frac{3}{2}r}}{(q; q)_{n+2r+1} (q; q)_{n-2r}} - \sum_{r=1}^{[n/2]} \frac{(1-q^{4r-1})q^{r^2-\frac{5}{2}r+1}}{(q; q)_{n-2r+1} (q; q)_{n+2r}} \\ &= \sum_{r=0}^n \frac{1}{(q; q)_{n-r} (q^2; q)_{n+r}} \frac{(1-q^{2r+1})(-1)^r q^{\frac{r^2-3r}{4}}}{1-q}.\end{aligned}$$

Hence, we have

$$\sum_{r=0}^n \frac{1}{(q; q)_{n-r} (q^2; q)_{n+r}} \frac{(1-q^{2r+1})(-1)^r q^{\frac{r^2-3r}{4}}}{1-q} = \frac{(-1)^n q^{\frac{n^2}{2}-n} (q^{\frac{1}{2}}; q)_n}{(q; q)_{2n}},$$

which yields the desired Bailey pair. This completes the proof. \blacksquare

To get an infinite family of Hecke-Rogers identities, we need the following limiting form of Bailey's lemma [4, 18].

Theorem 2.6. *Supposed (α_n, β_n) is a Bailey pair relative to a . Then (α'_n, β'_n) is a new Bailey pair relative to a , where*

$$\alpha'_n = a^n q^{n^2} \alpha_n$$

and

$$\beta'_n = \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(q; q)_{n-j}} \beta_j.$$

Iterating this lemma leads to the Bailey chain. We now produce the following infinite sequence of Bailey pairs by iterating the Bailey pair in Lemma 2.3 along this Bailey chain.

Lemma 2.7. *For $k \geq 2$, $(\alpha_n^{(k)}, \beta_n^{(k)})$ form Bailey pairs relative to q , where*

$$\begin{aligned} \alpha_{3n}^{(k)} &= q^{3(3k-2)n^2+(3k-5)n} \frac{1-q^{6n+1}}{1-q}, \quad \alpha_{3n+1}^{(k)} = 0, \\ \alpha_{3n+2}^{(k)} &= -q^{3(3k-2)n^2+(15k-13)n+6(k-1)} \frac{1-q^{6n+5}}{1-q}, \\ \beta_n^{(k)} &= \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\dots+N_{k-2}^2+2N_{k-1}^2+N_1+\dots+N_{k-2}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q; q)_{2n_{k-1}}}, \end{aligned}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

Proof. We proceed by induction on k . When $k = 2$. Applying Theorem 2.6 to the Bailey pair in Lemma 2.3, we obtain a new Bailey pair $(\alpha_n^{(2)}, \beta_n^{(2)})$ relative to q , where

$$\begin{aligned} \alpha_{3n}^{(2)} &= q^{9n^2+3n} \times q^{(3n-2)n} \frac{1-q^{6n+1}}{1-q} = q^{12n^2+n} \frac{1-q^{6n+1}}{1-q}, \quad \alpha_{3n+1}^{(2)} = 0, \\ \alpha_{3n+2}^{(2)} &= q^{(3n+2)(3n+3)} \times \left(-q^{(3n+2)n} \frac{1-q^{6n+5}}{1-q} \right) = -q^{12n^2+17n+6} \frac{1-q^{6n+5}}{1-q} \end{aligned}$$

and

$$\beta_n^{(2)} = \sum_{j=0}^n \frac{q^{2j^2}}{(q; q)_{n-j} (q; q)_{2j}} = \sum_{j=0}^{\infty} \frac{q^{2j^2}}{(q; q)_{n-j} (q; q)_{2j}},$$

as desired.

