

Long paths and cycles passing through specified vertices under the average degree condition

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Abstract Let G be a k -connected graph with $k \geq 2$. In this paper we first prove that: For two distinct vertices x and z in G , it contains a path connecting x and z which passes through its any $k - 2$ specified vertices with length at least the average degree of the vertices other than x and z . Further, with this result, we prove that: If G has n vertices and m edges, then it contains a cycle of length at least $2m/(n - 1)$ passing through its any $k - 1$ specified vertices. Our results generalize a theorem of Fan on the existence of long paths and a classical theorem of Erdős and Gallai on the existence of long cycles under the average degree condition.

Keywords Long path · Long cycle · Average degree

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1 Introduction

We use Bondy and Murty [2] for terminology and notations not defined here and consider finite simple graphs only.

Let G be a graph and H a subgraph of G . We use $V(H)$ and $E(H)$ to denote the set of vertices and edges of H , respectively, and use $e(H)$ for the number of the edges of H . For a vertex $v \in V(G)$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of neighbors of v in H . We call $d_H(v)$ the *degree* of v in H . Let x and z be two distinct vertices of G . A path connecting x and z is called an (x, z) -*path*. For a subset Y of $V(G)$, an (x, z) -path passing through all the vertices in Y is called an (x, Y, z) -*path*, and a cycle passing through all the vertices in Y is called a Y -*cycle*. If Y contains only one vertex y , an $(x, \{y\}, z)$ -path and a $\{y\}$ -cycle are simply denoted by an (x, y, z) -path and a y -cycle, respectively. The *distance* between x and z in H , denoted by $d_H(x, z)$, is the length of a shortest (x, z) -path with all its internal vertices in H . If no such path exists, we define $d_H(x, z) = \infty$. The *codistance* between x and z in H , denoted by $d_H^*(x, z)$, is the length of a longest (x, z) -path with all its internal vertices in H . If no such path exists, we define $d_H^*(x, z) = 0$. We remark that in the definitions of $d_H(x, z)$ and $d_H^*(x, z)$, the vertices x and z is not necessarily in H . When no confusion occurs, we use $N(v)$, $d(v)$, $d(x, z)$ and $d^*(x, z)$ instead of $N_G(v)$, $d_G(v)$, $d_G(x, z)$ and $d_G^*(x, z)$, respectively.

Long path and cycle problems are interesting and important in graph theory and have been deeply studied, see [1, 7]. The following theorem by Erdős and Gallai [5] opened the study on long paths with specified end vertices.

Theorem 1 (Erdős and Gallai [5]) *Let G be a 2-connected graph and x and z be two distinct vertices of G . If $d(v) \geq d$ for every vertex $v \in V(G) \setminus \{x, z\}$, then G contains an (x, z) -path of length at least d .*

Theorem 1 has a stronger extension due to Enomoto [4].

Theorem 2 (Enomoto [4]) *Let G be a 2-connected graph and x and z be two distinct vertices of G . If $d(v) \geq d$ for every vertex $v \in V(G) \setminus \{x, z\}$, then for every given vertex $y \in V(G) \setminus \{x, z\}$, G contains an (x, y, z) -path of length at least d .*

Another direction of extending Theorem 1 is to weaken the minimum degree condition to the average degree condition. Fan [6] finished this work as follows.

Theorem 3 (Fan [6]) *Let G be a 2-connected graph and x and z be two distinct vertices of G . If the average degree of the vertices other than x and z is at least r , then G contains an (x, z) -path of length at least r .*

The following graph shows that one cannot replace the minimum degree condition in Theorem 2 by the average degree condition. Let H be the complete graph on $n - 1$ vertices and $x, z \in V(H)$, and G be the graph obtained from H by adding a new vertex y and two edges xy, yz . Then the length of the longest

(x, y, z) -path in G is 2, less than the average degree of the vertices other than x and z when $n \geq 5$.

Our first result in this paper is a generalization of Theorem 3.

Theorem 4 *Let G be a k -connected graph with $k \geq 2$, and x and z be two distinct vertices of G . If the average degree of the vertices other than x and z is at least r , then for any subset Y of $V(G)$ with $|Y| = k - 2$, G contains an (x, Y, z) -path of length at least r .*

We remark here that the size of Y cannot be replaced by $k - 1$. Let H be a complete graph on $n - k + 1$ vertices with $n > 3k$ and $u_1 = x, u_2, \dots, u_k = z$ be k vertices of H , and $Y = \{y_1, y_2, \dots, y_{k-1}\}$ be a set of vertices not in $V(H)$. We construct a graph G with $V(G) = V(H) \cup Y$ and $E(G) = E(H) \cup \{u_i y_j : 1 \leq i \leq k, 1 \leq j \leq k - 1\}$. Then G is a k -connected graph and the longest (x, Y, z) -path has length $2k - 1$, which is less than

$$\begin{aligned} \frac{\sum_{v \in V(G) \setminus \{x, z\}} d(v)}{n - 2} &= \frac{(k - 1)k + (k - 2)(n - 1) + (n - 2k + 1)(n - k)}{n - 2} \\ &= \frac{n^2 - 2kn + n + 3k^2 - 3k}{n - 2}. \end{aligned}$$

Besides, the complete graph K_n with $n \geq k + 1$ shows that the bound r on the length of the (x, Y, z) -path is sharp.

There also exist results on long cycles passing through specified vertices in graphs. Theorem 5 shows the existence of long cycles in 2-connected graph under the minimum degree condition, and Theorem 6 extends Theorem 5 to graphs with higher connectivity.

