Harnack Inequality and Gradient Estimate for G-SDEs with Degenerate Noise*

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Abstract

In this paper, the Harnack inequality for G-SDEs with degenerate noise is derived by the method of coupling by change of measure. Moreover, for any bounded and continuous function f, the gradient estimate

$$|\nabla \bar{P}_t f| \leq c(p,t)(\bar{P}_t|f|^p)^{\frac{1}{p}}, \quad p > 1, t > 0$$

is obtained for the associated nonlinear semigroup $\bar{P}_t$. As an application of Harnack inequality, we prove the existence of weak solution for degenerate G-SDEs under some integrable condition. Finally, an example is presented. All of the above results extend the existing ones in the linear expectation setting.

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1 Introduction

Since Peng [11, 12, 13] established the fundamental theory of G-Brownian motion and stochastic differential equations driven by it (G-SDEs), the study of G-expectation has gained much attention. G-expectation has been applied in many areas, such as stochastic optimization [5, 6], financial markets with volatility uncertainty [2], the Feyman-Kac formula [7] and so on, see [14] and references within for more details.

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Recently, Song [15] studied the gradient estimate for nonlinear diffusion semigroups by using the method of coupling by change of measure introduced by Wang [16, Chapter 1]. In [15], the diffusion coefficient is assumed to be non-degenerate. Quite recently, the second author obtained the dimensional-free Harnack inequality for G-SDEs with non-degenerate noise in [22].

On the other hand, the stochastic Hamiltonian system, as a typical model of degenerate SDEs in linear probability space, has been investigated in [4, 17, 21]. One can also refer to [20] for the log-Harnack inequality of Gruschin semigroup, which is another degenerate diffusion model, see the references in [20] for more details.

In this paper, we intend to investigate the Harnack inequality and the gradient estimate for the stochastic Hamiltonian system driven by G-Brownian motion. Due to lack of additivity of G-expectation, the Bismut formula [16, (1.8), (1.14)] cannot be proved either by coupling by change of measure or by Malliavin calculus. Instead, to get the gradient estimate, we focus on estimating the local Lipschitz constant defined in (3.2) below. The coupling by change of measure and the Girsanov transform will play crucial roles. However, the Girsanov transform is different from the one in linear expectation case since the quadratic variation process of G-Brownian motion is random, see [7] and Theorem 2.2 below for more details. In addition, as an application of the Harnack inequality, we will prove the existence of weak solution for degenerate G-SDEs perturbed by a drift which only satisfies some integrable assumption with respect to a reference nonlinear expectation.

The paper is organized as follows. In Section 2, we first recall some knowledge on G-expectation space, including the G-Girsanov transform; In Section 3, we investigate the Harnack inequality and the gradient estimate; Finally, applying the Harnack inequality, the existence of weak solution for degenerate G-SDEs with integrable drift is studied in Section 4. Furthermore, an example is presented.

2 Preparations

2.1 G-Expectation and G-Brownian Motion

Before moving on, we recall some preliminary on G-expectation and G-Brownian motion. Let $\Omega = C_0([0,\infty);\mathbb{R}^d)$, the $\mathbb{R}^d$-valued and continuous functions on $[0,\infty)$ vanishing at zero, equipped with the metric

$$\rho(\omega^1,\omega^2) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left[\max_{t\in[0,n]} |\omega_t^1 - \omega_t^2| \wedge 1\right], \quad \omega^1,\omega^2 \in \Omega.$$

For any $T > 0$, set

$$L_{ip}(\Omega_T) = \{\Omega \ni \omega \mapsto \varphi(\omega_{t_1},\cdots,\omega_{t_n}) : n \in \mathbb{N}^+, t_1,\cdots,t_n \in [0,T], \varphi \in C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n)\},$$

and

$$L_{ip}(\Omega) = \cup_{T>0} L_{ip}(\Omega_T),$$

where $C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n)$ denotes the set of bounded and Lipschitz continuous functions on $\mathbb{R}^d \otimes \mathbb{R}^n$. Let $\mathbb{S}^d$ be the collection of all $d \times d$ symmetric matrices and $\mathbb{S}^d_+ \subset \mathbb{S}^d$ denote all $d \times d$ positive definite and symmetric matrices. Fix $\underline{\sigma}, \overline{\sigma} \in \mathbb{S}^d_+$ with $\underline{\sigma} < \overline{\sigma}$ and define

$$G(A) = \frac{1}{2} \sup_{\gamma \in [\underline{\sigma}, \bar{\sigma}] \cap \mathbb{S}^d} \mathrm{trace}(\gamma^2 A), \ A \in \mathbb{S}^d. \tag{2.1}$$

Then it is not difficult to see

$$G(A) - G(\bar{A}) \geq \frac{\lambda_0(\underline{\sigma}^2)}{2} \mathrm{trace}[A - \bar{A}], \ A \geq \bar{A}, A, \bar{A} \in \mathbb{S}^d, \tag{2.2}$$

where $\lambda_0(\underline{\sigma}^2) > 0$ is the minimal eigenvalue of $\underline{\sigma}^2$.

Let $\bar{\mathbb{E}}^G$ be the nonlinear expectation on Ω such that coordinate process $B = (B_t)_{t \geq 0}$, i.e. $B_t(\omega) = \omega_t, \omega \in \Omega$, is a d-dimensional G-Brownian motion on $(\Omega, L^1_G(\Omega), \bar{\mathbb{E}}^G)$, where $L^1_G(\Omega)$ is the completion of $L_{ip}(\Omega)$ under the norm $\bar{\mathbb{E}}^G(|\cdot|)$. See [15] for details on the construction of $\bar{\mathbb{E}}^G$. For any $p \geq 1$, let $L^p_G(\Omega)$ be the completion of $L_{ip}(\Omega)$ under the norm $(\bar{\mathbb{E}}^G|\cdot|^p)^{\frac{1}{p}}$. Similarly, we can define $L^p_G(\Omega_T)$ for any $T > 0$.

Let

$$M^{p,0}_G([0,T]) = \left\{ \eta_t = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t) : \ \xi_j \in L^p_G(\Omega_{t_j}), N \in \mathbb{N}^+, \ 0 = t_0 < t_1 < \cdots < t_N = T \right\},$$

and $M^p_G([0,T])$ be the completion of $M^{p,0}_G([0,T])$ under the norm

$$\|\eta\|_{M^p_G([0,T])} := \left(\bar{\mathbb{E}}^G \int_0^T |\eta_t|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Moreover, set

$$M^2_G([0,T])^d = \left\{ X = (X^1, X^2, \cdots, X^d)^* : \ X^i \in M^2_G([0,T]), 1 \leq i \leq d \right\},$$

where $*$ stands for the transpose of a matrix. Let $\mathcal{M}$ be the collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$. According to [1, 9], there exists a weakly compact subset $\mathcal{P} \subset \mathcal{M}$ such that

$$\bar{\mathbb{E}}^G[X] = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[X], \ X \in L^1_G(\Omega), \tag{2.3}$$

where $\mathbb{E}_{\mathbb{P}}$ denotes the linear expectation under probability measure $\mathbb{P} \in \mathcal{P}$. $\mathcal{P}$ is called a set that represents $\bar{\mathbb{E}}^G$. In fact, let W^0 be a standard d-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$, and $\mathbb{H}$ be the set of all progressively measurable stochastic processes valued in $[\underline{\sigma}, \bar{\sigma}]$. For any $\theta \in \mathbb{H}$, define $\mathbb{P}_\theta$ as the law of $\int_0^\cdot \theta_s \mathrm{d}W^0_s$. Then by [1, 9], we can take $\mathcal{P} = \{\mathbb{P}_\theta : \theta \in \mathbb{H}\}$, i.e.

$$\bar{\mathbb{E}}^G[X] = \sup_{\theta \in \mathbb{H}} \mathbb{E}_{\mathbb{P}_\theta}[X], \ X \in L^1_G(\Omega). \tag{2.4}$$

The associated Choquet capacity to $\bar{\mathbb{E}}^G$ is defined by

$$\mathcal{C}(A) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}(A),\ \ A \in \mathcal{B}(\Omega).$$

A set $A \subset \Omega$ is called polar if $\mathcal{C}(A) = 0$, and we say that a property holds $\mathcal{C}$-quasi-surely ($\mathcal{C}$-q.s.) if it holds outside a polar set, see [1] for more details on capacity.

