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A NOTE ON THE DIFFERENTIAL CALCULUS OF HOCHSCHILD THEORY FOR A_∞-ALGEBRAS

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ABSTRACT. We show by constructing explicit homotopy operators that the Hochschild (co)homology of an A_∞-algebra of Stasheff admits a differential calculus structure. As an application, we reproduce a result of Tradler which says that the Hochschild cohomology of a cyclic A_∞-algebra admits a Batalin-Vilkovisky algebra structure.

1. INTRODUCTION

In differential geometry, on a smooth manifold M, we have the following classical structures (*cf.* Kobayashi and Nomizu [13, Proposition 3.10]):

(a) the space of polyvector fields, under the wedge product and Schouten bracket, forms a Gerstenhaber (also called super-Poisson) algebra;
(b) the space of differential forms, together with the exterior differential and wedge product, forms a commutative differential graded algebra; and
(c) vector fields act on differential forms by Lie derivative and by contraction, which satisfies the following two identities:

$$L_X = d \circ \iota_X + \iota_X \circ d, \quad [\iota_X, L_Y] = \iota_{[X,Y]}, \tag{1.1}$$

where X, Y are vector fields on M, L_X is the Lie derivative and ι_X is the contraction.

There are analogous statements in the holomorphic, symplectic and even in the non-commutative geometry.

For instance, given an associative algebra A, which is viewed as a non-commutative "space", the Hochschild cohomology $\mathrm{HH}^\bullet(A, A)$ and Hochschild homology $\mathrm{HH}_\bullet(A, A)$ play the roles of polyvector fields and differential forms on this space, and the Connes cyclic operator on $\mathrm{HH}_\bullet(A, A)$ substitutes the de Rham differential. One may similarly define a version of contraction and Lie derivative as in the smooth manifolds case, which satisfy (1.1). This was first obtained by Daletskii-Gelfand-Tsygan [3], and summarized by Tamarkin-Tsygan in [22]. According to Tamarkin-Tsygan, a pair of spaces satisfying the above (a), (b) and (c) form a structure of *differential calculus*, a notion introduced in the same paper. In this note, we first show a similar result:

Theorem 1.1. *Let A be an A_∞-algebra over a field $\mathbb{K}$. Then the Hochschild cohomology and homology of A,*

$$(\mathrm{HH}^\bullet(A, A), \mathrm{HH}_\bullet(A, A), \cup, [-, -], \cap, B)$$

is a differential calculus, where $\cup$ is the cup product, $\cap$ is the cap product, $[-, -]$ is the Gerstenhaber Lie bracket and B is the Connes differential.

This result is known to experts Dolgushev-Tamarkin-Tsygan [5] and has been essentially laid out by Kontsevich in his article Formal (non)commutative symplectic geometry, and was also explained in Section 7 of his work Notes on A_∞-algebras, A_∞-categories and noncommutative geometry [14], joint with Soibelman. We here give all necessary calculations required to prove (1.1), which seems to be rarely found in the literature.

Another motivation for us to show the above result is that it is related to the study of Calabi-Yau algebras, a notion introduced by Ginzburg in [11], where he also showed that, for a Calabi-Yau algebra, say A, there is a *Batalin-Vilkovisky algebra* structure on its Hochschild cohomology. The proof is heavily based on the differential calculus structure on the Hochschild (co)homology of A (see also [4]).

On the other hand, for a Calabi-Yau algebra, if it is Koszul (see Priddy [20]), then its Koszul dual algebra is a cyclic associative algebra (that is, an associative algebra with a cyclically invariant non-degenerate pairing). Tradler showed in [24] that there is also a Batalin-Vilkovisky algebra structure on the Hochschild cohomology of such cyclic associative algebra. Recently in [2], Chen, the third author and Zhou proved that for a Koszul

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Calabi-Yau algebra the Batalin-Vilkovisky algebras on the Hochschild cohomology of A and on that of its Koszul dual are isomorphic.

To understand the Batalin-Vilkovisky algebra structure on Calabi-Yau algebra in a more general setting, such as N-Koszul Calabi-Yau algebras in the sense of Berger ([1]), or even more generally, exact complete Calabi-Yau algebras in the sense of Van den Bergh ([25]), one is led to understand the differential calculus structure on cyclic A_∞-algebras (that is, A_∞-algebras with a cyclically invariant non-degenerate pairing), since in both of these two cases, the "Koszul dual" of these types of Calabi-Yau algebras are cyclic A_∞-algebras.

Corollary 1.2 (Tradler [24]). *If A is a cyclic A_∞-algebra, that is, A is a finite dimensional A_∞-algebra with a cyclically invariant non-degenerate pairing, then the Hochschild cohomology $\mathrm{HH}^\bullet(A, A)$ has a Batalin-Vilkovisky algebra structure.*

This corollary is originally due to Tradler [24, Theorem 2]. Here we give an alternative proof, which is in the same spirit of Menichi [18] for finite dimensional symmetric algebras from the differential calculus point of view.

Convention. Throughout the note, we work over a ground field $\mathbb{K}$. All algebras are associative algebras over $\mathbb{K}$ with unit. All vector spaces, their tensors and morphisms etc. are over $\mathbb{K}$.

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2. Gerstenhaber algebras and differential calculi

In this section, we recall the definitions of differential calculus and some of its applications. Let us start with Gerstenhaber algebras.

Definition 2.1 (Gerstenhaber [6]). A *Gerstenhaber algebra* is a quaternion $(\mathrm{H}^\bullet, \cup, [-,-], \mathbb{1})$, where $\mathrm{H}^\bullet$ is a $\mathbb{N}$-graded vector space and $\mathbb{1} \in \mathrm{H}^0$, such that:

(1) $(\mathrm{H}^\bullet, \cup)$ is a graded commutative algebra with unit $\mathbb{1} \in \mathrm{H}^0$;
(2) $(\mathrm{H}^\bullet, [-,-])$ is a graded Lie algebra of degree -1, i.e.

$$[f,\, g] = -(-1)^{(|f|-1)(|g|-1)}[g,\, f]$$

and a graded Jacobi identity

$$(-1)^{(|f|-1)(|h|-1)}[[f,\, g],\, h] + (-1)^{(|g|-1)(|f|-1)}[[g,\, h],\, f] + (-1)^{(|h|-1)(|g|-1)}[[h,\, f],\, g] = 0;$$

(3) The Lie bracket $[-,-]$ is a derivation with respect to the product $\cup$, i.e.

$$[f, g \cup h] = [f, g] \cup h + (-1)^{|g|(|f|-1)} g \cup [f, h],$$

for arbitrary homogenous elements $f, g, h \in \mathrm{H}^\bullet$, where $|f|$ is the degree of the homogenous element f.

Definition 2.2 (Tamarkin-Tsygan [22]). Let $\mathrm{H}^\bullet$ be a $\mathbb{N}$-graded vector space and $\mathrm{H}_\bullet$ be $\mathbb{Z}$-graded vector space. A *differential calculus* is the data

$$(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, [-,-], \mathbb{1}, \cap, B)$$

such that:

(1) $(\mathrm{H}^\bullet, \cup, [-,-], \mathbb{1})$ is a Gerstenhaber algebra;
(2) $\mathrm{H}_\bullet$ is a graded module over $(\mathrm{H}^\bullet, \cup)$ through the map

$$\cap : \mathrm{H}^m \otimes \mathrm{H}_n \to \mathrm{H}_{n-m},\ f \otimes \mu \mapsto f \cap \mu,$$

for $\mu \in \mathrm{H}_n$ and $f \in \mathrm{H}^m$, i.e. if we define $\iota_f(\mu) := f \cap \mu$, then $\iota_{f \cup g} = \iota_f \iota_g$;

(3) there exists a map $B : \mathrm{H}_\bullet \to \mathrm{H}_{\bullet+1}$ such that $B^2 = 0$, and

$$[\iota_f, L_g]_{gr} = \iota_{[f,g]},$$

where $L_g := [B, \iota_g]_{gr} = B\iota_g - (-1)^{|g|}\iota_g B$, for f, g homogenous elements of $\mathrm{H}^\bullet$.

Hochschild [12] introduced the cohomology theory of associative algebras. But the Hochschild cohomology ring of a $\mathbb{K}$-algebra is a Gerstenhaber algebra, which was first discovered by Gerstenhaber in [6]. Given a $\mathbb{K}$-algebra A, its Hochschild cohomology groups are defined as $\mathrm{HH}^n(A, A) \cong \mathrm{Ext}^n_{A^e}(A, A)$ for $n \geq 0$, where $A^e = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$ is the enveloping algebra of A. There exists a projective resolution of A as A^e-module, the so called normalized bar resolution $\overline{\mathrm{Bar}}_\bullet(A)$ which is given by $\overline{\mathrm{Bar}}_r(A) = A \otimes \overline{A}^{\otimes r} \otimes A$, where $\overline{A} = A/(\mathbb{K} \cdot 1_A)$, that is,

$$\overline{\mathrm{Bar}}_\bullet(A)\colon \cdots \to A \otimes \overline{A}^{\otimes r} \otimes A \xrightarrow{d_r} A \otimes \overline{A}^{\otimes r-1} \otimes A \to \cdots \to A \otimes \overline{A} \otimes A \xrightarrow{d_1} A^{\otimes 2} (\xrightarrow{\mu} A),$$

where the map $\mu : A \otimes A \to A$ is the multiplication of A, and the differential d_r is given by

$$\begin{aligned} d_r(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1}) &= a_0 a_1 \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \\ &+ \sum_{i=1}^{r-1} (-1)^i a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_r} \otimes a_{r+1} \\ &+ (-1)^r a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{r-1}} \otimes a_r a_{r+1}. \end{aligned}$$

The Hochschild cohomology complex is $C^\bullet(A, A) = \mathrm{Hom}_{A^e}(\overline{\mathrm{Bar}}_\bullet(A), A)$. Note that $C^r(A, A) = \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{\otimes r} \otimes A, A) \cong \mathrm{Hom}_{\mathbb{K}}(\overline{A}^{\otimes r}, A)$ for each $r \geq 0$. We also identify $C^0(A, A)$ with A. Thus $C^\bullet(A, A)$ has the following form:

$$C^\bullet(A, A)\colon A \xrightarrow{\delta^0} \mathrm{Hom}_{\mathbb{K}}(\overline{A}, A) \to \cdots \to \mathrm{Hom}_{\mathbb{K}}(\overline{A}^{\otimes r}, A) \xrightarrow{\delta^r} \mathrm{Hom}_{\mathbb{K}}(\overline{A}^{\otimes (r+1)}, A) \to \cdots.$$

It is not difficult to give the definition of $\delta^\bullet$, in fact, for any f in $\mathrm{Hom}_{\mathbb{K}}(\overline{A}^{\otimes r}, A)$, the map $\delta^r(f)$ is defined by

$$\begin{aligned} \delta^r(f)(\overline{a_1} \otimes \cdots \otimes \overline{a_{r+1}}) &= (-1)^{r-1} a_1 \cdot f(\overline{a_2} \otimes \cdots \otimes \overline{a_{r+1}}) \\ &+ \sum_{i=1}^{r} (-1)^{i+r-1} f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_{r+1}}) \\ &+ f(\overline{a_1} \otimes \cdots \otimes \overline{a_r}) \cdot a_{r+1}. \end{aligned}$$

Moreover, the cup product $f \cup g \in C^{m+n}(A, A) = \mathrm{Hom}_{\mathbb{K}}(\overline{A}^{\otimes (m+n)}, A)$ for $f \in C^m(A, A)$ and $g \in C^n(A, A)$ is given by

$$(f \cup g)(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+n}}) := g(\overline{a_1} \otimes \cdots \otimes \overline{a_n}) \cdot f(\overline{a_{n+1}} \otimes \cdots \otimes \overline{a_{m+n}}).$$

One can prove that this cup product induces a well-defined product in Hochschild cohomology

$$\cup\colon \mathrm{HH}^m(A, A) \times \mathrm{HH}^n(A, A) \longrightarrow \mathrm{HH}^{m+n}(A, A).$$

As a consequence, the graded $\mathbb{K}$-vector space $\mathrm{HH}^\bullet(A, A) = \bigoplus_{n \geq 0} \mathrm{HH}^n(A, A)$ is a graded commutative algebra with unit 1_A([6, Corollary 1]).

