# Derivation of the Real-rootedness of Coordinator Polynomials from the Hermite–Biehler Theorem

Matthew H.Y. Xie[1] and Philip B. Zhang[2]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

Center for Applied Mathematics
Tianjin University, Tianjin 300072, P. R. China

[1]xiehongye@163.com, [2]zhangbiaonk@163.com

**Abstract.** By using the Hermite–Biehler theorem, we give a new proof of the real-rootedness of the coordinator polynomials of type $D$, which was recently established by Wang and Zhao. As a consequence, we also obtain the compatibility between the coordinator polynomials of type $D$ and those of type $C$.

*AMS Classification 2010:* 26C10, 30C15, 05A15

*Keywords:* coordinator polynomials, real-rootedness, the Hermite–Biehler theorem, compatibility.

*Corresponding author:* Philip B. Zhang, zhangbiaonk@163.com

# 1 Introduction

This paper is concerned with the real-rootedness of the following polynomials

$$\sum_{k=0}^{n} \binom{2n}{2k} z^k + 2nz(1+z)^{n-2}, \tag{1}$$

which arose in the theory of coordinator polynomials of Weyl group lattices developed by Conway and Sloane [6]. These polynomials are known as the coordinator polynomials of type $D_n$, denoted $h_{D_n}(z)$. Wang and Zhao [13] proved that for any $n \geq 2$ the polynomial $h_{D_n}(z)$ has only real roots. Their proof uses a technique of trigonometric substitution. The main objective of this paper is to give a new proof of the real-rootedness of $h_{D_n}(z)$ by using the Hermite–Biehler theorem. Our proof is motivated by the Hermite–Biehler theorem approach to the real-rootedness of the coordinator polynomials of type $C_n$ given by

$$h_{C_n}(z) = \sum_{k=0}^{n} \binom{2n}{2k} z^k. \tag{2}$$

As a result of our approach, we get the compatibility between $h_{C_n}(z)$ and $h_{D_n}(z)$ in the sense of Chudnovsky and Seymour [5].

Let us first review some background on the coordinator polynomials $h_{C_n}(z)$ and $h_{D_n}(z)$. For more information on the coordinator polynomials of root lattices, see [1, 2, 6] and references therein. Let $\mathbb{Z}$ be the ring of integers, and let $\mathbb{R}$ be the field of real numbers. Let

$$\begin{aligned} M_{C_n} &= \left\{\pm\mathbf{e}_i \pm \mathbf{e}_j \,\middle|\, 1 \le i < j \le n\right\} \cup \left\{\pm 2\mathbf{e}_i \,\middle|\, 1 \le i \le n\right\}, \\ M_{D_n} &= \left\{\pm\mathbf{e}_i \pm \mathbf{e}_j \,\middle|\, 1 \le i < j \le n\right\}, \end{aligned}$$

where $\mathbf{e}_i$ denotes the vector in $\mathbb{R}^n$ with the $i$th entry one and all other entries zero. It is clear that both $M_{C_n}$ and $M_{D_n}$ generate the same root lattice

$$\mathcal{L} = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \,\middle|\, \sum x_i \text{ is even}\right\}$$

as a monoid. For each $u \in \mathcal{L}$, let $w_{C_n}(u)$ denote the word length of $u$ with respect to $M_{C_n}$ given by

$$w_{C_n}(u) = \min\left\{\sum c_i \,\middle|\, u = \sum c_i\, \mathbf{a}_i, c_i \in \mathbb{N}, \mathbf{a}_i \in M_{C_n}\right\}.$$

In the same manner, we can define the word length of $u$ with respect to $M_{D_n}$, denoted $w_{D_n}(u)$. The coordinator polynomials are related to the generating functions for word lengths over the root lattice $\mathcal{L}$. Baake and Grimm [2] conjectured that

$$\sum_{u\in\mathcal{L}} z^{w_{C_n}(u)} = \frac{h_{C_n}(z)}{(1-z)^n},$$

and Conway and Sloane [6] conjectured that

$$\sum_{u\in\mathcal{L}} z^{w_{D_n}(u)} = \frac{h_{D_n}(z)}{(1-z)^n}.$$

Subsequently, Bacher et al. [3] confirmed these two conjectures. For other proofs, see Ardila et al. [1].