Finally, letting $\langle B\rangle$ be the quadratic variation process of B, then by (2.2) and [13, Chapter III, Corollary 5.7], we have $\mathcal{C}$-q.s.

$$\underline{\sigma}^2 < \frac{\mathrm{d}}{\mathrm{d}t}\langle B\rangle_t \le \bar{\sigma}^2. \tag{2.5}$$

2.2 Girsanov's Transform

The following Girsanov's transform is taken from [10, Proposition 5.10].

Theorem 2.1. *Let* $\{g_t\}_{t\le T} \in M_G^2([0,T])^d$*. Assume that there exists a constant* $\delta > 0$ *such that Novikov's condition holds, i.e.*

$$\bar{\mathbb{E}}^G \exp\left\{\left(\frac{1}{2}+\delta\right)\int_0^T \langle g_u, \mathrm{d}\langle B\rangle_u g_u\rangle\right\} < \infty. \tag{2.6}$$

Then

$$\bar{B} := B + \int_0^{\cdot} \mathrm{d}\langle B\rangle_u g_u$$

is a G*-Brownian motion on* $[0,T]$ *under* $\tilde{\mathbb{E}}[\cdot] = \bar{\mathbb{E}}^G[\tilde{R}_T(\cdot)]$*, where*

$$\tilde{R}_T = \exp\left[-\int_0^T \langle g_u, \mathrm{d}B_u\rangle - \frac{1}{2}\int_0^T \langle g_u, \mathrm{d}\langle B\rangle_u g_u\rangle\right].$$

According to [7, Remark 5.3], letting $\hat{\Omega} = C_0([0,\infty),\mathbb{R}^{2d})$, we can construct an auxiliary $\hat{G}$-expectation space $(\hat{\Omega}, L^1_{\hat{G}}(\hat{\Omega}), \hat{\mathbb{E}}^{\hat{G}})$ with

$$\hat{G}(A) := \frac{1}{2}\sup_{\gamma\in[\underline{\sigma},\bar{\sigma}]\cap\mathbb{S}^d} \mathrm{trace}\left[A\begin{pmatrix} \gamma^2 & \mathbf{I}_{\mathrm{d\times d}} \\ \mathbf{I}_{\mathrm{d\times d}} & \gamma^{-2}\end{pmatrix}\right],\ \ A \in \mathbb{S}^{2d}, \tag{2.7}$$

and a d-dimensional process B' such that $\begin{pmatrix} B \\ B' \end{pmatrix}$ is a $2d$-dimensional $\hat{G}$-Brownian motion with $\langle B, B'\rangle_t = t\mathbf{I}_{\mathrm{d\times d}}$ under $\hat{\mathbb{E}}^{\hat{G}}$. In addition, the distribution of B under $\bar{\mathbb{E}}^G$ is equal to that of B under $\hat{\mathbb{E}}^{\hat{G}}$. Moreover, letting

$$\tilde{G}(A) = \frac{1}{2}\sup_{\gamma\in[\underline{\sigma},\bar{\sigma}]\cap\mathbb{S}^d} \mathrm{trace}\left[A\gamma^{-2}\right],\quad A \in \mathbb{S}^d, \tag{2.8}$$

then B' is a $\tilde{G}$-Brownian motion under $\hat{\mathbb{E}}^{\hat{G}}$. Letting $\hat{\mathcal{C}}$ be the associated Choquet capacity to $\hat{\mathbb{E}}^{\hat{G}}$, we have $\hat{\mathcal{C}}$-q.s.

$$\overline{\sigma}^{-2} \leq \frac{\mathrm{d}\langle B'\rangle_t}{\mathrm{d}t} \leq \underline{\sigma}^{-2}. \tag{2.9}$$

This together with Theorem 2.1 implies the following Girsanov's theorem, which will be used frequently in the sequel.

Theorem 2.2. *Let* $\{g_t^i\}_{t\leq T} \in M_G^2([0,T])^d, i = 1, 2$. *If*

$$\hat{\mathbb{E}}^{\hat{G}} \exp\left\{ \left(\frac{1}{2} + \delta\right) \int_0^T \left(\langle g_s^1, \mathrm{d}\langle B'\rangle_s g_s^1\rangle + \langle g_s^2, \mathrm{d}\langle B\rangle_s g_s^2\rangle + 2\langle g_s^1, g_s^2\rangle \mathrm{d}s\right) \right\} < \infty, \tag{2.10}$$

then

$$\breve{B} := B + \int_0^\cdot g_u^1 \mathrm{d}u + \int_0^\cdot \mathrm{d}\langle B\rangle_u g_u^2$$

is a G*-Brownian motion on* $[0,T]$ *under* $\breve{\mathbb{E}}[\cdot] = \hat{\mathbb{E}}^{\hat{G}}[\breve{R}_T(\cdot)]$ *with*

$$\begin{aligned}\breve{R}_T = \exp\Bigg[&- \int_0^T \left\langle \begin{pmatrix} g_u^1 \\ g_u^2 \end{pmatrix}, \mathrm{d}\begin{pmatrix} B_u' \\ B_u \end{pmatrix} \right\rangle \\ &- \frac{1}{2}\int_0^T \left(\langle g_s^1, \mathrm{d}\langle B'\rangle_s g_s^1\rangle + \langle g_s^2, \mathrm{d}\langle B\rangle_s g_s^2\rangle + 2\langle g_s^1, g_s^2\rangle \mathrm{d}s\right) \Bigg].\end{aligned}$$

Proof. Letting $W = \begin{pmatrix} B \\ B' \end{pmatrix}$, we have

$$\langle W\rangle_t = \begin{pmatrix} \langle B\rangle_t & t\mathbf{I}_{\mathrm{d}\times\mathrm{d}} \\ t\mathbf{I}_{\mathrm{d}\times\mathrm{d}} & \langle B'\rangle_t \end{pmatrix} \tag{2.11}$$

and

$$\int_0^T \left\langle \begin{pmatrix} g^2 \\ g^1 \end{pmatrix}, \mathrm{d}\langle W\rangle \begin{pmatrix} g^2 \\ g^1 \end{pmatrix} \right\rangle = \int_0^T \left(\langle g_s^1, \mathrm{d}\langle B'\rangle_s g_s^1\rangle + \langle g_s^2, \mathrm{d}\langle B\rangle_s g_s^2\rangle + 2\langle g_s^1, g_s^2\rangle \mathrm{d}s\right). \tag{2.12}$$

Set

$$\tilde{W} = W + \int_0^\cdot \mathrm{d}\langle W\rangle \begin{pmatrix} g^2 \\ g^1 \end{pmatrix}. \tag{2.13}$$

In view of (2.10) and applying Theorem 2.1 with $\left(W, \begin{pmatrix} g^2 \\ g^1 \end{pmatrix}\right)$ in place of (B, g), we conclude that $\tilde{W}$ is a $2d$-dimensional $\hat{G}$-Brownian motion on $[0,T]$ under $\breve{\mathbb{E}}$. This combined with (2.11) and (2.7) implies that

$$\breve{B} := B + \int_0^\cdot g_u^1 \mathrm{d}u + \int_0^\cdot \mathrm{d}\langle B\rangle_u g_u^2$$

is a G-Brownian motion on $[0,T]$ under $\breve{\mathbb{E}}$. □

Remark 2.3. *Theorem 2.2 extends the result in [7, Theorem 5.2], where* g^1 *and* g^2 *are assumed to be bounded and in* $M_G^2([0,T])^d$.