Furthermore, the Lie bracket is defined as follows. Let $f \in C^m(A, A)$ and $g \in C^n(A, A)$. If $m, n \geq 1$, then for $1 \leq i \leq m$, define $f \circ_i g \in C^{m+n-1}(A, A)$ by

$$(f \circ_i g)(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+n-1}}) := f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{g(\overline{a_i} \otimes \cdots \otimes \overline{a_{i+n-1}})} \otimes \overline{a_{i+n}} \otimes \cdots \otimes \overline{a_{m+n-1}}),$$

if $m \geq 1$ and $n = 0$, then $g \in A$ and define

$$(f \circ_i g)(\overline{a_1} \otimes \cdots \otimes \overline{a_{m-1}}) := f(\overline{a_1} \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{g} \otimes \overline{a_i} \otimes \cdots \otimes \overline{a_{m-1}}),$$

for any other case, set $f \circ_i g$ to be zero. Now we define

$$f \bar{\circ} g := \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f \circ_i g$$

and

$$[f,\, g] := f \bar{\circ} g - (-1)^{(m-1)(n-1)} g \bar{\circ} f.$$

Such bracket $[\ ,\ ]$ induces a well-defined Lie bracket in Hochschild cohomology

$$[\ ,\ ]\colon \mathrm{HH}^m(A, A) \times \mathrm{HH}^n(A, A) \longrightarrow \mathrm{HH}^{m+n-1}(A, A).$$

It is well known that $(\mathrm{HH}^\bullet(A, A), \cup, [\ ,\ ], 1_A)$ is a Gerstenhaber algebra ([6, Page 267]).

Meanwhile, the Hochschild chain complex is defined by $C_\bullet(A,A) := A \otimes_{A^e} \overline{\mathrm{Bar}}_\bullet(A)$. Note that $C_r(A,A) = A \otimes_{A^e} (A \otimes \overline{A}^{\otimes r} \otimes A) \cong A \otimes \overline{A}^{\otimes r}$, and the differential is given by

$$b(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r}) = \sum_{i=0}^{r-1} (-1)^i a_0 \otimes \cdots \otimes \overline{a_{i-1}} \otimes \overline{a_i a_{i+1}} \otimes \overline{a_{i+2}} \otimes \cdots \otimes \overline{a_r} + (-1)^r a_r a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{r-1}}.$$

For $f \in C^m(A,A)$ and $a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n} \in C_n(A,A)$, the cap product is

$$f \cap (a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_n}) = a_0 f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \otimes \overline{a_{m+1}} \otimes \cdots \otimes \overline{a_n},$$

while the Connes differential is defined by

$$B(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_r}) = \sum_{i=0}^{r} (-1)^{ir} 1 \otimes \overline{a_i} \otimes \cdots \otimes \overline{a_r} \otimes \overline{a_0} \otimes \cdots \otimes \overline{a_{i-1}}.$$

Originally, the differential calculus on Hochschild cohomology and homology of associative algebras was obtained by Daletskii-Gelfand-Tsygan [3]; see also Tamarkin-Tsygan [22].

Theorem 2.3 (Daletskii-Gelfand-Tsygan, [3])**.** *Let A be an associative algebra. Denote by $\mathrm{HH}^\bullet(A,A)$ and $\mathrm{HH}_\bullet(A,A)$ the Hochschild cohomology and homology of A respectively. Then*

$$(\mathrm{HH}^\bullet(A,A), \mathrm{HH}_\bullet(A,A), \cup, [-,-], \cap, B)$$

is a differential calculus, where $\cup$ is the cup product, $\cap$ is the cap product, $[-,-]$ is the Gerstenhaber Lie bracket and B is the Connes differential.

Let $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, [-,-], \mathbb{1}, \cap, B)$ be a differential calculus. Consider $\mathrm{H}'_\bullet := \mathrm{Hom}_{\mathbb{K}}(\mathrm{H}_{-\bullet}, \mathbb{K})$ the graded dual space of $\mathrm{H}_\bullet$. Then we can define the following two operations:

$$\kappa_f : \mathrm{H}'_\bullet \longrightarrow \mathrm{H}'_{\bullet - |f|}, \quad B' : \mathrm{H}'_\bullet \longrightarrow \mathrm{H}'_{\bullet+1},$$
$$\kappa_f(\Omega)(\mu) := (-1)^{|f||\Omega|}\Omega(\iota_f \mu), \quad B'(\Omega)(\mu) := (-1)^{|\Omega|}\Omega(B\mu)$$

for any arbitrary homogenous elements $f \in \mathrm{H}^\bullet$, $\mu \in \mathrm{H}_\bullet$ and $\Omega \in \mathrm{H}'_\bullet$ defined as well as the map $\cap' : \mathrm{H}^m \otimes \mathrm{H}'_n \to \mathrm{H}'_{n-m}$ by $f \cap' \Omega := \kappa_f(\Omega)$. Then we have the following proposition.

Proposition 2.4. *The data $(\mathrm{H}^\bullet, \mathrm{H}'_\bullet, \cup, [-,-], \mathbb{1}, \cap', B')$ is a differential calculus.*

Proof. Note that $(\mathrm{H}^\bullet, \cup, [-,-], \mathbb{1})$ is a Gerstenhaber algebra and ${B'}^2 = 0$. First, we have

$$\begin{aligned}
(\kappa_f \kappa_g)(\Omega)(\mu) &= \kappa_f(\kappa_g(\Omega))(\mu) = (-1)^{|f|(|\Omega|-|g|)}\kappa_g(\Omega)(\iota_f \mu) \\
&= (-1)^{-|f||g|+|\Omega|(|f|+|g|)}\Omega(\iota_g \iota_f \mu) = (-1)^{|f||g|+|\Omega|(|f|+|g|)}\Omega(\iota_{g\cup f}\mu) \\
&= (-1)^{|f||g|}\kappa_{g\cup f}(\Omega)(\mu) = \kappa_{f\cup g}(\Omega)(\mu),
\end{aligned}$$

for arbitrary homogenous elements $f, g \in \mathrm{H}^\bullet$, $\mu \in \mathrm{H}_\bullet$ and $\Omega \in \mathrm{H}'_\bullet$.

Next, we verify the condition (3) of the Definition 2.2. Let $L'_g := [B', \kappa_g] = B'\kappa_g - (-1)^{|g|}\kappa_g B'$, then we have

$$\begin{aligned}
L'_g(\Omega)(\mu) &= (B'\kappa_g - (-1)^{|g|}\kappa_g B')(\Omega)(\mu) \\
&= (-1)^{|\Omega|-|g|}\kappa_g(\Omega)(B\mu) - (-1)^{|g|+|g|(|\Omega|+1)}B'(\Omega)(\iota_g \mu) \\
&= (-1)^{|\Omega|-|g|+|g||\Omega|}\Omega(\iota_g(B\mu)) - (-1)^{|g||\Omega|+|\Omega|}\Omega(B(\iota_g \mu)) \\
&= -(-1)^{|\Omega|+|g||\Omega|}\Omega(L_g \mu),
\end{aligned}$$

and from this we get that

$$\begin{aligned}
[\kappa_f, L'_g](\Omega)(\mu) &= (\kappa_f L'_g - (-1)^{(|g|+1)|f|}L'_g \kappa_f)(\Omega)(\mu) \\
&= (-1)^{|f|(|\Omega|-|g|+1)}L'_g(\Omega)(\iota_f \mu) + (-1)^{|\Omega|(1+|g|)}\kappa_f(\Omega)(L_g \mu) \\
&= (-1)^{(|f|+|g|+1)|\Omega|+|f|(|g|+1)+1}\Omega(L_g \iota_f \mu) + (-1)^{(|f|+|g|+1)|\Omega|}\Omega(\iota_f L_g \mu) \\
&= (-1)^{(|f|+|g|+1)|\Omega|}\Omega([\iota_f, L_g]\mu) \\
&= (-1)^{(|f|+|g|+1)|\Omega|}\Omega(\iota_{[f,g]}\mu) \\
&= \kappa_{[f,g]}(\Omega)(\mu).
\end{aligned}$$

The proposition now follows. ∎

For finite dimensional associative algebras, we have the following result.

Corollary 2.5 (Menichi [19], Remark 17)**.** *Let A be a finite dimensional algebra, and denote by* $\mathrm{HH}^\bullet(A, A')$ *the Hochschild cohomology of A with value in $A' := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$. Then the data (*$\mathrm{HH}^\bullet(A, A)$, $\mathrm{HH}^\bullet(A, A')$, $\cup$, $[-, -]$, 1_A, $\cap'$, B'*) is a differential calculus.*

Proof. For finite dimensional algebra A, we have $\mathrm{HH}^\bullet(A, A') \cong \mathrm{HH}_\bullet(A, A)'$. Thus by Proposition 2.4, this corollary holds. □

3. Differential calculi with duality and Batalin-Vilkovisky algebras

In this section we consider a refined version of differential calculus which is called *differential calculus with duality.*

Definition 3.1 (Lambre [16])**.** A differential calculus $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, [-, -], \mathbb{1}, \cap, B)$ is called *a differential calculus with duality* if there exists an element (called volume form) $\eta \in \mathrm{H}_d$ for some integer d such that $B(\eta) = 0$ and the map

$$\partial(-) := - \cap \eta : \mathrm{H}^\bullet \to \mathrm{H}_{d-\bullet}$$

is an $\mathrm{H}^\bullet$-module isomorphism. In this situation, the map ∂ is called the *Van den Bergh-Poincaré duality.*

This structure is strongly related to the so-called *Batalin-Vilkovisky algebras.*

Definition 3.2. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra $(\mathrm{H}^\bullet, \cup, [-, -], \mathbb{1})$ with a linear map $\Delta : \mathrm{H}^\bullet \to \mathrm{H}^{\bullet-1}$ such that $\Delta^2 = 0$, $\Delta(\mathbb{1}) = 0$ and

$$[f, g] = (-1)^{|f|} \Big(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)\Big),$$

for arbitrary homogeneous elements $f, g \in \mathrm{H}^\bullet$.

Given a differential calculus with duality $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, [-, -], \mathbb{1}, \cap, B, \eta)$, the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet & \xrightarrow{\ \Delta\ } & \mathrm{H}^{\bullet-1} \\ \cong \downarrow \partial & & \cong \downarrow \partial \\ \mathrm{H}_{d-\bullet} & \xrightarrow{\ -B\ } & \mathrm{H}_{d-\bullet+1} \end{array}$$

defines an operator $\Delta := -\partial^{-1} \circ B \circ \partial$, which is called the *Batalin-Vilkovisky operator.* In particular, we have the following result due to Lambre which is important in constructing the Batalin-Vilkovisky algebra from differential calculus structures (cf. Lambre [16, Theorem 1.6]).

Theorem 3.3 (Lambre [16])**.** *Let $(\mathrm{H}^\bullet, \mathrm{H}_\bullet, \cup, [-, -], \mathbb{1}, \cap, B, \eta)$ be a differential calculus with duality. Then the quintuple $(\mathrm{H}^\bullet, \cup, [\ ,\ ], \mathbb{1}, \Delta)$ is a Batalin-Vilkovisky algebra.*

Proof. Take $\Delta := -\partial^{-1} B \partial$, and let $f \in \mathrm{H}^p,\ g \in \mathrm{H}^q,\ z \in \mathrm{H}_n$.
Claim:

$$\begin{aligned}[f, g] \cup \partial^{-1}(z) = (-1)^{p(q-1)} \Delta(g \cup (f \cup \partial^{-1}(z))) + (-1)^q f \cup (g \cup \Delta \partial^{-1}(z)) \\ - f \cup \Delta(g \cup \partial^{-1}(z)) + (-1)^{(p-1)(q-1)} g \cup \Delta(f \cup \partial^{-1}(z)).\end{aligned} \tag{3.1}$$

Indeed, by the equation $[\iota_g, L_f]_{gr} = \iota_{[g,f]}$, we can obtain the equation:

$$\begin{aligned}[f, g] \cap z = (-1)^{p-1} B(f \cap (g \cap z)) + f \cap (B(g \cap z)) \\ - (-1)^{(q-1)(p-1)} g \cap B(f \cap z) - (-1)^{(p-1)q} g \cap (f \cap B(z)).\end{aligned} \tag{3.2}$$

Since ∂ is an $\mathrm{H}^\bullet$-module isomorphism and apply ∂^{-1} to the equation (3.2), we can obtain this claim.