Theorem 5 (Locke [8]) *Let G be a 2-connected graph. If the minimum degree of G is at least d , then for any two vertices y_1 and y_2 of G , G contains either a $\{y_1, y_2\}$ -cycle of length at least $2d$ or a Hamilton cycle.*

Theorem 6 (Egawa, Glas and Locke [3]) *Let G be a k -connected graph with $k \geq 2$. If the minimum degree of G is at least d , then for any subset Y of $V(G)$ with $|Y| = k$, G contains either a Y -cycle of length at least $2d$ or a Hamilton cycle.*

On the existence of long cycles in graphs with a given number of edges, Erdős and Gallai [5] gave the following result.

Theorem 7 (Erdős and Gallai [5]) *Let G be a 2-edge-connected graph on n vertices. Then G contains a cycle of length at least $2e(G)/(n - 1)$.*

In this paper, as an application of Theorem 4, we give the following theorem on long cycles passing through specified vertices of graphs with a given number of vertices and edges.

Theorem 8 *Let G be a k -connected graph on n vertices with $k \geq 2$. Then for any subset Y of $V(G)$ with $|Y| = k - 1$, G contains a Y -cycle of length at least $2e(G)/(n - 1)$.*

In Theorem 8, one cannot expect a cycle passing through k specified vertices of length at least $2e(G)/(n-1)$. Let H be a complete graph on $n-k$ vertices with $n > 3k$ and u_1, u_2, \dots, u_k be k vertices of H , and $Y = \{v_1, v_2, \dots, v_k\}$ be a set of vertices not in $V(H)$. We construct a graph G with $V(G) = V(H) \cup Y$ and $E(G) = E(H) \cup \{u_i v_j : 1 \leq i, j \leq k\}$. Then G is a k -connected graph and the longest Y -cycle has length $2k$, which is less than

$$\frac{2e(G)}{n-1} = \frac{(n-k)(n-k-1) + 2k^2}{n-1}.$$

In the following section we will give some further notations and preliminary results that will be used later. The proofs of Theorems 4 and 8 are given in Sections 3 and 4, respectively.

2 Preliminaries

Let G be a graph and P, H two disjoint subgraphs of G . We use $E(P, H)$ to denote the set, and $e(P, H)$ the number, of edges with one vertex in P and the other in H . If $E(P, H) \neq \emptyset$, then we call P and H are *joined*. We use $N_P(H)$ to denote the set of vertices in P which are joined to H . If x is a vertex in $G - P$, we say that x is *locally k -connected* to P (in G) if there are k paths connecting x and vertices in P such that any two of them have only the vertex x in common. We say that H is *locally k -connected* to P (in G) if for every vertex $x \in V(H)$, x is locally k -connected to P . Note that if H is locally k -connected to P , then H is locally l -connected to P for all l , $0 \leq l \leq k$; and, if G is k -connected and $|V(P)| \geq k$, then H is locally k -connected to P in G .

The following propositions on local k -connectedness are proved in [6].

Proposition 1 (Fan [6]) *Let H and P be two disjoint subgraphs of a graph G . If H is locally k -connected to P in the subgraph induced by $V(H) \cup V(P)$, then $E(P, H)$ contains an independent set of t edges, where $t \geq \min\{k, |V(H)|\}$.*

Proposition 2 (Fan [6]) *Let H and P be two disjoint subgraphs of a graph G . Let $u \in N_P(H)$ and G' be the graph obtained from G by deleting all edges from u to H . If H is locally k -connected to P in G , then H is locally $(k-1)$ -connected to P in G' .*

Proposition 3 (Fan [6]) *Let H and P be two disjoint subgraphs of a graph G , and B a block of H . Let H' be the subgraph obtained from H by contracting B . If H is locally k -connected to P in G , then H' is also locally k -connected to P in the resulting graph.*

Next we introduce the concept of local maximality for paths.

Let P be a path of a graph G , and $u, v \in V(P)$. We use $P[u, v]$ to denote the segment of P from u to v , and $P(u, v)$ the segment obtained from $P[u, v]$ by deleting the two end vertices u and v . Let H be a component of $G - P$. We say that P is a *locally longest path with respect to H* if we cannot obtain

a longer path than P by replacing the segment $P[u, v]$ by a (u, v) -path with all its internal vertices in H for any $u, v \in V(G)$. In other words, P is locally longest with respect to H if, for any $u, v \in V(P)$,

$$e(P[u, v]) \geq d_H^*(u, v).$$

If P is an (x, Y, z) -path of G , where $x, z \in V(G)$ and $Y \subset V(G)$, then we say that P is a *locally longest (x, Y, z) -path with respect to H* if we cannot obtain an (x, Y, z) -path longer than P by replacing the segment $P[u, v]$ with $Y \cap V(P(u, v)) = \emptyset$ by a (u, v) -path with all its internal vertices in H . Note that if P is a longest path (longest (x, Y, z) -path) in a graph G , then, of course, P is a locally longest path (locally longest (x, Y, z) -path) with respect to any component of $G - P$. If two vertices u and u' in $V(P)$ are joined to H by two independent edges, then we call $\{u, u'\}$ a *strong attached pair* of H to P . A *strong attachment* of H to P (in G) is a subset $T = \{u_1, u_2, \dots, u_t\} \subset N_P(H)$, where u_i , $1 \leq i \leq t$, are in order along P , such that each ordered pair $\{u_i, u_{i+1}\}$, $1 \leq i \leq t - 1$, is a strong attached pair of H to P . A strong attachment T of H to P is *maximum* if T has maximum cardinality over all strong attachments of H to P .

