DISCRETE YAMABE FLOWS WITH R-CURVATURE REVISITED

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Abstract. Using Ge-Jiang-Shen's extension method [8], we extend Ge-Xu's discrete Yamabe flows with R-curvature [11]. We show the solution to the extended flow is unique and exists for all time $t \geq 0$ if all vertex degree are no less than 23, which partly confirm the Conjecture 1 in [11]. We give two sufficient conditions (the "energy gap condition" and the "big regular triangulation" condition) for the convergence of the solution, which partly confirm the Conjecture 2 in [11].

1. Introduction

Thurston [18] once built a deep connection between circle packings and hyperbolic 3-manifolds. Inspired by Hamilton's Ricci flow [15] methods, Chow and Luo [2] introduced the combinatorial Ricci flow: $r_i' = -K_i r_i$ on surfaces to deform circle packings. For 3-manifolds, Cooper and Rivin [3] considered the ball packings. In 2005, Glickenstein [13, 14] introduced a combinatorial version of Yamabe flow which is a 3-dimensional analogue to Chow-Luo's combinatorial Ricci flows. In 2015, Ge and Xu [11] investigated a new normalized discrete Yamabe flow $r_i' = (R_{av} - R_i)r_i$ with ball packings, where the R-curvature $R_i = K_i/r_i^2$. This flow seems "right" because it shares formally similar properties with the smooth Yamabe flow such as the scaling property. There are some variations of these flows in 3-manifolds, one may refer [4, 9, 10, 12]. Recently, Ge, Jiang and Shen [8] extended the solution to Glickenstein's flow so as it exists for all time. They also got some deep convergence results by probing into the combinatorial curvatures K_i. In this article, we revisit Ge-Xu's discrete Yamabe flow $r_i' = (R_{av} - R_i)r_i$ by Ge-Jiang-Shen's methods.

Let $(M, \mathcal{T})$ be a triangulated closed 3-manifold with $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, where $\mathcal{T}_i$ is the set of i-simplices. Denote $V = \mathcal{T}_0$, $N = |V|$. A ball packing is a positive function on V, $\mathbf{r} : V \to \mathbb{R}_+$, such that for each $ijkl \in \mathcal{T}_3$, by defining the length $l_{ij} = r_i + r_j$, $ijkl$ can be realized as a geometric tetrahedron in $\mathbb{R}^3$. Denote

$$Q_{ijkl} = (\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_l})^2 - 2(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_l^2}).$$

The classical Descartes' circle theorem tells us $ijkl$ forms a geometric tetrahedron if and only if $Q_{ijkl} > 0$. Denote the space of all ball packings by

$$\mathcal{M}_{\mathcal{T}} = \{\mathbf{r} \in \mathbb{R}_+^N : \ Q_{ijkl} > 0, \forall ijkl \in \mathcal{T}_3\}.$$

Comparing with the smooth case, we may regard r_i^2 as a discrete version of Riemannian metric g, and regard $\mathrm{Vol} = \sum_{i=1}^{N} r_i^3$ as a discrete version of volume. Denote α_{ijkl} by the solid angle at the vertex i in a tetrahedron $ijkl$. The combinatorial curvature is defined as

$$K_i = 4\pi - \sum_{ijkl\in\mathcal{T}_3} \alpha_{ijkl}.$$

Denote $R_i = \frac{K_i}{r_i^2}$ and $R_{av} = \sum_{i=1}^{N} K_i r_i/\mathrm{Vol}$. R_i is called the R-curvature. It can be regarded as the scalar curvature due to the scaling property. The Cooper-Rivin functional is defined as $\mathcal{S}(r) = \sum_{i=1}^{N} K_i r_i$. It may be regarded as the Einstein-Hilbert functional $\mathcal{E}(g) = \int R dV$. Based on these analogue, Ge and Xu [11] introduced the following discrete Yamabe flow:

$$r_i' = -R_i r_i, \tag{1}$$

and the normalized discrete Yamabe flow:

$$r_i' = (R_{av} - R_i) r_i. \tag{2}$$

In [11], Ge and Xu showed that the flows (1) and (2) are similar to the smooth (normalized) Yamabe flow in many aspects.

Inspired by [1, 16][5]-[7], Xu [19] gave a natural extension $\tilde{\alpha}_{ijkl}$ of the solid angle α_{ijkl}. Roughly speaking, for $ijkl \in \mathcal{T}_3$, if r_i is small enough such that $Q_{ijkl} \leq 0$, then we define $\tilde{\alpha}_{ijkl} = 2\pi$ and 0 for others. Ge, Jiang and Shen [8] verified this extension is C^0-continuous in $\mathbb{R}_+^N$. We call $\mathbf{r}$ a real ball packing metric if $\mathbf{r} \in \mathcal{M}_{\mathcal{T}}$, a virtual ball packing metric if $\mathbf{r} \in \mathbb{R}_+^N \setminus \mathcal{M}_{\mathcal{T}}$. We will discuss the extension in more detail in Section 2.