Let $z = \eta$, then $\partial^{-1}(z) = \partial^{-1}(\eta) = \mathbb{1} \in \mathrm{H}^0$, so

$$\Delta(\mathbb{1}) = -\partial^{-1} B \partial \partial^{-1}(\eta) = -\partial^{-1} B(\eta) = 0,$$

and

$$\Delta \circ \Delta = \partial^{-1} B \partial \partial^{-1} B \partial = \partial^{-1} B B \partial = 0.$$

By the equation (3.1), we have that

$$[f, g] = (-1)^{p(q-1)} \Delta(g \cup f) - f \cup \Delta(g) + (-1)^{(p-1)(q-1)} g \cup \Delta(f)$$

$$= (-1)^{p(q-1)+pq}\Delta(f\cup g) - f\cup\Delta(g) + (-1)^{(p-1)(q-1)+q(p-1)}\Delta(f)\cup g$$
$$= (-1)^{p}(\Delta(f\cup g) - \Delta(f)\cup g - (-1)^{p} f\cup\Delta(g)).$$

Thus we have the theorem. □

Note that not all associative algebras admit the structure of differential calculus with duality on its Hochschild (co)homology. In the literature, there are two main classes of associative algebras having this property: *Calabi-Yau algebras* and finite dimensional *symmetric algebras*. The notion of Calabi-Yau algebras is introduced by Ginzburg [11]. More precisely, an algebra A is called *Calabi-Yau algebra of dimension d* if A has a finite length resolution of finitely generated projective A^e-modules, and there is an isomorphism $\mathrm{RHom}_{A^e}(A, A^e) \cong A[-d]$ in the derived category of A^e-modules. The following result is due to Ginzburg [11, Theorem 3.4.3]; see also Lambre [16].

Theorem 3.4 (Ginzburg [11])**.** *Let A be a Calabi-Yau algebra of dimension d. Then*

$$(\mathrm{HH}^{\bullet}(A, A), \mathrm{HH}_{\bullet}(A, A), \cup, [\ ,\ ], 1_A, \cap, B)$$

is differential calculus with duality, and therefore there is a Batalin-Vilkovisky algebra on $\mathrm{HH}^{\bullet}(A, A)$.

Another version of differential calculus with duality is defined on the Hochschild (co)homology of symmetric algebras. Recall a finite dimensional algebra A is *symmetric* if there exists a nondegenerate bilinear form $\langle -, -\rangle : A\otimes A \to \mathbb{K}$ such that $\langle ab, c\rangle = \langle a, bc\rangle$ and $\langle a, b\rangle = \langle b, a\rangle$ for arbitrary elements $a, b, c \in A$.

Theorem 3.5 (Tradler [24], Menichi [18])**.** *Let A be a symmetric algebra. Then*

$$(\mathrm{HH}^{\bullet}(A, A), \mathrm{HH}^{\bullet}(A, A'), \cup, [-, -], 1_A, \cap', B')$$

is differential calculus with duality, and therefore there is a Batalin-Vilkovisky algebra on $\mathrm{HH}^{\bullet}(A, A)$.

Following this line, there are some interesting relevant works. Indeed, there is a "twisted" version of Theorem 3.4 and Theorem 3.5, which are obtained recently by Kowalzig and Krähmer [15] and Lambre, Zhou and Zimmermann [17] respectively. More generally, Menichi [18] considered the algebras over a cyclic operad with multiplication and showed their cohomology gives rise to a Batalin-Vilkovisky algebra structure.

4. Hochschild (co)homology of A_∞-algebras

In this section, we review the definitions of Hochschild (co)homology for A_∞-algebras, and give a proof of the Gerstenhaber algebra structure on their Hochschild cohomology. Let us start with the definition of A_∞-algebras.

Definition 4.1 (Stasheff [21])**.** An A_∞*-algebra* over $\mathbb{K}$ is a graded vector space $A := \bigoplus_{i\in\mathbb{Z}} A_i$ with $\mathbb{K}$-linear maps $m_n : A^{\otimes n} \to A$ of degree $n-2$ for each $n \geq 1$, called the A_∞-operators, satisfying the following A_∞-relations

$$\sum_{\substack{n=j+k+l\geq 1\\ j,l\geq 0, k\geq 1}} (-1)^{j+kl} m_{j+1+l}(id^{\otimes j}\otimes m_k \otimes id^{\otimes l}) = 0,$$

that is,

$$\sum_{k=1}^{n}\sum_{j=0}^{n-k}(-1)^{k\eta_j+kj+j+kl} m_{j+1+l}(a_1, \cdots, a_j, m_k(a_{j+1}, \cdots, a_{j+k}), a_{j+k+1}, \cdots, a_n) = 0, \tag{4.1}$$

where $\eta_j = |a_1| + \cdots + |a_j| + j$ and $n = j+k+l$.

Recall that the *bar construction* $B(A)$ of A is the tensor coalgebra $\bigoplus_{n=0}^{\infty}(sA)^{\otimes n}$ with the coproduct

$$\Delta[a_1, \cdots, a_n] := \sum_{i=0}^{n}[a_1, \cdots, a_i]\otimes[a_{i+1}, \cdots, a_n],$$

where $[a_1, \cdots, a_n]$ denotes the element $(sa_1)\otimes\cdots\otimes(sa_n) \in (sA)^{\otimes n}$, and s is the suspension with degree $|s| = 1$. For simplicity, we also write $[a_{1,n}] := (sa_1)\otimes\cdots\otimes(sa_n)$ and $a_{1,n} := a_1\otimes\cdots\otimes a_n$ with some abuse of notation.

Given an A_∞-algebra $(A, \{m_n\}_{n\geq 1})$, we denote by $\mathrm{C}^{\bullet}(A, A) := \mathcal{H}om(B(A), A)$ the Hochchild cochain of A. Notice that we consider the graded-version $\mathcal{H}om$ and total degree. The standard algebraic structures on

Hochchild cochain complex (and induces on Hochchild cohomology) of A_∞-algebra may obtain from its *brace algebra* (see Gerstenhaber-Voronov [7]). Let us recall that the braces are the maps

$$\mathrm{C}^\bullet(A,A)\times\cdots\times\mathrm{C}^\bullet(A,A)\to\mathrm{C}^\bullet(A,A),\ (f,f_1,\cdots,f_k)\mapsto f\{f_1,\cdots,f_k\}$$

given as follows: for any homogeneous elements $[a_1,\cdots,a_n]\in B(A)$, $f\{f_1,\cdots,f_k\}[a_1,\cdots,a_n]$ is given by

$$\sum(-1)^{\sum\limits_{l=1}^{k}\eta_{i_l}(|f_l|+1)}f[a_1,\cdots,f_1[a_{i_1+1},\cdots,a_{j_1}],\cdots,f_k[a_{i_k+1},\cdots,a_{j_k}],\cdots,a_n],$$

where the sum run over all $0\le i_1\le j_1\le\cdots\le i_k\le j_k\le n$, and $\eta_i=\sum_{s=1}^{i}|a_s|+i$. Clearly, the degree of $f\{f_1,\cdots,f_k\}$ satisfies $|f\{f_1,\cdots,f_k\}|=|f|+|f_1|+\cdots+|f_k|+k$. If $k=1$, then we denote that $f\bar{\circ}g:=f\{g\}$, actually, it is just the pre-Lie operator introduced by Gerstenhaber [6]:

$$f\{g\}[a_{1,n}]=\sum_{0\le i\le j\le n}(-1)^{\eta_i(|g|+1)}f[a_{1,i},g[a_{i+1,j}],a_{j+1,n}].$$

For any homogeneous elements $f,g\in\mathrm{C}^\bullet(A,A)$, the *Gerstenhaber Lie bracket* of f,g is given by

$$[f,g]:=f\bar{\circ}g-(-1)^{(|f|+1)(|g|+1)}g\bar{\circ}f.$$

By [Getzler [9], Lemma 1.2], with the formula

$$(f\bar{\circ}g)\bar{\circ}h-f\bar{\circ}(g\bar{\circ}h)=f\{g,h\}+(-1)^{(|g|+1)(|h|+1)}f\{h,g\},\tag{4.2}$$

we know that $(\mathrm{C}^\bullet(A,A),[-,-])$ is a graded Lie algebra of degree 1. Note that the space of coderivations $\mathrm{Coder}(B(A))$ is a graded Lie algebra with bracket the graded commutator, and there is an isomorphisms of graded Lie algebra between $s\mathrm{C}^\bullet(A,A)$ and $\mathrm{Coder}(B(A))$ (see [Getzler-Jones [8], Proposition 1.2]). Since $B(A)$ is cofree, the coderivation is determined by its corestriction to degree 1, and we have the following equivalent definition:

Definition 4.2 (Stasheff [21])**.** An A_∞*-algebra* A is a graded vector space A equipped with a codifferential

$$D:B(A)\to B(A)$$

(i.e. a coderivation of degree $|D|=-1$ with $D\circ D=0$ and $D(1)=0$).

In the following, if $(A,\{m_n\}_{n\ge1})$ is an A_∞-algebra in Definition 4.1, then we denote by (A,m) its associated A_∞-algebra in Definition 4.2 with $|m|=-2$. In fact, we consider the following composition: $\mathrm{Coder}(B(A))\cong s\mathrm{C}^\bullet(A,A)\xrightarrow{s^{-1}}\mathrm{C}^\bullet(A,A)$, a codifferential $D\in\mathrm{Coder}(B(A))$ of degree -1 corresponds to a Hochschild cochain $m\in\mathrm{C}^\bullet(A,A)$ of degree -2. The condition $D^2=0$ corresponds to $m\bar{\circ}m=0$ which can be translated to the equation (4.1) in the definition of A_∞-algebras. The *cup product* on $\mathrm{C}^\bullet(A,A)$ is given by

$$f\cup g:=(-1)^{|g|(|f|+1)}m\{g,f\}.$$

for any homogeneous elements $f,g\in\mathrm{C}^\bullet(A,A)$. Clearly, $|f\cup g|=|f|+|g|$.

Definition 4.3. Let (A,m) be an A_∞-algebra. An A_∞*-bimodule* M of A is a graded vector space with operations

$$b_{i,j}:A^{\otimes i}\otimes M\otimes A^{\otimes j}\longrightarrow M,\ i,j\ge0$$

of degree $i+j-1$ such that for any integers k and l, any homogeneous element $\omega\in M$,

$$\begin{aligned}0=&\sum_{i=1}^{k}\sum_{j=1}^{k-i+1}(-1)^{\theta_1}b_{k-i+1,l}(a_{1,j-1},m_i(a_{j,i+j-1}),\cdots,w,\cdots,a_{k+l})\\&+\sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{\theta_2}b_{k-i,l-j}(a_{1,k-i},b_{i,j}(a_{k-i+1},\cdots,w,\cdots,a_{k+j}),\cdots,a_{k+l})\\&+\sum_{i=1}^{l}\sum_{j=1}^{l-i+1}(-1)^{\theta_3}b_{k,l-i+1}(a_1,\cdots,w,\cdots,a_{k+j-1},m_i(a_{k+j,k+i+j-1}),\cdots,a_{k+l}),\end{aligned}$$

where

$$\begin{aligned}\theta_1&=i\varepsilon_{j-1}+j-1+i(k+l-i-j),\\\theta_2&=(i+j-1)\varepsilon_{k-i}+k-i+(l-j)(i+j-1),\end{aligned}$$

$$\theta_3 = i(\varepsilon_{k+j-1} + |w|) + k - j + i(l - i - j + 1)$$

and $\varepsilon_i := \sum_{l=1}^{i} |a_l|$. We denote by (M, b) the A_∞-bimodule M.

Remark 4.4. An A_∞-algebra A itself is naturally an A_∞-bimodule of A. Meanwhile, for an A_∞-bimodule M of A, its dual M' of M is still an A_∞-bimodule of A.

Now we recall the definition of the Hochschild cohomology of A_∞-algebras with value in an A_∞-bimodules.

