Vacillating Hecke Tableaux and Linked Partitions

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Abstract

We introduce the structure of vacillating Hecke tableaux. By using the Hecke insertion algorithm developed by Buch, Kresch, Shimozono, Tamvakis and Yong, we establish a one-to-one correspondence between vacillating Hecke tableaux and linked partitions, which arise in free probability theory. We define a Hecke diagram as a Young diagram possibly with a marked corner. A vacillating Hecke tableau is defined as a sequence of Hecke diagrams subject to a certain condition on addition and deletion of rook strips. The notion of a rook strip was introduced by Buch in the study of the Littlewood-Richardson rule for stable Grothendieck polynomials. We show that the crossing number and the nesting number of a linked partition can be determined by the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. The proof relies on a theorem due to Thomas and Yong. As consequences, we confirm two conjectures on the distribution of the crossing number and the nesting number over linked partitions and ordinary partitions, respectively proposed by de Mier and Kim.

Keywords: Vacillating Hecke tableau, Hecke diagram, rook strip, the Hecke insertion algorithm, linked partition, crossing number, nesting number

AMS Classifications: 05A05, 05A17, 05C30

1 Introduction

In this paper, we introduce the structure of a vacillating Hecke tableau, which is a sequence of Hecke diagrams subject to a certain condition on addition and deletion of rook strips. A Hecke diagram is defined as a Young diagram possibly with a marked corner, which arises in the Hecke insertion algorithm developed by Buch, Kresch, Shimozono, Tamvakis and Yong [2]. Employing vacillating Hecke tableaux, we prove two combinatorial conjectures on linked partitions and ordinary partitions of the set $[n] = \{1, 2, \ldots, n\}$, respectively proposed by de Mier [6] and Kim [8].

The Hecke algorithm provides a combinatorial rule to expand a stable Grothendieck polynomial in terms of stable Grothendieck polynomials indexed by partitions [2]. Buch [1] showed that a stable Grothendieck polynomial can be expanded as a linear combination of stable Grothendieck polynomials indexed by partitions with integer coefficients. Based on the Hecke algorithm, Buch, Kresch, Shimozono, Tamvakis and Yong [2] found a combinatorial interpretation of these coefficients in terms of increasing tableaux satisfying certain conditions.

On the other hand, linked partitions were introduced by Dykema [7] in the study of unsymmetrized T-transforms in free probability theory. Using the Hecke algorithm, we establish a one-to-one correspondence between the set of vacillating Hecke tableaux of empty shape and length $2n$ and the set of linked partitions of $[n]$. We show that the crossing number and the nesting number of a linked partition can be determined by the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. The proof relies on the following property of the Hecke algorithm proved by Thomas and Yong [12]. Given a word w of positive integers, one can construct the insertion tableau of w by successively applying the Hecke algorithm. Thomas and Yong showed that the length of the longest strictly increasing subsequences and the length of the longest strictly decreasing subsequences in w can be read off from its insertion tableau.

The above correspondence implies that the crossing number and the nesting number have a symmetric joint distribution over linked partitions of $[n]$, as conjectured by de Mier. When restricted to the front representations of partitions of $[n]$, we are led to the fact that the crossing number and the nesting number have a symmetric joint distribution over the front representations of partitions of $[n]$. This proves a conjecture of Kim.

Recall that a linked partition of $[n]$ is a collection of nonempty subsets $B_1, B_2, \ldots, B_k$ of $[n]$, called blocks, such that the union of $B_1, B_2, \ldots, B_k$ is $[n]$ and any two distinct blocks are nearly disjoint. Two distinct blocks B_i and B_j are said to be nearly disjoint if for any $t \in B_i \cap B_j$, one of the following conditions holds:

(1) $t = \min(B_i)$, $|B_i| > 1$, and $t \neq \min(B_j)$,

(2) $t = \min(B_j)$, $|B_j| > 1$, and $t \neq \min(B_i)$.

The linear representation of a linked partition was defined by Chen, Wu and Yan [4]. For a linked partition P of $[n]$, list the n vertices $1, 2, \ldots, n$ in increasing order on a horizontal line. For a block $B_i = \{a_1, a_2, \ldots, a_m\}$ with $m \geq 2$ and $a_1 < a_2 < \cdots < a_m$, draw an arc from a_1 to a_j for $j = 2, \ldots, m$. For example, the linear representation of the linked partition $\{\{1, 3, 5\}, \{2, 6, 10\}, \{4\}, \{5, 8, 9\}, \{6, 7\}\}$ is illustrated in Figure 1.1. By definition, it is easily checked that the linear representation of a linked partition of

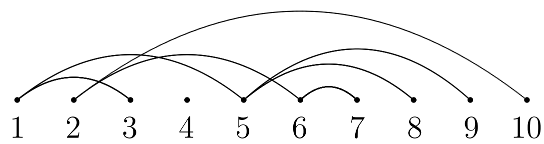

Figure 1.1: The linear representation of a linked partition.