The following result due to Fan is useful in our proofs.

Lemma 1 (Fan [6]) *Let G be a graph and P an (x, z) -path of G . Suppose that H is a component of $G - P$ and $T = \{u_1, u_2, \dots, u_t\}$ is a maximum strong attachment of H to P . Set $S = N_P(H) \setminus T$. Then the following statements are true:*

- (1) *Every vertex in S is adjacent to exactly one vertex in H .*
- (2) *For each segment $P[u_i, u_{i+1}]$, $1 \leq i \leq t - 1$, suppose that*

$$N_P(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \dots, a_q, a_{q+1}\},$$

where $a_0 = u_i$, $a_{q+1} = u_{i+1}$ and a_j , $0 \leq j \leq q + 1$, are in order along P . Then there is a subscript m , $0 \leq m \leq q$, such that

$$N_H(a_j) = N_H(a_0), \text{ for } 0 \leq j \leq m,$$

and

$$N_H(a_j) = N_H(a_{q+1}), \text{ for } m + 1 \leq j \leq q + 1.$$

Besides, if

$$N_P(H) \cap V(P[x, u_1]) = \{a_1, \dots, a_q, a_{q+1}\},$$

where, $a_{q+1} = u_1$, then

$$N_H(a_j) = N_H(a_{q+1}), \text{ for } 1 \leq j \leq q + 1;$$

and if

$$N_P(H) \cap V(P[u_t, z]) = \{a_0, a_1, \dots, a_q\},$$

where, $a_0 = u_t$, then

$$N_H(a_j) = N_H(a_0), \text{ for } 0 \leq j \leq q.$$

(3) If H is locally k -connected to P in G , then

$$t \geq \min\{k, h + d_2\},$$

where $h = |V(H)|$ and d_2 is the number of vertices in $N_P(H)$ each of which has at least two neighbors in H .

Lemma 1 (2) is somewhat different from that in [6], but the proofs of them are similar.

For a path P , we use $l(P)$ to denote the length of P .

Lemma 2 Let G be a graph, P an (x, Y, z) -path of G , where $x, z \in V(G)$ and $Y \subset V(G)$, H a component of $G - P$ and $T = \{u_1, u_2, \dots, u_t\}$ a maximum strong attachment of H to P . Set $S = N_P(H) \setminus T$ and $s = |S|$. Suppose that P is a locally longest (x, Y, z) -path with respect to H , and $\theta = |\{x, z\} \cap N_P(H)|$. Set

$$T_r = \{u_i \in T \setminus \{u_t\} : Y \cap V(P(u_i, u_{i+1})) = \emptyset\} \text{ and } t_r = |T_r|.$$

Then

$$l(P) \geq \sum_{u_i \in T_r} d_H^*(u_i, u_{i+1}) + 2(s + t - t_r) - \theta.$$

Proof If $t = 0$, then $s = 0$ and the statement is trivially true. Suppose now that $t \geq 1$.

Consider a segment $P[u_i, u_{i+1}]$, $1 \leq i \leq t - 1$. Suppose that

$$N_P(H) \cap V(P[u_i, u_{i+1}]) = \{a_0, a_1, \dots, a_q, a_{q+1}\},$$

where $q = |S \cap V(P[u_i, u_{i+1}])|$, $a_0 = u_i$, $a_{q+1} = u_{i+1}$, and a_j , $0 \leq j \leq q + 1$, are in order along P .

If $Y \cap V(P(u_i, u_{i+1})) = \emptyset$, then by Lemma 1 (2), there is a subscript m , $0 \leq m \leq q$, such that

$$N_H(a_0) = N_H(a_m) \text{ and } N_H(a_{q+1}) = N_H(a_{m+1}).$$

Therefore

$$d_H^*(a_m, a_{m+1}) = d_H^*(a_0, a_{q+1}) = d_H^*(u_i, u_{i+1}).$$

Since P is a locally longest (x, Y, z) -path with respect to H , we have

$$\begin{aligned} l(P[u_i, u_{i+1}]) &\geq \sum_{j=0}^q d_H^*(a_j, a_{j+1}) = d_H^*(a_m, a_{m+1}) + \sum_{\substack{j=0 \\ j \neq m}}^q d_H^*(a_j, a_{j+1}) \\ &= d_H^*(u_i, u_{i+1}) + \sum_{\substack{j=0 \\ j \neq m}}^q d_H^*(a_j, a_{j+1}). \end{aligned}$$

