Regularity and stability of finite energy weak solutions for the Camassa-Holm equations with nonlocal viscosity

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Abstract

We consider the n-dimensional ($n = 2, 3$) Camassa-Holm equations with nonlocal diffusion of type $(-\Delta)^s$, $\frac{n}{4} \le s < 1$. In [17], the global-in-time existence and uniqueness of finite energy weak solutions are established. In this paper, we show that with regular initial data, the finite energy weak solutions are indeed regular for all time. Moreover, the weak solutions are stable with respect to the initial data. The main difficulty lies in establishing higher order uniform estimates with the presence of the fractional Laplacian diffusion. To achieve this, we need to explore suitable fractional Sobolev type inequalities and bilinear estimates for fractional derivatives. The critical case $s = \frac{n}{4}$ contains extra difficulties and a smallness assumption on the initial data is imposed.

Key Words. Camassa-Holm equations; Nonlocal viscosity; Finite energy weak solutions; Regularity; Stability.

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1 Introduction

Recently a great attention has been devoted to the study of nonlocal problems driven by nonlocal operators in PDEs, which is not only for a pure academic interest, but also for various applications in different fields. In this paper, we shall consider the following Camassa-Holm equations with nonlocal viscosity:

$$\begin{cases} \mathbf{v}_t + \mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u}^T + \nabla p = -\nu(-\Delta)^s \mathbf{v}, \\ \mathbf{u} - \alpha^2 \Delta \mathbf{u} = \mathbf{v}, \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \tag{1.1}$$

where $1 - \alpha^2\Delta$ is the Helmholtz operator, $\mathbf{u} \cdot \nabla \mathbf{v}$ and $\mathbf{v} \cdot \nabla \mathbf{u}^T$ are vectors with the i^{th} components defined respectively as

$$(\mathbf{u} \cdot \nabla \mathbf{v})_i := \sum_{j=1}^{n} \mathbf{u}_j \partial_j \mathbf{v}_i, \quad \left(\mathbf{v} \cdot \nabla \mathbf{u}^T\right)_i := \sum_{j=1}^{n} \mathbf{v}_j \partial_i \mathbf{u}_j.$$

The system is subjected to the prescribed initial condition

$$\mathbf{v}(0, x) = \mathbf{v}_0(x), \quad x \in \mathbb{R}^n. \tag{1.2}$$

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Here, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$ with $n = 2, 3$. In the system (1.1), $\mathbf{u} - \alpha^2 \Delta \mathbf{u} = \mathbf{v}$ can be referred as the filter, $\mathbf{u}$ represents the filtered fluid velocity, α is a length scale parameter representing the width of the filter, and $\nu > 0$ is the viscosity coefficient. In particular, both α and ν will be assumed to be fixed positive constants in our discussion. Moreover, $\mathbf{v}$ denotes the fluid velocity field, and p the scalar pressure. The nonlocal operator $(-\Delta)^s$ and $\frac{n}{4} \leq s < 1$ is defined classically as a Fourier multiplier with symbol $|\xi|^{2s}$.

When $s = 1$, the sytem (1.1) becomes the classical Camassa-Holm equations, also known as the Navier-Stokes-α model, or the isotropic Lagrangian averaged Navier-Stokes ($LANS$-α) equations. In particular, the Lagrangian framework is a natural setting to study the behavior of solutions. It should be pointed out that the $LANS$-α equations are a system of partial differential equations for the mean velocity field. Compared to the Reynolds averaged Navier-Stokes or large-eddy simulation models that add artificial dissipation to the Navier-Stokes equations to filter small scales, the $LANS$-α equations do not add any artificial viscosity but a nonlinear dispersive mechanism filters the small scales. Hence the $LANS$-α equations serve as a nice model for turbulent flow (see [24]). As is known, the classical Camassa-Holm equations rise typically from the asymptotic studies on shallow water equations [5, 20]. Specifically, it was introduced in [20] as a natural mathematical generalization of the integrable inviscid one-dimensional Camassa-Holm equations discovered in [5] through a variational formulation and with a lagrangian averaging. It could be used as a closure model for the mean effects of subgrid excitations, and be also viewed as a filtered Navier-Stokes equations with the parameter α in the filter, which obeys a modified Kelvin circulation theorem along filtered velocities [20]. Numerical examples that seem to justify this intuition were given in [7].

There has been a vast literature focusing on investigating the existence, uniqueness and regularity issues for the classical Camassa-Holm equations ((1.1) with $s = 1$), see for example [1, 8, 13, 15, 18, 31, 36, 37, 39, 40]. The classical results on the existence, uniqueness, regularity and the decay estimates for the classical Cammassa-Holm equations were established in [1, 24]. Chen et al. in [8] investigated the oscillation-induced blow-up to the modified Camassa-Holm equations with linear dispersion. De Lellis et al. in [13] considered the low-regularity solutions for the periodic Camassa-Holm equations. Escher and Yin in [15] analyzed a kind of initial-boundary value problems of the Camassa-Holm equations. Hakkaev in [18] obtained the local well-posedness for a generalized Camassa-Holm equations. Misiolek in [31] discussed classical solutions of a periodic Camassa-Holm equations. Perrollaz in [36] dealt with an initial boundary value problem for the Camassa-Holm equations on an interval. Tan and Yin in [37] established the global periodic conservative solutions for a periodic modified two-component Camassa-Holm equations. Wu and Yin in [39] showed global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equations. Yan et al. in [40] took into account the Cauchy problem for a generalized Camassa-Holm equations in Besov space.

In [9, 22], the authors studied the Navier-Stokes equations with hyper-dissipation $(-\Delta)^\alpha$:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = -(-\Delta)^\alpha \mathbf{u}. \tag{$H-NS$}$$

Due to the close relationship between the Camassa-Holm equations and the Navier-Stokes equations, the Camassa-Holm equations with nonlocal viscosity naturally occur in hydrodynamics [3, 11, 12, 16, 23, 33, 34, 4, 6, 35, 2, 14, 32, 38]. Fractional diffusion arises naturally in many hydrodynamic problems, capturing nonlocal feature of certain dynamics, see for example Caffarelli-Silvestre [3], Córdoba-Córdoba-Fontelos [11, 12], Fujiwara-Georgiev-Ozawa [16], Kenig-Ponce-Vega [23], Musina-Nazarov [33], Nezza-Palarucci-Valdinoci [34], and the references therein. Moreover, fractional Laplacian has been utilized to model energy dissipation of acoustic propagation in human tissue [4], turbulence diffusion [6], contaminant transport in ground water [35], non-local heat conduction [2, 14, 32], and electromagnetic fields on fractals [38].

While, in contrast to those works for the classical Camassa-Holm equations mentioned as above in recent decades, little has been known concerning the Camassa-Holm equations with fractional viscosity despite that non-standard diffusions are naturally risen for these problems. With nonlocal diffusion, the problem is more challenging for achieving the existence, uniqueness and regularity of solutions, not only due to the integral expression and nonlocal property, but also due to the less regularity in the a priori estimates when s is strictly less than 1. One may be curious that up to which range of $s \in [0, 1]$, the existence, regularity and uniqueness can be shown as the classical case. Indeed, when $\frac{n}{4} \leq s < 1$, in [17] the authors showed the existence and uniqueness of global-in-time weak solutions satisfying energy inequalities (so called finite energy weak solutions). A nature question is that whether this weak solution is regular, when the initial datum is regular and smooth. This is the main concern in this paper.

We remark that the local existence of strong solutions issued from regular initial data can be obtained by applying the same method as in [17]. Due to the uniqueness of the finite energy weak solutions, this local-in-time strong solution coincides with the global-in-time finite energy weak solution as long as the previous one exists. Hence, to show the regularity of the finite energy weak solution issued from a smooth datum, it is sufficient to show higher order a priori regularity for smooth solutions with any given time interval. Then the local-in-time strong solution can be extended globally, which is certainly the weak one.

When $s < 1$, the study of the (1.1) is hindered by a lack of explicit information on the kernel of the nonlocal operator, and the main difficulty lies in proving uniform a prior estimates for the nonlocal viscosity. However, we can still obtain various estimates by using different tools in the study of the classical Cammasa-Holm equations: such as the Córdoba-Córdoba inequality [11, 12], a nonlinear lower bound in the spirit of [10], and commutator estimates [21]. Moreover, we will establish several fractional-type interpolation inequalities to obtain our desired uniform estimates: such as the nonlocal version of Ladyzhenskaya's inequalities [25, 26, 27, 28], the fractional Gagliardo-Nirenberg-Sobolev inequality and the fractional Leibniz rule. In particular, under the critical case $s = n/4$, the nonlocal version of Ladyzhenskaya's inequalities is skillfully used, and the smallness of initial data in several Sobolev spaces is required to gain the desired results.

1.1 Some notations

Before going further, let us describe the notation we shall use in this paper. We use $C_0^\infty(\mathbb{R}^n)$ to denote the set of C^∞ functions with compact support in $\mathbb{R}^n$, and $\mathscr{S}(\mathbb{R}^n)$ to denote the standard Schwartz class. Denote the space of smooth and divergence free functions as

$$\Sigma := \left\{\phi \in C_0^\infty(\mathbb{R}^n) | \nabla \cdot \phi = 0\right\}.$$

We denote by $L^p_\sigma(\mathbb{R}^n)$ the completion of Σ in the standard Lebesgue space $L^p(\mathbb{R}^n)$. The completion of Σ in classical Sobolev spaces $H^m(\mathbb{R}^n)$ is denoted by $H^m_\sigma(\mathbb{R}^n)$, and $(H^m_\sigma(\mathbb{R}^n))'$ denotes the corresponding dual space. We use $\langle \cdot, \cdot \rangle$ to denote the $L^2(\mathbb{R}^n)$. For $a \lesssim b$, we mean that there is a universal constant C, which may differ from line to line, such that $a \leq Cb$.

For the nonlocal operator $(-\Delta)^s$, known as the fractional Laplacian of order s in the whole space, there are several ways to define it [3]. Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz class. First of all, it is defined for any $g \in \mathscr{S}(\mathbb{R}^n)$ through the Fourier transform: $h = (-\Delta)^s g$ means

$$\widehat{h}(\xi) = |\xi|^{2s} \widehat{g}(\xi). \tag{1.3}$$

Let $\Lambda^s := (-\Delta)^{\frac{s}{2}}$ be the standard Riesz potential of order $s \in \mathbb{R}$.

It should be pointed out that if ψ and ϕ belong to the Schwartz class $\mathscr{S}(\mathbb{R}^n)$, definition (1.3) of the fractional Laplacian together with Plancherel's theorem yields

$$\int_{\mathbb{R}^n} (-\Delta)^s \psi \phi dx = \int_{\mathbb{R}^n} |\xi|^{2s} \widehat{\psi \phi} d\xi = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \psi (-\Delta)^{\frac{s}{2}} \phi dx. \tag{1.4}$$

We remark that, by density argument, the Definition in (1.3) and the equality (1.4) can be generalized to functions in Sobolev spaces.

