Cui Ning and Yaohong Wang*

Low-Regularity Integrator for the Davey–Stewartson System: Elliptic-Elliptic Case

Abstract: In this paper, we introduce a first-order low-regularity integrator for the Davey–Stewartson system in the elliptic-elliptic case. It only requires the boundedness of one additional derivative of the solution to be first-order convergent. By rigorous error analysis, we show that the scheme provides first-order accuracy in $H^\gamma(\mathbb{T}^d)$ for rough initial data in $H^{\gamma+1}(\mathbb{T}^d)$ with $\gamma > \frac{d}{2}$.

Keywords: Davey–Stewartson System, Low-Regularity Method, First-Order Accuracy

1 Introduction

The Davey–Stewartson (DS) systems, originally introduced by Davey and Stewartson [6] in the context of water waves, have extensive applications in ferromagnetism [16], plasma physics [32] and nonlinear optics [21]. Higher-dimensional cases are also attractive [22, 25, 33]. In dimensionless form, they are generally read as the following systems for the amplitude $u(t; x, y)$ and the mean velocity potential $\upsilon(t; x, y)$:

$$\begin{cases} iu_t + \lambda u_{xx} + \mu u_{yy} = -a|u|^2 u + b_1 u\upsilon_x, \\ \nu\upsilon_{xx} + \upsilon_{yy} = b_2(|u|^2)_x, \end{cases}$$

where $\lambda, \mu, \nu, a, b_1, b_2$ are real constants and $\lambda > 0$, $b_1 b_2 > 0$. According to the signs of μ and ν, the DS systems is classified as

(i) elliptic-elliptic or E-E: $\mu > 0$, $\nu > 0$;
(ii) elliptic-hyperbolic or DS-I: $\mu > 0$, $\nu < 0$;
(iii) hyperbolic-elliptic or DS-II: $\mu < 0$, $\nu > 0$;
(iv) hyperbolic-hyperbolic: $\mu < 0$, $\nu < 0$.

Note that the last case does not occur in the context of water surface waves.

The Cauchy problem for DS systems is widely studied. Ghidaglia and Saut [7] proved the existence, uniqueness and continuous dependence with respect to the data for E-E, DS-I and DS-II cases; see also [4, 5]. Tsutsumi [26] obtained the L^p-decay estimates of solutions of the DS-I case for $2 < p < \infty$; see also[8, 9]. Recently, Nachman, Regev and Tataru [20] proved the global well-posedness and scattering for defocusing DS-II using a Plancherel theorem. Linares and Ponce [17], Hayashi and Saut [10] used dispersive methods to establish the local well-posedness and global existence. Lu and Wu [18] used a variational approach to give a dichotomy of blow-up and scattering for the elliptic-elliptic generalized DS systems; see also [23].

In the context of numerical approaches, splitting methods are popular for smooth initial conditions. Besse, Mauser and Stimming [2] showed numerical results for the blow-up of focusing E-E and for exact soli-

***Corresponding author: Yaohong Wang,** Center for Applied Mathematics, Tianjin University, Tianjin, 300072, P. R. China, e-mail: yaohong@tju.edu.cn. https://orcid.org/0000-0002-5088-1673
Cui Ning, School of Financial Mathematics and Statistics, Guangdong University of Finance, Guangzhou, Guangdong 510521, P. R. China, e-mail: cuiningmath@gmail.com

ton type solutions of DS-II by time splitting spectral method; see also [27, 28]. Besides, Besse [1] introduced a new numerical scheme for the nonlinear Schrödinger equation and the Davey–Stewartson systems as the first source for the relaxation scheme. Klein and Stoilov [14] discussed spectral algorithms for the numerical scattering for the defocusing DS-II with initial data having compact support on a disk; see also [12, 13]. Muslu [19] studied blow-up solutions for the purely elliptic generalized Davey–Stewartson system by using a relaxation numerical method. White and Weideman [28] presented the numerical simulations of dromions for DS-I and DS-II by split-step Fourier method; see also [27].

Recently, the DS systems with rough data initial condition have also gained a lot of attention in the context of numerical integrations. The concept of the low-regularity integrator is introduced to express a series of numerical methods which bring down the regularity requirement in Sobolev space sense. The systems can be viewed as a family of nonlinear Schrödinger (NLS) type equations. For NLS equations, Ostermann and Schratz [24] proved first-order convergence in H^γ for solutions in $H^{\gamma+1}$ with $\gamma > d/2$ of the derived low-regularity exponential type integrators. For cubic NLS equations, Knöller, Ostermann and Schratz [15] presented a new type of integrator, which showed in one dimension and in high dimensions the second-order convergence in H^γ for solutions in $H^{\gamma+2}$ and $H^{\gamma+3}$ respectively. Wu and Yao [31] proposed a new scheme which provides the first-order accuracy without loss of regularity in the one-dimensional case, that is, the first-order convergence in H^γ for H^γ-data. For the KdV equation, Hofmanová and Schratz [11] gave a first-order numerical scheme in H^1 for initial value from H^3, and Wu and Zhao [29, 30] proposed a new class of embedded low-regularity integrators, which obtain the first-order and the second-order accuracy in H^γ for initial data in $H^{\gamma+1}$ and $H^{\gamma+3}$ respectively.

Inspired by the above low-regularity results [15, 24], we derive a numerical scheme which only requires the boundedness of one additional derivative of the solution with rough initial data to get the first-order convergence.