Assume that we have a Bailey pair $(\alpha_n^{(k-1)}, \beta_n^{(k-1)})$ relative to q , where

$$\begin{aligned} \alpha_{3n}^{(k-1)} &= q^{3(3k-5)n^2+(3k-8)n} \frac{1-q^{6n+1}}{1-q}, \quad \alpha_{3n+1}^{(k-1)} = 0, \\ \alpha_{3n+2}^{(k-1)} &= -q^{3(3k-5)n^2+(15k-28)n+6(k-2)} \frac{1-q^{6n+5}}{1-q}, \\ \beta_n^{(k-1)} &= \sum_{n'_1, \dots, n'_{k-2}=0}^{\infty} \frac{q^{N_1'^2+\dots+N_{k-3}'^2+2N_{k-2}'^2+N_1'+\dots+N_{k-3}'}}{(q; q)_{n-N_1'} (q; q)_{n'_1} \cdots (q; q)_{n'_{k-3}} (q; q)_{2n'_{k-2}}}, \end{aligned}$$

where $N'_j = n'_j + n'_{j+1} + \cdots + n'_{k-2}$.

When we apply Theorem 2.6 to this Bailey pair, we arrive at a new Bailey pair $(\alpha_n^{(k)}, \beta_n^{(k)})$ relative to q , where for $\alpha_n^{(k)}$, we have

$$\begin{aligned}\alpha_{3n}^{(k)} &= q^{9n^2+3n} \alpha_{3n}^{(k-1)} \\ &= q^{9n^2+3n+3(3k-5)n^2+(3k-8)n} \frac{1-q^{6n+1}}{1-q} \\ &= q^{3(3k-2)n^2+(3k-5)n} \frac{1-q^{6n+1}}{1-q}, \\ \alpha_{3n+1}^{(k)} &= 0, \\ \alpha_{3n+2}^{(k)} &= q^{3(n+1)(3n+2)} \alpha_{3n+2}^{(k-1)} \\ &= -q^{3(n+1)(3n+2)+3(3k-5)n^2+(15k-28)n+6(k-2)} \frac{1-q^{6n+5}}{1-q} \\ &= -q^{3(3k-2)n^2+(15k-13)n+6(k-1)} \frac{1-q^{6n+5}}{1-q}.\end{aligned}$$

For $\beta_n^{(k)}$, we have

$$\begin{aligned}\beta_n^{(k)} &= \sum_{n'_{k-1}=0}^{\infty} \frac{q^{n'_{k-1}{}^2+n'_{k-1}}}{(q; q)_{n-n'_{k-1}}} \beta_{n'_{k-1}}^{(k-1)} \\ &= \sum_{n'_{k-1}=0}^{\infty} \frac{q^{n'_{k-1}{}^2+n'_{k-1}}}{(q; q)_{n-n'_{k-1}}} \sum_{n'_1, \dots, n'_{k-2}=0}^{\infty} \frac{q^{N_1'^2+\cdots+N_{k-3}'^2+2N_{k-2}'^2+N_1'+\cdots+N_{k-3}'}}{(q; q)_{n'_{k-1}-N_1'} (q; q)_{n'_1} \cdots (q; q)_{n'_{k-3}} (q; q)_{2n'_{k-2}}}.\end{aligned}$$

Recall that $N'_j = n'_j + n'_{j+1} + \cdots + n'_{k-2}$ for $1 \leq j \leq k-2$. Let

$$n_1 = n'_{k-1} - N_1', \quad n_2 = n'_1, \quad n_3 = n'_2, \quad \dots, \quad n_{k-1} = n'_{k-2},$$

and set $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$ for $1 \leq j \leq k-1$. Obviously, we have

$$N_2 = N_1', \quad N_3 = N_2', \quad \dots, \quad N_{k-1} = N_{k-2}', \quad N_1 = n'_{k-1},$$

and so

$$\beta_n^{(k)} = \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-2}^2+2N_{k-1}^2+N_1+\cdots+N_{k-2}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q; q)_{2n_{k-1}}}.$$

This completes the proof. ■

The following infinite sequence of Bailey pairs can be obtained from the Bailey pair in Lemma 2.4 along the same line.