Secondly, for $0 < s < 1$ and a function $f \in \mathscr{S}(\mathbb{R}^n)$, using the rpresentation by means of a hypersingular kernel [29], it can be defined as

$$(-\Delta)^s f(x) := C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy = C_{n,s} \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{f(x+y)-f(x)}{|y|^{n+2s}} dy,$$

where $C_{n,s}$ is a normalization constant (see [29]) precisely given by

$$C_{n,s} := \left(\int_{\mathbb{R}^n} \frac{1-\cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1} = \frac{2^{2s} \Gamma\left((n+2s)/2\right)}{\pi^{\frac{n}{2}} \Gamma(1-s)}. \tag{1.5}$$

In the rest of this section, we collect some facts on the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, see [34]. We first give the definition:

Definition 1.1. *Let $s \in (0,1)$. For any $p \in [1,\infty)$, we define $W^{s,p}(\mathbb{R}^n)$ as follows*

$$W^{s,p}(\mathbb{R}^n) := \left\{ u \in L^p(\mathbb{R}^n) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\mathbb{R}^n \times \mathbb{R}^n) \right\},$$

i.e., an intermediary Banach space between $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}, \tag{1.6}$$

where the term

$$[u]_{W^{s,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}} \tag{1.7}$$

is the so-called Gagliardo (semi) norm of u. However, there is another case for $s \in (1,\infty)$ and s is not an integer. In this case, we write $s = m + m'$, where m is an integer and $m' \in (0,1)$. The space $W^{s,p}(\mathbb{R}^n)$ consists of those equivalence classes of functions $u \in W^{m,p}(\mathbb{R}^n)$ whose distributional derivatives $D^\alpha u$, with $|\alpha| = m$, belong to $W^{m',p}(\mathbb{R}^n)$, namely

$$W^{s,p}(\mathbb{R}^n) := \left\{ u \in W^{m,p}(\mathbb{R}^n) : \ \ D^\alpha u \in W^{m',p}(\mathbb{R}^n) \ \ for \ \ any \ \ \alpha \ s.t. \ |\alpha| = m \right\},$$

and this is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} := \left(\|u\|^p_{W^{m,p}(\mathbb{R}^n)} + \sum_{|\alpha|=m} \left\| D^\alpha u \right\|^p_{W^{m',p}(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$

Some more facts are collected in the following remark:

Remark 1.2. *If $s = m$ is an integer, the space $W^{s,p}(\mathbb{R}^n)$ coincides with the Sobolev space $W^{m,p}(\mathbb{R}^n)$.*

Since for any $s > 0$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$, we have $W_0^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$, where $W_0^{s,p}(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ in the space $W^{s,p}(\mathbb{R}^n)$.

There is an alternative definition of the space $H^s(\mathbb{R}^n)$ via the Fourier transform. For any real number $s \in \mathbb{R}$, we may define

$$\widehat{H}^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}u(\xi)|^2 d\xi < \infty \right\}. \tag{1.8}$$

On the other hand, let $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^n)$ defined in (1.6) coincides with $\widehat{H}^s(\mathbb{R}^n)$ defined in (1.8): for any $u \in H^s(\mathbb{R}^n)$,

$$[u]^2_{H^s(\mathbb{R}^n)} := [u]_{W^{s,2}(\mathbb{R}^n)} = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi = 2C(n,s)^{-1} \left\| (-\Delta)^{\frac{s}{2}} u \right\|^2_{L^2(\mathbb{R}^n)},$$

where $C(n, s)$ is defined by (1.5).

1.2 The main results

In this subsection, we will state our main results: the regularity and stability of the finite energy weak solutions to the Cauchy problem (1.1)-(1.2). Firstly, we give the definition of finite energy weak solutions of (1.1) following Leray [30].

Definition 1.3 (Finite energy weak solutions)**.** *Let $n = 2, 3$ and $T > 0$. We say $(\mathbf{v}, \mathbf{u})$ is a finite energy weak solution of (1.1)-(1.2) over time interval $[0, T]$ provided:*

- *There holds the estimates*

$$\mathbf{v} \in C\left([0,T]; L^2_\sigma(\mathbb{R}^n)\right) \cap L^2\left([0,T]; H^s_\sigma(\mathbb{R}^n)\right), \quad \mathbf{u} \in C\left([0,T]; H^2_\sigma(\mathbb{R}^n)\right) \cap L^2\left([0,T]; H^{2+s}_\sigma(\mathbb{R}^n)\right).$$

- *The equations(1.1) is satisfied in the weak sense, i.e. for all $\phi \in C_0^\infty([0,T]\times\mathbb{R}^n)$ with $\nabla \cdot \phi = 0$, there holds*

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{v}(t,x) \cdot \phi(t,x) dx = &\int_{\mathbb{R}^n} \mathbf{v}_0(x) \cdot \phi(0,x) dx - \int_0^t \int_{\mathbb{R}^n} (\mathbf{u} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u}^T) \cdot \phi(t',x) dx dt' \\ &- \nu \int_0^t \int_{\mathbb{R}^n} \mathbf{v}(t',x) \cdot (-\Delta)^s \phi(t',x) dx dt' + \int_0^t \int_{\mathbb{R}^n} \mathbf{v}(t',x) \cdot \phi_t(t',x) dx dt',\end{aligned}$$

  and

$$\int_{\mathbb{R}^n} \mathbf{u}(t,x) \cdot \phi(t,x) dx - \alpha^2 \int_{\mathbb{R}^n} \mathbf{u}(t,x) \cdot \Delta\phi(t,x) dx = \int_{\mathbb{R}^n} \mathbf{v}(t,x) \cdot \phi(t,x) dx.$$

- *In addition, there holds the following energy inequality*

$$\begin{aligned}&\|\mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \|\nabla \mathbf{u}(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + 2\nu \int_0^t \left\|\Lambda^s \mathbf{u}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^n)} dt + 2\nu\alpha^2 \int_0^t \left\|\nabla \Lambda^s \mathbf{u}(\cdot,t)\right\|^2_{L^2(\mathbb{R}^n)} dt \\ &\le \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \|\nabla \mathbf{u}_0\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{1.9}$$

In [17], by using a fixed point argument, it was shown that there exists a finite energy weak solution to the Cauchy problem (1.1)-(1.2) in the sense of Definition 1.3. We recall this result by adding some uniform estimate related to the time derivative:

Theorem 1.4. *Let $n = 2, 3$ and $T > 0$. Assume that $\frac{n}{4} \le s < 1$ and $\mathbf{v}_0 \in L^2_\sigma(\mathbb{R}^n)$. If $s = \frac{n}{4}$, we suppose in addition that $\|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)} \le \varepsilon^*(\alpha, \nu, n)$ which is sufficiently small. Then there exists a finite energy weak solution on $[0, T]$ to the Cauchy problem (1.1)-(1.2) in the sense of Definition 1.3. In particular, there hold:*

$$\|\mathbf{v}\|_{L^\infty([0,T],L^2_\sigma(\mathbb{R}^n))} + \left\|\Lambda^s \mathbf{v}\right\|_{L^2([0,T],L^2_\sigma(\mathbb{R}^n))} \le C\left(n, s, \alpha, \nu, T, \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right), \tag{1.10}$$

and

$$\|\partial_t \mathbf{v}\|_{L^2([0,T],[H^s_\sigma(\mathbb{R}^n)]')} \le C\left(n, s, \alpha, \nu, T, \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right). \tag{1.11}$$

Now we give our main result which is the regularity of the weak solutions:

Theorem 1.5. *Let $n = 2, 3$ and $T > 0$. Assume that $\frac{n}{4} \le s < 1$ and $\mathbf{v}_0 \in H^M_\sigma(\mathbb{R}^n)$, $M \ge 0$. If $s = \frac{n}{4}$, we suppose in addition that $\|\mathbf{v}_0\|_{H^M(\mathbb{R}^n)} \le \varepsilon^{**}(\alpha, \nu, n)$ which is sufficiently small. Then the finite energy weak solution to the Cauchy problem (1.1)-(1.2) obtained in Theorem 1.4 satisfies the following higher order estimates*

$$\left\|\partial_t^k \nabla^m \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \nu \int_0^T \left\|\partial_t^k \nabla^m \Lambda^s \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} dt \le C\left(n, s, \alpha, \nu, T, \|\mathbf{v}_0\|_{H^M(\mathbb{R}^n)}\right) \tag{1.12}$$

for all $m + 2ks \le M$ with m and k are both non-negative integers.

Thanks to Theorem 1.4 and Theorem 1.5, we have further the following uniform estimates:

Corollary 1.6. *Let $n = 2, 3$ and $\frac{n}{4} \le s < 1$. Let $(\mathbf{v}, \mathbf{u})$ be the finite energy weak solution to the Cauchy problem (1.1)-(1.2) constructed in Theorem 1.4 and Theorem 1.5. Then for all $m + 2ks \le M$ with m and k are both non-negative integers, there holds*

$$\begin{aligned}
&\left\|\partial_t^k \nabla^m \mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2 \left\|\partial_t^k \nabla^{m+1} \mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^4 \left\|\partial_t^k \nabla^{m+2} \mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} = \left\|\partial_t^k \nabla^m \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)},\\
&\left\|\partial_t^k \nabla^m \mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} + \left\|\partial_t^k \nabla^m \Lambda^s \mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} + \left\|\partial_t^k \nabla^{m+1} \mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} \lesssim \left\|\partial_t^k \nabla^m \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)},\\
&\left\|\partial_t^k \nabla^m \mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} + \nu \int_0^t \left\|\partial_t^k \nabla^m \Lambda^s \mathbf{u}\right\|^2_{L^n(\mathbb{R}^n)} ds \le C\left(n, s, \alpha, \nu, m, k, \|\mathbf{v}_0\|_{H^M(\mathbb{R}^n)}\right).
\end{aligned} \tag{1.13}$$

From Theorem 1.5 and Corollary 1.6, we see that the weak solutions issued from smooth initial data are actually strong ones. This indirectly shows the global existence of strong solutions to (1.1)-(1.2).

It has been shown in [17] the uniqueness of the finite energy weak solutions to (1.1)-(1.2). By the similar argument as the proof for the regularity of the weak solutions, we can prove the stability result by using energy method. The uniqueness follows as a corollary.

Theorem 1.7. *Let $n = 2, 3$ and $\frac{n}{4} \le s < 1$. Then the finite energy weak solution to the Cauchy problem (1.1)-(1.2) obtained in Theorem 1.4 is unique. Moreover, the finite energy weak solutions are stable in the sense that for any initial data $\mathbf{v}_0, \mathbf{w}_0 \in L^2_\sigma(\mathbb{R}^n)$ which satisfies the smallness assumption in Theorem 1.4 when $s = \frac{n}{4}$, the corresponding finite energy weak solutions $(\mathbf{v}, \mathbf{u})$ and $(\mathbf{w}, \mathbf{q})$ satisfy*

$$\|\mathbf{v}(t) - \mathbf{w}(t)\|_{L^2(\mathbb{R}^n)} + \|\mathbf{u}(t) - \mathbf{q}(t)\|_{H^2(\mathbb{R}^n)} \le e^{Ct}\|\mathbf{v}_0 - \mathbf{w}_0\|_{L^2(\mathbb{R}^n)}, \quad \forall\, t \ge 0, \tag{1.14}$$

where $C = C(s, n, \|(\mathbf{v}_0, \mathbf{w}_0)\|_{L^2(\mathbb{R}^n)})$ depends only on the initial data.

At the end of this section, we give a remark concerning the smallness assumption.