$[n]$ can be characterized as a simple graph on $[n]$ such that for each vertex i there is at most one vertex j such that $1 \leq j < i$ and j is connected to i.

The crossing number and the nesting number of a linked partition P are defined based on k-crossings and k-nestings in the linear representation of P, where k is a positive integer. We use a pair (i, j) with $i < j$ to denote an arc in the linear representation of P, and we call i and j the left-hand endpoint and the right-hand endpoint of (i, j), respectively. We say that k arcs (i_1, j_1),(i_2, j_2),...,(i_k, j_k) of P form a k-crossing if

$$i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k.$$

Similarly, we say that (i_1, j_1),(i_2, j_2),...,(i_k, j_k) form a k-nesting if

$$i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1.$$

The crossing number $\mathrm{cr}(P)$ of P is defined as the maximal number k such that P has a k-crossing, and the nesting number $\mathrm{ne}(P)$ of P is the maximal number k such that P has a k-nesting. For example, for the linked partition in Figure 1.1, we have $\mathrm{cr}(P) = 2$ and $\mathrm{ne}(P) = 3$. The distributions of the crossing number and the nesting number over graphs as well as over fillings of diagrams have been studied in [5, 9, 10].

de Mier [6] posed a conjecture on the equidistribution of the crossing number and the nesting number over simple graphs on $[n]$ such that for each vertex i there is at most one vertex j such that $1 \leq j < i$ and j is connected to i. Equivalently, we can state this conjecture in terms of linked partitions.

Conjecture 1.1 *For any positive integers i and j, the number of linked partitions P of $[n]$ with $\mathrm{cr}(P) = i$ and $\mathrm{ne}(P) = j$ equals the number of linked partitions P of $[n]$ with $\mathrm{cr}(P) = j$ and $\mathrm{ne}(P) = i$.*

Kim [8] conjectured that Conjecture 1.1 also holds when restricted to the set of partitions of $[n]$. A partition of $[n]$ is a collection of mutually disjoint nonempty subsets whose union is $[n]$. Clearly, a partition of $[n]$ is a special linked partition of $[n]$ such that any two distinct blocks are disjoint. For a partition P, the linear representation of P is also called the front representation of P by Kim [8].

For example, Figure 1.2 is the front representation of the partition

$$\{\{1,3,5,8\},\{2,6,9\},\{4\},\{7,10\}\}. \tag{1.1}$$

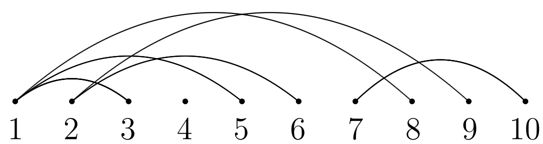

Figure 1.2: The front representation of a partition of [10].

Conjecture 1.2 *For any positive integers i and j, the number of front representations of partitions P of $[n]$ with $\mathrm{cr}(P) = i$ and $\mathrm{ne}(P) = j$ equals the number of front representations of partitions P of $[n]$ with $\mathrm{cr}(P) = j$ and $\mathrm{ne}(P) = i$.*

It should be noted that there is another representation of a partition. For a partition P of $[n]$, Chen, Deng, Du, Stanley and Yan [3] defined the standard representation of P as a graph with vertices $1, 2, \ldots, n$ drawn from left to right such that there is an edge between i and j if i and j are consecutive elements in the same block. For example, Figure 1.3 is the standard representation of the partition given in (1.1). By introducing the

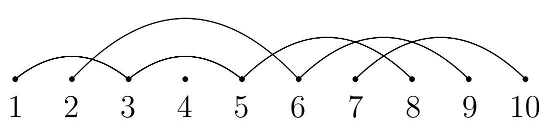

Figure 1.3: The standard representation of a partition of [10].

structure of vacillating tableaux, they established a correspondence between vacillating tableaux and the standard representations of partitions, which implies that the crossing number and the nesting number have a symmetric joint distribution over the standard representations of partitions of $[n]$.

This paper is organized as follows. In Section 2, we recall the notion of a rook strip, and give the definition of a vacillating Hecke tableau. In Section 3, we give an overview of the Hecke algorithm and obtain a property of this algorithm. Section 4 provides a bijection between vacillating Hecke tableaux and linked partitions based on the Hecke algorithm. Moreover, we show that the crossing number and the nesting number of a linked partition are equal to the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. As consequences, Conjecture 1.1 and Conjecture 1.2 are confirmed.