Definition 4.5. Let (A, m) be an A_∞-algebra and (M, b) be an A_∞-bimodule over (A, m). Then the *Hochschild cohomology* $\mathrm{HH}^\bullet(A, M)$ *of A with value in M* is given by the cohomology of the Hochschild cochain complex

$$(\mathrm{C}^\bullet(A, M), \delta)$$

where $\mathrm{C}^\bullet(A, M) := \mathcal{H}om(B(A), M)$, the differential $\delta(f) := b\bar{\circ}f - (-1)^{|f|+1} f\bar{\circ}m$, for $f \in \mathcal{H}om(B(A), M)$ and

$$b \circ f[a_1, \cdots, a_n] := \sum_{0 \le i \le j \le n} (-1)^{\eta_i(|f|+1)} b_{i,n-j}[a_1, \cdots, f[a_{i+1}, \cdots, a_j], \cdots, a_n],$$
$$f \circ m[a_1, \cdots, a_n] := \sum_{0 \le i \le j \le n} (-1)^{\eta_i} f[a_1, \cdots, m[a_{i+1}, \cdots, a_j], \cdots, a_n],$$

where $\eta_i = \sum_{r=1}^{i} (|a_r| + 1)$. Especially, when $M = A$, $\delta(f) = [m, f] = m\bar{\circ}f - (-1)^{|f|+1} f\bar{\circ}m$.

An important feature of the Hochschild cochain complex $\mathrm{C}^\bullet(A, A)$ of an A_∞-algebra A is that it also admits an A_∞-algebra structure (see [Getzler [9], Proposition 1.7]). From this A_∞-algebra structure we have the following equations:

$$m\{[m, f], g\} + (-1)^{|f|+1} m\{f, [m, g]\} + [m, m\{f, g\}] = 0, \tag{4.3}$$
$$\begin{aligned} &m\{[m, f], g, h\} + (-1)^{|f|+1} m\{f, [m, g], h\} + (-1)^{|f|+|g|} m\{f, g, [m, h]\} \\ &+ m\{m\{f, g\}, h\} + (-1)^{|f|+1} m\{f, m\{g, h\}\} + [m, m\{f, g, h\}] = 0, \end{aligned} \tag{4.4}$$

for any homogenous elements $f, g, h \in \mathrm{C}^\bullet(A, A)$. In particular, we have

Proposition 4.6. *Let (A, m) be an A_∞-algebra, then $(C^\bullet(A, A), [\ ,\ ], \delta)$ is a differential graded Lie algebra.*

Proof. By the equation (4.2), $(C^\bullet(A, A), [\ ,\ ])$ is a graded Lie algebra. We only need to show that

$$\delta[f, g] = [\delta f, g] + (-1)^{|f|+1}[f, \delta g],$$

for any $f, g \in C^\bullet(A, A)$. Equivalently, we need to prove

$$[m, [f, g]] = [[m, f], g] + (-1)^{|f|+1}[f, [m, g]].$$

Then the proposition follows by the graded Jacobi identity. □

Lemma 4.7. *Let $f, g \in C^\bullet(A, A)$, then we have that*

$$\delta f\bar{\circ}g - \delta(f\bar{\circ}g) + (-1)^{|f|+1} f\bar{\circ}\delta g = m\{f, g\} + (-1)^{(|f|+1)(|g|+1)} m\{g, f\}.$$

Proof. Using the equation (4.2), we have that

$$\begin{aligned} LHS &= [m, f]\bar{\circ}g - [m, f\bar{\circ}g] + (-1)^{|f|+1} f\bar{\circ}[m, g] \\ &= (m\bar{\circ}f)\bar{\circ}g - (-1)^{|f|+1}(f\bar{\circ}m)\bar{\circ}g - m\bar{\circ}(f\bar{\circ}g) \\ &\quad + (-1)^{|f|+|g|}(f\bar{\circ}g)\bar{\circ}m - (-1)^{|f|} f\bar{\circ}(m\bar{\circ}g) - (-1)^{|f|+|g|} f\bar{\circ}(g\bar{\circ}m) \\ &= [(m\bar{\circ}f)\bar{\circ}g - m\bar{\circ}(f\bar{\circ}g)] + (-1)^{|f|}[(f\bar{\circ}m)\bar{\circ}g - f\bar{\circ}(m\bar{\circ}g)] \\ &\quad + (-1)^{|f|+|g|}[(f\bar{\circ}g)\bar{\circ}m - f\bar{\circ}(g\bar{\circ}m)] \\ &= m\{f, g\} + (-1)^{(|f|+1)(|g|+1)} m\{g, f\} + (-1)^{|f|}[f\{m, g\} + (-1)^{|g|+1} f\{g, m\}] \\ &\quad + (-1)^{|f|+|g|}[f\{g, m\} + (-1)^{|g|+1} f\{m, g\}] \\ &= RHS. \end{aligned}$$

Hence we have the lemma. □

Corollary 4.8. *Let $f, g \in C^\bullet(A,A)$, then we have that*

$$\delta f \bar{\circ} g - \delta(f \bar{\circ} g) + (-1)^{|f|+1} f \bar{\circ} \delta g = (-1)^{|f|+1}(f \cup g - (-1)^{|f||g|} g \cup f).$$

Proof. By the Lemma 4.7, we have that

$$\begin{aligned} LHS &= (-1)^{|f|(|g|+1)} g \cup f - (-1)^{|f|} f \cup g \\ &= RHS \end{aligned}$$

and thus we have the corollary. □

Proposition 4.9. *Let (A, m) be an A_∞-algebra, then $(C^\bullet(A,A), \cup, \delta)$ is a differential graded algebra which is commutative up to homotopy.*

Proof. By the equation (4.3), we have that

$$\delta(f \cup g) = \delta f \cup g + (-1)^{|f|} f \cup \delta g,$$

for any $f, g \in C^\bullet(A,A)$. According to equation (4.4), we can obtain that

$$(f \cup g) \cup h - f \cup (g \cup h) = (-1)^\alpha(\delta(m\{h, g, f\}) + m\{\delta h, g, f\} + (-1)^{|h|-1} m\{h, \delta g, f\} + (-1)^{|g|+|h|} m\{h, g, \delta f\}),$$

where $\alpha = (-1)^{|h|(|g|-1)+(|g|+|h|)(|f|-1)}$. By the Corollary 4.8, the cup product is commutative up to homotopy and hence we have the proposition. □

In [6], Gerstenhaber proved that there is a Gerstenhaber algebra on Hochschid cohomology of an associative algebra. Analogously, there is a similar Gerstenhaber algebra on Hochschid cohomology of an A_∞ algebra, which was first observed by Getzler-Jones in [10].

Lemma 4.10. *Let (A, m) be an A_∞-algebra, then we have that*

$$\begin{aligned} f\{f_1, f_2\} \bar{\circ} f_3 = f\{f_1, f_2, f_3\} &+ (-1)^{(|f_2|-1)(|f_3|-1)} f\{f_1, f_3, f_2\} + (-1)^{(|f_1|+|f_2|)(|f_3|-1)} f\{f_3, f_1, f_2\} \\ &+ (-1)^{(|f_2|-1)(|f_3|-1)} f\{f_1 \bar{\circ} f_3, f_2\} + f\{f_1, f_2 \bar{\circ} f_3\} \end{aligned} \tag{4.5}$$

$$\begin{aligned} (f \bar{\circ} f_1)\{f_2, f_3\} = f\{f_1, f_2, f_3\} &+ (-1)^{(|f_1|-1)(|f_2|-1)} f\{f_2, f_1, f_3\} + (-1)^{(|f_2|+|f_3|)(|f_1|-1)} f\{f_2, f_3, f_1\} \\ &+ f\{f_1 \bar{\circ} f_2, f_3\} + (-1)^{(|f_1|-1)(|f_2|-1)} f\{f_2, f_1 \bar{\circ} f_3\} + f \bar{\circ} (f_1\{f_2, f_3\}) \end{aligned} \tag{4.6}$$

for any $f, f_1, f_2, f_3 \in C^\bullet(A,A)$.

Proof. It follows by straight-forward computation. □

Theorem 4.11 ([10], [7])**.** *Let (A, m) be an A_∞-algebra, then $(\mathrm{HH}^\bullet(A,A), [\ ,\ ], \cup)$ is a Gerstenhaber algebra.*

Now let us recall the definition of Hochschild homology of an A_∞-algebra and the *Connes differential*; for more details, we refer to Getzler-Jones [8].

Definition 4.12. Let (A, m) be an A_∞-algebra, and set $\mathrm{C}_\bullet(A,A) := A \otimes B(A)$. Then the *Hochschild homology* $\mathrm{HH}_\bullet(A,A)$ of A is the homology of the Hochschild chain complex $(\mathrm{C}_\bullet(A,A), b)$ where the differential b is given by

$$\begin{aligned} b(a_0[a_1, \cdots, a_n]) \;\; := & \sum_{0 \le j \le i \le n} (-1)^{\eta_i(\eta_n - \eta_i)} m[a_{i+1}, \cdots, a_n, a_0, \cdots, a_j][a_{j+1}, \cdots, a_i] \\ & + \sum_{0 \le i \le j \le n} (-1)^{\eta_i} a_0[a_1, \cdots, a_i, m[a_{i+1}, \cdots, a_j], a_{j+1}, \cdots, a_n], \end{aligned}$$

where $\eta_i = \sum_{s=0}^{i}(|a_s| + 1)$.

An element $e \in A_0$ is called a *strict unit* if $m_n[a_1, \cdots, a_i, e, a_{i+1}, \cdots, a_{n-1}] = 0$, for $n \neq 2$; and $m_2[e, a] = (-1)^{|a|} m_2[a, e] = a$.

Definition 4.13. Suppose that (A, m) is an A_∞-algebra with a strict unit. Then the *Connes differential*

$$B : \mathrm{C}_\bullet(A,A) \to \mathrm{C}_\bullet(A,A),$$

is given by

$$B(a_0[a_1, \cdots, a_n]) = \sum_{i=0}^{n} (-1)^{\eta_i(\eta_n - \eta_i)} e[a_{i+1}, \cdots, a_n, a_0, a_1, \cdots, a_i],$$

for any $a_0[a_1, \cdots, a_n] \in \mathrm{C}_\bullet(A,A)$, where $\eta_i = \sum_{s=0}^{i}(|a_s| + 1)$.

It is not difficult to see $B^2 = 0$ up to homotopy and $Bb + bB = 0$.

Remark 4.14. If we define

$$t(a_0[a_1, \cdots, a_n]) = (-1)^{\eta_0(\eta_n - \eta_0)} a_1[a_2, \cdots, a_n, a_0],$$

and

$$T(e[a_0, \cdots, a_n, e, e]) = (-1)^{\eta_0(\eta_n - \eta_0)} e[a_1, \cdots, a_n, e, e, a_0],$$

then $B^2 = 0$ up to homotopy, that is,

$$B_{n+1}B_n = b_{n+3}s_n + s_{n-1}b_n,$$

where

$$\begin{aligned}
s_n(a_0[a_1, \cdots, a_n]) &= \sum_{j=0}^{n+1} T^j (1 \otimes \sum_{i=0}^{n} t^i \otimes 1 \otimes 1)(e[a_0, \cdots, a_n, e, e]) \\
&= \sum_{j=0}^{n+1} T^j (\sum_{i=0}^{n} (-1)^{\eta_{i-1}(\eta_n - \eta_{i-1})} e[a_i, \cdots, a_n, a_0, \cdots, a_{i-1}, e, e]) \\
&= \sum_{i=0}^{n} (-1)^{\eta_{i-1}(\eta_n - \eta_{i-1})} \{ e[a_i, \cdots a_n, a_0, \cdots, a_{i-1}, e, e] \\
&\quad + \sum_{j=1}^{n-i} (-1)^{(\eta_n - 1)(\eta_{i+j-1} - \eta_{i-1})} e[a_{i+j}, \cdots a_n, a_0, \cdots, a_{i-1}, e, e, a_i, \cdots, a_{i+j-1}] \\
&\quad + (-1)^{\eta_{n-1}(|a_n|+1)} e[a_0, \cdots, a_{i-1}, e, e, a_i \cdots, a_n] \\
&\quad + \sum_{j=n-i+2}^{n+1} (-1)^{\eta_{i+j-n-2}(\eta_n - \eta_{i+j-n-2})} e[a_{i+j-n-1}, \cdots, a_{i-1}, e, e, a_i, \cdots, a_n, a_0, \cdots, a_{i+j-n-1}] \}.
\end{aligned}$$

5. Proof of the main theorem

5.1. **Differential calculus operators.** In the last part of previous section, we recalled the definition of the Connes differential. Now we give another two differential calculus operators: the *contraction* (or *cap product*) and the *Lie derivative*.