Remark 1.8. *In our argument, we will repeatedly use Lemma 2.4 about fractional Gagliardo-Nirenberg-Sobolev inequalities. When $s > \frac{n}{4}$, we can always gain a smallness constant by interpolation (see* (2.5) *in Lemma 2.4), and this is quite essential to us to close the energy estimates. While for the critical case $s = \frac{n}{4}$, the estimate constant is merely bounded, see* (2.6)*. In order to close the estimates, an extra smallness assumption is added.*

The paper is organized as follows. In Section 2 we collect some preliminaries. In Section 3 we present the proof of lower-order regularity of inite energy weak solution (Theorem 1.4). We prove the higher-order regularity of the finite energy weak solution (Theorem 1.5 and Corollary 1.6) in Section 4. The proof of Theorem 1.7 is given in Section 5.

2 Preliminaries

In this section, we collect some preliminaries. Direct calculation leads to:

Lemma 2.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two smooth divergence free functions with compact support. Then*

$$\begin{cases} \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u}^T = -\mathbf{u}\times(\nabla\times\mathbf{v}) + \nabla(\mathbf{v}\cdot\mathbf{u}), \\ \langle \mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{u}\rangle + \left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \mathbf{u}\right\rangle = 0, \\ \langle \mathbf{u}\times(\nabla\times\mathbf{v}), \mathbf{u}\rangle = 0. \end{cases}$$

Basic L^2 energy estimate yields:

Lemma 2.2. *Let* $(\mathbf{u}, \mathbf{v})$ *be a smooth solution of* (1.1)-(1.2) *with compact support. Then there holds*

$$\frac{1}{2}\frac{d}{dt}\Big(\langle \mathbf{u}, \mathbf{u}\rangle + \alpha^2\langle \nabla\mathbf{u}, \nabla\mathbf{u}\rangle\Big) + \nu\Big(\langle \Lambda^s\mathbf{u}, \Lambda^s\mathbf{u}\rangle + \alpha^2\langle \nabla\Lambda^s\mathbf{u}, \nabla\Lambda^s\mathbf{u}\rangle\Big) = 0,$$

and the energy inequality (1.9) *is satisfied (the energy equality is actually satisfied).*

We then claim the following estimates.

Lemma 2.3. *Let $n = 2, 3$ and $\frac{n}{4} \le s < 1$. Let $\mathbf{v} \in H^s_\sigma(\mathbb{R}^n)$ and $\mathbf{u} \in H^{2+s}_\sigma(\mathbb{R}^n)$ satisfy the Helmholtz equations*

$$\mathbf{u} - \alpha^2\Delta\mathbf{u} = \mathbf{v}. \tag{2.1}$$

This gives

$$\Lambda^s\mathbf{u} - \alpha^2\Delta\Lambda^s\mathbf{u} = \Lambda^s\mathbf{v}. \tag{2.2}$$

In addition, there holds

$$\begin{cases} \|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\|\Delta\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} = \|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}, \\ \|\Lambda^s\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + 2\alpha^2\|\nabla\Lambda^s\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + \alpha^4\|\Delta\Lambda^s\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} = \|\Lambda^s\mathbf{v}\|^2_{L^2(\mathbb{R}^n)}, \\ \|\mathbf{u}\|_{L^n(\mathbb{R}^n)} \le C\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}, \quad \|\nabla\mathbf{u}\|_{L^n(\mathbb{R}^n)} \le C\|\mathbf{v}\|_{L^2(\mathbb{R}^n)}, \\ \|\nabla\mathbf{u}\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \le C\|\Lambda^s\mathbf{v}\|_{L^2(\mathbb{R}^n)}, \quad \|\Delta\mathbf{u}\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \le C\|\Lambda^s\mathbf{v}\|_{L^2(\mathbb{R}^n)}. \end{cases} \tag{2.3}$$

Proof. Note that

$$\frac{s}{n} \geq \frac{1}{2} - \frac{s}{n}, \quad \text{for } \frac{n}{4} \leq s < 1 \text{ with } n = 2, 3.$$

By the Gagliardo-Nirenberg-Sobolev inequality, direct calculations yield the estimates in (2.3). □

The following Lemma concerns the nonlocal version of the known inequalities established in [25, 26, 27, 28] by Ladyzhenskaya, Shkoller and Seregin:

Lemma 2.4 (Fractional Gagliardo-Nirenberg-Sobolev inequalities)**.** *Let $n = 2, 3$, $\mathbf{u} \in H_0^1(\mathbb{R}^n)$ and let $\varepsilon > 0$. Firstly, the following inequalities hold:*

$$\begin{cases} \|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq \varepsilon \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + \varepsilon^{-1}\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} & \text{for } n = 2, \\ \\ \|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq \varepsilon \|\nabla \mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + C(\varepsilon)\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} & \text{for } n = 3. \end{cases} \tag{2.4}$$

The above inequalities (2.4) *can be generalized to the following nonlocal version (fractional power Sobolev-type):*

- *For $n/4 < s < 1$ and $\mathbf{u} \in H^s(\mathbb{R}^n)$, the following inequalities hold:*

$$\begin{cases} \|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq \varepsilon \,\|\Lambda^s \mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + C(\varepsilon)\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} & \text{for } n = 2, \\ \\ \|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq C(s)\varepsilon \,\|\Lambda^s \mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + C(\varepsilon)\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} & \text{for } n = 3. \end{cases} \tag{2.5}$$

  Here, ε, $C(s)$ and $C(\varepsilon)$ are constants; $C(s)$ depends only on spatial dimensions and s, and $C(\varepsilon) = O(\varepsilon^{-\frac{n}{4s-n}})$.

- *For the critical case $s = n/4$ and $\mathbf{u} \in H^{\frac{n}{4}}(\mathbb{R}^n)$, the following inequality holds:*

$$\|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq C\left(\left\|\Lambda^{\frac{n}{4}}\mathbf{u}\right\|^2_{L^2(\mathbb{R}^n)} + \|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)}\right). \tag{2.6}$$

  Here, C is a constant depending only on spatial dimensions n.

Proof. For $\frac{n}{4} < s < 1$, in view of Gagliardo-Nirenberg-Sobolev inequality, interpolation estimates give rise to

$$\|\mathbf{u}\|_{L^4(\mathbb{R}^n)} \leq C(s)\|\mathbf{u}\|^{1-\theta}_{L^2(\mathbb{R}^n)}\|\Lambda^s \mathbf{u}\|^{\theta}_{L^2(\mathbb{R}^n)}, \tag{2.7}$$

where $\frac{1}{4} = \frac{1-\theta}{2} + (\frac{1}{2} - \frac{s}{n})\theta$, i.e., $\theta = \frac{n}{4s}$. Together with Young's inequality, one has

$$\|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq C(\varepsilon)\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + C(s)\varepsilon\|\Lambda^s \mathbf{u}\|^2_{L^2(\mathbb{R}^n)},$$

where $C(\varepsilon) = O(\varepsilon^{-\frac{n}{4s-n}})$, $C(s) = 1$ for $n = 2$, and $C(s)$ depends only on spatial dimensions and s for $n = 3$. This leads to (2.5).

For the critical case $s = \frac{n}{4}$, by (2.7),

$$\|\mathbf{u}\|^2_{L^4(\mathbb{R}^n)} \leq C(n)\|\Lambda^{\frac{n}{4}}\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} \leq C(n)\left(\|\Lambda^{\frac{n}{4}}\mathbf{u}\|^2_{L^2(\mathbb{R}^n)} + \|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)}\right),$$

which is just the inequality (2.6). □

We now collect some known estimates for the vector-valued fractional Leibniz rule.

Lemma 2.5. *The following two conclusions hold:*

(i) *(see [16]) Let* $s_1, s_2 \in [0,1]$, $s = s_1 + s_2$, *and* $p,\ p_1,\ p_2 \in (1,\infty)$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Then the following bilinear estimate holds for all* $f, g \in \mathscr{S}(\mathbb{R}^n)$, $n \geq 1$:

$$\left\|\Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\Lambda^{s_1} f\right\|_{L^{p_1}(\mathbb{R}^n)} \left\|\Lambda^{s_2} g\right\|_{L^{p_2}(\mathbb{R}^n)}.$$

(ii) *(see [19]) Let* $s > \max\left(0, \frac{n}{p} - n\right)$, *or s be a positive even integer,* $\frac{1}{2} < p < \infty$, $1 < p_1, p_2 \leq \infty$, *and* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, *then*

$$\left\|\Lambda^s (fg)\right\|_{L^p(\mathbb{R}^n)} \leq C \left\|\Lambda^s f\right\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)} \left\|\Lambda^s g\right\|_{L^{p_2}(\mathbb{R}^n)}.$$

Remark 2.6. *For* $0 < s < 1$ *and* $n = 1$, *Kenig, Ponce, and Vega in [23] obtained the similar estimates for fractional derivatives as those in Lemma 2.5:*

$$\left\|\Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f\right\|_{L^p(\mathbb{R})} \leq C \left\|\Lambda^{s_1} f\right\|_{L^{p_1}(\mathbb{R})} \left\|\Lambda^{s_2} g\right\|_{L^{p_2}(\mathbb{R})},$$

$$\left\|\Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f\right\|_{L^p(\mathbb{R})} \leq C \|g\|_{L^\infty(\mathbb{R})} \left\|\Lambda^s f\right\|_{L^p(\mathbb{R})},$$

where $p, p_1, p_2 \in (1,\infty)$ *satisfying* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, *and* $0 < s = s_1 + s_2 < 1$ *with* $s_1, s_2 \geq 0$.

Direct calculation yields:

Lemma 2.7. *Let* $n/4 < s < 1$ *with* $n = 2, 3$ *and let* $r = n/2 + 1 - 2s$. *Then*

$$\begin{aligned}
& s \geq 1 - s, \quad \frac{n-2}{2n} < \frac{2s-1}{n} < \frac{1}{n}, \quad \frac{2s-1}{n} = \frac{1}{2} - \frac{r}{n}, \\
& \frac{n}{2} - 1 < r < 1 < \frac{n}{2} < 2 \ \textit{for} \ n = 3, \\
& \frac{n}{2} - 1 < r < 1 = \frac{n}{2} < 2 \ \textit{for} \ n = 2,
\end{aligned}$$

and thus there holds the continuous Sobolev embedding $\dot{H}^r(\mathbb{R}^n) \hookrightarrow L^{n/(2s-1)}(\mathbb{R}^n)$.

Assume that $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, or $\Omega = \mathbb{R}^n$ with $n = 2, 3$. Let $A := \mathcal{P}(-\Delta)$ be the Stokes operator with $\mathcal{P}$ being the Leray projection operator $\mathcal{P} : L^2(\Omega) \to \{v \in L^2(\Omega) : \operatorname{div} v = 0,\ v \cdot n = 0 \text{ on } \partial\Omega\}$. In [17], they studied the viscous Camassa-Holm equations with fractional diffusion

$$\begin{aligned}
& (1 - \alpha^2\Delta)\mathbf{u}_t + \mathbf{u} \cdot \nabla(1 - \alpha^2\Delta)\mathbf{u} - \alpha^2 \nabla \mathbf{u}^T \cdot \Delta\mathbf{u} + \nabla p = -\nu(1 - \alpha^2\Delta)A^s\mathbf{u}, \\
& \operatorname{div} \mathbf{u} = 0.
\end{aligned} \tag{2.8}$$

We claim the following conclusion:

Lemma 2.8. *Equation* (2.8) *is equivalent to the equations below:*

$$\begin{aligned}
& \mathbf{v}_t + \mathbf{u} \cdot \nabla\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{u}^T + \nabla\tilde{p} = -\nu A^s \mathbf{v}, \\
& \mathbf{v} = (1 + \alpha^2 A)\mathbf{u}, \\
& \operatorname{div} \mathbf{u} = 0.
\end{aligned}$$

As a result, what was shown in [17], in particular the existence and uniqueness of finite energy weak solutions, can apply to our case.