2 Vacillating Hecke Tableaux

In this section, we give the definition of a vacillating Hecke tableau. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of a positive integer n, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. A Young diagram of shape λ is a left-justified array of squares with λ_i squares in row i. By a Hecke diagram we mean a Young diagram possibly with a marked corner. The shape of a Hecke diagram is referred to as the shape of the underlying Young diagram. For example, Figure 2.1 gives four Hecke diagrams of shape $(4, 4, 2, 1)$, where we use a bullet to indicate a marked corner. We call a Hecke diagram an ordinary

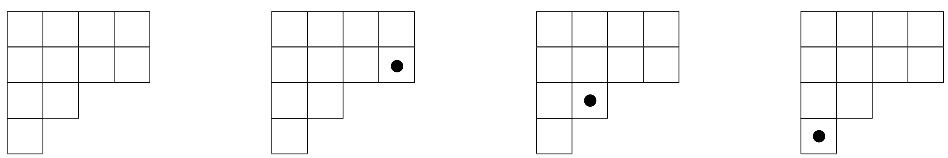

Figure 2.1: Hecke diagrams with underlying Young diagram $(4, 4, 2, 1)$.

diagram if it does not have a marked corner, and a marked diagram if it has a marked corner. When λ is a marked diagram with a marked corner c, we also write λ as a pair (μ, c), where μ is the underlying Young diagram of λ.

To define a vacillating Hecke tableau, we need the notion of a rook strip which was introduced by Buch [1] in the study of the Littlewood-Richardson rule for stable Grothendieck polynomials. For two Young diagrams λ and μ such that μ is contained in λ, the skew diagram λ/μ is the collection of squares of λ that are outside μ. A rook strip is a skew diagram with at most one square in each row and each column. For example, in Figure 2.2, the skew diagram (a) is a rook strip, but (b) is not a rook strip.

(a) (b)

Figure 2.2: Examples of skew diagrams.

A vacillating Hecke tableau of empty shape and length $2n$ is defined to be a sequence $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of Hecke diagrams such that

(i) $\lambda^0 = \lambda^{2n} = \emptyset$, and for $1 \leq i \leq n$, λ^{2i-1} is an ordinary diagram;

(ii) If λ^{2i} is an ordinary diagram, then λ^{2i-1} is an ordinary diagram contained in λ^{2i} such that $\lambda^{2i-1} = \lambda^{2i}$ or $\lambda^{2i}/\lambda^{2i-1}$ is a rook strip, and λ^{2i+1} is an ordinary diagram contained in λ^{2i} such that $\lambda^{2i+1} = \lambda^{2i}$ or $\lambda^{2i}/\lambda^{2i+1}$ is a square;

If $\lambda^{2i} = (\mu, c)$ is a marked diagram, then λ^{2i-1} is an ordinary diagram contained in μ such that $\lambda^{2i-1} = \mu$ or μ/λ^{2i-1} is a rook strip, and $\lambda^{2i+1} = \mu$.

As an example, Figure 2.3 illustrates a vacillating Hecke tableau of empty shape and length 14.

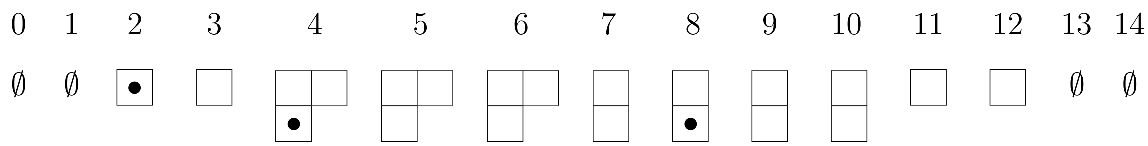

Figure 2.3: A vacillating Hecke tableau of empty shape and length 14.

Notice that a vacillating Hecke tableau reduces to a vacillating tableau defined in [3] if it does not contain any marked corner and each rook strip is a single square.

3 The Hecke insertion algorithm

In this section, we give an overview of the Hecke insertion algorithm developed by Buch, Kresch, Shimozono, Tamvakis and Yong [2] and a property of this algorithm proved by Thomas and Yong [12]. We also observe a property that will be used in the proof of the main theorem of this paper.

The Hecke algorithm is a procedure to insert a positive integer into an increasing tableau, resulting in a new increasing tableau. Let λ be a partition. By λ we also denote the Young diagram of shape λ. An increasing tableau T of shape λ is an assignment of positive integers to the squares of λ such that the numbers are strictly increasing in each row and each column. Suppose that U is the tableau obtained from T by inserting a positive integer x. Then U is either of the same shape as T or it has an extra square compared with T. In the case when U has the same shape as T, it also contains a special corner where the algorithm terminates and this corner needs to be recorded. A parameter $\alpha \in \{0, 1\}$ is used to distinguish these two cases. Thus the output of the Hecke algorithm when applied to T is a triple (U, c, α), where c is a corner of U.

The Hecke algorithm can be described as follows. Assume that T is an increasing tableau and x is a positive integer. To insert x into T, we begin with the first row of T. Roughly speaking, an element in this row may be bumped out and then inserted into the next row. The process is repeated until no more elements are bumped out. More precisely, let R be the first row of T. We have the following two cases.