Recently, the real-rootedness of the coordinator polynomials $h_{C_n}(z)$ and $h_{D_n}(z)$ has drawn attention. As pointed out by Wang and Zhao [13], there are at least two ways to prove that $h_{C_n}(z)$ has only real roots, one using the theory of total positivity, and the other using the theory of Sturm sequences. This paper is motivated by another proof of the real-rootedness of $h_{C_n}(z)$ by using the Hermite–Biehler theorem, which we shall recall below.

The Hermite–Biehler theorem is a basic result in the Routh–Hurwitz theory [11, 12], which provides a criterion for determining the Hurwitz stability of a polynomial. Recall that a polynomial $P(z)$ is said to be Hurwitz stable (respectively, weakly Hurwitz stable) if $P(z) \neq 0$ whenever $\mathrm{Re}(z) \geq 0$ (respectively, $\mathrm{Re}(z) > 0$), where $\mathrm{Re}(z)$ denotes the real part of $z$. Suppose that

$$P(z) = \sum_{k=0}^{n} a_k z^k.$$

Let

$$P^E(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{2k} z^k \quad \text{and} \quad P^O(z) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{2k+1} z^k. \tag{3}$$

As will be shown in the Hermite–Biehler theorem, the stability of $P(z)$ is closely related to the interlacing property between $P^E(z)$ and $P^O(z)$. Given two real-rooted polynomials $f(z)$ and $g(z)$ with positive leading coefficients, let $\{r_i\}$ be the set of zeros of $f(z)$ and $\{s_j\}$ the set of zeros of $g(z)$. We say that $g(z)$ interlaces $f(z)$, denoted $g(z) \preceq f(z)$, if either $\deg f(z) = \deg g(z) = n$ and

$$s_n \leq r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1, \tag{4}$$

or $\deg f(z) = \deg g(z) + 1 = n$ and

$$r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{5}$$

If all inequalities in (4) or (5) are strict, then we say that $g(z)$ strictly interlaces $f(z)$, denoted $g(z) \prec f(z)$. The Hermite–Biehler theorem is stated as follows.

**Theorem 1.1** ([4, Theorem 4.1])**.** *Let $P(z)$ be a polynomial with real coefficients, and let $P^E(z)$ and $P^O(z)$ be defined as in* (3)*. Suppose that $P^E(z)P^O(z) \not\equiv 0$. Then $P(z)$ is Hurwitz stable (respectively, weakly Hurwitz stable) if and only if $P^E(z)$ and $P^O(z)$ have only real and negative (respectively, non-positive) zeros, and $P^E(z) \prec P^O(z)$ (respectively, $P^E(z) \preceq P^O(z)$).*

The Hermite–Biehler theorem has been widely used to study the real-rootedness of polynomials. Csordas et al. [8] utilized the Hermite–Biehler theorem to confirm a conjecture on the real-rootedness of some polynomials related to a class of Jacobi polynomials, which was proposed while developing a numerical solution for the Navier-Stokes equations. Craven and Csordas [7] applied stability analysis, in conjunction with the Hermite-Biehler theorem, to proving that certain Mittag-Leffler-type functions have only real zeros. By using the Hermite-Biehler theorem, Brändén [4] gave characterizations of two non-linear operators which send polynomials with only real and non-positive zeros to polynomials of the same kind.

To apply the Hermite–Biehler theorem to proving the real-rootedness of $h_{C_n}(z)$, in view of (2), we only need to take

$$P(z) = (1+z)^{2n} = \sum_{k=0}^{n} \binom{2n}{k} z^k.$$

It is clear that $P(z)$ is Hurwitz stable and $h_{C_n}(z) = P^E(z)$.

Although the expression of $h_{D_n}(z)$ looks very similar to that of and $h_{C_n}(z)$, it is not an easy task to prove that $h_{D_n}(z)$ has only real zeros. By a technique of substituting the variable $z$ by a trigonometric function, Wang and Zhao [13] managed to prove the real-rootedness of $h_{D_n}(z)$. Considering the similarity of (1) and (2), it is natural to ask whether the real-rootedness of $h_{D_n}(z)$ has a proof using the Hermite–Biehler theorem. In the next section, we shall give such a proof.