By extending the solid angles, we extend the curvature K_i to $\tilde{K}_i$, and also to other terms, say $\tilde{R}_i, \tilde{R}_{av}$, $\tilde{\mathcal{S}}(\mathbf{r})$. Then we introduce the extended discrete Yamabe flow on $\mathbb{R}_+^N$

$$r_i' = -\tilde{R}_i r_i \tag{3}$$

and its normalization

$$r_i' = (\tilde{R}_{av} - \tilde{R}_i) r_i. \tag{4}$$

Remark 1.1. *Let $\boldsymbol{r}(t)$ be a solution to (3), then $\bar{\boldsymbol{r}}(\bar{t}) = c(t)\boldsymbol{r}(t)$, where $c(t) = e^{\int_0^t \tilde{R}_{av}(s)ds}$, $\bar{t} = \int_0^t c^2(s)ds$, gives a solution to (4). Conversely, let $\boldsymbol{r}(t)$ be a solution to 4), then $\bar{\boldsymbol{r}}(\bar{t}) = c(t)\boldsymbol{r}(t)$, where $c(t) = e^{-\int_0^t \tilde{R}_{av}(s)ds}, \bar{t} = \int_0^t c^2(s)ds$, gives a solution to (3).*

Due to the above remark, we mainly study the flow (4). First, we show the uniqueness.

Theorem 1.2. *Given an initial $\boldsymbol{r}(0)$, the solution to the flow (3) or (4) is unique.*

Remark 1.3. *The extension is just C^0, so we can't directly apply the standard ODE theory to get the uniqueness result. We use Ge-Hua's [4] mollifying method. We point that this method also yields an affirmative answer to the uniqueness problem proposed in [8].*

Next, we consider the long time behavior of (4). Let $r_i(t), t \in [0,T)$ be its solution, with maximal existence time $T \leq +\infty$. For the extended flow (4), the singularity happens only when r_i degenerates to 0. We show that if a singularity happens in finite time, then not only r_i degenerates, but also r_j degenerates simultaneously for some j adjacent to i.

Theorem 1.4. *Given a triangulated closed 3-manifold $(M, \mathcal{T})$. Let $\boldsymbol{r}(t)$ be a solution to the extended flow (4). Suppose the maximum existence time T is finite. Then there exists an edge ij, such that $r_i(t_n) \to 0, r_j(t_n) \to 0$ for some $t_n \to T$.*

For each vertex i, set $\deg_3(i) = \#\{ijkl \in \mathcal{T}_3\}$. If $\min\limits_{i \in V} \deg_3(i) \geq 23$, we show the solution to the extended flow (4) exists on $[0, +\infty)$, which partly confirm the Conjecture 1 in [11].

Theorem 1.5. *Given a triangulated closed 3-manifold $(M, \mathcal{T})$, suppose $\min\limits_{i \in V} \deg_3(i) \geq 23$. Let $\boldsymbol{r}(t)$ be a solution to the extended flow (4). Then the maximum existence time $T = +\infty$.*

Last, we consider the convergence behavior of the extended flow.

Definition 1.6. *Let $\boldsymbol{r}(t)$ be a solution to the flow (4), T be the maximal existence time. If $\boldsymbol{r}(t)$ is precompact in $\mathbb{R}^N_+$, we call $\boldsymbol{r}(t)$ a non-singular solution (in this case $T = +\infty$).*

We can show that for the non-singular solution, the flow sub-converges to a constant R-curvature metric $\hat{r}$. But unlike the non-extended case, we can't directly check whether $\hat{r}$ is an attractor by using the linearization since the extension is just C^0.

We give two sufficient conditions for a solution to (4) being non-singular. Denote

$$S_3 = \{\mathbf{r} \in \mathbb{R}^N_+ : \mathrm{Vol} = 1\}, \quad L_c = \{\mathbf{r} \in \mathbb{R}^N_+ : \tilde{\mathcal{S}}(\mathbf{r}) \leq c\}.$$

Definition 1.7. *Given a triangulated closed 3-manifold $(M, \mathcal{T})$ with ball packings. Assume $\{c : L_c \cap S_3$ is compact$\}$ is non-empty. Define*

$$\Theta(\mathcal{T}) = \sup\{c : L_c \cap S_3 \textit{ is compact}\}.$$

The invariant $\Theta(\mathcal{T})$ plays the similar role as $\chi(\hat{r}, \mathcal{T})$ in [8]. One may regard $\Theta(\mathcal{T})$ as an energy gap. The solution is non-singular under the *"energy gap"* condition, which reads as, the initial "energy", i.e. the extended Cooper-Rivin functional, is smaller than $\Theta(\mathcal{T})$.