Lemma 2.8. For $k \geq 2$, $(\alpha_n^{(k)}, \beta_n^{(k)})$ form Bailey pairs relative to q^2 , where

$$\alpha_{2n}^{(k)} = (-1)^n q^{(4k-3)n^2 + (4k-5)n} \frac{1 - q^{4n+2}}{1 - q^2}, \quad \alpha_{2n+1}^{(k)} = 0,$$

$$\beta_n^{(k)} = \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{k-2}^2 + \frac{3}{2}N_{k-1}^2 + 2(N_1 + \dots + N_{k-2}) + \frac{3}{2}N_{k-1}} (-q; q)_{n_{k-1}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q^2; q)_{2n_{k-1}}},$$

where $N_j = n_j + n_{j+1} + \dots + n_{k-1}$.

Similarly, we obtain the following infinite sequence of Bailey pairs from the Bailey pair in Lemma 2.5 which are used to establish the infinite family in Theorem 1.5.

Lemma 2.9. For $k \geq 2$, $(\alpha_n^{(k)}, \beta_n^{(k)})$ form Bailey pairs relative to q , where

$$\alpha_n^{(k)} = (-1)^n q^{\frac{4k-3}{4}n^2 + \frac{4k-7}{4}n} \frac{1 - q^{2n+1}}{1 - q},$$

$$\beta_n^{(k)} = \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{(-1)^{n_{k-1}} q^{N_1^2 + \dots + N_{k-2}^2 + \frac{3}{2}N_{k-1}^2 + N_1 + \dots + N_{k-2}} (q^{\frac{1}{2}}; q)_{n_{k-1}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q; q)_{2n_{k-1}}},$$

where $N_j = n_j + n_{j+1} + \dots + n_{k-1}$.

3 A two-variable Hecke-Rogers identity for $R(z; q)$ and generalization

In this section, we first give a proof of (1.12) in Theorem 1.1 by using the Bailey transform with the conjugate Bailey pair in Lemma 2.2 and the Bailey pair in Lemma 2.3. We then show that (1.11) can be derived from (1.12) with the aid of two identities given by Garvan [10]. Finally, we give a proof of Theorem 1.3.

Proof of Theorem 1.1, Eq. (1.12). Combining the Bailey transform with the Bailey pairs in Lemma 2.3 and the conjugate Bailey pair in Lemma 2.2 with $a = z$, and $b = z^{-1}q$, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n \delta_n &= (zq, z^{-1}q, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq; q)_n (z^{-1}q; q)_n} \\ &= \sum_{j=0}^{\infty} \alpha_{3j} \gamma_{3j} + \sum_{j=1}^{\infty} \alpha_{3j-1} \gamma_{3j-1} \\ &= \sum_{j=0}^{\infty} q^{j(3j-2)+3j} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} ((zq^{3j+1})^n + (z^{-1}q^{3j+1})^n) \right) \\ &\quad - \sum_{j=1}^{\infty} q^{(j-1)(3j-1)+(3j-1)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} ((zq^{3j})^n + (z^{-1}q^{3j})^n) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^n (z^n + z^{-n}) q^{\binom{n+1}{2} + 3j^2 + j + 3nj} + \sum_{j=0}^{\infty} q^{3j^2 + j} \\
&\quad + \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} (z^n + z^{-n}) q^{\binom{n}{2} + 3j^2 - j + 3nj} - \sum_{j=1}^{\infty} q^{3j^2 - j}. \tag{3.1}
\end{aligned}$$

Replace n by $n - 3j$ in the first sum of (3.1) and interchange the order of summation to get

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^n (z^n + z^{-n}) q^{\binom{n+1}{2} + 3j^2 + j + 3nj} \\
&= \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)}. \tag{3.2}
\end{aligned}$$

Replace n by $n - 3j + 1$ in the third sum of (3.1) and interchange the order of summation to yield

$$\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} (z^n + z^{-n}) q^{\binom{n}{2} + 3j^2 - j + 3nj} \\
&= \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n+j)}. \tag{3.3}
\end{aligned}$$

Plugging (3.2) and (3.3) into (3.1), we get the desired identity (1.12). \blacksquare

We are now in position to show (1.11) in Theorem 1.1 holds.