Throughout the paper, the letter C or c will denote a positive constant, and $C(\theta)$ or $c(\theta)$ stands for a constant depending on θ. The values of the constants may vary from line to line. For $k \in \mathbb{N}^+$, let $C_b(\mathbb{R}^k)$ ($C_b^+(\mathbb{R}^k)$) denote the bounded (non-negative and bounded) and continuous function on $\mathbb{R}^k$.

3 Harnack Inequality and Gradient Estimate

Consider the following G-SDE on $\mathbb{R}^{m+d}$:

$$\begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = b_1(X_t, Y_t)\mathrm{d}t + \mathrm{d}\langle B\rangle_t b_2(X_t, Y_t) + Q\mathrm{d}B_t, \end{cases} \tag{3.1}$$

where B_t is a d-dimensional G-Brownian motion defined in Section 2, A is an $m \times m$ matrix, M is an $m \times d$ matrix, Q is a $d \times d$ matrix and $b_1, b_2 : \mathbb{R}^{m+d} \to \mathbb{R}^d$ are measurable.

In this paper, we only consider $m = d = 1$, where $\underline{\sigma}$ and $\overline{\sigma}$ in (2.1) are two positive constants satisfying $\underline{\sigma} < \overline{\sigma}$, and the corresponding generating function is given by

$$G(a) = \frac{1}{2}\overline{\sigma}^2 a^+ - \frac{1}{2}\underline{\sigma}^2 a^-, \quad a \in \mathbb{R}^1.$$

In this case, $\mathrm{d}\langle B\rangle_t b_2(X_t, Y_t)$ can be rewritten as $b_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t$. The result can be extended to the case of $m \geq 1$ and $d \geq 1$, see Remark 3.5 below for more details. In this section, we study the Harnack inequality and the gradient estimate for (3.1). To this end, we make the following assumptions.

(A1) $MQ \neq 0$.

(A2) There exists $K > 0$ such that

$$|b_1(z) - b_1(\bar{z})| + |b_2(z) - b_2(\bar{z})| \leq K|z - \bar{z}|, \quad z, \bar{z} \in \mathbb{R}^2.$$

Remark 3.1. *According to [13, Theorem 1.2],* **(A2)** *implies that* (3.1) *has a unique strong solution* $(X_t^z, Y_t^z)^*$ *in* $M_G^2([0,T])^2$ *for any* $T > 0$ *and* $(X_0,\, Y_0)^* = z \in \mathbb{R}^2$. *Let* $\bar{P}_t$ *be the associated nonlinear semigroup to* (X_t^z, Y_t^z), *i.e.*

$$\bar{P}_t f(z) = \bar{\mathbb{E}}^G f(X_t^z, Y_t^z), \quad f \in C_b(\mathbb{R}^2).$$

For a real-valued function f defined on a metric sapce (H, ρ), set

$$|\nabla f(z)| := \limsup_{\rho(x,z)\to 0} \frac{|f(x) - f(z)|}{\rho(x,z)}, \quad z \in H. \tag{3.2}$$

$|\nabla f(z)|$ is called the local Lipschitz constant of f at point $z \in H$. Moreover, let $\|\nabla f\|_\infty := \sup_{z\in H} |\nabla f(z)|$.

Theorem 3.2. *Assume* **(A1)-(A2)**. *Then for any $T_0 > 0$, there exists some constant $C > 0$ depending on T_0, A, M, K and $|Q^{-1}|$ such that the following assertions hold for any $T \in (0, T_0]$.*

(1) For any $z = (z_1, z_2)^*, h = (h_1, h_2)^* \in \mathbb{R}^2$, $p > 1$, the Harnack inequality

$$(\bar{P}_T f)^p(z + h) \leq \bar{P}_T f^p(z) \exp\left[\frac{p}{2(p-1)}\Sigma(T)|h|^2\right], \quad f \in C_b^+(\mathbb{R}^2) \tag{3.3}$$

holds for

$$\Sigma(T) := C\left(\underline{\sigma}^{-2} + \overline{\sigma}^2\right)\left(T^3 + \frac{1}{(T \wedge 1)^3}\right). \tag{3.4}$$

(2) The gradient estimate

$$\|\nabla \bar{P}_T f\|_\infty \leq \|f\|_\infty \sqrt{\Sigma(T)}, \quad f \in C_b^+(\mathbb{R}^2) \tag{3.5}$$

holds.

(3) For any $p > 1$, there exists a constant $c(p) > 0$ such that

$$|\nabla \bar{P}_T f(z)| \leq c(p)\left(\bar{P}_T |f|^p(z)\right)^{\frac{1}{p}} \sqrt{\Sigma(T)}, \quad z \in \mathbb{R}^2, \quad f \in C_b^+(\mathbb{R}^2). \tag{3.6}$$

Remark 3.3. *With* (2.3) *in hand, one can easily derive* (3.3) *provided that for any $\mathbb{P} \in \mathcal{P}$, the Harnack inequality*

$$\left(\mathbb{E}_\mathbb{P} f(X_t^{z+h}, Y_t^{z+h})\right)^p \leq \left(\mathbb{E}_\mathbb{P} f^p(X_t^z, Y_t^z)\right) \exp\{\Phi(t, h, p)\}, \quad f \in C_b^+(\mathbb{R}^2) \tag{3.7}$$

holds. However, it is difficult to get (3.7) *by coupling by change of measure since B_t is a martingale rather than Brownian motion under $\mathbb{E}_\mathbb{P}$. Therefore, the results in Theorem 3.2 are non-trivial.*

Remark 3.4. *Compared to the SDEs in [15], the SDE* (3.1) *is allowed to contain an extra drift term $b_2 \mathrm{d}\langle B\rangle_t$. Taking $T_0 = 1$, we conclude that* (3.3) *holds for $\Sigma(T) = \frac{c^2}{T^3}, T \in (0, 1]$ with some constant $c > 0$ independent of T. This implies*

$$\lim_{T \to 0} \frac{\sqrt{\Sigma(T)}}{T^{-\frac{3}{2}}} = c. \tag{3.8}$$

It is different from the result in [15, Theorem 4.1], where the diffusion coefficient is assumed to be non-degenerate and the power of T in (3.8) *is equal to $-\frac{1}{2}$.*

Now, we are in the position to prove Theorem 3.2.