Case 1: The integer x is larger than or equal to all entries in R. If adding x as a new square to the end of R results in an increasing tableau, then U is the resulting tableau, c is the corner where x is added. We set $\alpha = 1$ to signify that the corner c is outside the shape of T, and the process terminates. As will be seen, although the corner c in the shape of U is easily recognized in comparison with the shape of T, it is necessary to keep a record of this case by setting $\alpha = 1$ to make the construction reversible. If adding x as a new square to the end of R does not result in an increasing tableau, then let $U = T$, and c be the corner at the bottom of the column of U containing the rightmost square of R. In this case, we set $\alpha = 0$ to indicate that the corner c is inside the shape of T, and the process terminates.

Case 2: The integer x is strictly smaller than some element in R. Let y be the leftmost entry in R that is strictly larger than x. If replacing y by x results in an increasing tableau, then y is bumped out by x and y will be inserted into the next row. If replacing y by x does not result in an increasing tableau, then keep the row R unchanged and the element y will also be inserted into the next row.

We iterate the above process to insert the element y into the next row, still denoted by R. Finally, we get the output (U, c, α) of the insertion algorithm, and we write $U = (T \xleftarrow{\text{H}} x)$ and $(U, c, \alpha) = H(T, x)$.

We give two examples to demonstrate the two cases $\alpha = 0$ and $\alpha = 1$ of the insertion algorithm. Let T be an increasing tableau of shape $(4, 3, 2, 2)$ as given in Figure 3.1. Let

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 5 & \\ \cline{1-3} 4 & 5 & & \\ \cline{1-2} 5 & 7 & & \\ \cline{1-2} \end{array}$$

Figure 3.1: An increasing tableau of shape $(4, 3, 2, 2)$.

$x = 1$. The process to insert x into T is illustrated in Figure 3.2, where an element in boldface represents the entry that is bumped out and is to be inserted into the next row. We see that the resulting tableau U has one more square than T, and so we have $\alpha = 1$.

For $x = 3$, we find that the resulting tableau U has the same shape as T, and so we have $\alpha = 0$, see Figure 3.3.

The Hecke algorithm is reversible [2]. In other words, given an increasing tableau U, a corner c of U, and the value of α, there exist a unique increasing tableau T and a unique positive integer x such that $U = (T \xleftarrow{\text{H}} x)$.

Thomas and Yong [12] showed that the Hecke algorithm can be used to determine the length of the longest strictly increasing and the length of the longest strictly decreasing subsequences of a word. Let $w = w_1 w_2 \cdots w_n$ be a word of positive integers. A subword of $w = w_1 w_2 \cdots w_n$ is a subsequence $w_{i_1} w_{i_2} \cdots w_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. A subword $w_{i_1} w_{i_2} \cdots w_{i_k}$ is said to be strictly increasing if $w_{i_1} < w_{i_2} < \cdots < w_{i_k}$, and strictly decreasing if $w_{i_1} > w_{i_2} > \cdots > w_{i_k}$. Let $\text{is}(w)$ (resp., $\text{de}(w)$) denote the length

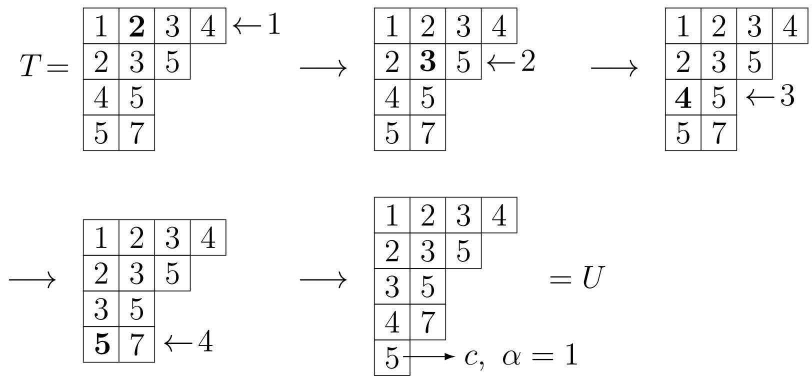

Figure 3.2: An example of the Hecke insertion algorithm for $\alpha = 1$.

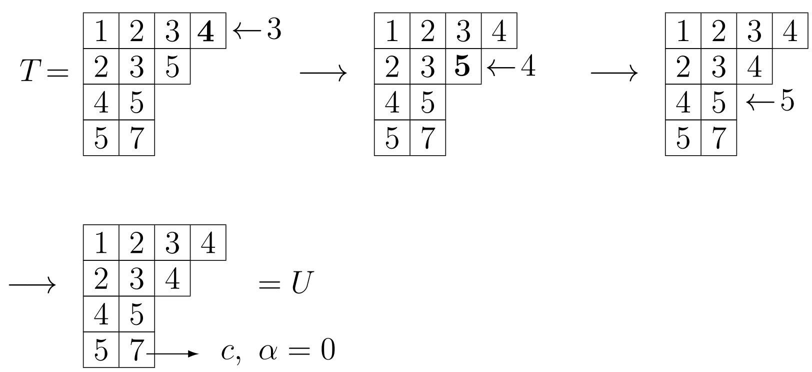

Figure 3.3: An example of the Hecke insertion algorithm for $\alpha = 0$.