Proof. Let $\mathbf{u}$ be sufficiently regular. Applying $\mathcal{P}$ to (2.8) yields that

$$\partial_t(1+\alpha^2 A)\mathbf{u} + \mathcal{P}[\mathbf{u}\cdot\nabla(1-\alpha^2\Delta)\mathbf{u} - \alpha^2\nabla\mathbf{u}^T\cdot\Delta\mathbf{u}] = -\nu(1+\alpha^2 A)A^s\mathbf{u}. \tag{2.9}$$

Let $-\Delta\mathbf{u} = A\mathbf{u} + \nabla q$ for some q and let $\mathbf{v} = (1+\alpha^2 A)\mathbf{u}$. Then (2.9) leads to

$$\partial_t\mathbf{v} + \mathcal{P}[\mathbf{u}\cdot\nabla\mathbf{v} + \alpha^2\mathbf{u}\cdot\nabla\nabla q - \alpha^2\nabla\mathbf{u}^T\cdot\Delta\mathbf{u}] = -\nu A^s\mathbf{v}, \tag{2.10}$$

where we used

$$\begin{aligned}\mathcal{P}[\mathbf{u}\cdot\nabla(1-\alpha^2\Delta)\mathbf{u} - \alpha^2\nabla\mathbf{u}^T\cdot\Delta\mathbf{u}] &= \mathcal{P}[\mathbf{u}\cdot\nabla(1+\alpha^2 A)\mathbf{u} + \alpha^2\mathbf{u}\cdot\nabla\nabla q - \alpha^2\nabla\mathbf{u}^T\cdot\Delta\mathbf{u}]\\ &= \mathcal{P}[\mathbf{u}\cdot\nabla\mathbf{v}] + \alpha^2\mathcal{P}[\mathbf{u}\cdot\nabla\nabla q - \nabla\mathbf{u}^T\cdot\Delta\mathbf{u}].\end{aligned}$$

On the other hand, the known fact $\mathcal{P}(\nabla) = 0$ as well as the identity

$$\nabla(\mathbf{u}\cdot\nabla q) = \nabla(\mathbf{u}^j\cdot\partial_j q) = \nabla\mathbf{u}^j\cdot\partial_j q + \mathbf{u}^j\cdot\nabla\partial_j q,$$

implies

$$\mathcal{P}[\mathbf{u}\cdot\nabla\nabla q] = -\mathcal{P}[\nabla u^j\cdot\partial_j q] = \mathcal{P}[-(\nabla\mathbf{u}^T)\cdot\nabla q].$$

In view of $\nabla\mathbf{u}^T\cdot\mathbf{u} = \frac{1}{2}\nabla(|\mathbf{u}|^2)$, direct verification leads to

$$\begin{aligned}\alpha^2\mathcal{P}[\mathbf{u}\cdot\nabla\nabla q - \nabla\mathbf{u}^T\cdot\Delta\mathbf{u}] &= \alpha^2\mathcal{P}[-\nabla\mathbf{u}^T\cdot\nabla q - \nabla\mathbf{u}^T\cdot\Delta\mathbf{u}]\\ &= \alpha^2\mathcal{P}[\nabla\mathbf{u}^T\cdot A\mathbf{u}] + \mathcal{P}[\nabla\mathbf{u}^T\cdot\mathbf{u}] = \mathcal{P}[\nabla\mathbf{u}^T\cdot\mathbf{v}].\end{aligned}$$

This together with (2.10) yields that for some $\tilde{p}$:

$$\partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} + \nabla\mathbf{u}^T\cdot\mathbf{v} + \nabla\tilde{p} = -\nu A^s\mathbf{v}.$$

□

3 Time derivative estimates for finite energy weak solutions

Follows from Theorem 3.1 in [17] and Lemma 2.8, there exists a unique finite energy weak solution $(\mathbf{v},\mathbf{u})$ in the sense of Definition 1.3 and the estimate (1.10) is satisfied. In this section, we shall prove the estimate (1.11) for the time derivative of finite energy weak solutions.

Let $\phi\in\Sigma$ be a smooth and divergence free function. Since $(\mathbf{v},\mathbf{u})$ is a weak solution in the sense of Definition 1.3, we have

$$\langle\partial_t\mathbf{v},\phi\rangle = -\langle\mathbf{u}\cdot\nabla\mathbf{v},\phi\rangle - \left\langle\mathbf{v}\cdot\nabla\mathbf{u}^T,\phi\right\rangle - \nu\left\langle(-\Delta)^s\mathbf{v},\phi\right\rangle =: A_1 + A_2 + A_3. \tag{3.1}$$

We then estimate A_1, A_2, A_3 one by one through considering two cases.

Case 1: $n/4 < s < 1$. In this case, a straightforward computation shows that

$$\begin{cases}\frac{n}{s} = \frac{2n}{n-2(n-2s)/2},\ \ \frac{1}{2} = \frac{n-2}{2} < \frac{n-2s}{2} < \frac{3}{4} & \text{for } n=3,\\ 0 = \frac{n-2}{2} < \frac{n-2s}{2} < \frac{1}{2} & \text{for } n=2,\\ B := \frac{n-2s}{2} + 1 - s = \frac{n}{2} + 1 - 2s,\ \ \frac{n}{2} - 1 < B < 1 & \text{for } n=2,3.\end{cases} \tag{3.2}$$

We first estimate A_1. Thanks to Lemma 2.3, Lemma 2.5 and Lemma 2.7, there holds

$$|A_1| = |\langle\mathbf{u}\cdot\nabla\mathbf{v},\phi\rangle| \lesssim \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{1-s}(\mathbf{u}\phi)\right\|_{L^2(\mathbb{R}^n)}, \tag{3.3}$$

and the term $\left\|\Lambda^{1-s}(\mathbf{u}\phi)\right\|_{L^2(\mathbb{R}^n)}$ in (3.3) can be estimated as

$$\left\|\Lambda^{1-s}(\mathbf{u}\phi)\right\|_{L^2(\mathbb{R}^n)} \lesssim \left\|\Lambda^{1-s}(\mathbf{u}\phi) - \mathbf{u}\Lambda^{1-s}\phi - \phi\Lambda^{1-s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} + \left\|\mathbf{u}\Lambda^{1-s}\phi\right\|_{L^2(\mathbb{R}^n)} + \left\|\phi\Lambda^{1-s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}. \tag{3.4}$$

Moreover, by (3.2) and observing

$$\frac{1}{2} = \frac{s}{n} + \frac{n-2s}{2n}, \quad 0 < 1-s < s < 1,$$

the three terms on the right hand side of (3.4) can be estimated as follows:

$$\begin{aligned}\left\|\Lambda^{1-s}(\mathbf{u}\phi) - \mathbf{u}\Lambda^{1-s}\phi - \phi\Lambda^{1-s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} &\lesssim \left\|\Lambda^{1-s}\mathbf{u}\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\ &\lesssim \left\|\Lambda^{\frac{n}{2}+1-2s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}} \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2s} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\left\|\mathbf{u}\Lambda^{1-s}\phi\right\|_{L^2(\mathbb{R}^n)} &\lesssim \|\mathbf{u}\|_{L^{\frac{n}{2s-1}}(\mathbb{R}^n)} \left\|\Lambda^{1-s}\phi\right\|_{L^{\frac{2n}{n-2(2s-1)}}(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{u}\|_{L^{\frac{2n}{n-2\left(\frac{n}{2}+1-2s\right)}}(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \left\|\Lambda^{\frac{n}{2}+1-2s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}} \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2s} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\left\|\phi\Lambda^{1-s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} &\lesssim \|\phi\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \left\|\Lambda^{1-s}\mathbf{u}\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \\ &\lesssim \left\|\Lambda^{\frac{n}{2}+1-2s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}} \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2s} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{3.7}$$

Here, $0 < 2s - \frac{n}{2} < 2 - \frac{n}{2}$, $\frac{n}{2} - 1 < \frac{n}{2} + 1 - 2s < 1$ for $\frac{n}{4} < s < 1$. By Young's inequality, combining (3.3) with (3.4), (3.5), (3.6) and (3.7) gives

$$\begin{aligned}|A_1| &\lesssim \|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}} \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2s} \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)} + \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right) \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{3.8}$$

We next estimate A_2. Thanks to (2.3) and (3.2), observing that $0 < 2 - \frac{n}{2s} < 2 - \frac{n}{2}$, $\frac{n}{2} - 1 < \frac{n}{2s} - 1 < 1$ for $\frac{n}{4} < s < 1$, we have

$$\begin{aligned}|A_2| = \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \phi\right\rangle\right| &\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \|\mathbf{v}\phi\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \|\mathbf{v}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{2}-s}\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)}^{2-\frac{n}{2s}} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2s}-1} \\ &\lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left(\|\phi\|_{L^2(\mathbb{R}^n)} + \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)}\right).\end{aligned} \tag{3.9}$$

Similarly, for A_3, we have

$$|A_3| = \nu\left|\left\langle(-\Delta)^s\mathbf{v}, \phi\right\rangle\right| \lesssim \left\|\Lambda^s\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\phi\right\|_{L^2(\mathbb{R}^n)}. \tag{3.10}$$

Collecting the estimates in(3.8), (3.9) and (3.10) yields that for $\frac{n}{4} < s < 1$,

$$|\langle \partial_t \mathbf{v}, \phi\rangle| \lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left(\|\phi\|_{L^2(\mathbb{R}^n)} + \left\|\Lambda^s \phi\right\|_{L^2(\mathbb{R}^n)}\right). \tag{3.11}$$

As $\phi \in \Sigma$ can be chosen arbitrarily, using density argument, by Hölder's inequality, (3.11), (1.9) and (1.10), we conclude that for $n/4 < s < 1$, there holds

$$\|\partial_t \mathbf{v}\|_{L^2\left([0,T],[H^s_\sigma(\mathbb{R}^n)]'\right)} \leq C\left(n, s, \alpha, \nu, T, \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right). \tag{3.12}$$

Case 2: $s = n/4$. In this case, direct calculation yields

$$|A_1| = |\langle \mathbf{u}\cdot\nabla\mathbf{v}, \phi\rangle| \lesssim \left\|\Lambda^{\frac{n}{4}} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\phi)\right\|_{L^2(\mathbb{R}^n)}. \tag{3.13}$$

In particular, $\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\phi)\right\|_{L^2(\mathbb{R}^n)}$ can be bounded as

$$\begin{aligned}\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\phi)\right\|_{L^2(\mathbb{R}^n)} &\lesssim \left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\phi) - u\Lambda^{1-\frac{n}{4}}\phi - \phi\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \\ &\quad + \left\|\mathbf{u}\Lambda^{1-\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)} + \left\|\phi\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{3.14}$$