Proof. (1) Fix $T_0>0$ and let $T\in(0,T_0]$. For any $\eta\in\mathbb{R}^2$, let (X_t^η,Y_t^η) solve (3.1) initial value η. Let

$$\Lambda_1(T):=\int_0^T\frac{s(T-s)}{T^2}\mathrm{e}^{-2sA}M^2\mathrm{d}s.$$

For $h=(h_1,h_2)^*\in\mathbb{R}^2$, define

$$\gamma_1(s)=\frac{T-s}{T}h_2+\alpha_1(s),\quad s\in[0,T]$$

with

$$\alpha_1(s)=-\frac{s(T-s)}{T^2}M\mathrm{e}^{-sA}\Lambda_1(T)^{-1}\Big(h_1+\int_0^T\frac{T-u}{T}\mathrm{e}^{-uA}Mh_2\mathrm{d}u\Big),\quad s\in[0,T].$$

Noting that $\mathrm{e}^{-sA}\geq\mathrm{e}^{-AT}\wedge 1$, $s\in[0,T]$, it holds that

$$\begin{aligned}\Lambda_1(T)=\int_0^T\frac{s(T-s)}{T^2}\mathrm{e}^{-2sA}M^2\mathrm{d}s&\geq\left(\mathrm{e}^{-2AT}\wedge 1\right)M^2\int_0^T\frac{s(T-s)}{T^2}\mathrm{d}s\\&=\frac{\left(\mathrm{e}^{-2AT}\wedge 1\right)M^2}{6}T,\quad T\in(0,T_0].\end{aligned}$$

This together with $M\neq 0$ due to **(A1)** yields

$$|\Lambda_1(T)^{-1}|\leq\frac{6}{(\mathrm{e}^{-2AT}\wedge 1)\,M^2}T^{-1}\leq CT^{-1},\quad T\in(0,T_0]\tag{3.9}$$

for some constant $C>0$ depending on T_0, A and M. Let $(\tilde{X}_t,\tilde{Y}_t)$ solve the equation

$$\begin{cases}\mathrm{d}\tilde{X}_t=\{A\tilde{X}_t+M\tilde{Y}_t\}\mathrm{d}t,\\ \mathrm{d}\tilde{Y}_t=b_1(X_t^z,Y_t^z)\mathrm{d}t+b_2(X_t^z,Y_t^z)\mathrm{d}\langle B\rangle_t+Q\mathrm{d}B_t+\gamma_1'(t)\mathrm{d}t\end{cases}\tag{3.10}$$

with $(\tilde{X}_0,\tilde{Y}_0)^*=z+h$. Set

$$\Theta_1(s)=\Big(\mathrm{e}^{As}h_1+\int_0^s\mathrm{e}^{(s-u)A}M\gamma_1(u)\mathrm{d}u,\ \gamma_1(s)\Big),\quad s\in[0,T].$$

It follows from (3.1) and (3.10) that

$$(\tilde{X}_s,\tilde{Y}_s)=(X_s^z,Y_s^z)+\Theta_1(s),\quad s\in[0,T],\tag{3.11}$$

and in particular, $(\tilde{X}_T,\tilde{Y}_T)=(X_T^z,Y_T^z)$. In view of (3.9), there exists a constant $C>0$ depending on T_0, A and M such that

$$\sup_{s\in[0,T]}|\gamma_1'(s)|\leq\frac{C}{(T\wedge 1)^2}|h|,\quad\sup_{s\in[0,T]}|\Theta_1(s)|\leq C\Big(1+T\Big)|h|,\quad T\in(0,T_0].\tag{3.12}$$

Let B' be defined in (2.8) and

$$\Phi_1(s) = Q^{-1}\{b_1(X_s^z, Y_s^z) - b_1(\tilde{X}_s, \tilde{Y}_s) + \gamma_1'(s)\},$$
$$\Phi_2(s) = Q^{-1}\{b_2(X_s^z, Y_s^z) - b_2(\tilde{X}_s, \tilde{Y}_s)\}, \quad s \in [0, T].$$

Then, (2.5), (2.9), (3.12) and (3.11) together with **(A1)**-**(A2)** imply $\hat{\mathcal{C}}$-q.s.

$$\begin{aligned}
&\int_0^T |\Phi_1(s)|^2 \mathrm{d}\langle B'\rangle_s + \int_0^T |\Phi_2(s)|^2 \mathrm{d}\langle B\rangle_s + 2\int_0^T \Phi_1(s)\Phi_2(s)\mathrm{d}s \\
&\le \int_0^T \underline{\sigma}^{-2}|\Phi_1(s)|^2\mathrm{d}s + \int_0^T \overline{\sigma}^2|\Phi_2(s)|^2\mathrm{d}s + 2\int_0^T \Phi_1(s)\Phi_2(s)\mathrm{d}s \\
&\le 2\int_0^T \underline{\sigma}^{-2}|\Phi_1(s)|^2\mathrm{d}s + 2\int_0^T \overline{\sigma}^2|\Phi_2(s)|^2\mathrm{d}s \\
&\le C\underline{\sigma}^{-2}\int_0^T \left(|\Theta_1(s)| + |\gamma_1'(s)|\right)^2 \mathrm{d}s + C\overline{\sigma}^2\int_0^T |\Theta_1(s)|^2\mathrm{d}s \\
&\le C\left(\underline{\sigma}^{-2}T\left(\frac{1}{(T\wedge 1)^2} + T\right)^2 |h|^2 + \overline{\sigma}^2 T\left(1+T\right)^2|h|^2\right) \\
&\le C\left(\underline{\sigma}^{-2} + \overline{\sigma}^2\right)\left(T^3 + \frac{1}{(T\wedge 1)^3}\right)|h|^2, \quad T \in (0, T_0]
\end{aligned} \tag{3.13}$$

for some constant $C > 0$ depending on T_0, A, M, K and $|Q^{-1}|$.
Applying Theorem 2.2, we conclude that

$$\tilde{B} := B + \int_0^\cdot \Phi_1(u)\mathrm{d}u + \int_0^\cdot \Phi_2(u)\mathrm{d}\langle B\rangle_u$$

is a G-Brownian motion on $[0, T]$ under $\mathbb{E}_1(\cdot) = \hat{\mathbb{E}}^{\hat{G}}(R_1(T)(\cdot))$, where

$$\begin{aligned}
R_1(T) = \exp\Bigg[&- \int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B_u' \\ B_u \end{pmatrix} \right\rangle \\
&- \frac{1}{2}\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\Bigg].
\end{aligned}$$

Noting that $\langle \tilde{B}\rangle = \langle B\rangle$, (3.10) reduces to

$$\begin{cases} \mathrm{d}\tilde{X}_t = \{A\tilde{X}_t + M\tilde{Y}_t\}\mathrm{d}t, \\ \mathrm{d}\tilde{Y}_t = b_1(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}t + b_2(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}\langle \tilde{B}\rangle_t + Q\mathrm{d}\tilde{B}_t. \end{cases}$$

This yields that the distribution of $(\tilde{X}_t, \tilde{Y}_t)$ under $\mathbb{E}_1$ coincides with that of (X_t^{z+h}, Y_t^{z+h}) under $\hat{\mathbb{E}}^{\hat{G}}$ (or $\bar{\mathbb{E}}^G$). Thus, for any $f \in C_b^+(\mathbb{R}^2)$ and $p > 1$, it follows from