Note that $d_H^*(a_j, a_{j+1}) \geq 2$, for every j , $0 \leq j \leq q$. We have

$$l(P[u_i, u_{i+1}]) \geq d_H^*(u_i, u_{i+1}) + 2q.$$

If $Y \cap V(P(u_i, u_{i+1})) \neq \emptyset$, then noting that $l(P[a_j, a_{j+1}]) \geq 2$, we have

$$l(P[u_i, u_{i+1}]) = \sum_{j=0}^q l(P[a_j, a_{j+1}]) \geq 2q + 2.$$

Besides, consider the two segments $P[x, u_1]$ and $P[u_t, z]$. Suppose that

$$N_P(H) \cap V(P[x, u_1]) = \{a_0, a_1, \dots, a_m\}$$

and

$$N_P(H) \cap V(P[u_t, z]) = \{a_{m+1}, a_{m+2}, \dots, a_{q+1}\},$$

where $m = |S \cap V(P[x, u_1])|$, $q - m = |S \cap V(P[u_t, z])|$, $a_m = u_1$, $a_{m+1} = u_t$, and a_j , $0 \leq j \leq q + 1$ are in order along P . Note that $l(P[x, a_0]) + l(P[a_{q+1}, z]) \geq 2 - \theta$ and $l(P[a_j, a_{j+1}]) \geq 2$, for every $0 \leq j \leq q$, and $j \neq m$, we have

$$l(P[x, u_1]) + l(P[u_t, z]) \geq 2q + 2 - \theta.$$

Thus summing over the lengths of all the segments, yields

$$\begin{aligned} l(P) &= l(P[x, u_1]) + \sum_{i=1}^{t-1} l(P[u_i, u_{i+1}]) + l(P[u_t, z]) \\ &\geq 2(|S \cap V(P[x, u_1])| + |S \cap V(P[u_t, z])|) + 2 - \theta \\ &\quad + \sum_{\substack{i=1 \\ u_i \in T_r}}^{t-1} (d_H^*(u_i, u_{i+1}) + 2|S \cap V(P[u_i, u_{i+1}])|) \\ &\quad + \sum_{\substack{i=1 \\ u_i \notin T_r}}^{t-1} (2|S \cap V(P[u_i, u_{i+1}])| + 2) \\ &= \sum_{u_i \in T_r} d_H^*(u_i, u_{i+1}) + 2(s + t - t_r) - \theta. \end{aligned}$$

This ends the proof.

For a strong attachment $T = \{u_1, u_2, \dots, u_t\}$, the pairs $\{u_j, u_{j+1}\}$, $1 \leq j \leq t - 1$, are called strong attached pairs *supported by* T , and we call a strong attached pair $\{u_j, u_{j+1}\}$ of H to P *transitive* if $Y \cap V(P(u_j, u_{j+1})) = \emptyset$.

A connected graph is *separable* if it has at least one cut-vertex.

Lemma 3 *Let G be a graph and P an (x, z) -path of G . Suppose that H is a separable component of $G - P$, B is an endblock of H , b is the cut vertex of H contained in B , and $M = B - b$. Let $T = \{u_1, u_2, \dots, u_t\}$ be a maximum strong attachment of H to P . If H is locally k -connected to P , then*

- (1) $|N_P(M) \cap T| \geq \min\{k - 1, m + d'_2\}$; and
- (2) *there exist at least $\min\{k - 1, m + d'_2\}$ strong attached pairs supported by T which are joined to M ,*
where $m = |V(M)|$ and d'_2 is the number of vertices in $N_P(M)$ each of which has at least two neighbors in H .

Proof Since H is locally k -connected to P , $|V(P)| \geq k$. It is easy to know that M is locally $(k-1)$ -connected to P in the subgraph induced by $V(P) \cup V(M)$. By Proposition 1, there are $\min\{k-1, m\}$ independent edges in $E(P, M)$. Let $v_i w_i$, $1 \leq i \leq \min\{k-1, m\}$ be such edges, where $v_i \in V(P)$ and $w_i \in V(M)$.

If v_i has at least two neighbors in H , then by Lemma 1 (1), $v_i \in T$. If v_i has only one neighbor w_i in H , then by Lemma 1 (2), there exists a vertex v'_i (maybe equal to v_i) in T which also has only one neighbor w_i in H . This implies that $|N_P(M) \cap T| \geq \min\{k-1, m\}$.

Now, we prove (1) by induction on d'_2 . If $d'_2 = 0$, then by the analysis above, the assertion is true. Thus we assume that $d'_2 \geq 1$.

Let u_j be a vertex in $N_P(M)$ which has at least two neighbors in H (u_j is of course in T by Lemma 1 (1)). Let G' be the graph obtained from G by deleting all edges from u_j to H . By Proposition 2, H is locally $(k-1)$ -connected to P in G' .

If $u_j = u_1$ or u_t , or $\{u_{j-1}, u_{j+1}\}$ is joined to H by two independent edges, then $T' = T \setminus \{u_j\}$ is a strong attachment of H to P in G' . Since u_j is joined to at least two vertices of H in G , any strong attachment of H to P in G' together with u_j is a strong attachment of H to P in G . Since $|T'| = t-1$, we see that T' is a maximum strong attachment of H to P in G' . By the induction hypothesis,

$$|N_P(M) \cap T'| \geq \min\{k-2, m + d'_2 - 1\}.$$

Therefore

$$|N_P(M) \cap T| \geq \min\{k-1, m + d'_2\},$$

as required.

If $u_j \in \{u_2, \dots, u_{t-1}\}$, and $\{u_{j-1}, u_{j+1}\}$ are not joined to H by two independent edges, i.e.,

$$N_H(u_{j-1}) = N_H(u_{j+1}) = \{w\},$$

for some $w \in V(H)$, then

$$T' = T \setminus \{u_j, u_{j+1}\} = \{u_1, \dots, u_{j-1}, u_{j+2}, \dots, u_t\}$$

is a strong attachment of H to P in G' . We prove now that T' is maximum by showing that any strong attachment of H to P in G' has cardinality at most $t-2 = |T'|$.