Theorem 1.8. *Suppose $\{c : L_c \cap S_3$ is compact$\}$ is non-empty. For an initial data $\boldsymbol{r}(0)$, if $\tilde{\mathcal{S}}(\boldsymbol{r}(0)) \leq \Theta(\mathcal{T})$, then $\boldsymbol{r}(t)$ is a non-singular solution to the flow (4).*

We say $\mathcal{T}$ is *regular*, if $\deg_3(i)$ are all equal. For a regular triangulation, the packing with all $r_i = 1$ has constant curvature $4\pi - \deg_3(i)(3\cos^{-1}\frac{1}{3} - \pi)$. In order for this curvature to be zero, $\deg_3(i) \approx 22.8$. If $\deg_3(i) \geq 23$ for all i, we say $\mathcal{T}$ is *big*. The following theorem says that a solution is non-singular under the *"big regular triangulation"* condition.

Theorem 1.9. *Assume $\deg_3(i)$ are all equal and no less than 23. Let $\boldsymbol{r}(t)$ be the unique solution to the extended flow (4). Then $\boldsymbol{r}(t)$ is non-singular. Moreover, $\boldsymbol{r}(t)$ converges exponentially fast to a ball packing with all radii equal (and hence constant curvature).*

Organzition: We organize the article as follows. In Section 2, we recall some facts on the extended solid angle and extended (normalized) discrete Yamabe flow. In Section 3, we prove our main results. In Section 3.1, we consider the uniqueness and show Theorem 1.2. In Section 3.2, we consider the long time existence, and show Theorem 1.4 and Theorem 1.5. In Section 3.3, we consider the convergence, and show Theorem 1.8 and Theorem 1.9.

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2. EXTENSIONS

Let $(M, \mathcal{T})$ be a triangulated closed 3-manifold with real or virtual ball packings $\mathbf{r} \in \mathbb{R}^N_+$. Recall $\mathcal{M}_{\mathcal{T}}$ is the set of all real ball packings, and Cooper and Rivin's functional $\mathcal{S}(\mathbf{r})$ is originally defined on $\mathcal{M}_{\mathcal{T}}$. One important feature of $\mathcal{S}$ is the locally convexity.

Proposition 2.1 (convexity; [3, 14, 17])**.** *Hess $\mathcal{S}$ is semi-positive of rank $N-1$. Its kernel is spanned by $\boldsymbol{r}$.*

From Proposition 2.1, $\mathcal{S}$ is convex locally. However, its definition domain $\mathcal{M}_{\mathcal{T}}$ is not a convex set. This is the main difficulty to get global properties for the energy $\mathcal{S}$. Ge, Jiang and Shen [8] overcame this difficulty by the extension method initialled in [1][16]. We recall the extension method and quote some useful results obtained in [8]. For $ijkl \in \mathcal{T}_3$, if $Q_{ijkl} > 0$, then define the extended solid angle $\tilde{\alpha}_{ijkl} = \alpha_{ijkl}$. If $Q_{ijkl} \leq 0$, then there must be a strictly minimal one among $\{r_i, r_j, r_k, r_l\}$, say r_i, then define $\tilde{\alpha}_{ijkl} = 2\pi$ and 0 for others. Together with Proposition 6 in [14], Ge, Jiang and Shen proved

Proposition 2.2 (extension; [8])**.** *The extended solid angle $\tilde{\alpha}_{ijkl}$ is a C^0 continuous extension of the solid angle α_{ijkl}.*

For the extended solid angle $\tilde{\alpha}_{ijkl}$, Ge, Jiang and Shen [8] obtained deep comparison principles. Denote $\bar{\alpha} = 3\cos^{-1}\frac{1}{3} - \pi$ by the solid angle of a regular tetrahedron (i.e. a tetrahedron whose six edge lengthes are all equal).

Proposition 2.3 (comparison principle; [8])**.** *For $ijkl \in \mathcal{T}_3$, if $r_i = \min\{r_i, r_j, r_k, r_l\}$, then $\tilde{\alpha}_{ijkl} \geq \bar{\alpha}$. Similarly, if $r_i = \max\{r_i, r_j, r_k, r_l\}$, then $\tilde{\alpha}_{ijkl} \leq \bar{\alpha}$.*

From the extension of the solid angles, the extended scalar curvature is defined as

$$\tilde{K} = 4\pi - \sum_{ijkl \in \mathcal{T}_3} \tilde{\alpha}_{ijkl}.$$

Similarly, we define $\tilde{R}_i$, $\tilde{R}_{av}$ and $\tilde{\mathcal{S}}$. Although the extension of $\tilde{\alpha}_{ijkl}$ is just C^0-continuous, the extended energy $\tilde{\mathcal{S}}$ is C^1-smooth.