Given an A_∞-algebra (A, m). For any homogeneous elements $x = a_0[a_1, \ldots, a_t] \in \mathrm{C}_\bullet(A, A)$ and $f_1, \ldots, f_k \in \mathrm{C}^\bullet(A, A)$, the *contraction* $\{f_1, \cdots, f_k\} \cap x$ is defined by

$$\begin{aligned}
\sum (-1)^{\eta_t(\eta_n - \eta_t) + \sum_{r=1}^{k} (\eta_n - \eta_t + \eta_{i_r} - 1)(|f_r|+1)} & m[a_{t+1}, \cdots, a_n, a_0, a_1, \cdots, \\
& f_1[a_{i_1+1}, \cdots, a_{j_1}], \cdots, f_k[a_{i_k+1}, \cdots, a_{j_k}], \cdots, a_s][a_{s+1}, \cdots, a_t],
\end{aligned}$$

where the sum runs over all $0 \leq s \leq t \leq n$ and $0 \leq i_1 \leq j_1 \leq \cdots \leq i_k \leq j_k \leq n$.

Definition 5.1. Assume that (A, m) is an A_∞-algebra. For any homogeneous element $f \in \mathrm{C}^\bullet(A, A)$, the *cap product* ι is given by

$$\iota_f(x) := f \cap x,$$

for any homogeneous elements $x \in \mathrm{C}_\bullet(A, A)$.

In fact, the cap product is well-defined in homology level by the following lemma.

Lemma 5.2. *Assume that (A, m) is an A_∞-algebra. Then we have*

$$\iota_{\delta f} = [b, \iota_f]_{gr} := b \circ \iota_f - (-1)^{|f|} \iota_f \circ b,$$

for any homogenous element $f \in \mathrm{C}^\bullet(A, A)$.

Proof. Denote

$$\begin{aligned}
\theta &= (\eta_k - \eta_t)\eta_t + (\eta_k - \eta_t + \eta_{i_1} - 1)|f|, \\
\xi &= (\eta_k - \eta_t)\eta_t + (\eta_k - \eta_t + \eta_{i_1} - 1)(|f| + 1).
\end{aligned}$$

Let $x := a_0[a_{1,k}]$. Then we have that

$$\iota_{\delta f} x = \iota_{[m,f]} x$$

$$= \sum(-1)^{\theta} m[a_{t+1,k}, a_0, \cdots, [m,f][a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}]$$
$$= \sum(-1)^{\theta+(\eta_{i_2}-\eta_{i_1})(|f|-1)} m[a_{t+1,k}, a_0, \cdots, m[a_{i_1+1}, \cdots, f[a_{i_2+1,j_2}], \cdots, a_{j_1}], \cdots, a_s][a_{s+1,t}]$$
$$+ \sum(-1)^{\theta+\eta_{i_2}-\eta_{i_1}+|f|} m[a_{t+1,k}, a_0, \cdots, f[a_{i_1+1}, \cdots, m[a_{i_2+1,j_2}], \cdots, a_{j_1}], \cdots, a_s][a_{s+1,t}].$$

We also have that

$$b\iota_f x = b(\sum(-1)^{\xi} m[a_{t+1,k}, a_0, \cdots, f[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}])$$
$$= \sum(-1)^{\xi+\eta_n-\eta_t+\eta_{i_2}+|f|} m[a_{t+1,k}, a_0, \cdots, f[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1}, \cdots, m[a_{i_2+1,j_2}], \cdots, a_t]$$
$$+ \sum(-1)^{\xi+(\eta_n-\eta_t+\eta_{i_2}+|f|)(\eta_t-\eta_{i_2})} m[a_{i_2+1,t}, m[a_{t+1,k}, a_0, \cdots, f[a_{i_1+1,j_1}], \cdots, a_s], \cdots, a_{j_2}][a_{j_2+1,i_2}]$$

and

$$\iota_f bx = \iota_f(\sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}} m[a_{i_1+1,k}, a_{0,j_1}][a_{j_1+1,i_1}] + \sum(-1)^{\eta_{i_1}} a_0[a_{1,i_1}, m[a_{i_1+1,j_1}], a_{j_1+1,k}])$$
$$= \sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}+(\eta_{i_1}-\eta_t)\eta_n+(\eta_k-\eta_t+\eta_{i_2})(|f|-1)} m[a_{t+1}, \cdots,$$
$$m[a_{i_1+1,k}, a_0, \cdots, a_{j_1}], \cdots, f[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}]$$
$$+ \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|f|-1)} m[a_{t+1,k}, a_0, \cdots,$$
$$f[a_{i_2+1,j_2}], \cdots, m[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}]$$
$$+ \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|f|-1)} m[a_{t+1,k}, a_0, \cdots,$$
$$f[a_{i_2+1}, \cdots, m[a_{i_1+1,j_1}], \cdots, a_{j_2}], \cdots, a_s][a_{s+1,t}]$$
$$+ \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2})(|f|-1)} m[a_{t+1,k}, a_0, \cdots,$$
$$m[a_{i_1+1,j_1}], \cdots, f[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}]$$
$$+ \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|f|-1)} m[a_{t+1,k}, a_0, \cdots,$$
$$f[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1}, \cdots, m[a_{i_1+1,j_1}], \cdots, a_t]$$
$$+ \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t-1)\eta_t+(\eta_k-\eta_t+\eta_{i_2})(|f|-1)} m[a_{t+1}, \cdots,$$
$$m[a_{i_1+1,j_1}], \cdots, a_k, a_0, \cdots, f[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}].$$

Then we can obtain that

$$\iota_{\delta f} - b\iota_f + (-1)^{|f|}\iota_f b = (-1)^{\xi}(m\bar{\circ}m)[a_{t+1,k}, a_0, \cdots, f[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}]$$
$$= 0.$$

Hence we prove this lemma. □

Proposition 5.3. *Let (A, m) be an A_∞-algebra, then the ($\mathrm{HH}_\bullet(A,A)$,$\cap$) is a graded module over ($\mathrm{HH}^\bullet(A,A)$,$\cup$), that is to say, there exists a linear map*

$$\cap : \mathrm{HH}^p(A,A) \otimes \mathrm{HH}_n(A,A) \longrightarrow \mathrm{HH}_{n-p}(A,A)$$
$$\varphi \otimes x \mapsto \varphi \cap x := \iota_\varphi(x),$$

satisfies

$$\iota_{\varphi\cup\psi} = \iota_\varphi \iota_\psi,$$

for $\varphi \in \mathrm{HH}^\bullet(A,A)$, $\psi \in \mathrm{HH}^\bullet(A,A)$ and $x := a_0[a_{1,n}] \in \mathrm{HH}_\bullet(A,A)$.

Proof. It suffices to verify the identity

$$\iota_{\varphi\cup\psi} - \iota_\varphi\iota_\psi = (-1)^{|\varphi||\psi|}[\iota_{\{\psi,\delta\varphi\}} - (-1)^{|\psi|}\iota_{\{\varphi,\delta\psi\}} + (-1)^{|\varphi|}\iota_{\{\psi,\varphi\}}b + (-1)^{|\psi|}b\iota_{\{\psi,\varphi\}}]. \quad (5.1)$$

We first compute the terms in equation (5.1) one by one. Denote

$$\xi := (\eta_k - \eta_t)\eta_t + (\eta_k - \eta_t + \eta_{i_1} - 1)(|\varphi| + |\psi| + 1).$$

Let $x := a_0[a_{1,k}]$. Then we first have

$$\iota_{\varphi\cup\psi}x = \sum(-1)^{\xi} m[a_{t+1,k}, a_0, \cdots, \varphi\cup\psi[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}]$$
$$= \sum(-1)^{\xi+(\eta_{i_2}-\eta_{i_1})(|\psi|-1)+(\eta_{i_3}-\eta_{i_1})(|\varphi|-1)+\psi(\varphi-1)} m[a_{t+1,k}, a_0, \cdots,$$
$$m[a_{i_1+1}, \cdots, \psi[a_{i_2+1,j_2}], \cdots, \psi[a_{i_3+1,j_3}], \cdots, a_{j_1}], \cdots, a_s][a_{s+1,t}].$$

Meanwhile, let

$$\theta := (\eta_k - \eta_t)\eta_t + (\eta_k - \eta_t + \eta_{i_1} - 1)(|\varphi| - 1).$$

Then we have that

$$\begin{aligned}
\iota_\varphi \iota_\psi x &= \iota_\varphi(\sum(-1)^{\theta} m[a_{t+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}]) \\
&= \sum(-1)^{\theta+(\eta_t-\eta_{i_2})(\eta_k-\eta_t+\eta_{i_2}+|\psi|)+(\eta_k-\eta_{i_2}+\eta_{i_3}+|\psi|-1)(|\varphi|-1)} m[a_{i_2+1,t}, m[a_{t+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_1+1,j_1}], \cdots, a_s], \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}].
\end{aligned}$$

Furthermore, we take

$$\zeta := (\eta_k - \eta_t)\eta_t + (\eta_k - \eta_t + \eta_{i_1} - 1)(|\psi| - 1) + (\eta_k - \eta_t + \eta_{i_2} - 1)|\varphi|,$$

and we have that

$$\begin{aligned}
\iota_{\{\psi,\delta\varphi\}} x &= \{\psi, \delta\varphi\} \cap x \\
&= \sum(-1)^{\zeta} m[a_{t+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, \delta\varphi[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}] \\
&= \sum(-1)^{\zeta+(\eta_{i_3}-\eta_{i_2})(|\varphi|-1)} m[a_{t+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, \\
&\qquad m[a_{i_2+1}, \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_{j_2}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\zeta+\eta_{i_3}-\eta_{i_2}+|\varphi|} m[a_{t+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, \\
&\qquad \varphi[a_{i_2+1}, \cdots, m[a_{i_3+1,j_3}], \cdots, a_{j_2}], \cdots, a_s][a_{s+1,t}].
\end{aligned}$$

Now we denote

$$\tau := (\eta_k - \eta_t)\eta_t + (\eta_k - \eta_t + \eta_{i_1} - 1)|\psi| + (\eta_k - \eta_t + \eta_{i_2} - 1)(|\varphi| - 1),$$

and we have that

$$\begin{aligned}
\iota_{\{\delta\psi,\varphi\}} x &= \{\delta\psi, \varphi\} \cap x \\
&= \sum(-1)^{\tau} m[a_{t+1,k}, a_0, \cdots, \delta\psi[a_{i_1+1,j_1}], \cdots, \varphi[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}] \\
&= \sum(-1)^{\tau+(\eta_{i_3}-\eta_{i_1})(|\psi|-1)} m[a_{t+1,k}, a_0, \cdots, m[a_{i_1+1}, \cdots, \\
&\qquad \psi[a_{i_3+1}, \cdots, a_{j_3}], \cdots, a_{j_1}], \cdots, \varphi[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\tau+\eta_{i_3}-\eta_{i_1}+|\psi|} m[a_{t+1,k}, a_0, \cdots, \psi[a_{i_1+1}, \cdots, \\
&\qquad m[a_{i_3+1}, \cdots, a_{j_3}], \cdots, a_{j_1}], \cdots, \varphi[a_{i_2+1,j_2}], \cdots, a_s][a_{s+1,t}].
\end{aligned}$$