In this paper, we only focus on the E-E case in which we set $\mu = \nu = 1$, under the rough initial data on a torus,

$$\begin{cases} iu_t + \Delta u = -a|u|^2 u + b_1 u v_{x_1}, & t > 0,\ \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{T}^d, \\ \quad -\Delta v = b_2(|u|^2)_{x_1}, \end{cases} \tag{1.1}$$

where $a > 0$, $b = b_1 b_2 > 0$, $\mathbb{T} = (0, 2\pi)$, $u = u(t, \boldsymbol{x})\colon \mathbb{R}^+ \times \mathbb{T}^d \to \mathbb{C}$ is the unknown and the given initial data is $u_0(\boldsymbol{x}) = u(0, \boldsymbol{x}) \in H^\gamma(\mathbb{T}^d)$ with some $0 \le \gamma < \infty$. Here u is the (complex) amplitude of the wave and v is the (real) velocity potential of the wave movement.

The paper is organized as follows. We construct the detailed numerical integrator and introduce the main convergence theorem in Section 2. The first-order convergence analysis is given in Section 3. The numerical experiments are presented to validate the numerical scheme in Section 4, and concluding remarks will be made in Section 5.

2 Numerical Integrator

2.1 Notation

Firstly, We present some notation and tools for future derivation and analysis. We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \le CB$ for some large absolute constant $C > 0$ which may vary from line to line, and we denote $A \sim B$ for $A \lesssim B \lesssim A$. For $\boldsymbol{k} := (k_1, \dots, k_d) \in \mathbb{T}^d$, $\boldsymbol{x} := (x_1, \dots, x_d) \in \mathbb{T}^d$, we denote

$$\boldsymbol{k} \cdot \boldsymbol{x} = k_1 x_1 + \dots + k_d x_d, \quad |\boldsymbol{k}|^2 = |k_1|^2 + \dots + |k_d|^2.$$

The Fourier transform of a function f on $\mathbb{T}^d$ is defined by

$$\hat{f}_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\, d\boldsymbol{x}.$$

For $f \in L^2(\mathbb{T}^d)$, we denote its Fourier expansion by $f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbf{Z}^d} \hat{f}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. Furthermore, we define the operator $(-\Delta)^{-1}$ for some function $f(\boldsymbol{x})$ on $\mathbb{T}^d$ as

$$\widehat{(-\Delta)^{-1}f} = \begin{cases} |\boldsymbol{k}|^{-2}\hat{f}_{\boldsymbol{k}} & \text{if } \boldsymbol{k} \neq 0, \\ 0 & \text{if } \boldsymbol{k} = 0. \end{cases}$$

Let $H^\gamma(\mathbb{T}^d)$ be the Sobolev space of H^γ functions defined on the d-dimensional torus $\mathbb{T}^d$, and its norm $\|\cdot\|_{H^\gamma}$ is defined as

$$\|f\|^2_{H^\gamma(\mathbb{T}^d)} = \sum_{\boldsymbol{k}\in\mathbf{Z}^d} (1+|\boldsymbol{k}|)^{2\gamma}|\hat{f}_{\boldsymbol{k}}|^2.$$

Throughout the paper, we will exploit the well known bilinear estimate $\|fg\|_{H^\gamma} \leq C_{\gamma,d}\|f\|_{H^\gamma}\|g\|_{H^\gamma}$, which holds for $\gamma > \frac{d}{2}$ and some constant $C_{\gamma,d} > 0$. Moreover, we will make frequent use of the isometric property of the free Schrödinger group $e^{it\Delta}$, $\|e^{it\Delta}f\|_{H^\gamma} = \|f\|_{H^\gamma}$ for all $f \in H^\gamma$ and $t \in \mathbb{R}$. Furthermore, we define a function

$$\varphi(z) = \begin{cases} \dfrac{e^z - 1}{z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{2.1}$$

2.2 Construction of the Numerical Integrator

For simplicity of notation, in the following, we shall omit the spatial variable $\boldsymbol{x}$ of the involved time-space dependent functions, e.g. $u(t) = u(t, \boldsymbol{x})$. In addition, we denote $\tau > 0$ as the time step and $t_n = n\tau$ as the time steps.

For the DS system in the E-E case (1.1), we can reduce the system to a non-local Schrödinger equation

$$\begin{cases} iu_t + \Delta u + uE(|u|^2) = 0, \\ \qquad\qquad\qquad v = -b_2\partial_{x_1}\Delta^{-1}|u|^2, \end{cases}$$

where the operator E is defined by

$$Ef = (a + b\partial^2_{x_1}\Delta^{-1})f, \quad b = b_1b_2. \tag{2.2}$$

Using Duhamel's formula, we have

$$u(t) = e^{it\Delta}u_0(\boldsymbol{x}) + i\int_0^t e^{i(t-s)\Delta}u(s)E(|u(s)|^2)\,ds.$$

We introduce the twisted variable

$$w(t) := \mathrm{e}^{-it\Delta}u(t). \tag{2.3}$$

Note that the twisted variable satisfies

$$\partial_t w(t) = i\mathrm{e}^{-it\Delta}\big[e^{it\Delta}w(t)\cdot E\big(|e^{it\Delta}w(t)|^2\big)\big], \quad w(0) = u_0, \tag{2.4}$$

with the mild solution given by

$$w(t_{n+1}) = w(t_n) + i\int_0^\tau e^{-i(t_n+s)\Delta}\big[e^{i(t_n+s)\Delta}w(t_n+s)\cdot E\big(|e^{i(t_n+s)\Delta}w(t_n+s)|^2\big)\big]\,ds. \tag{2.5}$$