Proposition 2.4 ([8])**.** *$\tilde{\mathcal{S}}$ is a C^1-smooth convex extension of $\mathcal{S}$ to $\mathbb{R}^N_+$ with $\frac{\partial \tilde{\mathcal{S}}}{\partial r_i} = \tilde{K}_i$.*

Next we focus on the extended flows and prove some facts. For more results on the non-extended flow, one may refer [11]. Recall the extended discrete Yamabe flow is $r_i' = -\tilde{R}_i r_i$, and the normalized one is $r_i' = (\tilde{R}_{av} - \tilde{R}_i)r_i$. Obviously, the extended flows coincide with the non-extended flows whenever the solution $\mathbf{r}(t)$ lies in $\mathcal{M}_{\mathcal{T}}$. Consider the volume Vol and the energy $\tilde{\mathcal{S}}$, we have

Lemma 2.5. *Along the the extended normalized discrete Yamabe flow (4), the volume* Vol *is invariant and the energy $\tilde{\mathcal{S}}$ is decreasing and uniformly bounded.*

Proof. By direct calculation $\frac{d}{dt}\mathrm{Vol} = \sum_{i=1}^{N} r_i^2(\tilde{R}_{av} - \tilde{R}_i)r_i = 0$. And

$$\frac{d\tilde{\mathcal{S}}}{dt} = \sum_{i=1}^{N} \tilde{K}_i(\tilde{R}_{av} - \tilde{R}_i)r_i = \sum_{i=1}^{N} r_i^3(\tilde{R}_{av} - \tilde{R}_i)\tilde{R}_i.$$

Since $\sum_{i=1}^{N} r_i^3(\tilde{R}_{av} - \tilde{R}_i) = 0$, we have $\frac{d\tilde{\mathcal{S}}}{dt} = -\sum_{i=1}^{N} r_i^3(\tilde{R}_{av} - \tilde{R}_i)^2 \le 0$. To see $|\mathcal{S}| \le C$, we assume $V \equiv 1$, then

$$|\tilde{\mathcal{S}}| = \Big|\sum_{i=1}^{N} \tilde{K}_i r_i\Big| \le \Big(\sum_{i=1}^{N} |\tilde{K}_i|^{\frac{3}{2}}\Big)^{\frac{2}{3}} \Big(\sum_{i=1}^{N} r_i^3\Big)^{\frac{1}{3}} \le C,$$

since the curvature $\tilde{K}_i$ are uniformly bounded only depending on $\mathcal{T}$. □

Remark 2.6. *The convexity of $\tilde{S}$ plays a crucial role in* [8] *since the restriction of $\tilde{S}$ on $\{r_i > 0 : \sum r_i = 1\}$ is also convex. But unfortunately, we don't have the convexity of $\tilde{S}$ on $\{r_i > 0 : \sum r_i^3 = 1\}$ or on $\{r_i > 0 : \sum r_i^2 = 1\}$.*

3. Main Results

3.1. **Uniqueness.** In this section, we show the uniqueness of the solution to the flow (3) and the flow (4). Notice that as a function of $\mathbf{r}$, R_{av} and R_i are C^0, not Lipschitz. So we cannot apply the standard ODE theory directly.

First we show the uniqueness of the (unnormalized) flow (3), we use the method in [4].

Theorem 3.1. *Given an initial $\mathbf{r}(0)$, the solution to the flow (3) is unique.*

Proof. Let $\mathbf{w} = \mathbf{w}(\mathbf{r})$ given by $w_i = \frac{2}{3}r_i^{\frac{3}{2}}$ and $\mathbf{r} = \mathbf{r}(\mathbf{w})$ be the inverse. Denote $\hat{\mathcal{S}}(\mathbf{w}) = \tilde{\mathcal{S}}(\mathbf{r}(\mathbf{w}))$. Then equaition (3) becomes

$$w_i' = -\nabla_{w_i}\hat{\mathcal{S}}. \tag{5}$$

Suppose $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are two solutions to the equation (3) with the same initial value for $t \in [0, t_0]$. Let $\mathbf{w}_1(t), \mathbf{w}_2(t)$ be the corresponding solutions to the equation (5). We claim there exists a positive constant A such that for $t \in [0, t_0]$,

$$(\nabla_{\mathbf{w}}\hat{\mathcal{S}}(\mathbf{w}_1) - \nabla_{\mathbf{w}}\hat{\mathcal{S}}(\mathbf{w}_2)) \cdot (\mathbf{w}_1 - \mathbf{w}_2) + A|\mathbf{w}_1 - \mathbf{w}_2|^2 \ge 0.$$

To see this, set $\phi_\epsilon(\mathbf{r}) = \frac{1}{\epsilon^N}\phi(\frac{\mathbf{r}}{\epsilon})$ be the standard mollifier with

$$\phi(\mathbf{r}) = \begin{cases} Ce^{-\frac{1}{1-|\mathbf{r}|^2}}, & |\mathbf{r}| < 1, \\ 0, & |\mathbf{r}| \geq 1, \end{cases}$$

where C is chosen such that $\int \phi = 1$. Set $\tilde{\mathcal{S}}_\epsilon = \tilde{\mathcal{S}} * \phi_\epsilon$ defined on $(\epsilon, +\infty)^N$. Since $[0, t_0]$ is compact, all the components of $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are away from zero. Hence for ϵ small enough, $\tilde{\mathcal{S}}_\epsilon$ is well defined along the flow for $t \in [0, t_0]$. From Proposition 2.4, $\tilde{S}$ is a C^1 convex function, then $\tilde{\mathcal{S}}_\epsilon$ is smooth convex on $(\epsilon, +\infty)^N$ and

$$\tilde{\mathcal{S}}_\epsilon \to \mathcal{S} \text{ in } C^1, \text{ on every compact set of } \mathbb{R}^N_+, \quad \epsilon \to 0.$$