of the longest strictly increasing (resp., strictly decreasing) subwords of w. As shown by Thomas and Yong, $\mathrm{is}(w)$ and $\mathrm{de}(w)$ are determined by the insertion tableau of w. The insertion tableau of w is defined by

$$(\cdots((\emptyset \overset{\mathrm{H}}{\longleftarrow} w_1) \overset{\mathrm{H}}{\longleftarrow} w_2) \overset{\mathrm{H}}{\longleftarrow} \cdots) \overset{\mathrm{H}}{\longleftarrow} w_n.$$

For example, let $w = 21131321$. The construction of the insertion tableau of w is given in Figure 3.4.

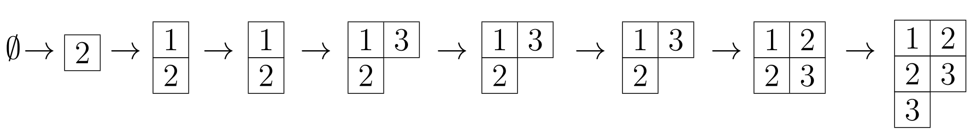

Figure 3.4: The insertion tableau of $w = 21131321$.

For an increasing tableau T, let $\mathrm{c}(T)$ and $\mathrm{r}(T)$ denote the number of columns and the number of rows of T, respectively. Using the jeu de taquin algorithm for increasing tableaux developed in [11], Thomas and Yong [12] established the following relation.

Theorem 3.1 *Let w be a word of positive integers, and T be the insertion tableau of w. Then* $\mathrm{is}(w) = \mathrm{c}(T)$ *and* $\mathrm{de}(w) = \mathrm{r}(T)$.

The following theorem gives a property of the insertion tableau.

Theorem 3.2 *Let $w = w_1w_2\cdots w_n$ be a word of positive integers, and k be the maximal element appearing in w. Let $w' = a_1a_2\cdots a_m$ be the word obtained from w by deleting the elements equal to k. Assume that T is the insertion tableau of w and T' is the insertion tableau of w'. Then T' can be obtained from T by deleting the squares occupied with k.*

Proof. Let Q denote the increasing tableau obtained from T by deleting the squares occupied with the maximal element k. We use induction to prove that $T' = Q$. The claim is obvious when $n = 1$. We now assume that $n > 1$ and that the claim holds for $n-1$. Let P be the insertion tableau of $w_1w_2\cdots w_{n-1}$. Here are two cases.

Case 1: $w_n = k$. By the induction hypothesis, T' is obtained from P by deleting the squares occupied with k. On the other hand, since $w_n = k$ is the maximal element of w, we see that $T = P$ or T is obtained from P by adding a square filled with k at the end of the first row. This yields that $T' = Q$.

Case 2: $w_n < k$. Let U be the insertion tableau of $a_1\cdots a_{m-1}$. By the induction hypothesis, U is obtained from P by deleting the squares occupied with k. In the process of inserting w_n into P, if no entry equal to k is bumped out and is inserted into the next row, then it is clear that $T' = Q$.

Otherwise, there is a unique entry k in P that is bumped out and is inserted into the next row. Let c be the square of P occupied with this entry. Note that c is a corner of P since P is increasing and k is a maximal entry. Keep in mind that U is obtained from P by deleting the squares occupied with k. Since $T = (P \stackrel{\mathrm{H}}{\longleftarrow} w_n)$ and $T' = (U \stackrel{\mathrm{H}}{\longleftarrow} w_n)$, for any square C in U, the entry of T' in C equals the entry of T in C. Consequently, to verify $T' = Q$, it suffices to consider the entry of T in the corner c. Assume that this entry is equal to i. Here are two subcases.

Case 2.1: $i = k$. In this case, T' has the same shape as U. On the other hand, any square of T outside U is occupied with k. So we have $T' = Q$.

Case 2.2: $i < k$. In this case, T' has the extra corner c compared with U. By the construction of the Hecke algorithm, we see that the entry of T' in the corner c also equals i. Notice also that except for the corner c, any square of T outside U is occupied with k. So we are led to the assertion $T' = Q$. This completes the proof. ∎

For example, let $w = 21131321$. Then we have $w' = 211121$. The insertion tableau T of w is given in Figure 3.4. Meanwhile, the insertion tableau T' of w' is constructed in Figure 3.5, which coincides with the tableau obtained from T by deleting the squares occupied with 3.

$$\emptyset \to \begin{array}{|c|}\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2\\\hline 2 \\\cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline 1 & 2\\\hline 2 \\\cline{1-1}\end{array}$$

Figure 3.5: The insertion tableau of $w' = 211121$.