Since we only need first-order convergent scheme, we can simplify the above scheme using $w(t_n+s) \approx w(t_n)$ and get

$$w(t_{n+1}) = w(t_n) + \Phi^n(w(t_n)) + \mathcal{R}_1^n,$$

where

$$\Phi^n(w(t_n)) = i\int_0^\tau e^{-i(t_n+s)\Delta}\big[e^{i(t_n+s)\Delta}w(t_n)\cdot E\big(|e^{i(t_n+s)\Delta}w(t_n)|^2\big)\big]\,ds$$

and

$$\mathcal{R}_1^n = i\int_0^\tau e^{-i(t_n+s)\Delta}\big[e^{i(t_n+s)\Delta}w(t_n+s)\cdot E(|e^{i(t_n+s)\Delta}w(t_n+s)|^2) - e^{i(t_n+s)\Delta}w(t_n)\cdot E(|e^{i(t_n+s)\Delta}w(t_n)|^2)\big]\,ds. \tag{2.6}$$

Here $\mathcal{R}_1^n$ can be treated as a high-order term without additional regularity assumption. By Fourier expansion, we get

$$\Phi^n(w(t_n)) = i\sum_{\boldsymbol{\Omega}\in\mathbf{z}^d}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbf{z}^d\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}}\Big[a+b\frac{(j_1+k_1)^2}{|\boldsymbol{j}+\boldsymbol{k}|^2}\Big]\bar{\hat{w}}_{\boldsymbol{j}}\hat{w}_{\boldsymbol{k}}\hat{w}_{\boldsymbol{l}}\,e^{i\boldsymbol{\Omega}\cdot\boldsymbol{x}}\int_0^\tau e^{i(t_n+s)(|\boldsymbol{\Omega}|^2+|\boldsymbol{j}|^2-|\boldsymbol{k}|^2-|\boldsymbol{l}|^2)}\,ds, \tag{2.7}$$

where we set $\bar{\hat{w}}_{\boldsymbol{j}} = \bar{\hat{w}}_{\boldsymbol{j}}(t_n)$, $\hat{w}_{\boldsymbol{k}} = \hat{w}_{\boldsymbol{k}}(t_n)$ and $\hat{w}_{\boldsymbol{l}} = \hat{w}_{\boldsymbol{l}}(t_n)$ for short. And we denote $\alpha = |\boldsymbol{\Omega}|^2 + |\boldsymbol{j}|^2 - |\boldsymbol{k}|^2 - |\boldsymbol{l}|^2$ and $\beta = 2\boldsymbol{j}\cdot\boldsymbol{k} + 2\boldsymbol{j}\cdot\boldsymbol{l} + 2\boldsymbol{k}\cdot\boldsymbol{l}$. Then we have $\alpha = 2|\boldsymbol{j}|^2 + \beta$.

The integration in (2.7) cannot be transformed into the physical space directly. Inspired by [24], we only choose the dominant quadratic term $2|\boldsymbol{j}|^2$ so that the integration can be carried out fully in Fourier space as

$$\int_0^\tau e^{2is|\boldsymbol{j}|^2}\,ds = \tau\varphi(2i\tau|\boldsymbol{j}|^2).$$

Hence we have

$$\begin{aligned}\Phi^n(w(t_n)) &= i\sum_{\boldsymbol{\Omega}\in\mathbf{z}^d}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbf{z}^d\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} e^{it_n\alpha}\Big[a+b\frac{(j_1+k_1)^2}{|\boldsymbol{j}+\boldsymbol{k}|^2}\Big]\bar{\hat{w}}_{\boldsymbol{j}}\hat{w}_{\boldsymbol{k}}\hat{w}_{\boldsymbol{l}}\,e^{i\boldsymbol{\Omega}\cdot\boldsymbol{x}}\tau\varphi(2i\tau|\boldsymbol{j}|^2) + \mathcal{R}_2^n\\ &= i\tau e^{-it_n\Delta}\big[e^{it_n\Delta}w(t_n)\cdot E(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w(t_n)}\cdot e^{it_n\Delta}w(t_n))\big] + \mathcal{R}_2^n,\end{aligned}$$

where

$$\mathcal{R}_2^n = i\sum_{\boldsymbol{\Omega}\in\mathbf{z}^d}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbf{z}^d\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} e^{it_n\alpha}\Big[a+b\frac{(j_1+k_1)^2}{|\boldsymbol{j}+\boldsymbol{k}|^2}\Big]\bar{\hat{w}}_{\boldsymbol{j}}\hat{w}_{\boldsymbol{k}}\hat{w}_{\boldsymbol{l}}\,e^{i\boldsymbol{\Omega}\cdot\boldsymbol{x}}\int_0^\tau e^{2is|\boldsymbol{j}|^2}(e^{is\beta}-1)\,ds \tag{2.8}$$

and $\mathcal{R}_2^n$ can also be treated as a high-order term. However, one loss of spatial derivative comes when we drop this term.