Hölder's inequality that

$$
\begin{aligned}
\bar{P}_T f(z+h) &= \mathbb{E}_1 f(\tilde{X}_T, \tilde{Y}_T) \\
&= \hat{\mathbb{E}}^{\hat{G}}\left[R_1(T) f(X_T^z, Y_T^z)\right] \\
&\le \left(\hat{\mathbb{E}}^{\hat{G}}\left[f^p(X_T^z, Y_T^z)\right]\right)^{\frac{1}{p}} \{\hat{\mathbb{E}}^{\hat{G}} R_1(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}} \\
&= (\bar{P}_T f^p(z))^{\frac{1}{p}} \{\hat{\mathbb{E}}^{\hat{G}} R_1(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}},
\end{aligned} \tag{3.14}
$$

here we used the fact that the distribution of B under $\bar{\mathbb{E}}^G$ is equal to that of B under $\hat{\mathbb{E}}^{\hat{G}}$. Recalling the definition of $R_1(T)$ and (3.13), we have

$$
\begin{aligned}
&\hat{\mathbb{E}}^{\hat{G}} R_1(T)^{\frac{p}{p-1}} \\
&= \hat{\mathbb{E}}^{\hat{G}} \Bigg\{ \exp\Bigg[-\frac{p}{p-1} \int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle \\
&\quad - \frac{1}{2}\frac{p^2}{(p-1)^2} \int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle B' \rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle B \rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s \right) \Bigg] \\
&\quad \times \exp\left[\frac{p}{2(p-1)^2} \int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle B' \rangle_s + |\Phi_2(s)|^2 \mathrm{d}\langle B \rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s \right) \right] \Bigg\} \\
&\le \exp\left[\frac{p}{2(p-1)^2} \Sigma(T) |h|^2 \right], \quad T \in (0, T_0].
\end{aligned}
$$

Combining this with (3.14), we derive the Harnack inequality (3.3).

(2) Now, we prove the gradient estimate (3.5). Since the distribution of B under $\bar{\mathbb{E}}^G$ is equal to that of B under $\hat{\mathbb{E}}^{\hat{G}}$, we have

$$
\bar{P}_T f(z) = \bar{\mathbb{E}}^G f(X_T^z, Y_T^z) = \hat{\mathbb{E}}^{\hat{G}} f(X_T^z, Y_T^z).
$$

This combined with (3.14) yields

$$
\begin{aligned}
|\bar{P}_T f(z+h) - \bar{P}_T f(z)| &= |\hat{\mathbb{E}}^{\hat{G}}\left[R_1(T) f(X_T^z, Y_T^z)\right] - \hat{\mathbb{E}}^{\hat{G}} f(X_T^z, Y_T^z)| \\
&\le \hat{\mathbb{E}}^{\hat{G}}\left(|f(X_T^z, Y_T^z)||R_1(T) - 1|\right).
\end{aligned} \tag{3.15}
$$

Applying $|x-1| \le (x+1)|\log x|, x > 0$, we have

$$
\begin{aligned}
&|\bar{P}_T f(z+h) - \bar{P}_T f(z)| \\
&\le \|f\|_\infty \hat{\mathbb{E}}^{\hat{G}} R_1(T) |\log R_1(T)| + \|f\|_\infty \hat{\mathbb{E}}^{\hat{G}} |\log R_1(T)| \\
&= \|f\|_\infty \left(\mathbb{E}_1 |\log R_1(T)| + \hat{\mathbb{E}}^{\hat{G}} |\log R_1(T)| \right).
\end{aligned} \tag{3.16}
$$

Let

$$
\tilde{B}' = B' + \int_0^\cdot \Phi_1(u) \mathrm{d}\langle B' \rangle_u + \int_0^\cdot \Phi_2(u) \mathrm{d}u.
$$

From Theorem 2.2, we know that $\{\tilde{B'}_t\}_{t\in[0,T}$ is a $\tilde{G}$-Brownian motion under $\mathbb{E}_1$. Noting that $\langle\tilde{B}'\rangle = \langle B'\rangle$ and $\langle\tilde{B}\rangle = \langle B\rangle$, applying the B-D-G inequality [3, Theorem 2.1], we obtain

$$
\begin{aligned}
&\mathbb{E}_1|\log R_1(T)| \\
&= \mathbb{E}_1\bigg| -\int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle \\
&- \frac{1}{2}\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\bigg| \\
&\le \mathbb{E}_1\left| \int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} \tilde{B}'_u \\ \tilde{B}_u \end{pmatrix} \right\rangle \right| \\
&\quad + \frac{1}{2}\mathbb{E}_1\left| \int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle \tilde{B}'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle \tilde{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right| \\
&\le \mathbb{E}_1\left(\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle \tilde{B}'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle \tilde{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right)^{\frac{1}{2}} \\
&\quad + \frac{1}{2}\mathbb{E}_1\left| \int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle \tilde{B}'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle \tilde{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right|.
\end{aligned}
$$

Since $\tilde{B}'$ and B' are $\tilde{G}$-Brownian motions under $\mathbb{E}_1$ and $\hat{\mathbb{E}}^{\hat{G}}$ respectively, it follows from (3.13) that

$$\mathbb{E}_1|\log R_1(T)| \le \Sigma(T)|h|^2 + \sqrt{\Sigma(T)}|h|,$$

where $\Sigma(T)$ is defined in (3.4). Similarly, we arrive at

$$\hat{\mathbb{E}}^{\hat{G}}|\log R_1(T)| \le \Sigma(T)|h|^2 + \sqrt{\Sigma(T)}|h|.$$

Thus, (3.16) yields

$$|\bar{P}_T f(z+h) - \bar{P}_T f(z)| \le \|f\|_\infty \left(\Sigma(T)|h|^2 + \sqrt{\Sigma(T)}|h|\right), \qquad z \in \mathbb{R}^2.$$

So (3.2) implies

$$|\nabla \bar{P}_T f(z)| \le \|f\|_\infty \sqrt{\Sigma(T)}, \quad z \in \mathbb{R}^2,$$

which deduces (3.5).

(3) In order to get (3.6), let

$$\tilde{R}_1(T) = \exp\bigg[-\frac{p}{p-1}\int_0^T \left\langle \begin{pmatrix} \Phi_1(u) \\ \Phi_2(u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle$$

$$- \frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^T \left(|\Phi_1(s)|^2 \mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\Bigg],$$

and

$$\begin{aligned} \hat{B}' &= B' + \int_0^{\cdot} \frac{p}{p-1}\Phi_1(u)\mathrm{d}\langle B'\rangle_u + \int_0^{\cdot}\frac{p}{p-1}\Phi_2(u)\mathrm{d}u, \\ \hat{B} &= B + \int_0^{\cdot}\frac{p}{p-1}\Phi_1(u)\mathrm{d}u + \int_0^{\cdot}\frac{p}{p-1}\Phi_2(u)\mathrm{d}\langle B\rangle_u. \end{aligned} \tag{3.17}$$