Proof of Theorem 1.1, Eq. (1.11). Obviously, it suffices to show that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)} + \sum_{j=0}^{\infty} q^{3j^2 + j} \\
&\quad + \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n+j)} - \sum_{j=1}^{\infty} q^{3j^2 - j} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{n-3j+1} q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n+j)} \right). \tag{3.4}
\end{aligned}$$

To this end, we first show that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)} + \sum_{j=0}^{\infty} q^{3j^2 + j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j)}. \tag{3.5}
\end{aligned}$$

Using the following identity given by Garvan [10, Eq. (2.14)],

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \\
&= \sum_{n=0}^{\infty} \sum_{n/3 < j \leq n/2} (-1)^{n+j} z^{-n+3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)}, \tag{3.6}
\end{aligned}$$

we get

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} + \sum_{j=0}^{\infty} q^{3j^2+j} \\
&= \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} \\
& \quad + \sum_{n=0}^{\infty} \sum_{n/3 < j \leq n/2} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)} + \sum_{j=0}^{\infty} q^{3j^2+j} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{[n/2]} (-1)^{n+j} z^{n-3j} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n-j)},
\end{aligned}$$

as desired.

Similarly, applying the following identity of Garvan [10, Eq. (2.15)]

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} z^{n-3j+1} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} \\
&= \sum_{n=0}^{\infty} \sum_{(n+1)/3 < j \leq n/2} (-1)^{n+j} z^{-n+3j-1} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)},
\end{aligned}$$

we can show that

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)} - \sum_{j=1}^{\infty} q^{3j^2-j} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{[n/2]} (-1)^{n+j} z^{n-3j+1} q^{\frac{1}{2}(n^2-3j^2)+\frac{1}{2}(n+j)}. \tag{3.7}
\end{aligned}$$

Combining (3.5) and (3.7), we obtain (3.4). Thus we show (1.11) holds. \blacksquare

For the remainder of this section, we show (1.15) holds by using the Bailey pair technology.

Proof of Theorem 1.3. Applying the Bailey transform with the conjugate Bailey pair in Lemma 2.2 by setting $a = z$ and $b = z^{-1}q$ and the Bailey pair in Lemma 2.7, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n^{(k)} \delta_n &= (zq, z^{-1}q, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(q; q)_{2n} q^n}{(zq; q)_n (z^{-1}q; q)_n} \\
&\quad \times \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{k-2}^2 + 2N_{k-1}^2 + N_1 + \dots + N_{k-2}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q; q)_{2n_{k-1}}} \\
&= \sum_{j=0}^{\infty} \alpha_{3j}^{(k)} \gamma_{3j} + \sum_{j=1}^{\infty} \alpha_{3j-1}^{(k)} \gamma_{3j-1} \\
&= \sum_{j=0}^{\infty} q^{(3k-2)j(3j+1)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n+1}{2}} ((zq^{3j})^n + (z^{-1}q^{3j})^n) \right) \\
&\quad - \sum_{j=1}^{\infty} q^{(3k-2)j(3j-1)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} ((zq^{3j})^n + (z^{-1}q^{3j})^n) \right) \\
&= \sum_{j=0}^{\infty} q^{(3k-2)j(3j+1)} + \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^n (z^n + z^{-n}) q^{\binom{n+1}{2} + 3jn + (3k-2)j(3j+1)} \\
&\quad - \sum_{j=1}^{\infty} q^{(3k-2)j(3j-1)} + \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} (z^n + z^{-n}) q^{\binom{n}{2} + 3jn + (3k-2)j(3j-1)}. \quad (3.8)
\end{aligned}$$

Replace n by $n - 3j$ in the second sum of (3.8) and reverse the order of summation to get

$$\begin{aligned}
&\sum_{j=0}^{\infty} \sum_{n=1}^{\infty} (-1)^n (z^n + z^{-n}) q^{\binom{n+1}{2} + 3nj + (3k-2)j(3j+1)} \\
&= \sum_{n=0}^{\infty} \sum_{0 \leq j < n/3} (-1)^{n+j} (z^{n-3j} + z^{3j-n}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n-j) + 3(k-1)j(3j+1)}. \quad (3.9)
\end{aligned}$$