4 Vacillating Hecke tableaux and linked partitions

In this section, we provide a bijection between vacillating Hecke tableaux and linked partitions. We prove that the crossing number and the nesting number of a linked partition are equal to the maximal number of rows and the maximal number of columns of diagrams in the corresponding vacillating Hecke tableau. As a consequence, we show that the crossing number and the nesting number have a symmetric joint distribution over linked partitions with fixed left-hand endpoints and right-hand endpoints. This leads to a proof of Conjecture 1.1. Specializing the bijection to linked partitions containing no vertex that is both a left-hand endpoint and a right-hand endpoint, we confirm Conjecture 1.2.

To describe the bijection, by a Hecke tableau we mean an increasing tableau possibly with a marked corner. In other words, a Hecke tableau can be viewed as an increasing tableau whose shape is a Hecke diagram. Let λ be a Hecke diagram, and let T be a Hecke tableau of shape λ. When $\lambda = (\mu, c)$ is a marked diagram, we also express T by a pair (T', c), where T' is the underlying increasing tableau of T.

Let V_{2n} be the set of vacillating Hecke tableaux of empty shape and length $2n$. We now give a description of a bijection ϕ from V_{2n} to the set of linked partitions of $[n]$.

Let $V = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ be a vacillating Hecke tableau of empty shape and length $2n$. First, we recursively define a sequence $(E_0, T_0), (E_1, T_1), \ldots, (E_{2n}, T_{2n})$, where for $0 \le i \le 2n$, E_i is a set of edges and T_i is a Hecke tableau of shape λ^i. Let $E_0 = \emptyset$, and let T_0 be the empty tableau. Assume that $i \ge 1$. If $\lambda^i = \lambda^{i-1}$, then we let $(E_i, T_i) = (E_{i-1}, T_{i-1})$. If $\lambda^i \ne \lambda^{i-1}$, then (E_i, T_i) is constructed according to the parity of i.

Case 1: i is odd. Let $i = 2k - 1$. By the definition of a vacillating Hecke tableau, λ^i is a diagram without any marked corner. Here are two subcases according to whether λ^{i-1} is an ordinary diagram.

Case 1.1: λ^{i-1} is an ordinary diagram. Then λ^i is obtained from λ^{i-1} by deleting a corner c. Setting $\alpha = 1$, there are a unique increasing tableau T and a unique positive integer

j such that $(T_{i-1}, c, \alpha) = H(T, j)$. Let $T_i = T$ and define E_i to be the set obtained from E_{i-1} by adding the edge (j, k).

Case 1.2: $\lambda^{i-1} = (\mu, c)$ is a marked diagram. So we have $\lambda^i = \mu$. Setting $\alpha = 0$, there are a unique increasing tableau T and a unique positive integer j such that $(T_{i-1}, c, \alpha) = H(T, j)$. Let $T_i = T$ and define E_i to be the set obtained from E_{i-1} by adding the edge (j, k).

Case 2: i is even. Let $i = 2k$. We set $E_i = E_{i-1}$. To define T_i, there are two subcases according to whether λ^i is an ordinary diagram.

Case 2.1: λ^i is an ordinary diagram. Then λ^i/λ^{i-1} is a rook strip. Define T_i to be the tableau obtained from T_{i-1} by filling the squares of λ^i/λ^{i-1} with k.

Case 2.2: $\lambda^i = (\mu, c)$ is a marked diagram. Then μ/λ^{i-1} is a rook strip. Let T be the tableau of shape μ that is obtained from T_{i-1} by filling the squares of μ/λ^{i-1} with k. Define $T_i = (T, c)$.

Finally, we define $\phi(V)$ to be the diagram with n vertices $1, 2, \ldots, n$ listed on a horizontal line such that there is an arc connecting j and k with $j < k$ if and only if (j, k) is an edge in E_{2n}. By the above construction, for each vertex $k \in [n]$, there is at most one vertex j with $j < k$ such that (j, k) is an arc in $\phi(V)$. Thus $\phi(V)$ is a linked partition of $[n]$.

Figure 4.1 gives an illustration of the map ϕ when applied to the vacillating Hecke tableau in Figure 2.3, where an entry in boldface indicates a marked corner of a Hecke tableau.