We now estimate the first term on the right hand side of inequality (3.14). By using (I) of Lemma 2.5, one achieves for $0 < s_1 < 1 - \frac{n}{4}$,

$$\begin{aligned}\left\|\Lambda^{1-\frac{n}{4}}(\mathbf{u}\phi) - \mathbf{u}\Lambda^{1-\frac{n}{4}}\phi - \phi\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} &\lesssim \left\|\Lambda^{1-\frac{n}{4}-s_1}\mathbf{u}\right\|_{L^{\frac{2n}{n-2\left(\frac{n}{4}+s_1\right)}}(\mathbb{R}^n)} \left\|\Lambda^{s_1}\phi\right\|_{L^{\frac{2n}{n-2\left(\frac{n}{4}-s_1\right)}}(\mathbb{R}^n)} \\ &\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.15}$$

where $s_1 \in \left(0, \frac{1}{2}\right), 1 - \frac{n}{4} - s_1 \in \left(0, \frac{1}{2}\right)$ and

$$p_1 := \frac{2n}{n - 2(\frac{n}{4} + s_1)} \in (1, \infty), \quad p_2 := \frac{2n}{n - 2(\frac{n}{4} - s_1)} \in (1, \infty), \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}.$$

In the same manner, by virtue of (1.9), Agmon's inequality and interpolation inequality, the second and the third terms on the right hand side of inequality (3.14) can be bounded as follows:

$$\begin{aligned}&\left\|\mathbf{u}\Lambda^{1-\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)} + \left\|\phi\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim \|\mathbf{u}\|_{L^\infty(\mathbb{R}^n)} \left\|\Lambda^{1-\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)} + \|\phi\|_{L^4(\mathbb{R}^n)} \left\|\Lambda^{1-\frac{n}{4}}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)} \\ &\quad\lesssim \|\mathbf{u}\|^{\frac{1}{2}}_{H^1(\mathbb{R}^n)} \|\mathbf{u}\|^{\frac{1}{2}}_{H^2(\mathbb{R}^n)} \|\phi\|^{2-\frac{4}{n}}_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}}\phi\right\|^{\frac{4}{n}-1}_{L^2(\mathbb{R}^n)} + \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)\frac{n}{4}} \left(\|\phi\|_{L^2(\mathbb{R}^n)} + \left\|\Lambda^{\frac{n}{4}}\phi\right\|_{L^2(\mathbb{R}^n)}\right).\end{aligned} \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.16), it follows that

$$A_1 \lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left(\|\phi\|^2_{L^2(\mathbb{R}^n)} + \left\|\Lambda^{\frac{n}{4}}\phi\right\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}. \tag{3.17}$$

Using the smallness assumption on initial data, by similar arguments like those employed in A_1 above, implies

$$\begin{aligned}A_2 = \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \phi\right\rangle\right| &\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \|\mathbf{v}\phi\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left(\|\phi\|^2_{L^2(\mathbb{R}^n)} + \left\|\Lambda^{\frac{n}{4}}\phi\right\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}},\end{aligned} \tag{3.18}$$

and

$$A_3 = \nu \left|\left\langle (-\Delta)^{\frac{n}{4}} \mathbf{v}, \phi \right\rangle\right| \lesssim \left\|\Lambda^{\frac{n}{4}} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \phi\right\|_{L^2(\mathbb{R}^n)}. \tag{3.19}$$

By (3.17), (3.18) and (3.19), we thus have

$$|\langle \partial_t \mathbf{v}, \phi\rangle| \lesssim \|\mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left(\|\phi\|^2_{L^2(\mathbb{R}^n)} + \left\|\Lambda^{\frac{n}{4}} \phi\right\|^2_{L^2(\mathbb{R}^n)}\right)^{\frac{1}{2}}. \tag{3.20}$$

As we can choose arbitrarily for $\phi \in \Sigma$, by (1.10), Hölder's inequality and the assumption of Theorem 1.4, we deduce from (3.20) that for $s = \frac{n}{4}$ that

$$\|\partial_t \mathbf{v}\|_{L^2\left([0,T],\left[H_0^{\frac{n}{4}}(\mathbb{R}^n)\right]'\right)} \le C\left(n, \alpha, \nu, T, \|\mathbf{v}_0\|_{L^2(\mathbb{R}^n)}\right). \tag{3.21}$$

□

4 The regularity of the finite energy weak solutions

This section is devoted to proving our main result Theorem 1.5 concerning the high-order regularity of the finite energy weak solutions.

In [17] the authors showed the existence and uniqueness of global-in-time finite energy weak solutions. The local existence of strong solutions issued from regular initial data can be obtained by applying the same method as in [17]. Due to the uniqueness of the finite energy weak solutions, this local-in-time strong solution coincides with the weak one as long as the previous one exists. Hence, to show the regularity of the weak solution, it is sufficient to prove uniform higher order a priori regularity for smooth solutions in any given time interval.

With this reason, we shall derive formally the higher-order regularity of the solutions constructed in Theorem 1.4. This can be divided into two parts. We first prove the high-order regularity with respect to spatial variables. We then verify the high-order regularity with respect to space-time variables. We shall finish the proof of Theorem 1.5 after proving Theorems 4.1 and 4.2.

4.1 High-order regularity in space variables

In this subsection, we will prove the following result concerning the high-order regularity in space variables.

Theorem 4.1. *Let $n = 2, 3$, $\frac{n}{4} \le s < 1$ and $K \in \mathbb{Z}_+$. Assume the initial datum $\mathbf{v}_0 \in H^K_\sigma(\mathbb{R}^n)$, and if $s = \frac{n}{4}$, we assume in addition that the initial datum is small: there exists $\varepsilon^* = \varepsilon^*(\alpha, \nu, n)$ sufficiently small such that $\|\mathbf{v}_0\|_{H^K_0(\mathbb{R}^n)} \le \varepsilon^*$. Then for any integer $M \le K$, the finite energy weak solutions to the Cauchy problem* (1.1)-(1.2) *constructed in Theorem 1.4 satisfy*

$$\left\|\nabla^M \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \nu \int_0^T \left\|\nabla^M \Lambda^s \mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} dt \le C\left(n, \alpha, \nu, \|\mathbf{v}_0\|_{H^K_0(\mathbb{R}^n)}\right). \tag{4.1}$$

Proof. We prove Theorem 4.1 by induction with three steps.

Step 1. We first give the inductive assumption. Let $\frac{n}{4} \le s < 1$. Assume that for each nonnegative integer $m < M$, the following inductive bound holds:

$$\|\nabla^m \mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \nu \int_0^T \|\nabla^m \Lambda^s \mathbf{v}\|^2_{L^2(\mathbb{R}^n)} dt \le C\left(n, \alpha, \nu, \|\mathbf{v}_0\|_{H^K_0(\mathbb{R}^n)}\right). \tag{4.2}$$

Step 2. By Theorem 1.4, it is easy to verify that the inductive assumption (4.2) holds for the base case $m = 0$.

Step 3. We will show that the inductive assumption (4.2) holds for $m = M$.

Multiplying the first equation in (1.1) by $\Delta^M \mathbf{v}$ and integrating in space yields, after some integration by parts,

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu\left\|\nabla^M \Lambda^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \le \left|\left\langle \mathbf{u}\cdot\nabla\mathbf{v}, \Delta^M \mathbf{v}\right\rangle\right| + \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \Delta^M \mathbf{v}\right\rangle\right| := I_M + J_M. \tag{4.3}$$

We shall estimate I_M and J_M in (4.3) through two cases:

Case (1) $\frac{n}{4} < s < 1$, $n = 2, 3$;

Case (2) $s = \frac{n}{4}$, $n = 2, 3$.

We first consider **Case (1)**. Recall that $\langle \mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{v}\rangle = 0$. Thanks to Hölder's inequality, Young's inequality and the fractional Gagliardo-Nirenberg-Sobolev inequality, we deduce that

$$\begin{aligned}
I_M &= \left|\left\langle \mathbf{u}\cdot\nabla\mathbf{v}, \Delta^M\mathbf{v}\right\rangle\right| \lesssim \left|\left\langle \mathbf{u}\cdot\nabla\nabla^M\mathbf{v}, \nabla^M\mathbf{v}\right\rangle\right| + \left|\sum_{m=1}^{M}\left\langle \nabla^m\mathbf{u}\cdot\nabla\nabla^{M-m}\mathbf{v}, \nabla^M\mathbf{v}\right\rangle\right| \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^m\mathbf{u}\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)}\left\|\nabla^M\mathbf{v}\right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{2}-s}\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\nabla^{m-1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}}\left\|\Lambda^s\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2s}\left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left(\left\|\Lambda^s\nabla^{m-1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)} + \left\|\Lambda^s\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \nu^{-1}\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\Lambda^s\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \frac{\nu}{4}\left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{4.4}$$

In the same manner, we have

$$\begin{aligned}
J_M = \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \Delta^M\mathbf{v}\right\rangle\right| &\lesssim \sum_{m=0}^{M}\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\nabla^m\mathbf{u}, \nabla^{M-m}\mathbf{v}\right\rangle\right| \\
&\lesssim \left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\mathbf{u}, \nabla^M\mathbf{v}\right\rangle\right| + \sum_{m=1}^{M}\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla^{m+1}\mathbf{u}, \nabla^{M-m}\mathbf{v}\right\rangle\right|.
\end{aligned} \tag{4.5}$$

By Lemma 2.4, for the first term in (4.5), we have

$$\begin{aligned}
\left|\left\langle \nabla^M\mathbf{v}\cdot\nabla\mathbf{u}, \nabla^M\mathbf{v}\right\rangle\right| &\lesssim \left\|\nabla^M\mathbf{v}\right\|_{L^4(\mathbb{R}^n)}^2\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \left(C(\varepsilon)\left\|\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon\left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\right)\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.6}$$

And for the second term in (4.5), we have

$$
\begin{aligned}
\left|\left\langle \nabla^M \mathbf{v}\cdot\nabla^{m+1}\mathbf{u}, \nabla^{M-m}\mathbf{v}\right\rangle\right| &\lesssim \sum_{m=1}^{M} \left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\nabla^{m+1}\mathbf{u}\cdot\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M} \left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\nabla^{m+1}\mathbf{u}\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \left\|\nabla^M\mathbf{v}\right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M} \left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \nu^{-1}\sum_{m=1}^{M} \left\|\nabla^{M-m}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\nabla^{m-1}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \frac{\nu}{4}\left\|\Lambda^s\nabla^M\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.7}
$$

By (1.9), under the assumptions of Theorem 4.1, by (4.3), (4.4), (4.5), (4.6) and (4.7), choosing

$$
\varepsilon \le \frac{\nu}{2\left\|\nabla\mathbf{u}_0\right\|^2_{L^2(\mathbb{R}^n)}},
$$

we deduce

$$
\begin{aligned}
\frac{d}{dt}\left\|\nabla^M\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \lesssim &\sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\nabla^{m-1}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} \\
&+\sum_{m=1}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\nabla^{m-1}\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\mathbf{v}_0\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^M\mathbf{v}\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.8}
$$

Together with Theorem 1.4 and the induction assumption (4.2), using Gronwall's inequality to (4.8) leads to our desired estimate (4.1).