Moreover,

$$\nabla\tilde{\mathcal{S}}_\epsilon = \nabla\tilde{\mathcal{S}} * \phi_\epsilon = \tilde{K} * \phi_\epsilon, \text{ on } (\epsilon, +\infty)^N.$$

Set $\hat{\mathcal{S}}_\epsilon(\mathbf{w}) = \tilde{\mathcal{S}}_\epsilon(\mathbf{r}(\mathbf{w}))$. Then

$$\nabla_{\mathbf{w}}\hat{\mathcal{S}}_\epsilon = \nabla_{\mathbf{r}}\tilde{\mathcal{S}}_\epsilon \frac{\partial \mathbf{r}}{\partial \mathbf{w}}.$$

$$\begin{aligned} \nabla_{w_i w_j}\hat{\mathcal{S}}_\epsilon &= \frac{\partial^2\tilde{\mathcal{S}}_\epsilon}{\partial r_i \partial r_j}\frac{dr_i}{dw_i}\frac{dr_j}{dw_j} + \frac{\partial\tilde{\mathcal{S}}_\epsilon}{\partial r_i}\frac{\partial^2 r_i}{\partial w_i \partial w_j} \\ &= \frac{\partial^2\tilde{\mathcal{S}}_\epsilon}{\partial r_i \partial r_j}\frac{dr_i}{dw_i}\frac{dr_j}{dw_j} + (\tilde{K}_i * \phi_\epsilon)\frac{d^2 r_i}{dw_i^2}\delta_{ij}. \end{aligned}$$

Since $[0, t_0]$ is compact, each term of $(\tilde{K}_i * \phi_\epsilon)\frac{d^2 r_i}{dw_i^2}\delta_{ij}$ is bounded. Then there exists a constant A such that

$$|(\tilde{K}_i * \phi_\epsilon)\frac{d^2 r_i}{dw_i^2}| \leq A.$$

From the convexity of $\tilde{\mathcal{S}}_\epsilon$, we have as quadratic forms, $\nabla^2\hat{\mathcal{S}}_\epsilon \geq -AI$. For $t \in [0, t_0]$, $\mathbf{w}_1, \mathbf{w}_2$ lie in a compact convex set. Then

$$(\nabla_{\mathbf{w}}\hat{\mathcal{S}}_\epsilon(\mathbf{w}_1) - \nabla_{\mathbf{w}}\hat{\mathcal{S}}_\epsilon(\mathbf{w}_2)) \cdot (\mathbf{w}_1 - \mathbf{w}_2) + A|\mathbf{w}_1 - \mathbf{w}_2|^2 \geq 0.$$

This is from the fact

$$\nabla_{\mathbf{w}}\hat{\mathcal{S}}_\epsilon(\mathbf{w}_1) - \nabla_{\mathbf{w}}\hat{\mathcal{S}}_\epsilon(\mathbf{w}_2) = \int_1^0 \frac{d}{dt}\nabla_{\mathbf{w}}\hat{\mathcal{S}}_\epsilon(t\mathbf{w}_1 + (1-t)\mathbf{w}_2)dt.$$

Let $\epsilon \to 0$, the claim follows.

Consider the function $h(t) = |\mathbf{w}_1(t) - \mathbf{w}_2(t)|^2$. Then

$$\begin{aligned} h'(t) &= -2(\mathbf{w}_1(t) - \mathbf{w}_2(t)) \cdot (\nabla_{\mathbf{w}}\hat{\mathcal{S}}(\mathbf{w}_1(t)) - \nabla_{\mathbf{w}}\hat{\mathcal{S}}(\mathbf{w}_2(t))) \\ &\leq 2A|\mathbf{w}_1(t) - \mathbf{w}_2(t)|^2 = 2Ah(t) \end{aligned}$$

Hence $h(t) \leq h(0)e^{2At}$ for $t \in [0, t_0]$. Since $h(0) = 0$, $h(t) \equiv 0$ on $[0, t_0]$. Since t_0 is arbitrary, we finish the proof. □

Next, we consider the normalized flow (4). The uniqueness is basically from the equivalence between the flow (3) and (4), which is explained in Remark 1.1.