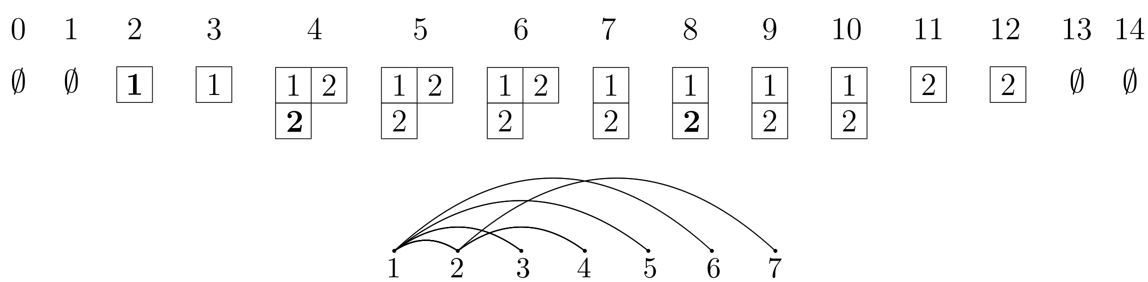

Figure 4.1: An illustration of the bijection ϕ.

It can be checked that ϕ is reversible, and hence it is a bijection. Let P be a linked partition of $[n]$. To recover the corresponding vacillating Hecke tableau, we first construct a sequence $(T_0, T_1, ..., T_{2n})$ of Hecke tableaux. Let T_{2n} be the empty tableau. Suppose that T_{2i} has been constructed, where $1 \le i \le n$. We proceed to construct T_{2i-1} and T_{2i-2}. To obtain T_{2i-1}, we assume that T is the underlying increasing tableau of T_{2i}. Let T_{2i-1} be the tableau obtained from T by deleting the squares (if any) filled with i.

Next we construct the Hecke tableau T_{2i-2}. If i is not a right-hand endpoint of any arc of P, then we set $T_{2i-2} = T_{2i-1}$. Otherwise, there is a unique arc (j,i) with $j < i$ of P. Assume that $(U,c,\alpha) = H(T_{2i-1}, j)$. We set $T_{2i-2} = U$ if $\alpha = 1$, and set $T_{2i-2} = (U,c)$ if $\alpha = 0$.

Let λ^i be the shape of T_i. Finally, the vacillating Hecke tableau $\phi^{-1}(P)$ is given by

$$(\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^{2n} = \emptyset).$$

The following theorem shows that the crossing number and the nesting number of a linked partition are determined by the diagrams in the corresponding vacillating Hecke tableau. For a vacillating Hecke tableau V, let $\mathrm{r}(V)$ be the greatest number of rows in any diagram λ^i of V. Similarly, let $\mathrm{c}(V)$ be the greatest number of columns in any diagram λ^i of V. We have the following relations.

Theorem 4.1 *Let V be a vacillating Hecke tableau in V_{2n}, and let $P = \phi(V)$. Then we have $\mathrm{c}(V) = \mathrm{ne}(P)$ and $\mathrm{r}(V) = \mathrm{cr}(P)$.*

Proof. Let $V = (\lambda^0, \lambda^1, \ldots, \lambda^{2n})$, and let $(T_0, T_1, \ldots, T_{2n})$ be the sequence of Hecke tableaux in the construction of ϕ. For $0 \le i \le 2n$, let U_i be the underlying increasing tableau of T_i. We shall generate a sequence $(w^{(0)}, w^{(1)}, \ldots, w^{(2n)})$ of words such that U_i is the insertion tableau of $w^{(i)}$.

Let $w^{(2n)}$ be the empty word. Suppose that $w^{(i)}$ has been constructed. Then $w^{(i-1)}$ is constructed as follows. If $T_{i-1} = T_i$, then we let $w^{(i-1)} = w^{(i)}$. If $T_{i-1} \neq T_i$, we have two cases.

Case 1: i is odd. Let $i = 2k-1$. By the construction of ϕ, we see that U_{i-1} is obtained from U_i by inserting a unique integer j. Define $w^{(i-1)} = w^{(i)}\, j$.

Case 2: i is even. Let $i = 2k$. Again, by the construction of ϕ, we find that U_{i-1} is obtained from U_i by deleting the squares (if any) filled with k. Define $w^{(i-1)}$ to be the word obtained from $w^{(i)}$ by removing the elements (if any) equal to k.

We proceed by induction to show that U_i is the insertion tableau of $w^{(i)}$. The claim is obvious for $i = 2n$. Assume that the claim is true for i, where $1 \le i \le 2n$. We wish to prove that it holds for $i-1$. If $T_{i-1} = T_i$, then the claim is evident. Let us now consider the case when $T_{i-1} \neq T_i$. If $w^{(i-1)}$ is generated according to Case 1, then the claim follows directly from the construction of ϕ. If $w^{(i-1)}$ is generated according to Case 2, then the claim is a consequence of Theorem 3.2. This proves the claim.