We then consider **Case (2)** $s = \frac{n}{4}$. In this case, unlike the previous case when $s > \frac{n}{4}$, we see from (2.6) in Lemma 2.4 that we cannot obtain a small constant in the interpolation inequality, so a smallness assumption on the initial data is needed. Recall again that $\langle \mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{v}\rangle = 0$. By Hölder's inequality and the Gagliardo-Nirenberg-Sobolev inequality, and Lemma 2.4, we have

$$
\begin{aligned}
I_M = \left|\left\langle \mathbf{u}\cdot\nabla\mathbf{v}, \Delta^M\mathbf{v}\right\rangle\right| &\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^m\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\left\|\nabla^M\mathbf{v}\right\|_{L^4(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^m\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{4.9}
$$

and

$$
\begin{aligned}
J_M = \left|\left\langle \mathbf{v}\cdot\nabla\mathbf{u}^T, \Delta^M\mathbf{v}\right\rangle\right| &\lesssim \sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\nabla^{m+1}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}\left\|\nabla^M\mathbf{v}\right\|_{L^4(\mathbb{R}^n)} \\
&\lesssim \sum_{m=0}^{M}\left\|\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_{m=1}^{M}\left\|\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^{m-1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{4}}\nabla^M\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{4.10}
$$

Combining (4.9) with (4.10) yields

$$\begin{aligned} I_M + J_M \lesssim& \left\|\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ &+ \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \nabla^{M-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ &+ \sum_{m=2}^{M-1} \left\|\nabla^{M+1-m} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \nabla^{m-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{\frac{n}{4}} \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)} \\ \lesssim& \left\|\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\Lambda^{\frac{n}{4}} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\Lambda^{\frac{n}{4}} \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\ &+ \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^n)}^2 \left\|\Lambda^{\frac{n}{4}} \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\Lambda^{\frac{n}{4}} \nabla^{M-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\ &+ \sum_{m=2}^{M-1} \left(\left\|\nabla^{M+1-m} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\Lambda^{\frac{n}{4}} \nabla^{m-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\Lambda^{\frac{n}{4}} \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\right). \end{aligned} \tag{4.11}$$

By the induction assumption (4.2), we obtain for $M \leq K$,

$$\sum_{m=1}^{M-1} \left\|\Lambda^{\frac{n}{4}} \nabla^{m-1} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \|\mathbf{v}\|_{H_0^{\frac{n}{4}+M-2}(\mathbb{R}^n)}^2 \lesssim \|\mathbf{v}_0\|_{H_0^K(\mathbb{R}^n)}^2. \tag{4.12}$$

Choosing ε^* sufficiently small such that

$$\|\mathbf{v}_0\|_{H_0^K(\mathbb{R}^n)}^2 \leq \varepsilon^* < \frac{\nu}{2},$$

and applying interpolation inequality and Gronwall's inequality to (4.3) and (4.11) leads to our desired result (4.1). This completes the proof of Theorem 4.1. □

4.2 High-order regularity in space-time variables

We further prove the following result:

Theorem 4.2. *Let $n = 2, 3$, $\frac{n}{4} \leq s < 1$ and $K \in \mathbb{Z}_+$. Assume the initial datum $\mathbf{v}_0 \in H_\sigma^K(\mathbb{R}^n)$, and if $s = \frac{n}{4}$, we additionally assume the initial datum is small: there exists an $\varepsilon^{**} = \varepsilon^{**}(\alpha, \nu, n) > 0$ such that $\|\mathbf{v}_0\|_{H_0^K(\mathbb{R}^n)} \leq \varepsilon^{**}$. Then for all nonnegative integers M, P such that $M + 2Ps \leq K$, the solutions to the Cauchy problem* (1.1)-(1.2) *constructed in Theorem 1.4 admits the bound*

$$\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu \int_0^T \left\|\partial_t^P \nabla^M \Lambda^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 dt \leq C\left(n, \alpha, \nu, \|\mathbf{v}_0\|_{H_0^K(\mathbb{R}^n)}\right). \tag{4.13}$$

Proof. Applying $\partial_t^P \nabla^M$ to the first equation in (1.1), we have

$$\partial_t^{P+1} \nabla^M \mathbf{v} + \partial_t^P \nabla^M (\mathbf{u} \cdot \nabla \mathbf{v}) + \partial_t^P \nabla^M \left(\mathbf{v} \cdot \nabla \mathbf{u}^T\right) + \partial_t^P \nabla^M \nabla p = -\nu \partial_t^P \nabla^M (-\Delta)^s \mathbf{v}. \tag{4.14}$$

Direct calculation gives

$$\partial_t^{P+1} \nabla^M \mathbf{v} + \partial_t^P \nabla^M \nabla p = -\partial_t^P \nabla^M (\mathbf{u} \cdot \nabla \mathbf{v}) - \partial_t^P \nabla^M \left(\mathbf{v} \cdot \nabla \mathbf{u}^T\right) - \nu \partial_t^P \nabla^M (-\Delta)^s \mathbf{v},$$

which yields that

$$\begin{aligned} \left\|\partial_t^{P+1} \nabla^M \mathbf{v} + \partial_t^P \nabla^M \nabla p\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim& \left\|-\partial_t^P \nabla^M (\mathbf{u} \cdot \nabla \mathbf{v}) - \partial_t^P \nabla^M \left(\mathbf{v} \cdot \nabla \mathbf{u}^T\right) - \nu \partial_t^P \nabla^M (-\Delta)^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\ \lesssim& \left\|\partial_t^P \nabla^M \Lambda^{2s} \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\partial_t^P \nabla^M (\mathbf{u} \cdot \nabla \mathbf{v})\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\partial_t^P \nabla^M \left(\mathbf{v} \cdot \nabla \mathbf{u}^T\right)\right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

Note that

$$\int_{\mathbb{R}^n} \left|\partial_t^{P+1}\nabla^M \mathbf{v} + \partial_t^P \nabla^M \nabla p\right|^2 dx = \left\|\partial_t^{P+1}\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\partial_t^P \nabla^M \nabla p\right\|_{L^2(\mathbb{R}^n)}^2 + 2\int_{\mathbb{R}^n} \partial_t^{P+1}\nabla^M \mathbf{v}\cdot \partial_t^P \nabla^M \nabla p dx,$$

and

$$\int_{\mathbb{R}^n} \partial_t^{P+1}\nabla^M \mathbf{v}\cdot \partial_t^P \nabla^M \nabla p dx = -\int_{\mathbb{R}^n} \partial_t^{P+1}\nabla^M \nabla\cdot \mathbf{v}\cdot \partial_t^P \nabla^M p dx = 0,$$

from div $\mathbf{v} = 0$, we get

$$\left\|\partial_t^{P+1}\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \left\|\partial_t^P \nabla^M \Lambda^{2s}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\partial_t^P \nabla^M (\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\partial_t^P \nabla^M \left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right)\right\|_{L^2(\mathbb{R}^n)}^2. \quad (4.15)$$

We shall estimate the last two terms on the right hand side of (4.15) through considering two cases:

Case (1) $\frac{n}{4} < s < 1, n = 2, 3;$

Case (2) $s = \frac{n}{4}, n = 2, 3.$

We first consider **Case (1)**. In this case, direct calculation gives $0 < \frac{n}{2} - 2s + 1 < 1$. Thanks to Hölder's inequality and the Gagliardo-Nirenberg-Sobolev inequality, we can deduce

$$\begin{aligned}
\left\|\partial_t^P \nabla^M (\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}^2 &\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^m \mathbf{u}\cdot \partial_t^{P-p}\nabla^{M-m+1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^m \mathbf{u}\right\|_{L^{\frac{n}{2s-1}}(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p}\nabla^{M-m+1}\mathbf{v}\right\|_{L^{\frac{2n}{n-2(2s-1)}}(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^m \Lambda^{\frac{n}{2}-2s+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{2s}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \qquad (4.16)\\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^m \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{4s-n} \left\|\partial_t^p \nabla^{m+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{n-4s+2} \left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{2s}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{2s}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned}$$

Similarly, observing that $m+1+\frac{n}{2}-s = m+s+1+\frac{n}{2}-2s$ and $0 < 1+\frac{n}{2}-2s$, by Young's inequality and the Gagliardo-Nirenberg-Sobolev inequality, direct calculation implies

$$\begin{aligned}
&\left\|\partial_t^P \nabla^M \left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right)\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P} \left\|\partial_t^p \nabla^{m+1}\mathbf{u}\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^{m+1}\Lambda^{\frac{n}{2}-s}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^2 \qquad (4.17)\\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left[\left\|\partial_t^p \nabla^m \Lambda^s \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{1+\frac{n}{2}-2s} \left\|\partial_t^p \nabla^{m+1}\Lambda^s \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}}\right]^2 \left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M} \left\|\partial_t^p \nabla^m \Lambda^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .
\end{aligned}$$

Furthermore, according to interpolation inequality, by (1.10), combining (4.15) with (4.16) and (4.17), we obtain

$$\left\|\partial_t^{P+1}\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \left\|\partial_t^P \mathbf{v}\right\|_{H_0^{M+2s}(\mathbb{R}^n)}^2.$$

By an induction argument, this implies for all positive integers M, P such that $M + 2Ps \leq K$ that

$$\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \|\mathbf{v}\|_{H_0^K(\mathbb{R}^n)}^2. \tag{4.18}$$

We next deal with **Case (2)** $s = \frac{n}{4}$. In this case, using Young's inequality, the Gagliardo-Nirenberg-Sobolev inequality, and Agmon's inequality, by Lemma 2.2 and Lemma 2.4, we obtain

$$\begin{aligned}
&\left\|\partial_t^P \nabla^M (\mathbf{u}\cdot\nabla\mathbf{v})\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^m\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m+1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^m\mathbf{u}\right\|_{L^\infty(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-m+1}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^{m+1}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^p\nabla^{m+2}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}\left\|\partial_t^{P-p}\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M+1-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\quad\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left(\left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{1-\frac{2}{n}}\cdot\left\|\partial_t^{P-p}\nabla^{M-m+\frac{n}{2}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^{\frac{2}{n}}\right)^2 \\
&\quad\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^m\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left(\left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2+\left\|\partial_t^{P-p}\nabla^{M-m+\frac{n}{2}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\right).
\end{aligned} \tag{4.19}$$

The same argument as above leads to

$$\begin{aligned}
\left\|\partial_t^P\nabla^M\left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right)\right\|_{L^2(\mathbb{R}^n)}^2 &\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^{m+1}\mathbf{u}\cdot\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^{m+1}\mathbf{u}\right\|_{L^4(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-m}\mathbf{v}\right\|_{L^4(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^{m+1}\Lambda^{\frac{n}{4}}\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\
&\lesssim \sum_{p=0}^{P}\sum_{m=0}^{M}\left\|\partial_t^p\nabla^{m}\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2\left\|\partial_t^{P-p}\nabla^{M-m}\Lambda^{\frac{n}{4}}\mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{4.20}$$

Combining (4.15) with (4.19) and (4.20) gives

$$\left\|\partial_t^{P+1}\nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \left\|\partial_t^P \mathbf{v}\right\|_{H_0^{M+\frac{n}{2}}(\mathbb{R}^n)}^2. \tag{4.21}$$

Again by an induction discussion, for all positive integers M, P such that $M + \frac{nP}{2} \le K$, we get

$$\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \lesssim \|\mathbf{v}\|_{H_0^K(\mathbb{R}^n)}^2 . \tag{4.22}$$

By (4.18) with (4.22), we can derive our desired estimates for both cases. Applying L^2 estimate to (4.14) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|\partial_t^P \nabla^M \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 + \nu\left\|\partial_t^P \nabla^M \Lambda^s \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad \le \left|\left\langle \partial_t^p \nabla^M (\mathbf{u}\cdot\nabla\mathbf{v}), \partial_t^p \nabla^M \mathbf{v}\right\rangle\right| + \left|\left\langle \partial_t^p \nabla^M \left(\mathbf{v}\cdot\nabla\mathbf{u}^T\right), \partial_t^p \nabla^M \mathbf{v}\right\rangle\right| . \end{aligned} \tag{4.23}$$

By the similar argument as in the proof of Theorem 4.1, in particular for $s = \frac{n}{4}$ we need the smallness assumption $\|\mathbf{v}_0\|_{H_0^K(\mathbb{R}^n)} \le \varepsilon^{**}$ with ε^{**} sufficiently small, combining (4.18) with (4.22), applying Grownwall's inequality to (4.23) implies our desired estimate (4.13). We finish the proof of Theorem 4.2. □

By Theorem 4.1 and Theorem 4.2, we complete the proof of Theorem 1.5. At the end of this section, we present only the sketch proof of Corollary 1.6, where the details are similar as the proofs of Theorems 4.1 and 4.2.