Again using Theorem 2.2, we know that $\hat{B}'$ is a $\tilde{G}$-Brownian motion under $\tilde{\mathbb{E}}_2(\cdot) = \hat{\mathbb{E}}^{\hat{G}}(\tilde{R}_1(T)(\cdot))$. Combining (3.13) with $|x-1| \le (x+1)|\log x|, x > 0$, we have

$$\begin{aligned} &\hat{\mathbb{E}}^{\hat{G}}\left[|R_1(T)-1|^{\frac{p}{p-1}}\right] \\ &\le \hat{\mathbb{E}}^{\hat{G}}\left[(R_1(T)+1)^{\frac{p}{p-1}}|\log R_1(T)|^{\frac{p}{p-1}}\right] \\ &\le c(p)\hat{\mathbb{E}}^{\hat{G}}\left[R_1(T)^{\frac{p}{p-1}}|\log R_1(T)|^{\frac{p}{p-1}}\right] + c(p)\hat{\mathbb{E}}^{\hat{G}}\left[|\log R_1(T)|^{\frac{p}{p-1}}\right] \\ &\le c(p)\exp\left[\frac{p}{2(p-1)^2}\Sigma(T)|h|^2\right]\hat{\mathbb{E}}^{\hat{G}}\left(\tilde{R}_1(T)|\log R_1(T)|^{\frac{p}{p-1}}\right) \\ &\quad + c(p)\hat{\mathbb{E}}^{\hat{G}}\left[|\log R_1(T)|^{\frac{p}{p-1}}\right] \end{aligned} \tag{3.18}$$

for some constants $c(p) > 0$. On the other hand, (3.17), the B-D-G inequality [3, Theorem 2.1], $\langle \hat{B}'\rangle = \langle B'\rangle$ and $\langle \hat{B}\rangle = \langle B\rangle$ imply

$$\begin{aligned} &\hat{\mathbb{E}}^{\hat{G}}\left(\tilde{R}_1(T)|\log R_1(T)|^{\frac{p}{p-1}}\right) \\ &= \tilde{\mathbb{E}}_2\Bigg| -\int_0^T \left\langle \begin{pmatrix}\Phi_1(u)\\ \Phi_2(u)\end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u\end{pmatrix}\right\rangle \\ &\qquad - \frac{1}{2}\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle B'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle B\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\Bigg|^{\frac{p}{p-1}} \\ &= \tilde{\mathbb{E}}_2\Bigg| -\int_0^T \left\langle \begin{pmatrix}\Phi_1(u)\\ \Phi_2(u)\end{pmatrix}, \mathrm{d}\begin{pmatrix} \hat{B}'_u \\ \hat{B}_u\end{pmatrix}\right\rangle \\ &\qquad + \left(\frac{p}{p-1}-\frac{1}{2}\right)\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle \hat{B}'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle \hat{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\Bigg|^{\frac{p}{p-1}} \\ &\le c(p)\tilde{\mathbb{E}}_2\left|\int_0^T \left\langle \begin{pmatrix}\Phi_1(u)\\ \Phi_2(u)\end{pmatrix}, \mathrm{d}\begin{pmatrix} \hat{B}'_u \\ \hat{B}_u\end{pmatrix}\right\rangle\right|^{\frac{p}{p-1}} \\ &+ c(p)\tilde{\mathbb{E}}_2\left|\left(\frac{p}{p-1}-\frac{1}{2}\right)\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle \hat{B}'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle \hat{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right|^{\frac{p}{p-1}} \\ &\le c(p)\tilde{\mathbb{E}}_2\left|\int_0^T \left(|\Phi_1(s)|^2\mathrm{d}\langle \hat{B}'\rangle_s + |\Phi_2(s)|^2\mathrm{d}\langle \hat{B}\rangle_s + 2\Phi_1(s)\Phi_2(s)\mathrm{d}s\right)\right|^{\frac{p}{2(p-1)}} \end{aligned}$$

$$
\begin{aligned}
&+ c(p)\tilde{\mathbb{E}}_2\left|\left(\frac{p}{p-1}-\frac{1}{2}\right)\int_0^T\Big(|\Phi_1(s)|^2\mathrm{d}\langle\hat{B}'\rangle_s+|\Phi_2(s)|^2\mathrm{d}\langle\hat{B}\rangle_s+2\Phi_1(s)\Phi_2(s)\mathrm{d}s\Big)\right|^{\frac{p}{p-1}}\\
&=: I_1+I_2.
\end{aligned}
$$

Let $\tilde{\mathcal{C}}$ be the Choquet capacity associated to $\tilde{\mathbb{E}}_2$. Since $\hat{B}'$ is a $\tilde{G}$-Brownian motion under $\tilde{\mathbb{E}}_2$, we conclude that $\tilde{\mathcal{C}}$-q.s. (3.13) holds with $(\hat{B},\hat{B}')$ in place of (B,B'). As a result, we get

$$
I_1 \le c(p)\left(\Sigma(T)|h|^2\right)^{\frac{p}{2(p-1)}},
$$

and

$$
I_2 \le c(p)\left(\Sigma(T)|h|^2\right)^{\frac{p}{p-1}}.
$$

Therefore, it holds that

$$
\left(\hat{\mathbb{E}}^{\hat{G}}\left(\tilde{R}_1(T)|\log R_1(T)|^{\frac{p}{p-1}}\right)\right)^{\frac{p-1}{p}} \le c(p)\left(\Sigma(T)|h|^2+\sqrt{\Sigma(T)}|h|\right). \tag{3.19}
$$

Similarly, by the B-D-G inequality and (3.13), we arrive at

$$
\left(\hat{\mathbb{E}}^{\hat{G}}|\log R_1(T)|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \le c(p)\left(\Sigma(T)|h|^2+\sqrt{\Sigma(T)}|h|\right).
$$

This together with (3.15), (3.18), (3.19) and Hölder's inequality yields

$$
\begin{aligned}
|\nabla \bar{P}_T f(z)| &= \limsup_{|h|\to 0}\frac{|\bar{P}_T f(z+h)-\bar{P}_T f(z)|}{|h|}\\
&\le \left(\bar{P}_T|f|^p(z)\right)^{\frac{1}{p}}\limsup_{|h|\to 0}\frac{\left(\hat{\mathbb{E}}^{\hat{G}}|R_1(T)-1|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}}{|h|}\\
&\le c(p)\left(\bar{P}_T|f|^p(z)\right)^{\frac{1}{p}}\sqrt{\Sigma(T)},\quad z\in\mathbb{R}^2.
\end{aligned}
$$

This completes the proof.

□

Remark 3.5. *When $m \ge 1$ and $d \ge 1$, we can replace* **(A1)** *with the Hörmander-type rank condition, i.e. there exists an integer $0 \le k \le m-1$ such that*

$$
\mathrm{Rank}[M, AM, \cdots, A^k M] = m.
$$

In this case, we can define

$$
\Lambda_1(T) := \int_0^T \frac{s(T-s)}{T^2}\mathrm{e}^{-sA}MM^*\mathrm{e}^{-sA^*}\mathrm{d}s.
$$

According to [16, (4.61)] with $r_0 = 0$ (see also [21, Proof of Theorem 4.2(1)]), when $m \ge 1$ the matrix $\Lambda_1(T)$ is invertible with

$$
\|\Lambda_1(T)^{-1}\| \le cT(T\wedge 1)^{-2(k+1)} \tag{3.20}
$$

for some constant $c > 0$. Therefore, combining this with (2.5) *and* (2.9) *and repeating the proof of Theorem 3.2, we can extend the results there to the case of $m \ge 1$ and $d \ge 1$.*

4 Applications of Harnack Inequality

As an application of the Harnack inequality, in this section, we will prove the existence of weak solution for SDEs perturbed by an integrable drift with respect to an invariant nonlinear expectation of a regular G-SDE. To this end, we assume that the Harnack inequality for the regular G-SDE holds, which is crucial to prove Novikov's condition. We should point out that the following procedure can also be applied in non-degenerate G-SDEs. However, to make the framework consistent, we only consider the stochastic Hamiltonian system. One can refer to [18] and [19] for the linear expectation case. Let A, M, Q, b_1, b_2 and B_t be introduced in Section 3 and $\bar{b}_1, \bar{b}_2 : \mathbb{R}^{m+d} \to \mathbb{R}^d$ be measurable. For simplicity, we still consider $m = d = 1$ and stochastic Hamiltonian system:

$$\begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = \bar{b}_1(X_t, Y_t)\mathrm{d}t + \bar{b}_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t \\ \qquad +b_1(X_t, Y_t)\mathrm{d}t + b_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}B_t. \end{cases} \tag{4.1}$$

The reference SDE is

$$\begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = b_1(X_t, Y_t)\mathrm{d}t + b_2(X_t, Y_t)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}B_t. \end{cases} \tag{4.2}$$

Assume that (4.2) has a unique non-explosive strong solution $(X_t^z, Y_t^z)^*$ in $M_G^2([0, T])^2$ with initial value $z \in \mathbb{R}^2$ for any $T > 0$. Let P_t^0 be the associated nonlinear semigroup to (4.2) which is defined by

$$P_t^0 f(z) = \bar{\mathbb{E}}^G f(X_t^z, Y_t^z), \quad f \in C_b(\mathbb{R}^2).$$

Before formulating the result, we first introduce the definition of weak solution.