We also have that

$$\begin{aligned}
\iota_{\{\psi,\varphi\}} bx &= \iota_{\{\psi,\varphi\}}(\sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}} m[a_{i_1+1,k}, a_{0,j_1}][a_{j_1+1,i_1}] \\
&\qquad + \sum(-1)^{\eta_{i_1}} a_0[a_{1,i_1}, m[a_{i_1+1,j_1}], a_{j_1+1,k}]) \\
&= \sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}+(\eta_{i_1}-\eta_t)\eta_n+(\eta_k-\eta_t+\eta_{i_2})(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3})(|\varphi|-1)} m[a_{t+1}, \cdots, \\
&\qquad m[a_{i_1+1,k}, a_0, \cdots, a_{j_1}], \cdots, \psi[a_{i_2+1,j_2}], \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3}-1)(|\varphi|-1)} m[a_{t+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_2+1,j_2}], \cdots, \varphi[a_{i_3+1,j_3}], \cdots, m[a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3}-1)(|\varphi|-1)} m[a_{t+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_2+1,j_2}], \cdots, \varphi[a_{i_3+1}, \cdots, m[a_{i_1+1,j_1}], \cdots, a_{j_3}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2})(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3})(|\varphi|-1)} m[a_{t+1,k}, a_0, \cdots, \\
&\qquad m[a_{i_1+1,j_1}], \cdots, \psi[a_{i_2+1,j_2}], \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3}-1)(|\varphi|-1)} m[a_{t+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_2+1,j_2}], \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_s][a_{s+1}, \cdots, m[a_{i_1+1,j_1}], \cdots, a_t] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t-1)\eta_t+(\eta_k-\eta_t+\eta_{i_2})(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3})(|\varphi|-1)} m[a_{t+1}, \cdots, \\
&\qquad m[a_{i_1+1,j_1}], \cdots, a_k, a_0, \cdots, \psi[a_{i_2+1,j_2}], \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_s][a_{s+1,t}].
\end{aligned}$$

$$+\sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3})(|\varphi|-1)}m[a_{t+1,k},a_0,\cdots,$$
$$\varphi[a_{i_2+1},\cdots,m[a_{i_1+1,j_1}],\cdots,a_{j_2}],\cdots,\varphi[a_{i_3+1,j_3}],\cdots,a_s][a_{s+1,t}]$$
$$+\sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_3})(|\varphi|-1)}m[a_{t+1,k},a_0,\cdots,$$
$$\psi[a_{i_2+1,j_2}],\cdots,m[a_{i_1+1,j_1}],\cdots,\varphi[a_{i_3+1,j_3}],\cdots,a_s][a_{s+1,t}].$$

Finally, we denote

$$\rho:=(\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\varphi|-1),$$

and we can also compute that

$$\begin{aligned}b\iota_{\{\psi,\varphi\}}x&=b(\sum(-1)^{\rho}m[a_{t+1,k},a_0,\cdots,\psi[a_{i_1+1,j_1}],\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1,t}])\\&=\sum(-1)^{\rho+\eta_k-\eta_t+\eta_{i_3}+|\varphi|+|\psi|-1}m[a_{t+1,k},a_0,\cdots,\\&\qquad\psi[a_{i_1+1,j_1}],\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1},\cdots,m[a_{i_3+1,j_3}],\cdots,a_t]\\&\quad+\sum(-1)^{\rho+(\eta_k-\eta_t+\eta_{i_3}+|\varphi|+|\psi|-1)(\eta_k+|\varphi|+|\psi|)}m[a_{i_3+1,t},m[a_{t+1,k},a_0,\cdots,\\&\qquad\psi[a_{i_1+1,j_1}],\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_s],\cdots,a_{j_3}][a_{j_3+1,i_3}].\end{aligned}$$

Hence we can obtain that

$$\begin{aligned}\iota_{\varphi\cup\psi}-\iota_\varphi\iota_\psi&=(-1)^{|\varphi||\psi|}[\iota_{\{\psi,\delta\varphi\}}-(-1)^{|\psi|}\iota_{\{\varphi,\delta\psi\}}+(-1)^{|\varphi|}\iota_{\{\psi,\varphi\}}b+(-1)^{|\psi|}b\iota_{\{\psi,\varphi\}}]\\&\quad+\sum(-1)^{(\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)(|\psi|-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\varphi|-1)+|\psi|(|\varphi|-1)+1}m\bar{\circ}m[a_{t+1,k},a_0,\cdots,\\&\qquad\psi[a_{i_1+1,j_1}],\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1,t}]\\&=(-1)^{|\varphi||\psi|}[\iota_{\{\psi,\delta\varphi\}}-(-1)^{|\psi|}\iota_{\{\varphi,\delta\psi\}}+(-1)^{|\varphi|}\iota_{\{\psi,\varphi\}}b+(-1)^{|\psi|}b\iota_{\{\psi,\varphi\}}].\end{aligned}$$

This finishes the proof of the proposition. □

Next, we give the definition of Lie derivative acting on Hochschild chain complex of A_∞-algebras.

Definition 5.4. Let (A,m) be an A_∞-algebra. The *Lie derivative* is given by

$$\begin{aligned}L_f(a_0[a_1,\cdots,a_n]):&=\sum_{0\le i\le j\le n}(-1)^{(\eta_i-1)(|f|+1)}a_0[a_1,\cdots,a_i,f[a_{i+1},\cdots,a_j],\cdots,a_n]\\&+\sum_{0\le j\le i\le n}(-1)^{\eta_i(\eta_n-\eta_i)+|f|-1}f[a_{i+1},\cdots,a_n,a_0,\cdots,a_j][a_{j+1},\cdots,a_i],\end{aligned}$$

for any homogenous elements $f\in \mathrm{C}^\bullet(A,A)$ and $x=a_0[a_1,\dots,a_n]\in \mathrm{C}_\bullet(A,A)$. In particular, taking $f=m$, then $L_f=-b$.

Proposition 5.5. *Let (A,m) be an A_∞-algebra, then we have that*

$$L_\varphi=[B,\iota_\varphi]_{gr}:=B\iota_\varphi-(-1)^{|\varphi|}\iota_\varphi B,$$

for any $\varphi\in \mathrm{HH}^\bullet(A,A)$.

Proof. We only need to prove that

$$L_\varphi x-B\iota_\varphi x+(-1)^{|\varphi|}\iota_\varphi Bx=bS_\varphi x-S_{\delta\varphi}x-(-1)^{|\varphi|}S_\varphi bx,\tag{5.2}$$

for $x:=a_0[a_{1,k}]\in \mathrm{HH}_\bullet(A,A)$, where

$$S_\varphi x=\sum(-1)^{\xi}e[a_{t+1,k},a_0,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_t],$$

the sum runs over $0\le i_1\le j_1\le t\le k$, and

$$\xi:=(\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)(|\varphi|-1).$$

We compute the terms in equation (5.2) one by one. Firstly,

$$\begin{aligned}B\iota_\varphi x&=B(\sum(-1)^{\xi}m[a_{t+1,k},a_0,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_s][a_{s+1,t}])\\&=\sum(-1)^{\xi+(\eta_k-\eta_t+\eta_{i_2}+|\varphi|)(\eta_t-\eta_{i_2})}e[a_{i_2+1,t},m[a_{t+1,k},a_0,\cdots,\\&\qquad\varphi[a_{i_1+1,j_1}],\cdots,a_s],\cdots,a_{i_2}][a_{i_2+1,j_2}].\end{aligned}$$

Secondly,

$$\begin{aligned}\iota_\varphi Bx &= \iota_\varphi(\sum(-1)^{(\eta_k-\eta_i)\eta_i}e[a_{i+1,k},a_{0,i}])\\ &=\sum(-1)^{(\eta_k-\eta_i)\eta_i}\varphi[a_{i+1,k},a_{0,j}][a_{j+1,i}]\\ &\qquad+\sum(-1)^{(\eta_k-\eta_i)\eta_i}\varphi[a_{i+1,j}][a_{j+1,k},a_{0,i}].\end{aligned}$$

Thirdly, denote

$$\zeta := (\eta_k-\eta_t+\eta_{i_2})(|\varphi|-1).$$

Then we have that

$$\begin{aligned}S_\varphi bx = S_\varphi(&\sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}}m[a_{i_1+1,k},a_0,\cdots,a_{j_1}][a_{j_1+1,i_1}]\\ &+(-1)^{\eta_{i_1}}a_0[a_{1,i_1},m[a_{i_1+1,j_1}],\cdots,a_k])\\ =\sum&(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}+(\eta_k-\eta_{i_1}+\eta_t-1)(\eta_{i_1}-\eta_t)+\zeta}e[a_{t+1,i_1},m[a_{i_1+1,k},a_0,\cdots,a_{j_1}],\cdots,\\ &\varphi[a_{i_2+1,j_2}],\cdots,a_t]\\ +&\sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+\zeta+|\varphi|-1}e[a_{t+1,k},a_0,\cdots,\\ &\varphi[a_{i_2+1,j_2}],\cdots,m[a_{i_1+1,j_1}],\cdots,a_t]\\ +&\sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t)(\eta_t-1)+\zeta+|\varphi|-1}e[a_{t+1,k},a_0,\cdots,\\ &\varphi[a_{i_2+1},\cdots,m[a_{i_1+1,j_1}],\cdots a_{j_2}],\cdots,a_t]\\ +&\sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t-1)(\eta_t-1)+\zeta}e[a_{t+1,k},a_0,\cdots,\\ &m[a_{i_1+1,j_1}],\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_t]\\ +&\sum(-1)^{\eta_{i_1}+(\eta_k-\eta_t-1)\eta_t+\zeta}e[a_{t+1},\cdots,\\ &m[a_{i_1+1,j_1}],\cdots,a_k,a_0,\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_t].\end{aligned}$$

We set

$$\tau := (\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)(|\varphi|-1),$$

and then we have that

$$\begin{aligned}bS_\varphi x &= \sum(-1)^\tau b(e[a_{t+1,k},a_0,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_t])\\ &=\sum(-1)^{\tau+(\eta_k+|\varphi|-1)(\eta_k-\eta_{j_1}+\eta_{i_1}-1)}m[\varphi[a_{i_1+1,j_1}],e][a_{j_1+1,k},a_{0,i_1}]\\ &\quad+\sum(-1)^\tau m[e,a_0][a_1,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_k]\\ &\quad+\sum(-1)^{\tau+\eta_{i_2}-\eta_t-1}e[a_{t+1},\cdots,m[a_{i_2+1,j_2}],\cdots,\\ &\qquad a_k,a_0,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_t]\\ &\quad+\sum(-1)^{\tau+\eta_{i_2}-\eta_t-1}e[a_{t+1},\cdots,m[a_{i_2+1,k},a_{0,j_2}],\cdots,\\ &\qquad\varphi[a_{i_1+1,j_1}],\cdots,a_t]\\ &\quad+\sum(-1)^{\tau+\eta_{i_2}-\eta_t-1}e[a_{t+1},\cdots,m[a_{i_2+1,k},a_0,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_{j_2}],\cdots,a_t]\\ &\quad+\sum(-1)^{\tau+\eta_k-\eta_t+\eta_{i_2}-1}e[a_{t+1,k},a_0,\cdots,m[a_{i_2+1,j_2}],\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_t]\\ &\quad+\sum(-1)^{\tau+\eta_k-\eta_t+\eta_{i_2}-1}e[a_{t+1,k},a_0,\cdots,m[a_{i_2+1},\cdots,\varphi[a_{i_1+1,j_1},\cdots,a_{j_2}],\cdots,a_t]\\ &\quad+\sum(-1)^{\tau+\eta_k-\eta_t+\eta_{i_2}+|\varphi|}e[a_{t+1,k},a_0,\cdots,\varphi[a_{i_1+1,j_1}],\cdots,m[a_{i_2+1,j_2}],\cdots,a_t].\end{aligned}$$

Lastly, we take

$$\rho := (\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)|\varphi|,$$

and we have that

$$\begin{aligned}S_{\delta\varphi}x &= \sum(-1)^\rho m[a_{t+1,k},a_0,\cdots,\delta\varphi[a_{i_1+1,j_1}],\cdots,a_t]\\ &=\sum(-1)^\rho m[a_{t+1,k},a_0,\cdots,m\{\varphi\}[a_{i_1+1,j_1}],\cdots,a_t]\end{aligned}$$

$$+\sum(-1)^{\rho+|\varphi|}m[a_{t+1,k},a_0,\cdots,\varphi\{m\}[a_{i_1+1,j_1}],\cdots,a_t]$$
$$=\sum(-1)^{\rho+(\eta_{i_2}-\eta_{i_1})(|\varphi|-1)}m[a_{t+1,k},a_0,\cdots,m[a_{i_1+1},\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_{j_1}],\cdots,a_t]$$
$$+\sum(-1)^{\rho+\eta_{i_2}-\eta_{i_1}+|\varphi|}m[a_{t+1,k},a_0,\cdots,\varphi[a_{i_1+1},\cdots,m[a_{i_2+1,j_2}],\cdots,a_{j_1}],\cdots,a_t].$$

It is not difficult to check the equation (5.2) according to the above computation. □

5.2. **Proof of Theorem 1.1.** We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 4.11, Proposition 5.3 and Proposition 5.5, it is sufficient to show that the identity

$$[L_\varphi,\iota_\psi]_{gr}=(-1)^{|\varphi|-1}\iota_{[\varphi,\psi]},\tag{5.3}$$

holds for any $\varphi,\psi\in \mathrm{HH}^\bullet(A,A)$.