Replace n by $n - 3j + 1$ in the fourth sum of (3.8) and interchange the order of summation. This gives

$$\begin{aligned}
&\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n-1} (z^n + z^{-n}) q^{\binom{n}{2} + 3nj + (3k-2)j(3j-1)} \\
&= \sum_{n=0}^{\infty} \sum_{1 \leq j < (n+1)/3} (-1)^{n+j} (z^{n-3j+1} + z^{3j-n-1}) q^{\frac{1}{2}(n^2 - 3j^2) + \frac{1}{2}(n+j) + 3(k-1)j(3j-1)}. \quad (3.10)
\end{aligned}$$

Substituting (3.9) and (3.10) in (3.8), we are led to the desired identity (1.15). \blacksquare

4 A two-variable Hecke-Rogers identity for $H(z; q)$ and generalization

In this section, we first use the Bailey transform to show Theorem 1.2. We then derive Garvan's identity (1.5) from Theorem 1.2. Last, we give a proof of Theorem 1.4.

Proof of Theorem 1.2. Apply the Bailey transform with the conjugate Bailey pair in Lemma 2.2 by setting $a = z^{-1}q$ and $b = zq$ and the Bailey pair in Lemma 2.4 to yield

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n \delta_n &= (zq, z^{-1}q, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-q; q)_n q^{\binom{n+1}{2}}}{(z^{-1}q; q)_{n+1} (zq; q)_n} \\
&= (1-z)(zq, z^{-1}q, q; q)_{\infty} g_2(z; q) \\
&= \sum_{m=0}^{\infty} \alpha_{2m} \gamma_{2m} \\
&= \sum_{m=0}^{\infty} (-1)^m q^{m^2+m} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n+1}{2}} ((z^{-1}q^{2m+1})^n + (zq^{2m})^n) \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2}+2mn+m^2+m} \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2}+(2m+1)n+m^2+m}. \tag{4.1}
\end{aligned}$$

Shift n to $n - 2m$ in the first sum of (4.1) and interchange the order of summation. This gives

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2}+2mn+m^2+m} = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^{m+n} z^{n-2m} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}. \tag{4.2}$$

Shift n to $n - 2m - 1$ in the second sum of (4.1), and then replace m by $m - 1$, change the order of summation. This summation becomes

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2}+(2m+1)n+m^2+m} = \sum_{n=0}^{\infty} \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^{m+n} z^{2m-n-1} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}. \tag{4.3}$$

Plugging (4.2) and (4.3) into (4.1), we get the desired identity (1.14). Thus we complete the proof of Theorem 1.2. \blacksquare

Now it's an easy matter to prove (1.5).

Proof of Garvan's identity (1.5). By the relation (1.4), we see that

$$\begin{aligned}
&(1+z)(zq, z^{-1}q, q; q)_{\infty} H(z; q) \\
&= (1-z)(zq, z^{-1}q, q; q)_{\infty} + 2z(1-z)(zq, z^{-1}q, q; q)_{\infty} g_2(z; q). \tag{4.4}
\end{aligned}$$

Invoking Jacobi's triple product identity (2.4) and Theorem 1.2 in (4.4), we find that

$$\begin{aligned}
& (1+z)(zq, z^{-1}q, q; q)_\infty H(z; q) \\
&= \sum_{n=0}^{\infty} (-1)^n z^{-n} q^{\binom{n+1}{2}} + \sum_{n=1}^{\infty} (-1)^n z^n q^{\binom{n}{2}} + 2 \sum_{n=0}^{\infty} \sum_{m=0}^{[n/2]} (-1)^{m+n} z^{n-2m+1} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} \\
&\quad + 2 \sum_{n=0}^{\infty} \sum_{m=1}^{[n/2]} (-1)^{m+n} z^{2m-n} q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}.
\end{aligned}$$