Definition 4.1. $(\Omega, \tilde{\mathbb{E}}, (\tilde{X}_t, \tilde{Y}_t), \tilde{B})$ is called a weak solution to (4.1) with initial value $z = (x, y)^* \in \mathbb{R}^2$, if $\tilde{B}$ is a G-Brownian motion on some nonlinear space $(\Omega, \tilde{\mathbb{E}})$ and it holds that

$$\begin{cases} \tilde{X}_s = x + \int_0^s \{A\tilde{X}_t + M\tilde{Y}_t\}\mathrm{d}t, \\ \tilde{Y}_s = y + \int_0^s \bar{b}_1(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}t + \int_0^s \bar{b}_2(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}\langle \tilde{B}\rangle_t \\ \qquad + \int_0^s b_1(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}s + \int_0^s b_2(\tilde{X}_t, \tilde{Y}_t)\mathrm{d}\langle \tilde{B}\rangle_t + Q\mathrm{d}\tilde{B}_t, \quad s \geq 0. \end{cases}$$

Next, we recall the definition of invariant expectation, see [8] for more details. Let $C_{Lip}(\mathbb{R}^2)$ denote all the Lipschitz continuous functions ϕ, that is: there exists a constant $C > 0$ depending on ϕ such that

$$|\phi(z) - \phi(\bar{z})| \leq C|z - \bar{z}|, \quad z, \bar{z} \in \mathbb{R}^2.$$

Definition 4.2. Let $(\mathbb{R}^2, C_{Lip}(\mathbb{R}^2), \mathbb{E}_0)$ be a sublinear expectation space. $\mathbb{E}_0$ is called an invariant sublinear expectation of P_t^0, if for any $f \in C_{Lip}(\mathbb{R}^2)$, it holds

$$\mathbb{E}_0(P_t^0 f) = \mathbb{E}_0 f, \quad t \geq 0. \tag{4.3}$$

Theorem 4.1. *Assume that P_t^0 has a unique invariant sublinear expectation $\mathbb{E}_0$ satisfying that (4.3) holds for any $f \in C_b(\mathbb{R}^2)$, and for any $p > 1$, the Harnack inequality*

$$(P_t^0|f|)^p(z) \le (P_t^0|f|^p)(\bar z)\mathrm{e}^{\Phi_p(t,z,\bar z)}, \quad f \in C_b(\mathbb{R}^2), z, \bar z \in \mathbb{R}^2, t > 0 \tag{4.4}$$

holds for some continuous function $\Phi_p : (0,\infty) \times \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ with

$$\int_0^t \frac{\mathrm{d}s}{\{\mathbb{E}_0 \mathrm{e}^{-\Phi_p(s,z,\cdot)}\}^{\frac{1}{p}}} < \infty, \quad t > 0, z \in \mathbb{R}^2. \tag{4.5}$$

If in addition, $\bar b_1$ and $\bar b_2$ are continuous and there exists some $\varepsilon > 0$ such that

$$\mathbb{E}_0 \mathrm{e}^{\varepsilon(|\bar b_1|^2 + |\bar b_2|^2)} < \infty,$$

then for any $z \in \mathbb{R}^2$, the stochastic Hamiltonian system (4.1) has a weak solution with initial value z.

Proof. Let (X_t, Y_t) be the solution to (4.2) with initial value z. Define

$$\bar B_s = B_s - \int_0^s Q^{-1}[\bar b_1(X_t,Y_t)\mathrm{d}t + \bar b_2(X_t,Y_t)\mathrm{d}\langle B\rangle_t].$$

Then (4.2) can be rewritten as

$$\begin{cases} \mathrm{d}X_t = \{AX_t + MY_t\}\mathrm{d}t, \\ \mathrm{d}Y_t = \bar b_1(X_t,Y_t)\mathrm{d}t + \bar b_2(X_t,Y_t)\mathrm{d}\langle B\rangle_t \\ \qquad\quad + b_1(X_t,Y_t)\mathrm{d}t + b_2(X_t,Y_t)\mathrm{d}\langle B\rangle_t + Q\mathrm{d}\bar B_t. \end{cases}$$

It is sufficient to find out a constant $t_0 > 0$ such that $\{\bar B_s\}_{s\in[0,t_0]}$ is a G-Brownian motion under $\tilde{\mathbb{E}}[\cdot] = \hat{\mathbb{E}}^{\hat G}[\tilde R(t_0)(\cdot)]$, where

$$\begin{aligned} \tilde R(t_0) = \exp\Bigg[&\int_0^{t_0} \left\langle \begin{pmatrix} \bar b_1(X_u,Y_u) \\ \bar b_2(X_u,Y_u) \end{pmatrix}, \mathrm{d}\begin{pmatrix} B'_u \\ B_u \end{pmatrix} \right\rangle \\ &- \frac{1}{2}\int_0^{t_0} \left(|\bar b_1(X_u,Y_u)|^2 \mathrm{d}\langle B'\rangle_u + |\bar b_2(X_u,Y_u)|^2 \mathrm{d}\langle B\rangle_u + 2\bar b_1(X_u,Y_u)\bar b_2(X_u,Y_u)\mathrm{d}u\right)\Bigg]. \end{aligned}$$

According to Theorem 2.2, we only need to prove

$$\begin{aligned} &\hat{\mathbb{E}}^{\hat G} \exp\left\{ \left(\frac{1}{2} + \delta\right)\left(\int_0^{t_0} \left(|\bar b_1(X_t,Y_t)|^2 \mathrm{d}\langle B'\rangle_t + |\bar b_2(X_t,Y_t)|^2 \mathrm{d}\langle B\rangle_t + 2(\bar b_1 \bar b_2)(X_t,Y_t)\mathrm{d}t\right)\right)\right\} \\ &\le \hat{\mathbb{E}}^{\hat G} \exp\left\{(1+2\delta)\left(\int_0^{t_0} \underline{\sigma}^{-2}|\bar b_1(X_t,Y_t)|^2 \mathrm{d}t + \int_0^{t_0} \bar\sigma^2 |\bar b_2(X_t,Y_t)|^2 \mathrm{d}t\right)\right\} \\ &\le \bar{\mathbb{E}}^{G} \exp\left\{(1+2\delta)(\underline{\sigma}^{-2} + \bar\sigma^2)\int_0^{t_0} \left(|\bar b_1(X_t,Y_t)|^2 + |\bar b_2(X_t,Y_t)|^2\right)\mathrm{d}t\right\} < \infty \end{aligned}$$

for some $\delta, t_0 > 0$. Firstly, noting that (4.3) holds for any $f \in C_b(\mathbb{R}^2)$, applying (4.4) for $f = \mathrm{e}^{\frac{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}{p}} \wedge N$ for any $N \geq 1$, we obtain

$$\left(P_t^0\left(\mathrm{e}^{\frac{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}{p}} \wedge N\right)\right)^p(z)\mathbb{E}_0\mathrm{e}^{-\Phi_p(t,z,\cdot)} \leq \mathbb{E}_0\left(P_t^0\left(\mathrm{e}^{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)} \wedge N^p\right)\right)$$
$$= \mathbb{E}_0\left(\mathrm{e}^{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)} \wedge N^p\right).$$