For convenience, let us denote

$$\Psi^n(f) = i\tau e^{-it_n\Delta}\big[e^{it_n\Delta}f\cdot E(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}f}\cdot e^{it_n\Delta}f)\big]. \tag{2.9}$$

By dropping high-order terms, we would get

$$w(t_{n+1}) \approx w(t_n) + \Psi^n(w(t_n)).$$

Hence we finish the construction of the first-order numerical integrator

$$w^{n+1} = w^n + i\tau e^{-it_n\Delta}\big[e^{it_n\Delta}w^n\cdot E(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w^n}\cdot e^{it_n\Delta}w^n)\big], \quad n\ge 0,\ w^0 = u_0, \tag{2.10}$$

where $w^n = w^n(\boldsymbol{x})$.

By reversing the twisted variable (2.3) in (2.10), we deduce the scheme of the first-order low-regularity integrator (LRI) for solving DS system (1.1): denoting $u^n = u^n(\boldsymbol{x})$ as the numerical solution,

$$u^n = e^{i\tau\Delta}u^{n-1} + i\tau e^{i\tau\Delta}\big[u^{n-1}\cdot E(\varphi(-2i\tau\Delta)\overline{u^{n-1}}\cdot u^{n-1})\big] \tag{2.11}$$

for $n = 1, 2, 3, \dots$, with φ in (2.1) and E in (2.2).

Based on DS system (1.1), we can write the numerical solution of v: denoting $v^n = v^n(\boldsymbol{x})$ as the numerical solution, for $n = 1, 2, 3, \dots$,

$$v^n = -b_2\partial_{x_1}\Delta^{-1}|u^n|^2. \tag{2.12}$$

The proposed schemes (2.11) and (2.12) are fully explicit and efficient.

2.3 Main Convergence Results

We now state the main result for the convergence of the proposed LRI schemes.

Theorem 2.1. *Let u^n and v^n be the numerical solution of DS system* (1.1) *obtained from LRI schemes* (2.11) *and* (2.12) *up to some fixed time $T > 0$. Under the assumption $u_0(\boldsymbol{x}) \in H^{\gamma+1}(\mathbb{T}^d)$ for some $\gamma > \frac{d}{2}$, there exist constants $\tau_0 > 0$ and $C > 0$ such that, for any $0 < \tau \le \tau_0$, we have*

$$\|u(t_n) - u^n\|_{H^\gamma} \le C\tau \quad \text{and} \quad \|v(t_n) - v^n\|_{H^{\gamma+1}} \le C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}, \tag{2.13}$$

where the constants τ_0 and C depend only on T and $\sup_{0\le t\le T}\|u(t)\|_{H^{\gamma+1}}$.

Remark 2.2. For initial data in $H^{\gamma+1}$, we see that LRI scheme (2.11) for u can get the first-order convergence rate in H^γ, while LRI scheme (2.12) for v can get first-order convergence rate in $H^{\gamma+1}$.

Remark 2.3. As shown above, the construction of the numerical integrator is independent of the non-local operator E. Our scheme can be easily extended to the Cauchy problem

$$\begin{cases} iu_t + \Delta u + g(u) = 0, \\ \qquad\qquad u(0) = u(0, \boldsymbol{x}), \end{cases} \tag{2.14}$$

where $g(u)$ is chosen to be a combined nonlinearity, defined as

$$g(u) = \lambda_1 E(|u|^2)u + \lambda_2|u|^2u + \lambda_3(W * |u|^2)u,$$

$\lambda_j \in \mathbb{R}$ for $j = 1, 2, 3$, and the third term is a Hartree type nonlinear term. Here W is an even, real-valued potential, and $W \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $q \ge 1$. It was proved in [3] that problem (2.14) is locally well-posed in $H^s(\mathbb{R}^d)$ when $s \ge \frac{d}{2} - 1$.

Let us denote

$$f(|u|^2) = \lambda_1 E(|u|^2) + \lambda_2|u|^2 + \lambda_3 W * |u|^2;$$

then we have $g(u) = f(|u|^2)u$. Based on the reduction above, we can construct a similar numerical integrator for (2.14),

$$u^n = e^{i\tau\Delta}u^{n-1} + i\tau e^{i\tau\Delta}[u^{n-1} \cdot f(\varphi(-2i\tau\Delta)\overline{u^{n-1}} \cdot u^{n-1})],$$

with the first-order accuracy

$$\|u(t_n) - u^n\|_{H^\gamma} \le C\tau, \quad n = 0, 1, \ldots, \frac{T}{\tau}.$$

Remark 2.4. Schemes (2.11), (2.12) work for the whole space problem with decaying functions too. Taking the solitary wave case, we can choose a large torus so that boundary errors are negligible.

3 The First-Order Convergence Analysis

In this section, we will provide the rigorous proof of the convergence result. Since the twisted variable in (2.3) is isometric in the Sobolev space H^γ, we will prove the convergence result (2.13) for the mild solution $w(t_{n+1})$ (2.5) and w^{n+1} (2.10).

Recalling the derivation of the scheme in Section 2, we write that

$$w(t_{n+1}) = w(t_n) + \Psi^n(w(t_n)) + \mathcal{R}_1^n + \mathcal{R}_2^n \quad \text{and} \quad w^{n+1} = w^n + \Psi^n(w^n).$$

Then we have $w(t_{n+1}) - w^{n+1} = \mathcal{L}^n + \mathcal{S}^n$, where

$$\mathcal{L}^n = \mathcal{R}_1^n + \mathcal{R}_2^n \quad \text{and} \quad \mathcal{S}^n = w(t_n) + \Psi^n(w(t_n)) - w^n - \Psi^n(w^n).$$

Then we will analyze the local error and the stability of the numerical propagator in the following. We start with some useful lemmas.