Replacing n by $n+1$ in the second term, and subtracting the terms with $m=0$ in the third sum, we obtain

$$\begin{aligned}
& (1+z)(zq, z^{-1}q, q; q)_\infty H(z; q) \\
&= \sum_{n=0}^{\infty} (-1)^n z^{-n} q^{\binom{n+1}{2}} - \sum_{n=0}^{\infty} (-1)^n z^{n+1} q^{\binom{n+1}{2}} + 2 \sum_{n=0}^{\infty} (-1)^n z^{n+1} q^{\binom{n+1}{2}} \\
&\quad + 2 \sum_{n=0}^{\infty} \sum_{m=1}^{[n/2]} (-1)^{m+n} (z^{n-2m+1} + z^{2m-n}) q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} \\
&= \sum_{n=0}^{\infty} (-1)^n (z^{n+1} + z^{-n}) q^{\binom{n+1}{2}} + 2 \sum_{n=0}^{\infty} \sum_{m=1}^{[n/2]} (-1)^{m+n} (z^{n-2m+1} + z^{2m-n}) q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n} \\
&= \sum_{n=0}^{\infty} \sum_{|m| \leq [n/2]} (-1)^{n+m} (z^{n-2|m|+1} + z^{2|m|-n}) q^{\frac{1}{2}(n^2-2m^2)+\frac{1}{2}n}.
\end{aligned}$$

Thus we obtain Garvan's identity (1.5). ■

We proceed to prove Theorem 1.4.

Proof of Theorem 1.4. Substituting the Bailey pair in Lemma 2.4 with the conjugate Bailey pair in Lemma 2.2 with $a = z^{-1}q$, and $b = zq$ into the Bailey transform, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n^{(k)} \delta_n &= (zq, z^{-1}q, q; q)_\infty \sum_{n=0}^{\infty} \frac{(q^2; q)_{2n} q^n}{(zq; q)_n (z^{-1}q; q)_{n+1}} \\
&\quad \times \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{q^{N_1^2 + \dots + N_{k-2}^2 + \frac{3}{2}N_{k-1}^2 + 2(N_1 + \dots + N_{k-2}) + \frac{3}{2}N_{k-1}} (-q; q)_{n_{k-1}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q^2; q)_{2n_{k-1}}} \\
&= \sum_{m=0}^{\infty} \alpha_{2m}^{(k)} \gamma_{2m} \\
&= \sum_{m=0}^{\infty} (-1)^m q^{(4k-3)(m^2+m)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n+1}{2}} ((z^{-1}q^{2m+1})^n + (zq^{2m})^n) \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2} + 2mn + (4k-3)(m^2+m)} \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2} + (2m+1)n + (4k-3)(m^2+m)}. \tag{4.5}
\end{aligned}$$

Replacing n by $n - 2m$ in the first term of (4.5) and interchanging the order of summation, we find that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2} + 2mn + (4k-3)(m^2+m)} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^{m+n} z^{n-2m} q^{\frac{1}{2}(n^2-2m^2) + \frac{1}{2}n + 4(k-1)(m^2+m)}. \tag{4.6}
\end{aligned}$$

Replace n by $n - 2m - 1$ in the second sum of (4.5), and then shift m to $m - 1$, reverse the order of summation to yield

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2} + (2m+1)n + (4k-3)(m^2+m)} \\
&= \sum_{n=0}^{\infty} \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^{m+n} z^{2m-n-1} q^{\frac{1}{2}(n^2-2m^2) + \frac{1}{2}n + 4(k-1)(m^2-m)}. \tag{4.7}
\end{aligned}$$

Substituting (4.6) and (4.7) into (4.5), we get the desired identity (1.16). This completes the proof. \blacksquare

5 A two-variable Hecke-Rogers identity for $K(z; q)$ and generalization

In this section, we give a proof of Garvan's identity (1.9) by using Bailey pair technology. We then give a proof of Theorem 1.5, which is a generalization of (1.9).