Proof of Corollary 1.6. Differentiating the second equation in (1.1) with respect to x and t shows

$$\partial_t^k \nabla^m \mathbf{u} - \alpha^2 \partial_t^k \nabla^m \Delta \mathbf{u} = \partial_t^k \nabla^m \mathbf{v}.$$

Squaring this equations and integrating in space yields, after some integration by parts,

$$\left\|\partial_t^k \nabla^m \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 + 2\alpha^2 \left\|\partial_t^k \nabla^{m+1} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 + \alpha^4 \left\|\partial_t^k \nabla^{m+2} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 = \left\|\partial_t^k \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 .$$

This is the identity $(1.13)_1$.

Thanks to the Gagliardo-Nirenberg-Sobolev inequality and interpolation inequality, it follows from (1.13) that

$$\begin{aligned} \left\|\partial_t^k \nabla^m \mathbf{u}\right\|_{L^n(\mathbb{R}^n)}^2 &\lesssim \left\|\partial_t^k \nabla^m \Lambda^{\frac{n-2}{2}} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 \\ &\lesssim \left(\left\|\partial_t^k \nabla^m \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2-\frac{n}{2}} \left\|\partial_t^k \nabla^{m+1} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}-1}\right)^2 \lesssim \left\|\partial_t^k \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 , \\ \left\|\partial_t^k \nabla^{m+1} \mathbf{u}\right\|_{L^n(\mathbb{R}^n)}^2 &\lesssim \left\|\partial_t^k \nabla^{m+1} \Lambda^{\frac{n-2}{2}} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 \\ &\lesssim \left(\left\|\partial_t^k \nabla^{m+1} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{2-\frac{n}{2}} \left\|\partial_t^k \nabla^{m+2} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}-1}\right)^2 \lesssim \left\|\partial_t^k \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 , \\ \left\|\partial_t^k \nabla^m \Lambda^s \mathbf{u}\right\|_{L^n(\mathbb{R}^n)}^2 &\lesssim \left\|\partial_t^k \nabla^m \Lambda^{\frac{n-2}{2}+s} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 \\ &\lesssim \left(\left\|\partial_t^k \nabla^m \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{3}{2}-\frac{n}{4}-\frac{s}{2}} \left\|\partial_t^k \nabla^{m+2} \mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^{\frac{n}{4}+\frac{s}{2}-\frac{1}{2}}\right)^2 \lesssim \left\|\partial_t^k \nabla^m \mathbf{v}\right\|_{L^2(\mathbb{R}^n)}^2 . \end{aligned} \tag{4.24}$$

This yields the estimate $(1.13)_2$.

Finally, combining the first inequality in (4.24) with the regularity estimates (1.12) yields the estimate $(1.13)_3$. □

5 Stability of the finite energy weak solutions

This section is devoted to proving Theorem 1.7. Again, as the case for proving the regularity of the finite energy weak solutions, it is sufficient to prove (1.14) for smooth solutions with a uniform estimate constant C depending only on $\|(\mathbf{v}_0, \mathbf{w}_0)\|_{L^2(\mathbb{R}^n)}$.

Let $(\mathbf{v}, \mathbf{u})$ and $(\mathbf{w}, \mathbf{q})$ be two finite energy weak solutions as in Theorem 1.7. Then:

$$\begin{cases} \partial_t(\mathbf{v}-\mathbf{w}) + \nu(-\Delta)^s(\mathbf{v}-\mathbf{w}) + \nabla\tau + \mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{q}\cdot\nabla\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{u}^T - \mathbf{w}\cdot\nabla\mathbf{q}^T = 0, \\ (\mathbf{u}-\mathbf{q}) - \alpha^2\Delta(\mathbf{u}-\mathbf{q}) = \mathbf{v}-\mathbf{w}, \\ \nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{w} = \nabla\cdot\mathbf{q} = 0, \\ (\mathbf{v}-\mathbf{w})(0,x) = \mathbf{v}_0(x) - \mathbf{w}_0(x). \end{cases} \tag{5.1}$$

Here, $\nabla\tau$ denotes the difference of the pressures corresponding to $\mathbf{v}$ and $\mathbf{w}$. Recall

$$\mathbf{u} - \alpha^2\Delta\mathbf{u} = \mathbf{v}, \quad \mathbf{q} - \alpha^2\Delta\mathbf{q} = \mathbf{w}. \tag{5.2}$$

Then

$$\|\mathbf{v}-\mathbf{w}\|_{L^2(\mathbb{R}^n)}^2 = \|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2 + 2\alpha^2\|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2 + \alpha^4\|\Delta(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2. \tag{5.3}$$

We now estimate (5.3) term by term.

5.1 The estimate for $\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2$

Multiplying the first equation in (5.1) by $\mathbf{u}-\mathbf{q}$ and integrating in space yields, after some integration by parts,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2 + \alpha^2\|\nabla(\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2\right) + \nu\left(\|\Lambda^s(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2 + \alpha^2\|\nabla\Lambda^s(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2\right) \\ &\quad = -\int_{\mathbb{R}^n}\left(\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{q}\cdot\nabla\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{u}^T - \mathbf{w}\cdot\nabla\mathbf{q}^T\right)(\mathbf{u}-\mathbf{q})\,dx \\ &\quad \le \left|\int_{\mathbb{R}^n}(\mathbf{u}-\mathbf{q})\cdot\nabla\left(\mathbf{u}-\alpha^2\Delta\mathbf{u}\right)(\mathbf{u}-\mathbf{q})\,dx\right| \\ &\qquad + \left|\int_{\mathbb{R}^n}\mathbf{q}\cdot\nabla\left[(\mathbf{u}-\mathbf{q}) - \alpha^2\Delta(\mathbf{u}-\mathbf{q})\right](\mathbf{u}-\mathbf{q})\,dx\right| \\ &\qquad + \left|\int_{\mathbb{R}^n}\left(\mathbf{u}-\alpha^2\Delta\mathbf{u}\right)\cdot\nabla\left(\mathbf{u}^T-\mathbf{q}^T\right)(\mathbf{u}-\mathbf{q})\,dx\right| \\ &\qquad + \left|\int_{\mathbb{R}^n}\left[(\mathbf{u}-\mathbf{q}) - \alpha^2\Delta(\mathbf{u}-\mathbf{q})\right]\cdot\nabla\mathbf{q}^T(\mathbf{u}-\mathbf{q})\,dx\right| \\ &\quad =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.4}$$

Again we consider two cases for estimating (5.4): **Case (I)** $\frac{n}{4} < s < 1$; **Case (II)** $s = \frac{n}{4}$.

We first deal with **Case (I)** $\frac{n}{4} < s < 1$. Thanks to Hölder's inequality, the Gagliardo-Nirenberg-Sobolev inequality, Lemmas 2.3 and 2.4, and the third equation in (5.1), we deduce the following a priori estimates for I_1:

$$\begin{aligned}
I_1 &= \left|\int_{\mathbb{R}^n} (\mathbf{u}-\mathbf{q})\,\nabla\left(\mathbf{u}-\alpha^2\Delta\mathbf{u}\right)(\mathbf{u}-\mathbf{q})\,dx\right| \\
&\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\mathbf{u}-\mathbf{q}\|^2_{L^4(\mathbb{R}^n)} + \|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|(\mathbf{u}-\mathbf{q})\,\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\left(\varepsilon\left\|\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)} + C(\varepsilon)\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\mathbf{u}-\mathbf{q}\|_{L^4(\mathbb{R}^n)}\|\nabla(\mathbf{u}-\mathbf{q})\|_{L^4(\mathbb{R}^n)} \\
&\le C(\varepsilon)\left(\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\cdot\left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \varepsilon\left(\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left(\left\|\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\left\|\nabla\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned} \tag{5.5}$$

For I_2, similarly as the estimate for I_1, together with the divergence free condition, we obtain

$$\begin{aligned}
I_2 &= \left|\int_{\mathbb{R}^n} \mathbf{q}\cdot\nabla\left[(\mathbf{u}-\mathbf{q})-\alpha^2\Delta(\mathbf{u}-\mathbf{q})\right](\mathbf{u}-\mathbf{q})\,dx\right| = \left|\alpha^2\int_{\mathbb{R}^n}\mathbf{q}\cdot\nabla\Delta(\mathbf{u}-\mathbf{q})(\mathbf{u}-\mathbf{q})\,dx\right| \\
&= \alpha^2\left|\int_{\mathbb{R}^n}\nabla\mathbf{q}\cdot\nabla(\mathbf{u}-\mathbf{q})\,\nabla(\mathbf{u}-\mathbf{q})\,dx\right| \lesssim \|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^4(\mathbb{R}^n)} \\
&\le \|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}\left(\varepsilon\left\|\nabla\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+C(\varepsilon)\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned} \tag{5.6}$$

Similarly as the estimates for I_1 and I_2, we have for I_3 and I_4 that

$$\begin{aligned}
I_3 &\le C(\varepsilon)\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\cdot\left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \varepsilon\left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left(\left\|\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\left\|\nabla\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right),
\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}
I_4 &\le C(\varepsilon)\left(\|\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{q}\|_{L^2(\mathbb{R}^n)}\right)\left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \varepsilon\left(\|\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{q}\|_{L^2(\mathbb{R}^n)}\right)\left(\left\|\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\left\|\nabla\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned} \tag{5.8}$$

Thanks to (1.9), (1.10) and (2.1), choosing ε small sufficiently such that

$$\varepsilon\left(\|\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)<\frac{\nu}{2},$$

combining (5.4) with (5.5)-(5.8) yields that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right)+\frac{\nu}{2}\left(\left\|\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\left\|\nabla\Lambda^s(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\qquad\lesssim \|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{5.9}$$