Lemma 3.1. *Let $\gamma > \frac{d}{2}$. Assume that $u_0 \in H^{\gamma+1}$; then there exist constants τ_0 and $C > 0$ such that, for any $0 < \tau \le \tau_0$, the estimate $\|\mathcal{R}_1^n\|_{H^\gamma} + \|\mathcal{R}_2^n\|_{H^\gamma} \le C\tau^2$ holds, where τ_0 and C depend only on T and $\sup_{0\le t\le T}\|w(t)\|_{H^{\gamma+1}}$.*

Proof. Firstly, by the definition of $\mathcal{R}_1^n$ in (2.6), we have

$$\begin{aligned}\|\mathcal{R}_1^n\|_{H^\gamma} \le \int_0^\tau \big\| & e^{i(t_n+s)\Delta} w(t_n+s)\cdot E(|e^{i(t_n+s)\Delta}w(t_n+s)|^2) \\ & - e^{i(t_n+s)\Delta}w(t_n)\cdot E(|e^{i(t_n+s)\Delta}w(t_n)|^2)\big\|_{H^\gamma}\,ds \\ \le \int_0^\tau & \|w(t_n+s)-w(t_n)\|_{H^\gamma}\cdot\|E(|e^{i(t_n+s)\Delta}w(t_n+s)|^2)\|_{H^\gamma}\,ds \\ & + \int_0^\tau \|w(t_n)\|_{H^\gamma}\cdot\|E(|e^{i(t_n+s)\Delta}w(t_n+s)|^2) - E(|e^{i(t_n+s)\Delta}w(t_n)|^2)\|_{H^\gamma}\,ds.\end{aligned}$$

Since $\|Ef\|_{H^\gamma} \le \|f\|_{H^\gamma}$, then for $\gamma > \frac{d}{2}$, we have

$$\begin{aligned}\|\mathcal{R}_1^n\|_{H^\gamma} &\le \int_0^\tau \|w(t_n+s)-w(t_n)\|_{H^\gamma}\|w(t_n+s)\|_{H^\gamma}^2\,ds \\ &\quad + \int_0^\tau \|w(t_n)\|_{H^\gamma}\||w(t_n+s)|^2-|w(t_n)|^2\|_{H^\gamma}\,ds \\ &\le \int_0^\tau \|w(t_n+s)-w(t_n)\|_{H^\gamma}\|w(t_n+s)\|_{H^\gamma}^2\,ds \\ &\quad + \int_0^\tau \|w(t_n)\|_{H^\gamma}(\|w(t_n+s)\|_{H^\gamma}+\|w(t_n)\|_{H^\gamma})\|w(t_n+s)-w(t_n)\|_{H^\gamma}\,ds.\end{aligned}$$

By (2.4), we have

$$\begin{aligned}\|w(t_n+s)-w(t_n)\|_{H^\gamma} &\le \int_0^s \|\partial_t w(t_n+t)\|_{H^\gamma}\,dt \\ &\le \int_0^s \big\|e^{i(t_n+t)\Delta}w(t_n+t)\cdot E(|e^{i(t_n+t)\Delta}w(t_n+t)|^2)\big\|_{H^\gamma}\,dt \\ &\le \int_0^s \|w(t_n+t)\|_{H^\gamma}\|w(t_n+t)\|_{H^\gamma}^2\,dt \\ &\le Cs\sup_{0\le t\le s}\|w(t_n+t)\|_{H^\gamma}^3. \qquad (3.1)\end{aligned}$$

Together with (3.1), we obtain

$$\|\mathcal{R}_1^n\|_{H^\gamma} \le C\tau^2 \sup_{0\le t\le T}\|w(t)\|_{H^\gamma}^5. \qquad (3.2)$$

Next, according to the definition of $\mathcal{R}_2^n$ in (2.8), we get

$$\|\mathcal{R}_2^n\|_{H^\gamma}^2 = \sum_{\boldsymbol{\Omega}\in\mathbf{Z}^d}(1+|\boldsymbol{\Omega}|)^{2\gamma}\Bigg|\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbf{Z}^d\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} e^{it_n\alpha}\bigg[a+b\frac{(j_1+k_1)^2}{|\boldsymbol{j}+\boldsymbol{k}|^2}\bigg]\bar{\hat{w}}_{\boldsymbol{j}}\hat{w}_{\boldsymbol{k}}\hat{w}_{\boldsymbol{l}}\,e^{i\boldsymbol{\Omega}\cdot\boldsymbol{x}}\int_0^\tau e^{2is|\boldsymbol{j}|^2}(e^{is\beta}-1)\,ds\Bigg|^2.$$

Since $a > 0$, $b > 0$, then we know that

$$\bigg|a+b\frac{(j_1+k_1)^2}{|\boldsymbol{j}+\boldsymbol{k}|^2}\bigg| \le a+b \le C.$$