Proof of Garvan's identity (1.9). Applying the Bailey transform with the conjugate Bailey pair in Lemma 2.2 by setting $a = z$ and $b = z^{-1}q$ and the Bailey pair in Lemma 2.5, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n \delta_n &= (zq, z^{-1}q, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2} (q^{1/2}; q)_n}{(zq; q)_n (z^{-1}q; q)_n} \\
&= \sum_{n=0}^{\infty} \alpha_n \gamma_n \\
&= \sum_{m=0}^{\infty} (-1)^m q^{(m^2+m)/4} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n}{2}} ((zq^{m+1})^n + (z^{-1}q^{m+1})^n) \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2} + \frac{m^2+m}{4} + mn} \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2} + \frac{m^2+m}{4} + mn}. \tag{5.1}
\end{aligned}$$

Shifting n to $n - m$ in the first sum of (5.1), and interchanging the order of summation, we get

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2} + \frac{m^2+m}{4} + mn} = \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^n z^{m-n} q^{\frac{1}{4}(2n^2 - m^2) + \frac{1}{4}(2n-m)}. \tag{5.2}$$

For the second sum of (5.1), we first shift n to $n - m$, then replace m by $m - 1$, and reverse the order of summation, thus the second sum of (5.1) becomes

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2} + \frac{m^2+m}{4} + mn} = \sum_{n=0}^{\infty} \sum_{m=1}^n (-1)^n z^{n-m+1} q^{\frac{1}{4}(2n^2 - m^2) + \frac{1}{4}(2n+m)}. \tag{5.3}$$

Substituting (5.2) and (5.3) into (5.1) and replacing q by q^2 , we recover (1.9). This completes the proof. \blacksquare

Using the similar argument, we could show (1.17) is valid.

Proof of Theorem 1.5. Applying the Bailey transform with Bailey pair in Lemma 2.9 and the conjugate Bailey pair in Lemma 2.2 with $a = z$ and $b = z^{-1}q$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_n^{(k)} \delta_n &= (zq, z^{-1}q, q; q)_{\infty} \sum_{n=0}^{\infty} \frac{(q; q)_{2n} q^n}{(zq; q)_n (z^{-1}q; q)_n} \\
&\quad \times \sum_{n_1, \dots, n_{k-1}=0}^{\infty} \frac{(-1)^{n_{k-1}} q^{N_1^2 + \dots + N_{k-2}^2 + \frac{3}{2}N_{k-1}^2 + N_1 + \dots + N_{k-2}} (q^{\frac{1}{2}}; q)_{n_{k-1}}}{(q; q)_{n-N_1} (q; q)_{n_1} \cdots (q; q)_{n_{k-2}} (q; q)_{2n_{k-1}}} \\
&= \sum_{n=0}^{\infty} \alpha_n^{(k)} \gamma_n \\
&= \sum_{m=0}^{\infty} (-1)^m q^{\frac{4k-3}{4}(m^2+m)} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{\binom{n+1}{2}} ((zq^m)^n + (z^{-1}q^m)^n) \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2} + mn + \frac{4k-3}{4}(m^2+m)} \\
&\quad + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2} + mn + \frac{4k-3}{4}(m^2+m)}. \tag{5.4}
\end{aligned}$$

Replace n by $n - m$ in the first sum of (5.4) and interchange the order of summation to get

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} z^{-n} q^{\binom{n+1}{2} + mn + \frac{4k-3}{4}(m^2+m)} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^n z^{m-n} q^{\frac{1}{4}(2n^2-m^2) + \frac{1}{4}(2n-m) + (k-1)(m^2+m)}. \end{aligned} \quad (5.5)$$

Shift n to $n - m$ in the second sum of (5.4), replace m by $m - 1$, and then reverse the order of summation. This gives

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} z^n q^{\binom{n+1}{2} + mn + \frac{4k-3}{4}(m^2+m)} \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^n (-1)^n z^{n-m+1} q^{\frac{1}{4}(2n^2-m^2) + \frac{1}{4}(2n+m) + (k-1)(m^2-m)}. \end{aligned} \quad (5.6)$$

Hence (1.17) is obtained upon substituting (5.5) and (5.6) into (5.4) and replacing q by q^2 . This completes the proof. \blacksquare

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