Theorem 3.2. *Given an initial* $\boldsymbol{r}(0)$*, the solution to the flow (4) is unique.*

Proof. Let $\mathbf{r}_1(t), \mathbf{r}_2(t)$ be two solutions of (4) with the same initial data. We use $r_{1,i}$ to denote the value of $\mathbf{r}_1$ at the vertex i, and also for other terms. As in Remark 1.1, set

$$\bar{\mathbf{r}}_1(\bar{t}) = c_1(t_1(\bar{t}))\mathbf{r}_1(t_1(\bar{t})), \quad \bar{\mathbf{r}}_2(\bar{t}) = c_2(t_2(\bar{t}))\mathbf{r}_2(t_2(\bar{t}))$$

where

$$c_1(t) = e^{-\int_0^t \tilde{R}_{1,av}(t)dt}, \quad c_2(t) = e^{-\int_0^t \tilde{R}_{2,av}(t)dt}, \quad \bar{t} = \int_0^{t_1} c_1^2(s)ds = \int_0^{t_2} c_2^2(s)ds$$

Then $\bar{\mathbf{r}}_1(\bar{t})$ and $\bar{\mathbf{r}}_2(\bar{t})$ are two solutions of the equation (3). From Theorem 3.1, $\bar{\mathbf{r}}_1(\bar{t}) = \bar{\mathbf{r}}_2(\bar{t})$. Then we have $r_{1,i}(t_1(\bar{t})) = f(\bar{t})r_{2,i}(t_2(\bar{t}))$, where $f(\bar{t})$ is independent of i. By taking the $\bar{t}$ derivative, we obtain

$$\frac{1}{c_1^2}(\tilde{R}_{1,av} - \tilde{R}_{1,i})r_{1,i} = f'r_{2,i} + f\frac{1}{c_2^2}(\tilde{R}_{2,av} - \tilde{R}_{2,i})r_{2,i}.$$

Then

$$f' + (\frac{\tilde{R}_{2,av} - \tilde{R}_{2,i}}{c_2^2} - \frac{\tilde{R}_{1,av} - \tilde{R}_{1,i}}{c_1^2})f = 0.$$

Notice that for fixed $\bar{t}$, $\mathbf{r}_1$ is a multiple of $\mathbf{r}_2$, so $\tilde{K}_{1,i} = \tilde{K}_{2,i}$. And $\tilde{R}_{1,i} = \frac{1}{f^2}\tilde{R}_{2,i}$, Then

$$f' + (\frac{1}{c_2^2} - \frac{1}{(c_2^2)^{\frac{1}{f^2}} f^2})(\tilde{R}_{2,av} - \tilde{R}_{2,i})f = 0.$$

Since $\tilde{R}_{2,av} = (\sum \tilde{R}_{2,i}r_{2,i}^3)/(\sum r_{2,i}^3)$, we obtain $f' = 0$. Since $f(0) = 1$, we have $f(t) \equiv 1$. So we finish the proof. $\square$

Remark 3.3. *In* [8]*, Ge, Jiang and Shen proposed the uniqueness problem on the extended combinatorial Yamabe flow* $r_i' = (\tilde{\lambda} - \tilde{K}_i)r_i$*. One can show this uniqueness by using the similar method in the proof of Theorem 3.1 and Theorem 3.2.*

3.2. **Long time existence.** In this section, we consider the long time behavior of the flow (3) and the flow (4). First we investigate the singularities in finite time. In [11], Ge and Xu introduced two kinds of singularities for the (non-extended) flow (1) and (2).

Definition 3.4. *We say flow (1) or (2) develops a removable singularity if there exists a sequence of* $t_n \to T$ *and a tetrahedron* $\{ijkl\} \in \mathcal{T}_4$ *such that* $Q_{ijkl}(t_n) \to 0$*. We say flow (1) or (2) develops an essential singularity if there exists a sequence of* $t_n \to T$ *and a vertex* i *such that* $r_i(t_n) \to 0$*.*

Similarly, we define the essential singularity for the extended flow (3) and (4). Notice that for the extended flow, the "removable singularity" is not a singularity. We only need to study the formation of the essential singularities for the extended flows.

Theorem 3.5. *Given a triangulated closed* 3*-manifold* $(M, \mathcal{T})$*. Let* $\boldsymbol{r}(t)$ *be a solution to the flow (3) or (4). Suppose the maximum existence time* T *is finite. Then there exists an edge* ij*, such that* $r_i(t_n) \to 0, r_j(t_n) \to 0$ *for some* $t_n \to T$*.*

Proof. Since for the extended flow, there is no removable singularity, if the maximum existence time T is finite, it must develop an essential singularity, i.e. there is a vertex i such that $\underline{\lim}_{t\to T} r_i(t) = 0$. If for all vertices j adjacent to i, $\underline{\lim}_{t\to T}(r_i + r_j)(t) > 0$, then for r_i small enough, r_i must be the unique smallest one in $\{r_i, r_j, r_k, r_l\}$ for $\{ijkl\} \in \mathcal{T}_3$. Then from [8], $Q_{ijkl} < 0$ and by the definition of extended solid angle, $\tilde{\alpha}_{ijkl} = 2\pi$. Since the manifold is without boundary, we have $\deg_3 \geq 2$. So the curvature $K_i = 4\pi - \deg_3(i)2\pi \leq 0$, which means

$$\text{for (3)} \ -\tilde{R}_i \geq 0, \quad \text{for (4)} \ \tilde{R}_{av,i} - \tilde{R}_i \geq -C, \ C > 0$$