We only need to prove that

$$\begin{aligned}&L_\varphi\iota_\psi x-(-1)^{(|\varphi|-1)|\psi|}\iota_\psi L_\varphi x-(-1)^{|\varphi|-1}\iota_{[\varphi,\psi]}\\&=bH_{\varphi,\psi}x-(-1)^{|\varphi|+|\psi|}H_{\varphi,\psi}bx-H_{\delta\varphi,\psi}x-(-1)^{|\varphi|}H_{\varphi,\delta\psi}x,\end{aligned}\tag{5.4}$$

for $x:=a_0[a_{1,k}]\in \mathrm{HH}_\bullet(A,A)$, where

$$H_{\varphi,\psi}x=\sum(-1)^{\zeta}\varphi[a_{i_2+1,k},a_0,\cdots,\psi[a_{i_1+1,j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}].$$

Here the sum runs over $0\le i_1\le j_1\le j_2\le i_2\le k$, and

$$\zeta:=(\eta_k-\eta_{i_2})\eta_{i_2}+(\eta_k-\eta_{i_2}+\eta_{i_1}-1)(|\psi|-1).$$

We compute the terms in equation (5.4) one by one. Firstly, denote

$$\theta:=(\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)(|\psi|-1),$$

and then we have that

$$\begin{aligned}L_\varphi\iota_\psi x&=L_\varphi(\sum(-1)^{\theta}m[a_{t+1,k},a_0,\cdots,\psi[a_{i_1+1,j_1}],\cdots,a_s][a_{s+1,t}])\\&=\sum(-1)^{\theta+(\eta_k-\eta_t+\eta_{i_2}+|\psi|-1)(|\varphi|-1)}m[a_{t+1,k},a_0,\cdots,\psi[a_{i_1+1,j_1}],\cdots,a_s][a_{s+1},\cdots,\varphi[a_{i_2+1,j_2}],\cdots,a_t]\\&\quad+\sum(-1)^{\theta+(\eta_k-\eta_t+\eta_{i_2}+|\psi|)(\eta_t-\eta_{i_2})+|\varphi|-1}\varphi[a_{i_2+1,t},m[a_{t+1,k},a_0,\cdots,\psi[a_{i_1+1,j_1}],\cdots,a_s],\cdots,a_{j_2}][a_{j_2+1,i_2}].\end{aligned}$$

Secondly, we have that

$$\begin{aligned}\iota_\psi L_\varphi x&=\iota_\psi(\sum(-1)^{(\eta_{i_1}-1)(|\varphi|-1)}a_0[a_{1,i_1},\varphi[a_{i_1+1,j_1}],\cdots,a_k]\\&\qquad+(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}+|\varphi|-1}\varphi[a_{i_1+1,k},a_0,\cdots,a_{j_1}][a_{j_1+1,i_1}])\\&=\sum(-1)^{(\eta_{i_1}-1)(|\varphi|-1)+(\eta_k-\eta_t)(\eta_t+|\varphi|-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)}m[a_{t+1,k},a_0,\cdots,\\&\qquad\psi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1},\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_t]\\&\quad+\sum(-1)^{(\eta_{i_1}-1)(|\varphi|-1)+(\eta_k-\eta_t)(\eta_t+|\varphi|-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)}m[a_{t+1,k},a_0,\cdots,\\&\qquad\psi[a_{i_2+1},\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_{j_2}],\cdots,a_s][a_{s+1,t}]\\&\quad+\sum(-1)^{(\eta_{i_1}-1)(|\varphi|-1)+(\eta_k-\eta_t)(\eta_t+|\varphi|-1)+(\eta_k-\eta_t+\eta_{i_2}+|\varphi|)(|\psi|-1)}m[a_{t+1,k},a_0,\cdots,\\&\qquad\varphi[a_{i_1+1,j_1}],\cdots,\psi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1,t}]\\&\quad+\sum(-1)^{(\eta_{i_1}-1)(|\varphi|-1)+(\eta_k-\eta_t)(\eta_t+|\varphi|-1)+(\eta_k-\eta_t+\eta_{i_2}-1)(|\psi|-1)}m[a_{t+1,k},a_0,\cdots,\\&\qquad\psi[a_{i_2+1,j_2}],\cdots,\varphi[a_{i_1+1,j_1}],\cdots,a_s][a_{s+1,t}]\\&\quad+\sum(-1)^{(\eta_{i_1}-1)(|\varphi|-1)+(\eta_k-\eta_t+|\varphi|-1)(\eta_t+|\varphi|-1)+(\eta_k-\eta_t+\eta_{i_2}+|\varphi|)(|\psi|-1)}m[a_{t+1},\cdots,\\&\qquad\varphi[a_{i_1+1,j_1}],\cdots,a_k,a_0,\cdots,\psi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1,t}]\\&\quad+\sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}+|\varphi|-1+(\eta_k-\eta_{i_1}+\eta_t+|\varphi|-1)(\eta_{i_1}-\eta_t)+(\eta_k-\eta_t+\eta_{i_2}+|\varphi|)(|\psi|-1)}m[a_{t+1,i_1},\cdots,\\&\qquad\varphi[a_{i_1+1,k},a_{0,j_1}],\cdots,\psi[a_{i_2+1,j_2}],\cdots,a_s][a_{s+1,t}].\end{aligned}$$

Thirdly, denote

$$\xi:=(\eta_k-\eta_t)\eta_t+(\eta_k-\eta_t+\eta_{i_1}-1)(|\varphi|+|\psi|),$$

and then we have that

$$\begin{aligned}
\iota_{[\varphi,\psi]}x &= \sum(-1)^{\xi} m[a_{t+1,k}, a_0, \cdots, [\varphi,\psi][a_{i_1+1,j_1}], \cdots, a_s][a_{s+1,t}] \\
&= \sum(-1)^{\xi+(\eta_{i_2}-\eta_{i_1})(|\psi|-1)} m[a_{t+1,k}, a_0, \cdots, \varphi[a_{i_1+1}, \cdots, \psi[a_{i_2+1,j_2}], \cdots, a_{j_1}], \cdots, a_s][a_{s+1,t}] \\
&\quad + \sum(-1)^{\xi+(\eta_{i_2}-\eta_{i_1})(|\varphi|-1)+(|\varphi|-1)(|\psi|-1)+1} m[a_{t+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_1+1}, \cdots, \varphi[a_{i_2+1,j_2}], \cdots, a_{j_1}], \cdots, a_s][a_{s+1,t}].
\end{aligned}$$

Next, we have that

$$\begin{aligned}
bH_{\varphi,\psi}x = &\sum(-1)^{\zeta+(\eta_{i_2}-\eta_{i_3})(\eta_k-\eta_{i_2}+\eta_{i_3}+|\varphi|+|\psi|)} m[a_{i_3+1,i_2}, \varphi[a_{i_2+1,k}, a_0, \cdots, \\
&\quad \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}], \cdots, a_{j_3}][a_{j_3+1,i_3}] \\
&+ \sum(-1)^{\zeta+\eta_k-\eta_{i_2}+\eta_{i_3}+|\varphi|+|\psi|} \varphi[a_{i_2+1,k}, a_0, \cdots, \\
&\quad \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1} \cdots, m[a_{j_3+1,i_3}], \cdots, a_{i_2}].
\end{aligned}$$

Continually, we have that

$$\begin{aligned}
H_{\varphi,\psi}bx &= H_{\varphi,\psi}(\sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}} m[a_{i_1+1,k}, a_0, \cdots, a_{j_1}][a_{j_1+1,i_1}] \\
&\qquad + (-1)^{\eta_{i_1}} a_0[a_{1,i_1}, m[a_{i_1+1,j_1}], \cdots, a_k]) \\
&= \sum(-1)^{(\eta_k-\eta_{i_1})\eta_{i_1}+(\eta_{i_1}-\eta_{i_2})(\eta_k-\eta_{i_1}+\eta_{i_2}-1)+(\eta_k-\eta_{i_1}+\eta_{i_3})(|\psi|-1)} \varphi[a_{i_2+1,i_1} m[a_{i_1+1,k}, a_0, \cdots, a_{j_1}], \cdots, \\
&\qquad \psi[a_{i_3+1,j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_{i_2})(\eta_{i_2}-1)+(\eta_k-\eta_{i_2}+\eta_{i_3}-1)(|\psi|-1)} \varphi[a_{i_2+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_3+1,j_3}], \cdots, a_{j_2}][a_{j_2+1}, \cdots, m[a_{i_1+1,j_1}], \cdots, a_{i_2}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_{i_2})(\eta_{i_2}-1)+(\eta_k-\eta_{i_2}+\eta_{i_3}-1)(|\psi|-1)} \varphi[a_{i_2+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_3+1,j_3}], \cdots, m[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_{i_2})(\eta_{i_2}-1)+(\eta_k-\eta_{i_2}+\eta_{i_3}-1)(|\psi|-1)} \varphi[a_{i_2+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_3+1}, \cdots, , m[a_{i_1+1,j_1}], \cdots, a_{j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_{i_2})(\eta_{i_2}-1)+(\eta_k-\eta_{i_2}+\eta_{i_3})(|\psi|-1)} \varphi[a_{i_2+1,k}, a_0, \cdots, \\
&\qquad m[a_{i_1+1,j_1}], \cdots, \psi[a_{i_3+1,j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\eta_{i_1}+(\eta_k-\eta_{i_2}-1)\eta_{i_2}+(\eta_k-\eta_{i_2}+\eta_{i_3})(|\psi|-1)} \varphi[a_{i_2+1}, m[a_{i_1+1,j_1}], \cdots, a_k, a_0 \cdots, \\
&\qquad \psi[a_{i_3+1,j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}].
\end{aligned}$$

We can also obtain that

$$\begin{aligned}
H_{\delta\varphi,\psi}x &= \sum(-1)^{\zeta}\delta\varphi[a_{i_2+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&= \sum(-1)^{\zeta} m\{\varphi\}[a_{i_2+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\zeta+|\varphi|}\varphi\{m\}[a_{i_2+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&= \sum(-1)^{\zeta+(\eta_{i_3}-\eta_{i_2})(|\varphi|-1)} m[a_{i_2+1}, \cdots, \varphi[a_{i_3+1,j_3}], \cdots, a_k, a_0, \cdots, \\
&\qquad \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\zeta+(\eta_{i_3}-\eta_{i_2})(|\varphi|-1)} m[a_{i_2+1}, \cdots, \varphi[a_{i_3+1,k}, a_0, \cdots, a_{j_3}], \cdots, \\
&\qquad \psi[a_{i_1+1,j_1}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\zeta+(\eta_{i_3}-\eta_{i_2})(|\varphi|-1)} m[a_{i_2+1}, \cdots, \varphi[a_{i_3+1,k}, a_0, \cdots, \\
&\qquad \psi[a_{i_1+1,j_1}], \cdots, a_{j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\zeta+(\eta_k-\eta_{i_2}+\eta_{i_3}+|\psi|-1)(|\varphi|-1)} m[a_{i_2+1,k}, a_0, \cdots, \psi[a_{i_1+1,j_1}], \cdots, \\
&\qquad \varphi[a_{i_3+1,j_3}], \cdots, a_{j_2}][a_{j_2+1,i_2}] \\
&\quad + \sum(-1)^{\zeta+(\eta_k-\eta_{i_2}+\eta_{i_3})(|\varphi|-1)} m[a_{i_2+1,k}, a_0, \cdots, \varphi[a_{i_3+1,j_3}], \cdots,
\end{aligned}$$

$$
\begin{aligned}
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+(\eta_k-\eta_{i_2}+\eta_{i_3})(|\varphi|-1)}m[a_{i_2+1,k},a_0,\cdots,\varphi[a_{i_3+1},\cdots,\\
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_3}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+|\varphi|+\eta_{i_3}-\eta_{i_2}}\varphi[a_{i_2+1},\cdots,m[a_{i_3+1,j_3}],\cdots,a_k,a_0,\cdots,\\
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+|\varphi|+\eta_{i_3}-\eta_{i_2}}\varphi[a_{i_2+1},\cdots,m[a_{i_3+1,k},a_0,\cdots,a_{j_3}],\cdots,\\
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+|\varphi|+\eta_{i_3}-\eta_{i_2}}\varphi[a_{i_2+1},\cdots,m[a_{i_3+1,k},a_0,\cdots,\\
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_3}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+|\varphi+\eta_k-\eta_{i_2}+\eta_{i_3}+|\psi|-1}\varphi[a_{i_2+1,k},a_0,\cdots,\psi[a_{i_1+1,j_1}],\cdots,\\
&m[a_{i_3+1,j_3}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+\eta_k-\eta_{i_2}+\eta_{i_3}}\varphi[a_{i_2+1,k},a_0,\cdots,m[a_{i_3+1,j_3}],\cdots,\\
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&+\sum(-1)^{\zeta+\eta_k-\eta_{i_2}+\eta_{i_3}}\varphi[a_{i_2+1,k},a_0,\cdots,m[a_{i_3+1},\cdots,\\
&\psi[a_{i_1+1,j_1}],\cdots,a_{j_3}],\cdots,a_{j_2}][a_{j_2+1,i_2}].
\end{aligned}
$$