Let v_1, v_2 ($\neq u_j$) be the two vertices in $N_P(H)$ which are closest to u_j on P , say v_1 preceding, and v_2 following, u_j on P (but not necessarily adjacent to u_j on P). Since $|N_H(u_j)| \geq 2$, it follows from Lemma 1 (2) that

$$N_H(v_1) = N_H(u_{j-1}) = \{w\} = N_H(u_{j+1}) = N_H(v_2).$$

By the choices of v_1 and v_2 , for any maximum strong attachment $\{a_1, a_2, \dots, a_p\}$ of H to P in G' , there is an integer l , $0 \leq l \leq p$, such that $v_1, v_2 \in V(P[a_l, a_{l+1}])$, where $a_0 = x$ and $a_{p+1} = z$. Since $N_H(v_1) = \{w\} = N_H(v_2)$, it follows from Lemma 1 (2) that either $N_H(a_l) = \{w\}$ or $N_H(a_{l+1}) = \{w\}$. The former implies a

strong attachment $\{a_1, \dots, a_l, u_j, v_2, a_{l+1}, \dots, a_p\}$, the latter a strong attachment $\{a_1, \dots, a_l, v_1, u_j, a_{l+1}, \dots, a_p\}$, of H to P in G ; in either case we have that $p + 2 \leq t$, that is, $p \leq t - 2 = |T'|$. This shows that T' is a maximum strong attachment of H to P in G' , as claimed. As before, by the induction hypothesis,

$$|N_P(M) \cap T'| \geq \min\{k - 2, m + d'_2 - 1\}.$$

Consequently

$$|N_P(M) \cap T| \geq \min\{k - 1, m + d'_2\},$$

which completes the proof of (1).

Now we prove (2). Clearly for every vertex $u_j \in N_P(M) \cap T \setminus \{u_t\}$, the strong attached pair $\{u_j, u_{j+1}\}$ supported by T is joined to M . If $|N_P(M) \cap T \setminus \{u_t\}| \geq \min\{k - 1, m + d'_2\}$, then the assertion is true. By (1), we assume that $|N_P(M) \cap T| = \min\{k - 1, m + d'_2\}$ and $u_t \in N_P(M) \cap T$.

By Lemma 1 (3), $t \geq \min\{k, h + d_2\} \geq \min\{k - 1, m + d'_2\} + 1$. This implies that there exists at least one vertex in $T \setminus N_P(M)$. We chose a vertex $u_i \in T \setminus N_P(M)$ such that $u_{i+1} \in N_P(M) \cap T$. Then $\{u_i, u_{i+1}\}$ together with $\{u_j, u_{j+1}\}$ for $u_j \in N_P(M) \cup T \setminus \{u_t\}$ are $\min\{k - 1, m + d'_2\}$ strong attached pairs supported by T joined to M .

Let P, H, B, M be defined as in Lemma 3. In the following, we call a strong attached pair which is joined to M a *good pair* (with respect to M). Let $\{u_j, u_{j+1}\}$ be a strong attached pair. If one of the vertices in $\{u_j, u_{j+1}\}$ is joined to M , and the other to $H - M$, then we call it a *better pair* (with respect to M); and if one of the vertices in $\{u_j, u_{j+1}\}$ is joined to M , and the other to $H - B$, then we call it a *best pair* (with respect to M).

3 Proof of Theorem 4

If $k = 2$, then the assertion is Theorem 3. So we assume that $k \geq 3$. Since G is k -connected and $|Y| = k - 2$, G contains an (x, Y, z) -path. In order to prove the theorem, we choose a longest (x, Y, z) -path P in G . Clearly $|V(P)| \geq |Y| + 2 = k$. Moreover, by the k -connectedness of G , for each component H of $G - P$, H is locally k -connected to P , and P is a locally longest (x, Y, z) -path with respect to H . So it is sufficient to prove that:

Proposition 4 *Let G be a graph, P an (x, Y, z) -path of G , where $x, z \in V(G)$, $Y \subset V(G)$, and $|Y| = k - 2$. Suppose that the average degree of vertices in $V(G) \setminus \{x, z\}$ is r . If for each component H of $G - P$, H is locally k -connected to P , and P is a locally longest (x, Y, z) -path with respect to H , then $l(P) \geq r$.*

Proof We prove this proposition by induction on $|V(G - P)|$. If $V(G - P) = \emptyset$, noting that $r \leq |V(G)| - 1$, the result is trivially true. So we assume that $V(G - P) \neq \emptyset$. Let H be a component of $G - P$.

Let $d = |N_P(H)|$, $\theta = |\{x, z\} \cap N_P(H)|$ and $N_P(H) = \{v_1, v_2, \dots, v_d\}$, where v_i , $1 \leq i \leq d$, are in order along P . Then, we have

$$l(P) = l(P[x, v_1]) + \sum_{i=1}^{d-1} l(P[v_i, v_{i+1}]) + l(P[v_d, z]).$$

It is easy to know that $l(P[x, v_1]) + l(P[v_d, z]) \geq 2 - \theta$ and $l(P[v_i, v_{i+1}]) \geq 2$ for $1 \leq i \leq d - 1$. Thus, we have

$$l(P) \geq 2d - \theta.$$

Note that $d \geq k$ by the local k -connectedness of H to P and clearly $\theta \leq 2$. If $r \leq 2k - 2$, then we have $l(P) \geq 2k - 2 \geq r$, and the proof is complete. Thus we assume that

$$r > 2k - 2. \quad (1)$$

Besides, if $d \geq (r + \theta)/2$, then $l(P) \geq r$, and we complete the proof. Thus, we assume that

$$d < (r + \theta)/2. \quad (2)$$

Let $T = \{u_1, u_2, \dots, u_t\}$ be a maximum strong attachment of H to P . Set $S = N_P(H) \setminus T$ and $s = |S|$ (note that $s + t = d$). Let $T_r = \{u_i \in T \setminus \{u_t\} : Y \cap V(P(u_i, u_{i+1})) = \emptyset\}$ and $t_r = |T_r|$.