where $-C$ is the lower bound of $\tilde{R}_{av,i}$ only depending on $\mathcal{T}$, see Lemma 2.5. To summarize, there is a constant ϵ, $\epsilon \leq r_i(0)$, such that if $r_i(t) \leq \epsilon$ then $r_i'(t) \geq -Cr_i(t)$ at t. Then

$$r_i(t) \geq \epsilon e^{-CT} \text{ for } t \in [0, T).$$

In fact, for $t < T$, suppose $r_i(t) < \epsilon$. Denote $t_1 = \inf\{s : 0 \leq s < t, \ r_i(s) \leq \epsilon\}$, then $r_i(t_1) = \epsilon$. So $(r_i(t)e^{Ct})' \geq 0$ for $t \in [t_1, t]$, which implies $r_i(t) \geq r_i(t_1)e^{-Ct_1} \geq \epsilon e^{-CT}$. It is a contradiction. □

Remark 3.6. *Theorem 3.5 tells us when a singularity is developed, the formation must be a tetrahedron collapsing to a segment, but not to a triangle.*

Remark 3.7. *Theorem 3.5 is obvious for the non-extended flows. In fact, if only* r_i *degenerates but no* r_j *degenerates for* $ij \in \mathcal{T}_1$*, then the flow must meet the removable singulary before the essential singularity.*

Next, we give a combinatorial condition to ensure the long time existence to the flow (4).

Theorem 3.8. *Given a triangulated closed* 3*-manifold* $(M, \mathcal{T})$*, suppose* $\min_{i\in V} \deg_3(i) \geq 23$*. Let* $\boldsymbol{r}(t)$ *be a solution of the extended flow (4). Then the maximum existence time* T *is* $+\infty$*.*

Proof. Consider $f(t) = \min_i r_i(t)$. Then $f(t)$ is piecewise C^1. Suppose on the interval $[t_1, t_2]$, $f(t) = r_i(t)$. For $ijkl \in \mathcal{T}_3$, since r_i is the minimal, from Proposition 2.3, $\tilde{\alpha}_{ijkl} \geq \bar{\alpha} = 3\cos^{-1}\frac{1}{3} - \pi$. Then $\tilde{K}_i = 4\pi - \sum \tilde{\alpha}_{ijkl} \leq 4\pi - \deg_3 \bar{\alpha} < 0$. Then $r_i'(t) \geq -Cr_i(t)$. Hence we obtain $f(t) \geq f(0)e^{-Ct}$. So we finish the proof. □

In [11], Ge and Xu conjectured

Conjecture 3.9. *For the discrete Yamabe flow (2), it won't develop an essential singularity in finite time.*

As a corollary of Theorem 3.8, we confirm this conjecture in the case $\min_{i\in V}\deg_3(i)\geq 23$.

Corollary 3.10. *Given a triangulated closed* 3*-manifold* $(M,\mathcal{T})$*, suppose* $\min_{i\in V}\deg_3(i)\geq 23$*. Then the flow (2) won't develop an essential singularity in finite time.*

Proof. In fact, if $\mathbf{r}(t)$ develops an essential singularity in finite time T, it then cannot develop a removable singularity before T. So the flow (2) coincide with the extended flow (4), then Theorem 3.8 applies. □

3.3. **Convergence.** In the last part we study the convergence of the flow. First we introduce the notion of the non-singular solution.

Definition 3.11. *Let* $\boldsymbol{r}(t)$ *be a solution to the flow (4),* T *be the maximal existence time. If* $\boldsymbol{r}(t)$ *is precompact in* $\mathbb{R}^N_+$*, we call* $\boldsymbol{r}(t)$ *a non-singular solution. Obviously, in this case,* T *must be* $+\infty$*.*

Proposition 3.12. *Let* $\boldsymbol{r}(t)$ *be a non-singular solution to the flow (4). Then there exists a subsequence* $t_n\to+\infty$*, such that* $\boldsymbol{r}(t_n)$ *converges to a real or virtual ball packing metric* $\boldsymbol{r}_\infty$ *with constant extended* R*-curvature.*

Proof. From Lemma 2.5, since $\tilde{\mathcal{S}}$ is decreasing and bounded, it must have a limit $\tilde{\mathcal{S}}_\infty$. Then there exists t_n such that $\tilde{\mathcal{S}}(n+1)-\tilde{\mathcal{S}}(n)=\tilde{\mathcal{S}}'(t_n)\to 0$. Since $\tilde{\mathcal{S}}'=-\sum_{i=1}^N r_i^3(\tilde{R}_{av}-\tilde{R}_i)^2$, we have $\tilde{R}_{av}(t_n)-\tilde{R}_i(t_n)\to 0$. Since $\mathbf{r}(t_n)$ is bounded, we may choose a subsequence of t_n, also denoted as t_n, such that $\mathbf{r}(t_n)$ converges, then $\mathbf{r}(t_n)$ converges to a ball packing metric with constant extended R-curvature. □