We then deal with **Case (II)** $s=\frac{n}{4}$. Similar argument as the previous case gives

$$\begin{aligned}
I_1 &\lesssim \left(\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left(\left\|\nabla\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\left\|\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right), \\
I_2 &\lesssim \|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}\left\|\nabla\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}, \\
I_3 &\lesssim \left(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{u}\|_{L^2(\mathbb{R}^n)}\right)\left(\left\|\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\left\|\nabla\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right), \\
I_4 &\lesssim \left(\|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\Delta\mathbf{q}\|_{L^2(\mathbb{R}^n)}\right)\left(\left\|\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}+\left\|\nabla\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned} \tag{5.10}$$

Recall (1.10) and (2.2), and observe that

$$\left(\|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^n)} + \|\nabla \mathbf{q}\|_{L^2(\mathbb{R}^n)} + \|\Delta \mathbf{u}\|_{L^2(\mathbb{R}^n)} + \|\Delta \mathbf{q}\|_{L^2(\mathbb{R}^n)}\right) \lesssim \|\mathbf{v}\|^2_{L^2(\mathbb{R}^n)} + \|\mathbf{w}\|^2_{L^2(\mathbb{R}^n)} \lesssim \|\mathbf{v}_0\|^2_{L^2(\mathbb{R}^n)}.$$

Then by the smallness assumption of Theorem 1.4 for $s = \frac{n}{4}$: $\|\mathbf{v}_0\|^2_{L^2(\mathbb{R}^n)} \lesssim \varepsilon^*$ for ε^* sufficiently small, in particular, choosing $\varepsilon^* \leq \frac{\nu}{8}$, we infer from (5.4) and (5.10) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \frac{\nu}{2}\left(\left\|\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \left\|\nabla\Lambda^{\frac{n}{4}}(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\lesssim \left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right).
\end{aligned} \tag{5.11}$$

5.2 The estimate for $\|\Delta(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}$

Multiplying the first equation in (5.1) by $\Delta(\mathbf{u}-\mathbf{q})$ and integrating in space yields, after some integration by parts,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \|\Delta(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right) \\
&\quad + \nu\left(\left\|\Lambda^s\nabla(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)} + \alpha^2 \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}\right) \\
&= \int_{\mathbb{R}^n}\left(\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{q}\cdot\nabla\mathbf{w} + \mathbf{v}\cdot\nabla\mathbf{u}^T - \mathbf{w}\cdot\nabla\mathbf{q}^T\right)\Delta(\mathbf{u}-\mathbf{q})\,dx \\
&\leq \left|\int_{\mathbb{R}^n}(\mathbf{u}-\mathbf{q})\cdot\nabla\left(\mathbf{u}-\alpha^2\Delta\mathbf{u}\right)\Delta(\mathbf{u}-\mathbf{q})\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}^n}\mathbf{q}\cdot\nabla\left[(\mathbf{u}-\mathbf{q}) - \alpha^2\Delta(\mathbf{u}-\mathbf{q})\right]\Delta(\mathbf{u}-\mathbf{q})\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}^n}\left(\mathbf{u}-\alpha^2\Delta\mathbf{u}\right)\cdot\nabla\left(\mathbf{u}^T-\mathbf{q}^T\right)\Delta(\mathbf{u}-\mathbf{q})\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}^n}\left[(\mathbf{u}-\mathbf{q}) - \alpha^2\Delta(\mathbf{u}-\mathbf{q})\right]\cdot\nabla\mathbf{q}^T\Delta(\mathbf{u}-\mathbf{q})\,dx\right| \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{5.12}$$

We first consider the case with $\frac{n}{4} < s < 1$. For A_1,

$$A_1 \leq \left|\int_{\mathbb{R}^n}(\mathbf{u}-\mathbf{q})\cdot\nabla\mathbf{u}\Delta(\mathbf{u}-\mathbf{q})\,dx\right| + \alpha^2\left|\int_{\mathbb{R}^n}(\mathbf{u}-\mathbf{q})\cdot\nabla\Delta\mathbf{u}\Delta(\mathbf{u}-\mathbf{q})\,dx\right| =: A_{11} + A_{12}. \tag{5.13}$$

For the first term A_{11} on the right-hand side of (5.13), by virtue of Hölder's inequality, the Gagliardo-Nirenberg-Sobolev inequality and interpolation inequality, we have

$$\begin{aligned}
A_{11} &\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\mathbf{u}-\mathbf{q}\|_{L^{\frac{n}{s}}(\mathbb{R}^n)}\|\Delta(\mathbf{u}-\mathbf{q})\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\
&\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^{\frac{n}{2}-s}(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\|\mathbf{u}-\mathbf{q}\|^{1-\frac{n}{2}+s}_{L^2(\mathbb{R}^n)}\|\nabla(\mathbf{u}-\mathbf{q})\|^{\frac{n}{2}-s}_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}\left(\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)} + \|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}\right)\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\
&\lesssim \|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^n)}\left(\|\mathbf{u}-\mathbf{q}\|^2_{L^2(\mathbb{R}^n)} + \|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\right) + \frac{\nu}{16}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned} \tag{5.14}$$

For A_{12}, by Lemma 2.3, Lemma 2.5 and Lemma 2.7, we have

$$\begin{aligned} A_{12} &\lesssim \left\|\Lambda^s \Delta \mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^{1-s}\left[(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\right]\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \left\|\Lambda^s \Delta \mathbf{u}\right\|_{L^2(\mathbb{R}^n)} \Big(\big\|\Lambda^{1-s}\left[(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\right] \\ &\qquad - (\mathbf{u}-\mathbf{q})\Lambda^{1-s}\Delta(\mathbf{u}-\mathbf{q}) - \Lambda^{1-s}(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\big\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \left\|(\mathbf{u}-\mathbf{q})\Lambda^{1-s}\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} + \left\|\Lambda^{1-s}(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \Big). \end{aligned} \tag{5.15}$$

By Lemma 2.5, Lemma 2.7 and (3.2), note that $\frac{1}{2} = \frac{s}{n} + \frac{n-2s}{2n}$ and $0 < 1-s < s < 1$, we have the following estimates for the right hand side of (5.15):

$$\begin{aligned} &\left\|\Lambda^{1-s}\left[(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\right] - (\mathbf{u}-\mathbf{q})\Lambda^{1-s}\Delta(\mathbf{u}-\mathbf{q}) - \Lambda^{1-s}(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim \left\|\Lambda^{1-s}(\mathbf{u}-\mathbf{q})\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \|\Delta(\mathbf{u}-\mathbf{q})\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\ &\quad\lesssim \left\|\Lambda^{\frac{n}{2}+1-2s}(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim \|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)}^{2s-\frac{n}{2}} \|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2}+1-2s} \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\ &\quad\lesssim \Big(\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)} + \|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}\Big) \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \left\|(\mathbf{u}-\mathbf{q})\Lambda^{1-s}\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} &\lesssim \|\mathbf{u}-\mathbf{q}\|_{L^{\frac{n}{2s-1}}(\mathbb{R}^n)} \left\|\Lambda^{1-s}\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^{\frac{2n}{n-2(2s-1)}}(\mathbb{R}^n)} \\ &\lesssim \|\mathbf{u}-\mathbf{q}\|_{L^{\frac{2n}{n-2(n/2+1-2s)}}(\mathbb{R}^n)} \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \left\|\Lambda^{n/2+1-2s}(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \Big(\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)} + \|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}\Big) \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}, \\ \left\|\Lambda^{1-s}(\mathbf{u}-\mathbf{q})\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)} &\lesssim \left\|\Lambda^{1-s}(\mathbf{u}-\mathbf{q})\right\|_{L^{\frac{n}{s}}(\mathbb{R}^n)} \|\Delta(\mathbf{u}-\mathbf{q})\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \\ &\lesssim \Big(\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)} + \|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}\Big) \left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{5.17}$$

By (5.13)-(5.17), we deduce the estimate for A_1:

$$A_1 \lesssim \Big(\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 + \left\|\Lambda^s\Delta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2\Big)\Big(\|\mathbf{u}-\mathbf{q}\|_{L^2(\mathbb{R}^n)} + \|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}\Big) + \frac{\nu}{8}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}^2. \tag{5.18}$$

In the same manner, together with (1.9), the third equation in (5.1) and Lemma 2.4, A_2, A_3 and A_4 can be estimated as follows:

$$\begin{aligned} A_2 &\lesssim \Big(\|\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2\Big)\|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2 + \frac{\nu}{8}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}^2, \\ A_3 &\lesssim \Big(\|\mathbf{u}\|_{L^2(\mathbb{R}^n)}^2 + \left\|\Lambda^s\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2 + \left\|\Lambda^s\Delta\mathbf{u}\right\|_{L^2(\mathbb{R}^n)}^2\Big)\|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2 + \frac{\nu}{8}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}^2, \\ A_4 &\lesssim \Big(\|\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}\Big)\Big(\|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2 + \alpha^2\|\Delta(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2\Big) \\ &\quad + \frac{\nu}{16}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon\|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \tag{5.19}$$

By (1.9), choosing ε sufficiently small such that $\varepsilon\|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)} \le \frac{\nu}{16}$ leads to

$$\begin{aligned} A_4 &\lesssim \Big(\|\mathbf{q}\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}\Big)\Big(\|\nabla(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2 + \alpha^2\|\Delta(\mathbf{u}-\mathbf{q})\|_{L^2(\mathbb{R}^n)}^2\Big) \\ &\quad + \frac{\nu}{8}\left\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \tag{5.20}$$

Hence, by (1.9), (2.3), together with (5.12), (5.18)-(5.20), we finally deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\Big(\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\Delta(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\Big)\\&+\frac{\nu}{2}\Big(\big\|\Lambda^s\nabla(\mathbf{u}-\mathbf{q})\big\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\big\|\Lambda^s\Delta(\mathbf{u}-\mathbf{q})\big\|^2_{L^2(\mathbb{R}^n)}\Big)\\&\lesssim c^*\Big(\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\Delta(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\Big),\end{aligned}\tag{5.21}$$

with

$$c^*:=\|\nabla\mathbf{q}\|^2_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{q}\|_{L^2(\mathbb{R}^n)}+\|\mathbf{u}\|^2_{L^2(\mathbb{R}^n)}+\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^n)}+\big\|\Lambda^s\Delta\mathbf{u}\big\|_{L^2(\mathbb{R}^n)}.$$

For the case $s=\frac{n}{4}$, again by tedious but rather similar argument, together with the smallness assumptions in Theorem 1.5 with the choice $\varepsilon^{**}\leq\frac{\nu}{8}$, we can deduce

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\Big(\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\Delta(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\Big)\\&+\frac{\nu}{2}\Big(\big\|\Lambda^{\frac{n}{4}}\nabla(\mathbf{u}-\mathbf{q})\big\|^2_{L^2(\mathbb{R}^n)}+\big\|\Lambda^{\frac{n}{4}}\Delta(\mathbf{u}-\mathbf{q})\big\|^2_{L^2(\mathbb{R}^n)}\Big)\\&\lesssim\Big(1+\big\|\Lambda^{\frac{n}{4}}\mathbf{v}\big\|^2_{L^2(\mathbb{R}^n)}\Big)\Big(\|\nabla(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}+\alpha^2\|\Delta(\mathbf{u}-\mathbf{q})\|^2_{L^2(\mathbb{R}^n)}\Big).\end{aligned}\tag{5.22}$$

Now we can conclude our result. By the basic energy estimates (1.10), by (5.9), (5.11), (5.21) and (5.22), applying Gronwall's inequality implies our desired stability result (1.14). This completes the proof of Theorem 1.7. □

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