Lastly, denote

$$\zeta' := (\eta_k-\eta_{i_2})\eta_{i_2}+(\eta_k-\eta_{i_2}+\eta_{i_1}-1)|\psi|,$$

we compute that

$$
\begin{aligned}
H_{\varphi,\delta\psi}x &= \sum(-1)^{\zeta'}\varphi[a_{i_2+1,k},a_0,\cdots,\delta\psi[a_{i_1+1,j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&= \sum(-1)^{\zeta'+(\eta_{i_3}-\eta_{i_1}(|\psi|-1)}\varphi[a_{i_2+1,k},a_0,\cdots,m[a_{i_1+1},\cdots,\\
&\qquad \psi[a_{i_3+1,j_3}],\cdots,a_{j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}]\\
&\quad+\sum(-1)^{\zeta'+\eta_{i_3}-\eta_{i_1}+|\psi|}\varphi[a_{i_2+1,k},a_0,\cdots,\psi[a_{i_1+1},\cdots,\\
&\qquad m[a_{i_3+1,j_3}],\cdots,a_{j_1}],\cdots,a_{j_2}][a_{j_2+1,i_2}].
\end{aligned}
$$

Through comparing the two sides in equation (5.4) according the above computations, we have done. □

Corollary 5.6. *For any homogenous elements $\varphi,\psi\in \mathrm{HH}^\bullet(A,A)$, $[B,L_\varphi]_{gr}=0$ and $[L_\varphi,L_\psi]_{gr}=L_{[\varphi,\psi]}$.*

Proof. By Proposition 5.5, we obtain $B\circ L_\varphi=(-1)^{|\varphi|+1}B\circ\iota_\varphi\circ B=(-1)^{|\varphi|+1}L_\varphi\circ B$, and then by equation (5.3), we get $[L_\varphi,L_\psi]_{gr}=[[B,\iota_\varphi]_{gr},L_\psi]_{gr}=(-1)^{|\varphi|(|\psi|+1)}[[B,L_\psi]_{gr},L_\varphi]_{gr}+[B,[\iota_\varphi,L_\psi]_{gr}]_{gr}=[B,\iota_{[\varphi,\psi]}]_{gr}=L_{[\varphi,\psi]}$. □

6. Hochschild cohomology of cyclic A_∞-algebras

We start with the definition of cyclic A_∞-algebras.

Definition 6.1. Let (A,m) be a finite dimensional A_∞-algebra with strict unit. An *A_∞-cyclic structure of degree d* on A is a non-degenerate bilinear form

$$\langle -,-\rangle : A[1]\otimes A[1]\to\mathbb{K}$$

of degree $-d$ (i.e. $|a|+|b|=d-2$, if $\langle sa,sb\rangle\neq 0$), such that

$$\langle sa,sb\rangle=-(-1)^{(|a|+1)(|b|+1)}\langle sb,sa\rangle,$$

and

$$\langle sm[a_1,\cdots,a_n],sa_{n+1}\rangle=(-1)^{(|a_1|+1)\sum\limits_{i=2}^{n+1}(|a_i|+1)}\langle sm[a_2,\cdots,a_{n+1}],sa_1\rangle$$

for any homogenous elements $a_i\in A$ and integer $n\geq 0$.

Remark 6.2. There exists a non-shifted version of cyclic A_∞-algebra, see [25, Section 11]. An *A_∞-cyclic structure of degree d* on A is a non-degenerate bilinear form

$$\langle -,-\rangle' : A\otimes A \to \mathbb{K}$$

of degree $-d$, such that

$$\langle a,b\rangle' = (-1)^{|a||b|}\langle b,a\rangle',$$

and

$$\langle m[a_1,\cdots,a_n],a_{n+1}\rangle' = (-1)^{n+|a_1|\cdot\sum\limits_{i=2}^{n+1}(|a_i|)}\langle m[a_2,\cdots,a_{n+1}],a_1\rangle'$$

for any homogenous elements $a_i\in A$ and integer $n\geq 0$. In fact, in the Definition 6.1, if we take $\langle a,b\rangle' := (-1)^{|a|}\langle sa,sb\rangle$, then we can obtain the non-shifted version. In this note, we adopt the shifted version since the sign rules in this case are just the Koszul sign convention.

Proposition 6.3. *Let A be a cyclic A_∞-algebra of degree d. Then the Hochschild data*

$$(\mathrm{HH}^\bullet(A,A),\mathrm{HH}^\bullet(A,A'),\cup,\cap',[-,-],B',\Omega)$$

is a differential calculus with duality, where $\mathrm{HH}^\bullet(A,A')$ is the Hochschild cohomology of A with value in A_∞-bimodule A', and $\Omega\in\mathrm{HH}^d(A,A')$ is a volume form.

Proof. By Theorem 1.1 and Proposition 2.4, we obtain that $(\mathrm{HH}^\bullet(A,A),\mathrm{HH}^\bullet(A,A'),\cup,\cap',[-,-],B')$ is a differential calculus. The only thing left is to show the existence of the duality. That is:

Claim 6.4. *There exists an element $\Omega\in\mathrm{HH}^d(A,A')$, such that*

$$\mathrm{HH}^\bullet(A,A)\to\mathrm{HH}^{d-\bullet}(A,A'),\ f\mapsto\kappa_f(\Omega)$$

is an isomorphism.

In fact, the cyclic structure of A induces an isomorphism $\Phi: A\longrightarrow A'$ given by

$$\Phi(a)(b) = (-1)^{|a|}\langle sa,sb\rangle,$$

and an isomorphism of complexes $\varphi:\mathrm{C}^\bullet(A,A)\to\mathrm{C}^\bullet(A,A')$ given by $\varphi(f):=\Phi\circ f$, and a duality $\mathrm{C}^\bullet(A,A')\cong(\mathrm{C}_\bullet(A,A))'$ given by $\varphi(f)(x):=(-1)^{|a_0|(|f|+1)}\langle sa_0,sf[a_1,\cdots,a_n]\rangle$. Now we show that

$$\varphi(g\cup f) = (-1)^{|f|(|g|+d)}f\cap'\varphi(g).$$

Given any $x=a_0[a_1\cdots,a_n]\in\mathrm{C}_\bullet(A,A)$ and $f,g\in\mathrm{C}^\bullet(A,A)$, we have

$$\begin{aligned}
&\varphi(g\cup f)(x)\\
&=\sum(-1)^{|a_0|(|f|+|g|+1)+|f|(|g|-1)+\eta'_i(|f|+1)+\eta'_s(|g|+1)}\langle sa_0,sm[a_1,\cdots,f[a_{i+1},\cdots,a_j],\cdots,g[a_{s+1},\cdots,a_t],\cdots,a_n]\rangle\\
&=\sum(-1)^{|g|(|f|-1)+\eta_i(|f|+1)+\eta_s(|g|+1)+|a_0|}\langle a_0,sm[a_1,\cdots,f[a_{i+1},\cdots,a_j],\cdots,sg[a_{s+1},\cdots,a_t],\cdots,a_n]\rangle\\
&=\sum(-1)^{|g|(|f|-1)+\eta_i(|f|+1)+\eta_s(|g|+1)+(|a_0|+1)(\eta_n+|f|+|g|)}\langle sm[a_1,\cdots,f[a_{i+1},\cdots,a_j],\cdots,g[a_{s+1},\cdots,a_t],\cdots,a_n],sa_0\rangle
\end{aligned}$$

where $\eta'_i=\sum\limits_{i=1}^{i}(|a_i|+1)$, and

$$\begin{aligned}
&(f\cap'\varphi(g))(x) = (-1)^{|f|(|g|+d)}\varphi(g)(\iota_f x)\\
=\ &(-1)^{|f|(|g|+d)+(\eta_n-\eta_t)\eta_t+(\eta_n-\eta_t+\eta_i-1)(|f|+1)}\cdot\\
&\varphi(g)(\sum m[a_{t+1},\cdots,a_n,a_0,\cdots,f[a_{i+1},\cdots,a_j],\cdots,a_s][a_{s+1},\cdots,a_t])\\
=\ &\sum(-1)^{|f|(|g|+d)+(\eta_n-\eta_t)\eta_t+(\eta_n-\eta_t+\eta_i-1)(|f|+1)+(|g|+1)(\eta_n-\eta_t+\eta_s+|f|+1)}\cdot\\
&\langle sm[a_{t+1},\cdots,a_n,a_0,\cdots,f[a_{i+1},\cdots,a_j],\cdots,a_s],sg[a_{s+1},\cdots,a_t]\rangle\\
=\ &\sum(-1)^{|f|(|g|+d)+(\eta_n-\eta_t)\eta_t+(\eta_n-\eta_t+\eta_i-1)(|f|+1)+(|g|+1)(\eta_n-\eta_t+\eta_s+|f|+1)+(|a_{t+1}|+1)(\eta_n+|f|+|g|+1)}\cdot\\
&\langle sm[a_t,\cdots,a_n,a_0,\cdots,f[a_{i+1},\cdots,a_j],\cdots,g[a_{s+1},\cdots,a_t]],sa_{t+1}\rangle\\
=\ &\sum(-1)^{|f|(|g|+d)+(\eta_n-\eta_t)\eta_t+(\eta_n-\eta_t+\eta_i-1)(|f|+1)+(|g|+1)(\eta_n-\eta_t+\eta_s+|f|+1)+(\eta_n-\eta_t+|a_0|+1)(\eta_n+|f|+|g|+1)}\cdot\\
&\langle sm[a_1,\cdots,f[a_{i+1},\cdots,a_j],\cdots,g[a_{s+1},\cdots,a_t],\cdots,a_n],sa_0\rangle\\
=\ &(-1)^{|f|(|g|+d)}\varphi(g\cup f)(x).
\end{aligned}$$

Picking the map $id : k \to ke$ lying in $\mathrm{C}^0(A, A)$, where e is the strict unit, then we have

$$\varphi(f) = (-1)^{|f|d} f \cap' \Omega = (-1)^{|f|d} \kappa_f(\Omega),$$

where Ω denotes the element $\varphi(id)$ lying in $\mathrm{C}^d(A, A')$. This proves the claim, and Proposition 6.3 follows. □

Theorem 6.5 (Tradler [24])**.** *If A is a cyclic A_∞-algebra, that is, A is a finite dimensional A_∞-algebra with a cyclically invariant non-degenerate pairing, then the Hochschild cohomology $\mathrm{HH}^\bullet(A, A)$ has a Batalin-Vilkovisky algebra structure.*

Proof. This theorem is direct from Theorem 3.3 and Proposition 6.3. □

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