Letting N go to infinity and utilizing the monotone convergence theorem, we have

$$\left(P_t^0\mathrm{e}^{\frac{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}{p}}\right)^p(z)\mathbb{E}_0\mathrm{e}^{-\Phi_p(t,z,\cdot)} \leq \mathbb{E}_0\left(\mathrm{e}^{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}\right).$$

Thus, taking t_0 satisfying $\frac{\varepsilon}{pt_0} > (\underline{\sigma}^{-2}+\overline{\sigma}^2)$ and $\delta = \frac{\frac{\varepsilon}{pt_0}}{2(\underline{\sigma}^{-2}+\overline{\sigma}^2)} - \frac{1}{2} > 0$, by Jensen's inequality and (4.5), we arrive at

$$\begin{aligned}
&\bar{\mathbb{E}}^G \exp\left\{(1+2\delta)(\underline{\sigma}^{-2}+\overline{\sigma}^2)\int_0^{t_0}\left(|\bar{b}_1(X_t,Y_t)|^2+|\bar{b}_2(X_t,Y_t)|^2\right)\mathrm{d}t\right\}\\
&= \bar{\mathbb{E}}^G \exp\left\{\frac{\varepsilon}{pt_0}\int_0^{t_0}\left(|\bar{b}_1(X_t,Y_t)|^2+|\bar{b}_2(X_t,Y_t)|^2\right)\mathrm{d}t\right\}\\
&\leq \frac{1}{t_0}\int_0^{t_0}\bar{\mathbb{E}}^G \exp\left\{\frac{\varepsilon}{p}\left(|\bar{b}_1(X_t,Y_t)|^2+|\bar{b}_2(X_t,Y_t)|^2\right)\right\}\mathrm{d}t\\
&= \frac{1}{t_0}\int_0^{t_0} P_t^0\mathrm{e}^{\frac{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}{p}}(z)\mathrm{d}t\\
&\leq \frac{1}{t_0}\int_0^{t_0}\frac{\mathrm{d}t}{\{\mathbb{E}_0\mathrm{e}^{-\Phi_p(t,z,\cdot)}\}^{\frac{1}{p}}}\left(\mathbb{E}_0\mathrm{e}^{\varepsilon(|\bar{b}_1|^2+|\bar{b}_2|^2)}\right)^{\frac{1}{p}} < \infty, \quad z \in \mathbb{R}^2.
\end{aligned}$$

The proof is completed. □

Finally, we give an example in which (4.4) and (4.5) hold and (4.3) holds for any $f \in C_b(\mathbb{R}^2)$.

Example 4.2. *In (4.2), let $A = 0$, $M = Q = 1$, $b_2 = 0$ and $b_1(x,y) = -x-y$. Then (4.2) reduces to*

$$\begin{cases}\mathrm{d}X_t = Y_t\mathrm{d}t,\\ \mathrm{d}Y_t = (-X_t - Y_t)\mathrm{d}t + \mathrm{d}B_t.\end{cases} \tag{4.6}$$

Firstly, according to Theorem 3.2 (1), (4.4) holds for

$$\Phi_p(t,z,\bar{z}) = c\frac{|z-\bar{z}|^2}{t^3}, \quad z,\bar{z} \in \mathbb{R}^2, t \in (0,1)$$

with some constant $c > 0$ independent of t.

Next, by [8, Theorem 3.12], P_t^0 has a unique invariant sublinear expectation $\mathbb{E}_0$ and there exists a family of weakly compact probability measures $\{m_\gamma\}_{\gamma\in\Gamma}$ defined on $\mathbb{R}^2$ such that

$$\mathbb{E}_0 f = \sup_{\gamma\in\Gamma}\int_{\mathbb{R}^2} f\mathrm{d}m_\gamma, \quad f\in C_{Lip}(\mathbb{R}^2).$$

[13, Theorem 6.1.16] implies that $\mathbb{E}_0$ is regular. On the other hand, for any $f\in C_b(\mathbb{R}^2)$ and $n\geq 1$, define $f_n:\mathbb{R}^2\to\mathbb{R}$ by

$$f_n(z) := \inf_{z'\in\mathbb{R}^2}\{f(z') - n|z-z'|\}, \quad z\in\mathbb{R}^2.$$

According to [9, Lemma 4.3], we know that $\{f_n\}_{n\geq 1}\subset C_{Lip}(\mathbb{R}^2)$ and $f_n\uparrow f$ as $n\uparrow\infty$. This together with the regularity of $\mathbb{E}_0$ and [13, 6.1.15] implies that $\mathbb{E}_0(f-f_n)\downarrow 0$ as $n\uparrow\infty$. Thus, $C_b(\mathbb{R}^2)$ is contained in the completion of $C_{Lip}(\mathbb{R}^2)$ under the norm $\mathbb{E}_0(|\cdot|)$. Furthermore, $\mathbb{E}_0(P_tf)$ is well defined for any $f\in C_b(\mathbb{R}^2)$ due to the gradient estimate (3.5). *As a result, the monotone convergence theorem yields that* (4.3) *holds for any $f\in C_b(\mathbb{R}^2)$.*

Finally, let $\theta_s^0=\underline{\sigma}, s\geq 0$ and $\mathbb{P}_{\theta^0}$ be the corresponding probability measure as represented in (2.4), *which implies*

$$\bar{\mathbb{E}}^G f(X_t,Y_t)\geq \mathbb{E}_{\mathbb{P}_{\theta^0}} f(X_t,Y_t), \quad f\in C_{Lip}(\mathbb{R}^2). \tag{4.7}$$

On the other hand, by [17, Theorem 3.1(1)], under the probability $\mathbb{P}_{\theta^0}$, (4.6) *has a unique invariant measure*

$$\mu_0(\mathrm{d}x,\mathrm{d}y) = \frac{1}{2\pi\underline{\sigma}^2}\mathrm{e}^{-\frac{|x|^2+|y|^2}{2\underline{\sigma}^2}}\mathrm{d}x\mathrm{d}y.$$

Combining this with [8, Theorem 3.3, Theorem 3.12] and letting the time t go to infinity in (4.7), *we arrive at*

$$\mathbb{E}_0 f\geq \mu_0(f), \quad f\in C_{Lip}(\mathbb{R}^2).$$

This together with [18, Example 4.3] yields

$$\mathbb{E}_0\left(\mathrm{e}^{-\Phi_p(t,z,\cdot)}\right)\geq \mu_0\left(\mathrm{e}^{-\Phi_p(t,z,\cdot)}\right)\geq \mathrm{e}^{-c}\mu_0(B(z,1\wedge t^{\frac{3}{2}}))\geq \alpha(z)(1\wedge t)^{\frac{3}{2}}, \quad t>0, z\in\mathbb{R}^2$$

for some $\alpha\in C(\mathbb{R}^2)$. Consequently, (4.5) *holds for $p>\frac{3}{2}$.*

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