Clearly, for every transitive strong attached pair $\{u_j, u_{j+1}\}$, where $u_j \in T_r$, we have

$$d_H^*(u_j, u_{j+1}) \geq 2. \quad (3)$$

We distinguish two cases:

Case 1 H is nonseparable.

Let $h = |V(H)|$ and r' the average degree of vertices in $V(H)$. If $r'h + e(P - \{x, z\}, H) \leq rh$, then we consider the graph G' obtained from G by deleting the component H . Note that

$$\begin{aligned} \sum_{v \in V(G') \setminus \{x, z\}} d_{G'}(v) &= r(|V(G)| - 2) - r'h - e(P - \{x, z\}, H) \\ &\geq r(|V(G)| - 2) - rh \\ &= r(|V(G')| - 2). \end{aligned}$$

By the induction hypothesis, we have $l(P) \geq r$, and the proof is complete. Thus we assume that

$$r'h + e(P - \{x, z\}, H) > rh. \quad (4)$$

We use d_1 to denote the number of vertices in $N_P(H)$ which have only one neighbor in $V(H)$, $d_2 = d - d_1$, θ_1 to denote the number of vertices in $\{x, z\}$ which have only one neighbor in $V(H)$ and $\theta_2 = \theta - \theta_1$.

Clearly,

$$r'h \leq h(h-1+d_2) + d_1 \text{ and } e(P - \{x, z\}, H) \leq h(d_2 - \theta_2) + d_1 - \theta_1.$$

Thus, by (4), we have

$$h(h-1+2d_2-\theta_2) + 2d_1 - \theta_1 \geq r'h + e(P - \{x, z\}, H) > rh.$$

Note that $d_1 = d - d_2$ and $\theta_1 = \theta - \theta_2$, we have

$$h(h-1+2d_2-\theta_2) + 2d - 2d_2 - \theta + \theta_2 \geq rh.$$

By (2), we have

$$h(h-1+2d_2-\theta_2) + (r+\theta) - 2d_2 - \theta + \theta_2 > rh.$$

Thus,

$$(h-1)(h+2d_2-r-\theta_2) > 0.$$

This implies that $h \geq 2$ and $h+2d_2 > r+\theta_2 \geq r$, and then $2h+2d_2 > r+2$.

By (1), we have $2h+2d_2 > 2k$, that is

$$h+d_2 > k. \quad (5)$$

By (5) and Lemma 1 (3), $t \geq k$. Since $|Y| \leq k-2$, there exists at least one transitive strong attached pair (u_p, u_{p+1}) supported by T , where $u_p \in T_r$.

Let G' be the subgraph induced by $V(H) \cup \{u_p, u_{p+1}\}$. If $u_p u_{p+1} \notin E(G)$, we add the edge $u_p u_{p+1}$ in G' . Thus G' is 2-connected, and by (4),

$$\begin{aligned} \sum_{v \in V(G') \setminus \{u_p, u_{p+1}\}} d_{G'}(v) &= \sum_{v \in V(H)} d(v) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, H) \\ &= r'h - e(N_P(H) \setminus \{u_p, u_{p+1}\}, H) \\ &\geq rh - e(P - \{x, z\}, H) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, H). \end{aligned}$$

Noting that

$$\begin{aligned} e(P - \{x, z\}, H) &\leq (s+t-\theta)h, \text{ and} \\ e(N_P(H) \setminus \{u_p, u_{p+1}\}, H) &\leq (s+t-2)h, \end{aligned}$$

we have

$$\begin{aligned} \sum_{v \in V(G') \setminus \{u_p, u_{p+1}\}} d_{G'}(v) &\geq rh - (s+t-\theta)h - (s+t-2)h \\ &= (r-2s-2t+\theta+2)h. \end{aligned}$$

By Theorem 3, G' contains a (u_p, u_{p+1}) -path of length at least $r-2s-2t+\theta+2$, which implies that

$$d_H^*(u_p, u_{p+1}) \geq r-2s-2t+\theta+2. \quad (6)$$

Substituting (6) for $d_H^*(u_p, u_{p+1})$ in Lemma 2 and (3) for the other terms, we have

$$l(P) \geq (r-2s-2t+\theta+2) + 2(t_r-1) + 2(s+t-t_r) - \theta \geq r.$$

Case 2 H is separable.

Let B be an endblock of H , b the cut vertex of H contained in B , $M = B - b$, $m = |V(M)|$, and r'' the average degree of the vertices in $V(M)$.