Combining the above claim and Theorem 3.1, we obtain that

$$\mathrm{c}(V) = \max\left\{\mathrm{is}(w^{(i)}) \,|\, 0 \le i \le 2n\right\}$$

and

$$\mathrm{r}(V) = \max\left\{\mathrm{de}(w^{(i)}) \,|\, 0 \le i \le 2n\right\}.$$

It remains to show that P has a k-crossing (resp., k-nesting) if and only if there exists a word $w^{(i)}$ that contains a strictly decreasing (resp., increasing) subword of length k. We shall only give the proof of the statement concerning the relationship between a k-crossing and a strictly decreasing subword of length k. The same argument applies to the k-nesting case. Suppose that $w^{(i)} = a_1 a_2 \cdots a_t$ contains a strictly decreasing subsequence $a_{i_1} \cdots a_{i_k}$ of length k, where $1 \le i_1 < \cdots < i_k \le t$. By the construction of ϕ and the construction of the sequence $(w^{(0)}, w^{(1)}, \ldots, w^{(2n)})$, we deduce that the vertices $a_{i_1}, \ldots, a_{i_k}$ are left-hand endpoints of P. For $1 \le s \le k$, let b_{j_s} be the right-hand endpoint connected to a_{i_s}. Again, by the construction of $(w^{(0)}, w^{(1)}, \ldots, w^{(2n)})$, we see that

$$b_{j_1} > b_{j_2} > \cdots > b_{j_k}. \tag{4.1}$$

Hence the arcs $(a_{i_1}, b_{j_1}), \ldots, (a_{i_k}, b_{j_k})$ form a k-crossing of P.

On the other hand, suppose that P has a k-crossing consisting of arcs

$$(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k),$$

where $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. By the construction of the sequence $(w^{(0)}, w^{(1)}, \ldots, w^{(2n)})$, it is easily checked that $i_1 i_2 \cdots i_k$ forms a strictly decreasing subword of $w^{(2j_1-1)}$. This completes the proof. ∎

Conjecture 1.1 and Conjecture 1.2 are consequences of the correspondence ϕ and Theorem 4.1. In fact, based on the construction of ϕ and Theorem 4.1 we can deduce a symmetric distribution property of the crossing number and the nesting number over linked partitions with fixed sets of left-hand endpoints and right-hand endpoints. For two subsets S and T of $[n]$, let $L_n(S,T)$ be the set of linked partitions of $[n]$ such that S is the set of left-hand endpoints and T is the set of right-hand endpoints. Note that $L_n(S,T)$ may be empty.

Let $f_{n,S,T}(i,j)$ be the number of linked partitions P in $L_n(S,T)$ with $\mathrm{cr}(P) = i$ and $\mathrm{ne}(P) = j$. We have the following symmetry property.

Theorem 4.2 *Let S and T be two subsets of $[n]$. For any positive integers i and j, we have*

$$f_{n,S,T}(i,j) = f_{n,S,T}(j,i).$$

To prove Theorem 4.2, we establish an involution on the set $L_n(S,T)$ that exchanges the crossing number and the nesting number of a linked partition.

Proof of Theorem 4.2. Define the conjugate of a Hecke diagram as the transpose of the diagram. Taking the conjugate of every Hecke diagram leads to an involution on the set of vacillating Hecke tableaux $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ of empty shape and length $2n$. This yields an involution, denoted ψ, on the set of linked partitions of $[n]$. By Theorem 4.1, we find that ψ exchanges the crossing number and the nesting number of a linked partition.

It remains to show that ψ preserves the left-hand endpoints and the right-hand endpoints of a linked partition. Let P be a linked partition of $[n]$, and let $(\lambda^0, \lambda^1, \ldots, \lambda^{2n})$ be the corresponding vacillating Hecke tableau. By the construction of ϕ, we observe that a vertex i of P is a left-hand endpoint if and only if λ^{2i} has at least one more square than λ^{2i-1}, and it is a right-hand endpoint if and only if $\lambda^{2i-2} \neq \lambda^{2i-1}$. Hence the involution ψ preserves the left-hand and the right-hand endpoints. Restricting ψ to $L_n(S,T)$ gives an involution on $L_n(S,T)$. This completes the proof. ∎

Here is an example for the involution ψ. Let P be the linked partition given in Figure 4.1. Then $\psi(P)$ is the linked partition in Figure 4.2.

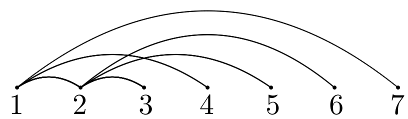

Figure 4.2: An example for the involution ψ.

To conclude, we note that Conjecture 1.1 follows from Theorem 4.2. Theorem 4.2 also implies Conjecture 1.2. Let P be a linked partition of $[n]$, and let S and T be the sets of left-hand endpoints and right-hand endpoints of P, respectively. It can be seen that P is the front representation of a partition of $[n]$ if and only if $S \cap T = \emptyset$. Hence the set of front representations of partitions of $[n]$ is the disjoint union of $L_n(S,T)$, where (S,T) ranges over pairs of disjoint subsets of $[n]$. For any two subsets S and T of $[n]$ with $S \cap T = \emptyset$, we can apply Theorem 4.2 to deduce that the crossing number and the nesting number have a symmetric joint distribution over $L_n(S,T)$. Thus we have proved Conjecture 1.2.

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