Combining with $|e^{is\beta}-1| \sim |s\beta|$, we have

$$\|\mathcal{R}_2^n\|_{H^\gamma}^2 \le C\sum_{\boldsymbol{\Omega}\in\mathbf{Z}^d}(1+|\boldsymbol{\Omega}|)^{2\gamma}\sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbf{Z}^d\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}}\bigg|\bar{\hat{w}}_{\boldsymbol{j}}\hat{w}_{\boldsymbol{k}}\hat{w}_{\boldsymbol{l}}\int_0^\tau s\beta\,ds\bigg|^2.$$

Recall that $\beta = 2\boldsymbol{j}\cdot\boldsymbol{k} + 2\boldsymbol{j}\cdot\boldsymbol{k} + 2\boldsymbol{k}\cdot\boldsymbol{l}$; then we have

$$|\beta| \le C(1+|\boldsymbol{j}|)(1+|\boldsymbol{k}|)(1+|\boldsymbol{l}|).$$

Furthermore, we obtain

$$\begin{aligned}
\|\mathcal{R}_2^n\|_{H^\gamma}^2 &\le C\tau^4 \sum_{\boldsymbol{\Omega}\in\mathbf{z}^d}(1+|\boldsymbol{\Omega}|)^{2\gamma} \sum_{\substack{\boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\in\mathbf{z}^d\\ \boldsymbol{\Omega}=\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}}} [(1+|\boldsymbol{j}|)(1+|\boldsymbol{k}|)(1+|\boldsymbol{l}|)|\hat{w}_{\boldsymbol{j}}|\cdot|\hat{w}_{\boldsymbol{k}}|\cdot|\hat{w}_{\boldsymbol{l}}|]^2\\
&\le C\tau^4 \sup_{0\le t\le T}\|w(t)\|_{H^{\gamma+1}}^6. \qquad (3.3)
\end{aligned}$$

Estimates (3.2) and (3.3) give the desired estimate and finish the proof of the lemma. □

By Lemma 3.1, we will show the local error $\mathcal{L}^n$ and have the following lemma.

Lemma 3.2 (Local Error). *Let $\gamma > \frac{d}{2}$. Assume that $u_0 \in H^{\gamma+1}$; then there exist constants τ_0 and $C > 0$ such that, for any $0 < \tau \le \tau_0$, the inequality $\|\mathcal{L}^n\|_{H^\gamma} \le C\tau^2$ holds, where τ_0 and C depend only on T and $\sup_{0\le t\le T}\|w(t)\|_{H^{\gamma+1}}$.*

Proof. Based on the definition of $\mathcal{L}^n$, we have $\|\mathcal{L}^n\|_{H^\gamma} \le \|\mathcal{R}_1^n\|_{H^\gamma} + \|\mathcal{R}_2^n\|_{H^\gamma}$. Using Lemma 3.1, $\|\mathcal{L}^n\|_{H^\gamma} \le C\tau^2$, where C depends on $\sup_{0\le t\le T}\|w(t)\|_{H^{\gamma+1}}$. This finishes the proof of lemma. □

For the numerical propagator $\Psi^n(f)$ defined in (2.9), we have the following stability result.

Lemma 3.3 (Stability). *Let $\gamma > \frac{d}{2}$. Assume that $u_0 \in H^{\gamma+1}$; then there exist constants τ_0 and $C > 0$ such that, for any $0 < \tau \le \tau_0$, the following inequality holds:*

$$\|\mathcal{S}^n\|_{H^\gamma} \le (1+C\tau)\|w(t_n) - w^n\|_{H^\gamma} + C\tau\|w(t_n) - w^n\|_{H^\gamma}^3,$$

where τ_0 and C depend only on T and $\sup_{0\le t\le T}\|w(t)\|_{H^\gamma}$.

Proof. According to the definition of $\mathcal{S}^n$, we have

$$\|\mathcal{S}^n\|_{H^\gamma} \le \|w(t_n) - w^n\|_{H^\gamma} + \|\Psi^n(w(t_n)) - \Psi^n(w^n)\|_{H^\gamma}.$$

Recall that

$$\Psi^n(f) = i\tau e^{-it_n\Delta}\big[e^{it_n\Delta}f\cdot E\big(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}f}\cdot e^{it_n\Delta}f\big)\big].$$