If $r''m + e(P - \{x, z\}, M) + d_M(b) \leq rm$, then we consider the graph G' obtained from G by contracting B . Let H' be the component of $G' - P$ obtained from H by contracting B . By Proposition 3, H' is locally k -connected to P . Clearly P is a locally longest (x, Y, z) -path with respect to H' , and

$$\begin{aligned} \sum_{v \in V(G') \setminus \{x, z\}} d_{G'}(v) &\geq \sum_{v \in V(G) \setminus \{x, z\}} d(v) - r''m - e(P - \{x, z\}, M) - d_M(b) \\ &\geq r(|V(G)| - 2) - rm \\ &= r(|V(G')| - 2). \end{aligned}$$

By the induction hypothesis, $l(P) \geq r$, and the proof is complete. Thus we assume that

$$r''m + e(P - \{x, z\}, M) + d_M(b) > rm. \quad (7)$$

Let $d'_0 = |N_P(H) \setminus N_P(M)|$, d'_1 be the number of vertices in $N_P(M)$ which have only one neighbor in $V(H)$, $d'_2 = d - d'_0 - d'_1$; $\theta'_0 = |\{x, z\} \cap N_P(H) \setminus N_P(M)|$, θ'_1 be the number of vertices in $\{x, z\} \cap N_P(M)$ which have only one neighbor in $V(H)$ and $\theta'_2 = \theta - \theta'_0 - \theta'_1$.

Now we prove that

$$m + d'_2 \geq k - 1. \quad (8)$$

Let B' be an endblock of H other than B , b' the cut vertex of H contained in B' , $M' = B' - b'$ and $m' = |V(M')|$.

By the local k -connectedness of H to P , $|N_P(M')| \geq k - 1$. If $|N_P(M') \setminus N_P(M)| \leq m$, then $d'_2 \geq |N_P(M) \cap N_P(M')| \geq k - 1 - m$, and $m + d'_2 \geq k - 1$, and (8) holds. Thus we assume that $|N_P(M') \setminus N_P(M)| \geq m + 1$. So we have

$$d'_0 \geq m + 1. \quad (9)$$

Clearly,

$$\begin{aligned} r''m &\leq m(m + d'_2) + d'_1, \\ e(P - \{x, z\}, M) &\leq m(d'_2 - \theta'_2) + d'_1 - \theta'_1, \text{ and} \\ d_M(b) &\leq m. \end{aligned}$$

Thus, by (7),

$$m(m + 2d'_2 + 1 - \theta'_2) + 2d'_1 - \theta'_1 \geq r''m + e(P - \{x, z\}, M) + d_M(b) > rm.$$

Noting that $d'_1 = d - d'_0 - d'_2$ and $\theta'_1 = \theta - \theta'_0 - \theta'_2$, we have

$$m(m + 2d'_2 + 1 - \theta'_2) + 2d - 2d'_0 - 2d'_2 - \theta + \theta'_0 + \theta'_2 > rm.$$

By (2) and (9), we have

$$m(m + 2d'_2 + 1 - \theta'_2) + (r + \theta) - 2(m + 1) - 2d'_2 - \theta + \theta'_0 + \theta'_2 > rm.$$

Thus,

$$(m - 1)(m + 2d'_2 - r - \theta'_2) > 2 - \theta'_0 \geq 0.$$

This implies that $m \geq 2$ and $m + 2d'_2 > r + \theta'_2 \geq r$, and then $2m + 2d'_2 > r + 2$. By (1), $2m + 2d'_2 > 2k$, that is $m + d'_2 > k$, and (8) holds.

By Lemma 3 (2), there exist at least $k - 1$ good pairs supported by T with respect to M . Since $|Y| = k - 2$, there exists at least one transitive good pair $\{u_p, u_{p+1}\}$ with respect to M . Similarly there exists at least one transitive good pair $\{u_q, u_{q+1}\}$ with respect to M' .

First we assume that there is a transitive best pair supported by T with respect to M or M' . Without loss of generality, we assume that $\{u_p, u_{p+1}\}$ is a best pair, where $u_p \in N_P(M)$ and $u_{p+1} \in N_P(H - B)$. Consider the subgraph G' induced by $V(B) \cup \{u_p\}$. If $u_p b \notin E(G)$, we add the edge $u_p b$ in G' . Thus G' is 2-connected, and by (7),

$$\begin{aligned} \sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) &= \sum_{v \in V(M)} d(v) - e(N_P(H) \setminus \{u_p\}, M) \\ &= r''m - e(N_P(H) \setminus \{u_p\}, M) \\ &\geq rm - e(P - \{x, z\}, M) - d_M(b) - e(N_P(H) \setminus \{u_p\}, M). \end{aligned}$$

Note that

$$\begin{aligned} e(P - \{x, z\}, M) &\leq (s + t - \theta)m, \\ d_M(b) &\leq m, \text{ and} \\ e(N_P(H) \setminus \{u_p\}, M) &\leq (s + t - 1)m. \end{aligned}$$

We have

$$\begin{aligned} \sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) &\geq rm - (s + t - \theta)m - m - (s + t - 1)m \\ &= (r - 2s - 2t + \theta)m. \end{aligned}$$

By Theorem 3, G' contains a (u_p, b) -path of length at least $r - 2s - 2t + \theta$. It is clear that there is a (b, u_{p+1}) -path in $H - M$ of length at least 2, which implies that

$$d_H^*(u_p, u_{p+1}) \geq r - 2s - 2t + \theta + 2. \quad (10)$$

Substituting (10) for $d_H^*(u_p, u_{p+1})$ in Lemma 2 and (3) for the other terms, we have

$$l(P) \geq (r - 2s - 2t + \theta + 2) + 2(t_r - 1) + 2(s + t - t_r) - \theta \geq r,$$

as required.

So, we assume that there are no transitive best pairs supported by T with respect to M or M' .