## 2 Real-rootedness and compatibility

The main objective of this section is to prove the following result by using the Hermite–Biehler theorem.

**Theorem 2.1** ([13, Theorem 2.1]). *For any $n \geq 2$, the polynomial $h_{D_n}(z)$ has only real zeros.*

*Proof.* To use the Hermite–Biehler theorem, as indicated in the proof of the real-rootedness of $h_{C_n}(z)$, we shall take

$$\begin{aligned} P(z) &= (1+z)^{2n} - 2nz^2(1+z^2)^{n-2} \\ &= \sum_{k=0}^{n} \binom{2n}{k} z^k - 2nz^2(1+z^2)^{n-2}, \end{aligned}$$

and whence $h_{D_n}(z) = P^E(z)$.

We proceed to show the Hurwitz stability of $P(z)$. It is clear that $P(0) \neq 0$. Without loss of generality, we may assume that $z \neq 0$. Note that

$$\begin{aligned} P(z) &= (1+z)^{2n} - 2nz^2(1+z^2)^{n-2} \\ &= (1+2z+z^2)^n - 2nz^2(1+z^2)^{n-2} \\ &= 2^n z^n \left( \left( \frac{z+1/z}{2} + 1 \right)^n - \frac{n}{2} \left( \frac{z+1/z}{2} \right)^{n-2} \right). \end{aligned}$$

Moreover, it is routine to verify that $\mathrm{Re}((z+1/z)/2) \geq 0$ if and only if $\mathrm{Re}(z) \geq 0$. Therefore, it suffices to prove the Hurwitz stability of the polynomial

$$Q(z) = (z+1)^n - \frac{n}{2} z^{n-2}.$$

Suppose that $\mathrm{Re}(z) \geq 0$. We need to show that $Q(z) \neq 0$. By the triangle inequality, we have

$$|Q(z)| \geq |z+1|^n - \frac{n}{2}|z|^{n-2}.$$

Note that the assumption $\mathrm{Re}(z) \geq 0$ implies that

$$|z+1| \geq \sqrt{|z|^2+1}.$$

Thus, we get

$$|Q(z)| \geq \left( \sqrt{|z|^2+1} \right)^n - \frac{n}{2}|z|^{n-2}.$$

Now it suffices to prove that

$$\left(\left(\sqrt{|z|^2+1}\right)^n\right)^2 > \left(\frac{n}{2}|z|^{n-2}\right)^2,$$

namely,

$$(|z|^2+1)^n > \frac{n^2}{4}|z|^{2n-4}.$$

Expanding the left hand side by the binomial theorem, we find that for $n \geq 2$,

$$(|z|^2+1)^n = \sum_{k=0}^{n} \binom{n}{k} |z|^{2k} > \binom{n}{n-2} |z|^{2(n-2)} \geq \frac{n^2}{4}|z|^{2n-4}.$$

Therefore, $|Q(z)| > 0$ if $\mathrm{Re}(z) \geq 0$. This means that $Q(z)$ is Hurwitz stable, so is $P(z)$. By the Hermite–Biehler theorem, we obtain the real-rootedness of $h_{D_n}(z)$. This completes the proof. $\square$

*Remark.* Following the lines of the above proof, it is easy to show that, for any $n \geq 2$ and $|r| \leq 2\sqrt{2n(n-1)}$, the polynomial

$$\sum_{k=0}^{n} \binom{2n}{2k} z^k + rz(1+z)^{n-2}, \tag{6}$$

has only real zeros. In this case, we only need to take

$$P(z) = \sum_{k=0}^{n} \binom{2n}{k} z^k + rz^2(1+z^2)^{n-2}.$$

The Hermite–Biehler theorem approach to the real-rootedness of $h_{C_n}(z)$ and $h_{D_n}(z)$ also leads us to the discovery of their compatibility. The notion of compatibility was introduced by Chudnovsky and Seymour [5] in the study of the real-rootedness of independence polynomials of claw-free graphs. Given two real-rooted polynomials $f(z)$ and $g(z)$ with positive leading coefficients, they are said to be compatible if for all real $a, b \geq 0$, the polynomial $af(z) + bg(z)$ has only real zeros. The compatibility also has a characterization in terms of certain interlacing property of polynomials. We say that $f(z)$ and $g(z)$ have a *common interleaver* if there exists another real-rooted polynomial $h(z)$ such that $f(z) \preceq h(z)$ and $g(z) \preceq h(z)$. The following lemma is a special case of a result of Chudnovsky and Seymour [5].