Then we have

$$\begin{aligned}
&\|\Psi^n(w(t_n)) - \Psi^n(w^n)\|_{H^\gamma}\\
&\quad\le \tau\big\|e^{it_n\Delta}w(t_n)\cdot E\big(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w(t_n)}\cdot e^{it_n\Delta}w(t_n)\big) - e^{it_n\Delta}w^n\cdot E\big(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w^n}\cdot e^{it_n\Delta}w^n\big)\big\|_{H^\gamma}\\
&\quad\le \tau\|w(t_n)-w^n\|_{H^\gamma}\cdot\big\|E\big(\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w(t_n)}\cdot e^{it_n\Delta}w(t_n)\big)\big\|_{H^\gamma}\\
&\qquad\quad + \tau\big\|e^{it_n\Delta}w^n\cdot E\big[\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w(t_n)}\cdot e^{it_n\Delta}w(t_n) - \varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w^n}\cdot e^{it_n\Delta}w^n\big]\big\|_{H^\gamma}\\
&\quad\le C\tau\|w(t_n)-w^n\|_{H^\gamma}\cdot\|\overline{e^{it_n\Delta}w(t_n)}\cdot e^{it_n\Delta}w(t_n)\|_{H^\gamma}\\
&\qquad\quad + C\tau\|w^n\|_{H^\gamma}\big\|\varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w(t_n)}\cdot e^{it_n\Delta}w(t_n) - \varphi(-2i\tau\Delta)\,\overline{e^{it_n\Delta}w^n}\cdot e^{it_n\Delta}w^n\big\|_{H^\gamma}\\
&\quad\le C\tau\|w(t_n)-w^n\|_{H^\gamma}\|w(t_n)\|_{H^\gamma}^2 + C\tau\big(\|w(t_n)-w^n\|_{H^\gamma} + \|w(t_n)\|_{H^\gamma}\big)\\
&\qquad\quad \cdot\big(\|w(t_n)-w^n\|_{H^\gamma}\|w(t_n)\|_{H^\gamma} + \|w^n\|_{H^\gamma}\|w(t_n)-w^n\|_{H^\gamma}\big)\\
&\quad\le C\tau\|w(t_n)-w^n\|_{H^\gamma}\|w(t_n)\|_{H^\gamma}^2 + C\tau\big(\|w(t_n)-w^n\|_{H^\gamma} + \|w(t_n)\|_{H^\gamma}\big)\\
&\qquad\quad \cdot\big(\|w(t_n)-w^n\|_{H^\gamma}\|w(t_n)\|_{H^\gamma} + \|w(t_n)-w^n\|_{H^\gamma}^2\big)\\
&\quad\le C\tau\|w(t_n)-w^n\|_{H^\gamma} + C\tau\|w(t_n)-w^n\|_{H^\gamma}^3,
\end{aligned}$$

where C depends on $\sup_{0\le t\le T}\|w(t)\|_{H^\gamma}$.

Hence we have

$$\|\mathcal{S}^n\|_{H^\gamma} \le (1+C\tau)\|w(t_n) - w^n\|_{H^\gamma} + C\tau\|w(t_n) - w^n\|_{H^\gamma}^3.$$

This proves this lemma. □

Together with the local error estimate and the stability result, we give the proof of Theorem 2.1. From Lemma 3.2 and 3.3, we have

$$\|w(t_{n+1}) - w^{n+1}\|_{H^\gamma} \le C\tau^2 + (1 + C\tau)\|w(t_n) - w^n\|_{H^\gamma} + C\tau\|w(t_n) - w^n\|^3_{H^\gamma},$$

where $n = 0, 1, 2, \dots, \frac{T}{\tau} - 1$ and C depends on T and $\sup_{0\le t\le T}\|w(t)\|_{H^{\gamma+1}}$.

By iteration and Gronwall's inequality, we get

$$\|w(t_{n+1}) - w^{n+1}\|_{H^\gamma} \le C\tau^2 \sum_{j=0}^{n}(1 + C\tau)^j \le C\tau, \quad n = 0, 1, 2, \dots, \frac{T}{\tau} - 1.$$

Since the twisting of variable (2.3) is isometric, we get the first estimate in (2.13).

From DS system (1.1), we know that $v(t) = -b_2\partial_{x_1}\Delta^{-1}|u(t)|^2$. Meanwhile, by using the first estimate in (2.13), we have

$$\begin{aligned}\|v(t_n) - v^n\|_{H^{\gamma+1}} &\le \|-b_2\partial_{x_1}\Delta^{-1}(|u(t_n)|^2 - |u^n|^2)\|_{H^{\gamma+1}} \\ &\le C\||u(t_n)|^2 - |u^n|^2\|_{H^\gamma} \le C\|u(t_n) - u^n\|_{H^\gamma}\big(\|u(t_n) - u^n\|_{H^\gamma} + \|u(t_n)\|_{H^\gamma}\big) \le C\tau,\end{aligned}$$

where C depends on T and $\sup_{0\le t\le T}\|u(t)\|_{H^{\gamma+1}}$. This proves Theorem 2.1.

4 Numerical Experiments

In this section, we present the numerical experiments of scheme (2.11) to justify the convergence theorem. Without loss of generality, we only test the 2-dimensional case, i.e. $d = 2$. Since v_n is calculated via equation (2.12), which will not lose any regularity and keeps first-order convergence (see Theorem 2.1), we only need to test u_n in this section. To get an initial datum with the desired regularity, we construct $u_0(\boldsymbol{x})$ by the following strategy [24]. Choose N as an even integer, and discretize the spatial domain $\mathbb{T}$ with grid points $x_l^j = l\frac{2\pi}{N}$ for $l = 0, \dots, N$ and $j = 1, 2$. Take a uniformly distributed random vector $\mathrm{rand}(N, 1) \in [0, 1]^N$ and an $N \times 2$ vector $\mathcal{U}$ whose elements are defined as

$$\mathcal{U}^{l,j} = \mathrm{rand}(N, 1) + i\,\mathrm{rand}(N, 1) \quad (l = 0, \dots, N,\ j = 1, 2).$$

In our numerical experiments, we set

$$u_0(\boldsymbol{x}) := \frac{|\partial_{\boldsymbol{x},N}|^{-\gamma}\mathcal{U}}{\||\partial_{\boldsymbol{x},N}|^{-\gamma}\mathcal{U}\|_{L^\infty}}, \quad \boldsymbol{x} \in \mathbb{T}^2,$$

where the pseudo-differential operator $|\partial_{\boldsymbol{x},N}|^{-\gamma}$ for $\gamma \ge 0$ reads as follows:

$$(|\partial_{\boldsymbol{x},N}|^{-\gamma})_{\boldsymbol{k}} = \begin{cases} |\boldsymbol{k}|^{-\gamma} & \text{if } \boldsymbol{k} \ne \boldsymbol{0}, \\ 0 & \text{if } \boldsymbol{k} = \boldsymbol{0}, \end{cases}$$

for Fourier modes $\boldsymbol{k} = (k_1, k_2)$, and $k_j = -\frac{N}{2}, \dots, \frac{N}{2} - 1$ for $j = 1, 2$. Thus we get $u_0 \in H^\gamma(\mathbb{T}^2)$ for any $\gamma \ge 0$.