Remark 3.13. *Notice that the non-singular solution only gives the subsequence convergence. For the convergence on time* t*, in the non-extended setting, if the limit of the subsequence is an attractor, then the subsequence convergence will become the convergence. One way to show the limit is an attractor is to show the linearized operator of* $(R_{av}-R_i)r_i$ *is negative definite, as in* [8]*. But in the extended setting,* $(\tilde{R}_{av}-\tilde{R}_i)r_i$ *is just* C^0*, we can't do linearization. So we don't know whether the subsequence convergence can become the convergence.*

Now we give two sufficient conditions for $\mathbf{r}(t)$ being non-singular. Denote

$$S_3=\{\mathbf{r}\in\mathbb{R}^N_+:\sum_{i=1}^N r_i^3=1\},\quad L_c=\{\mathbf{r}\in\mathbb{R}^N_+:\tilde{\mathcal{S}}(\mathbf{r})\leq c\}.$$

Definition 3.14. *Given a triangulated closed three manifold* $(M,\mathcal{T})$*. Suppose the set* $\{c: L_c\cap S_3$ *is compact*$\}$ *is non-empty. Define*

$$\Theta(\mathcal{T})=\sup\{c: L_c\cap S_3 \textit{ is compact.}\}.$$

Remark 3.15. *The invariant $\Theta(\mathcal{T})$ plays the similar role as $\chi(\hat{r},\mathcal{T})$ in [8], which can be regarded as an energy gap for the energy $\tilde{\mathcal{S}}$.p*

Theorem 3.16. *Suppose $\{c : L_c \cap S_3$ is compact$\}$ is non-empty. For an initial data $\boldsymbol{r}(0)$, if $\tilde{\mathcal{S}}(\boldsymbol{r}(0)) \leq \Theta(\mathcal{T})$, then $\boldsymbol{r}(t)$ is a non-singular solution to flow (4).*

Proof. If $\frac{d}{dt}\Big|_{t=0}\tilde{S} = 0$, then from the proof of Lemma 2.5, we see $\tilde{R}_i = \tilde{R}_{av}$ for every i, which is a constant. And from the uniqueness $\mathbf{r}(t) \equiv \mathbf{r}(0)$. Now we assume $\frac{d}{dt}\Big|_{t=0}\tilde{S} \neq 0$, that means $\frac{d}{dt}\tilde{S}|_{t=0} < 0$. So after a short while t_1, $\tilde{\mathcal{S}}(\mathbf{r}(t_1)) < \Theta(\mathcal{T})$, which implies $L_{\tilde{\mathcal{S}}(\mathbf{r}(t_1))}$ is compact. From Lemma 2.5, $\mathbf{r}(t)$ always lie in $L_{\tilde{\mathcal{S}}(\mathbf{r}(t_1))}$ for $t > t_1$. So $\mathbf{r}(t)$ is a non-singular solution. □

From Theorem 3.8, we see if $\deg_3(i) \geq 23$ for all $i \in V$, the we have the long time existence. Furthermore if all the $\deg_3$ are all equal, then we obtain the non-singularity.

Theorem 3.17. *Given a triangulated closed 3-manifold $(M,\mathcal{T})$, suppose $\deg_3(i)$ are all equal and not less than 23. Let $\boldsymbol{r}(t)$ be a solution to the flow (4). Then $\boldsymbol{r}(t)$ is non-singular.*

Proof. Consider $f(t) = \frac{\min\limits_i r_i(t)}{\max\limits_j r_j(t)}$. Then $f(t)$ is piecewise C^1. Suppose on the interval $[t_1,t_2]$, $f(t) = \frac{r_i(t)}{r_j(t)}$. Consider $\ln f$, we have

$$f'(t) = f(t)\Big(\frac{\tilde{K}_j(t)}{r_j^2(t)} - \frac{\tilde{K}_i(t)}{r_i^2(t)}\Big).$$

Since r_i is minimal and r_j is maximal, from [8], for $iabc, jpqr \in \mathcal{T}_3$, we have solid angle $\tilde{\alpha}_{iabc} \geq \bar{\alpha} \geq \tilde{\alpha}_{jpqr}$, where $\bar{\alpha} = 3\cos^{-1}\frac{1}{3} - \pi$. Hence

$$\tilde{K}_j = 4\pi - \sum \tilde{\alpha}_{jpqr} \geq 4\pi - \sum \tilde{\alpha}_{iabc} = \tilde{K}_i, \quad \tilde{K}_i \leq 4\pi - \deg_3 \bar{\alpha} < 0.$$

So $f' \geq 0$. Since $\mathrm{Vol} = \sum \mathrm{r}_\mathrm{i}^3$ is invariant, $\max r_j \geq (\frac{V}{N})^{\frac{1}{3}}$. So $\min r_i$ has a uniformly positive lower bound. □

Remark 3.18. *One may study the extension of the α-flow $r_i'(t) = s_\alpha r_i^\alpha - K_i$ introduced in [12], where $s_\alpha = \mathcal{S}/\|r\|_{\alpha+1}$, $\alpha \in \mathbb{R}$. The results are similar to those in this paper.*

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