**Lemma 2.2.** *Suppose that $f(z)$ and $g(z)$ have only real zeros. Then $f(z)$ and $g(z)$ are compatible if and only if they have a common interleaver.*

It should be mentioned that in the special case $\deg f(z) = \deg g(z)$, the above result has been proved by Dedieu [9]; see also Fisk [10, Chapter 1].

With the above results, we now proceed to show the compatibility between $h_{C_n}(z)$ and $h_{D_n}(z)$.

**Corollary 2.3.** *For $n \geq 2$, the polynomials $h_{C_n}(z)$ and $h_{D_n}(z)$ are compatible.*

*Proof.* Let

$$g(z) = \sum_{k=0}^{n-1} \binom{2n}{2k+1} z^k.$$

As before, applying the Hermite–Biehler theorem to $P(z) = (1+z)^{2n}$, we obtain that $h_{C_n}(z) \prec g(z)$. If $P(z)$ is taken to be

$$(1+z)^{2n} - 2nz^2(1+z^2)^{n-2},$$

then we get that $h_{D_n}(z) \prec g(z)$. Therefore, $h_{C_n}(z)$ and $h_{D_n}(z)$ have a common interleaver $g(z)$. By Lemma 2.2, these two polynomials are compatible. This completes the proof. □

**Acknowledgments.** This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education and the National Science Foundation of China.

# References

[1] F. Ardila, M. Beck, S. Hoşten, J. Pfeifle, and K. Seashore, Root polytopes and growth series of root lattices, SIAM J. Discrete Math., 25 (2011), 360–378.

[2] M. Baake and U. Grimm, Coordination sequences for root lattices and related graphs, Z. Krist., 212 (1997), 253–256.

[3] R. Bacher, P. de la Harpe, and B. Venkov, Séries de croissance et séries d'Ehrhart associées aux réseaux de racines, C. R. Acad. Sci. Paris Sér. I Math., 325 (1997), 1137–1142.

[4] P. Brändén, Iterated sequences and the geometry of zeros, J. Reine Angew. Math., 658 (2011), 115–131.

[5] M. Chudnovsky and P. Seymour, The roots of the independence polynomial of a clawfree graph, J. Combin. Theory Ser. B, 97 (2007), 350–357.

[6] J. H. Conway and N. J. A. Sloane, Low-dimensional lattices. VII. Coordination sequences, Proc. Roy. Soc. London Ser. A, 453 (1997), 2369–2389.

[7] T. Craven and G. Csordas, The Fox-Wright functions and Laguerre multiplier sequences, J. Math. Anal. Appl., 314 (2006), 109–125.

[8] G. Csordas, M. Charalambides, and F. Waleffe, A new property of a class of Jacobi polynomials, Proc. Amer. Math. Soc., 133 (2005), 3551–3560 (electronic).

[9] J.-P. Dedieu, Obreschkoff's theorem revisited: what convex sets are contained in the set of hyperbolic polynomials?, J. Pure Appl. Algebra, 81 (1992), 269–278.

[10] S. Fisk, Polynomials, Roots, and Interlacing, arXiv:math/0612833 [math.CA].

[11] O. Holtz, Hermite-Biehler, Routh-Hurwitz, and total positivity, Linear Algebra Appl., 372 (2003), 105–110.

[12] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, vol. 26 of London Mathematical Society Monographs. New Series, The Clarendon Press Oxford University Press, Oxford, 2002.

[13] D. G. L. Wang and T. Zhao, The real-rootedness and log-concavities of coordinator polynomials of Weyl group lattices, European J. Combin., 34 (2013), 490–494.