We present the numerical error $u^n - u_{\mathrm{ref}}$ in the L^2-norm and $H^{\gamma-1}$-norm at the final time $t_n = T = 2.0$ under initial data with different regularities $\gamma = 3$ and $\gamma = 4$, and the space resolutions N are chosen to be 2^6, 2^7 and 2^8, where the reference solution u_{ref} is obtained numerically by scheme (2.11) with $\tau = 10^{-5}$. Figure 1 and Figure 2 validate the first-order convergence in both L^2-norm and $H^{\gamma-1}$-norm.

5 Conclusion

In this work, we have numerically studied the DS system in the E-E case on the torus under rough initial data. By some rigorous tools from harmonic analysis, we established the sharp convergence theorem of the low-regularity integrator. The theoretical result shows that the presented integrator can reach first-order accuracy in the space H^γ with initial data from $H^{\gamma+1}$ for any $\gamma > \frac{d}{2}$.

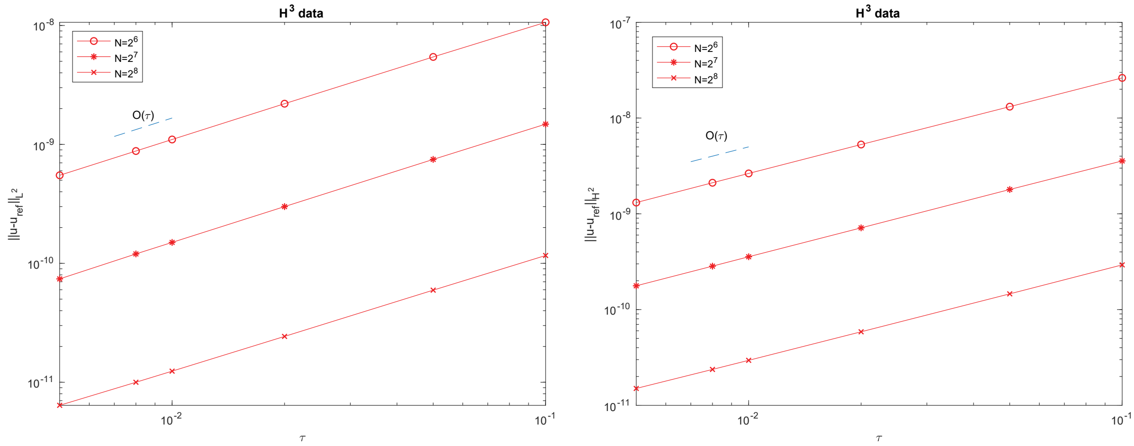

Figure 1: Convergence of scheme (2.11): relative error $\|u^n - u_{\mathrm{ref}}\|_{L^2}$ (left) and $\|u^n - u_{\mathrm{ref}}\|_{H^2}$ (right) at $t_n = T = 2.0$ with initial condition $\gamma = 3$

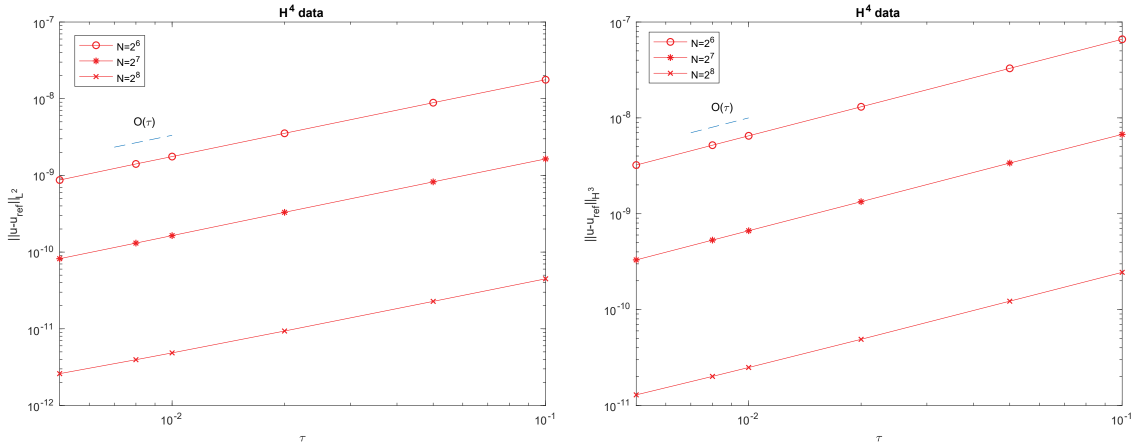

Figure 2: Convergence of scheme (2.11): relative error $\|u^n - u_{\mathrm{ref}}\|_{L^2}$ (left) and $\|u^n - u_{\mathrm{ref}}\|_{H^3}$ (right) at $t_n = T = 2.0$ with initial condition $\gamma = 4$

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