Now we assume that there is a transitive better pair (but not best pair) supported by T with respect to M or M' . Without loss of generality, we assume that $\{u_p, u_{p+1}\}$ is a better pair, where $u_p \in N_P(M)$ and $u_{p+1} \in N_P(b)$. Consider the subgraph G' induced by $V(B) \cup \{u_p\}$. If $u_p b \notin E(G)$, we add the edge $u_p b$ in G' . Thus G' is 2-connected and

$$\sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) \geq rm - e(P - \{x, z\}, M) - d_M(b) - e(N_P(H) \setminus \{u_p\}, M).$$

Note that

$$\begin{aligned} e(P - \{x, z\}, M) &\leq (s + t - \theta)m, \text{ and} \\ d_M(b) &\leq m, \end{aligned}$$

and since at least one vertex of u_q and u_{q+1} is not joined to M (otherwise, $\{u_q, u_{q+1}\}$ will be a best pair), we have

$$e(N_P(H) \setminus \{u_p\}, M) \leq (s + t - 2)m.$$

Thus we have

$$\begin{aligned} \sum_{v \in V(G') \setminus \{u_p, b\}} d_{G'}(v) &\geq rm - (s + t - \theta)m - m - (s + t - 2)m \\ &= (r - 2s - 2t + \theta + 1)m. \end{aligned}$$

By Theorem 3, G' contains a (u_p, b) -path of length at least $r - 2s - 2t + \theta + 1$, and then, by $bu_{p+1} \in E(G)$,

$$d_H^*(u_p, u_{p+1}) \geq r - 2s - 2t + \theta + 2.$$

Thus we also have $l(P) \geq r$.

So, we assume that there are no transitive better pairs supported by T with respect to M or M' . Thus $\{u_p, u_{p+1}\} \cap \{u_q, u_{q+1}\} = \emptyset$, and $\{u_p, u_{p+1}\}$ and $\{u_q, u_{q+1}\}$ are two distinct strong attached pairs.

Note that u_p and u_{p+1} are joined to M by two independent edges. Consider the subgraph G' induced by $V(B) \cup \{u_p, u_{p+1}\}$. If $u_p u_{p+1} \notin E(G)$, we add the edge $u_p u_{p+1}$ in G' . Thus G' is 2-connected, and by (7),

$$\begin{aligned} &\sum_{v \in V(G') \setminus \{u_p, u_{p+1}\}} d_{G'}(v) \\ &= \sum_{v \in V(M)} d(v) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, M) + d_M(b) + |\{u_p, u_{p+1}\} \cap N_P(b)| \\ &= r''m + d_M(b) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, M) + |\{u_p, u_{p+1}\} \cap N_P(b)| \\ &\geq rm - e(P - \{x, z\}, M) - e(N_P(H) \setminus \{u_p, u_{p+1}\}, M). \end{aligned}$$

Note that

$$e(P - \{x, z\}, M) \leq (s + t - \theta)m,$$

and since neither u_q and u_{q+1} has neighbors in M (otherwise $\{u_q, u_{q+1}\}$ will be a better pair), we have

$$e(N_P(H) \setminus \{u_p, u_{p+1}\}, M) \leq (s + t - 4)m.$$

Thus, we have

$$\begin{aligned} \sum_{v \in V(G') \setminus \{u_p, u_{p+1}\}} d_{G'}(v) &\geq rm - (s + t - \theta)m - (s + t - 4)m \\ &= (r - 2s - 2t + \theta + 4)m. \end{aligned}$$

By Theorem 3, G' contains a (u_p, u_{p+1}) -path of length at least $(r - 2s - 2t + \theta + 4)m / (1 + m) \geq (r - 2s - 2t + \theta + 4)/2$ (note that $m \geq 1$), which implies that

$$d_H^*(u_p, u_{p+1}) \geq (r - 2s - 2t + \theta + 4)/2.$$

and similarly,

$$d_H^*(u_q, u_{q+1}) \geq (r - 2s - 2t + \theta + 4)/2.$$

Then,

$$d_H^*(u_p, u_{p+1}) + d_H^*(u_q, u_{q+1}) \geq r - 2s - 2t + \theta + 4.$$

Thus, by Lemma 2, we have

$$l(P) \geq (r - 2s - 2t + \theta + 4) + 2(t_r - 2) + 2(s + t - t_r) - \theta \geq r.$$

The proof is complete.

4 Proof of Theorem 8

By the k -connectedness of G , it contains a Y -cycle. If $2e(G)/(n-1) \leq 3$, then the result is trivially true. Thus we assume that $2e(G)/(n-1) > 3$.

We choose a vertex $y \in Y$, and construct a graph G' such that $V(G') = V(G) \cup \{y'\}$, where $y' \notin V(G)$ and $E(G') = E(G) \cup \{vy' : v \in N_G(y)\}$. Clearly, G' is k -connected. Besides, we have that

$$e(G') = e(G) + d_G(y) \text{ and } d_{G'}(y) = d_{G'}(y') = d_G(y),$$

and the order of G' is $n + 1$. Now, by Theorem 4, there exists a $(y, Y \setminus \{y\}, y')$ -path P of length at least

$$\frac{2e(G') - d_{G'}(y) - d_{G'}(y')}{(n+1) - 2} = \frac{2(e(G) + d_G(y)) - 2d_G(y)}{n-1} = \frac{2e(G)}{n-1}.$$

Let uy' be the last edge of P . Then $uy \in E(G)$ and $C = P[y, u]uy$ is a cycle of G passing through all the vertices in Y of length at least $2e(G)/(n-1)$, which completes the proof.

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