Multiple positive solutions for coupled Schrödinger equations with perturbations

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Abstract. For coupled Schrödinger equations with nonhomogeneous perturbations we give several results on the existence of multiple positive solutions. In particular in one case we consider perturbations of the permutation symmetry.

Keywords: Coupled Schrödinger equations; Nonhomogeneous perturbations; Multiple positive solutions.

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1. Introduction

1.1. **Main results.** In this note, we give some multiplicity results of positive solutions on the following coupled nonlinear Schrödinger equations with nonhomogeneous perturbations:

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta u v^2 + f_1 & \text{in } \Omega, \\ -\Delta v + v = \mu_2 v^3 + \beta v u^2 + f_2 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary for $1 \leq n \leq 3$, and $\mu_1 > 0$, $\mu_2 > 0$ and β are constants. The perturbation terms f_1 and f_2 are assumed to be non-negative L^2 functions. We define the energy for u, $v \in H_0^1(\Omega)$

$$I(u,v) = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{\mu}{4}\int u^4 + v^4 - \frac{\beta}{2}\int u^2 v^2 - \int f \cdot (u+v),$$

where $\|u\| = (\int |\nabla u|^2 + \int |u|^2)^{\frac{1}{2}}$ is the Sobolev norm. We use $|\cdot|_p$ for L^p norm $1 \leq p < \infty$. It is easy to see that the functional I is of class C^2. We say a solution (u,v) is positive if both u and v are positive. We prove the following existence theorem and multiplicity result on positive solutions.

Theorem 1.1. *Let $\mu_1 > 0$, $\mu_2 > 0$ and $\beta \in \mathbb{R}$ be fixed and assume $f_i \geq 0$ for $i = 1, 2$. Then there is $\varepsilon_0 > 0$ such that if $0 < |f_i|_2 \leq \varepsilon_0$ for $i = 1, 2$, Problem (1.1) possesses two positive solutions (u_1, v_1) and (u_2, v_2) with $I(u_1, v_1) > 0$ and*

$I(u_2, v_2) < 0$. Moreover, the solution (u_2, v_2) is the only solution to Problem (1.1) with a small norm.

Theorem 1.2. *Assume $\mu := \mu_1 = \mu_2 > 0$ and $f := f_1 = f_2 \geq 0$. Then there is $\varepsilon_0 > 0$, such that if $|f|_2 \leq \varepsilon_0$, the following conclusions hold.*

(1). If $\beta \leq -\mu$, Problem (1.1) possesses an infinite sequence of positive solutions (u_k, v_k) such that $u_k \neq v_k$ for any k.

(2). For any positive integer k, there is a $\delta_k > 0$ such that for any $\beta \in (-\mu, -\mu + \delta_k)$, Problem (1.1) has at least k positive solutions.

Remark 1.3. *(i). Under the assumption of Theorem 1.2, due to Theorem 1.1, the small solution with negative energy is unique. Moreover, it is in the form of (u, u), where u is the unique solution to the equation $-\Delta u + u = (\mu + \beta)u^3 + f$ with negative energy.*

(ii). The results still hold when $\Omega = \mathbb{R}^n$ with $n = 2, 3$. In this case, we work in the subspace of radial functions, $H_r^1(\mathbb{R}^n)$, which embeds compactly into L^p for $2 < p < \frac{2n}{n-2}$. Our method works through with little modifications.

(i). We assumed the coefficients of u and v in (1.1) are the same. The first theorem still holds with different positive coefficients, and the second theorem the same coefficients are essential.

1.2. **Historical Notes.** Semilinear elliptic problems with non-homogeneous term, as an immediate generalization of the usual semilinear problems (c.f. [11, 27, 32]), are widely studied via many methods. For elliptic equations, the problem is usually in the form of

$$\begin{cases} -\Delta u + u = f(u) + p(x, u) & \text{in } \Omega, \\ u(x) \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

Here the nonlinearity $f(u)$ is assumed to be odd. One can find a traditional treatment in [4, 5, 7, 8, 27, 26, 29, 28] under the name of perturbation from symmetry or Bahri-Berestycki method. Infinitely many solutions can be obtained via this method. Two key steps of this method are to estimate the growth of critical level for the energy function of the problem (1.2) with $p(x, u) \equiv 0$ and to construct a perturbation of the linking. Although it is a powerful method, the range of the growth of the nonlinear term is incomplete. For example, for $f(u) = u^3$, the method fails for $p(x, u) = p(x) \in L^2$, a pure nonhomogeneous perturbation (see [5, 8] for the optimal range of the nonlinearity). The problem on validity of the full range is still open. In [33] this method was used for coupled systems like (1.1) for dimensions $n = 1, 2$, since $n = 3$ is already out of the range of the Bahri-Berestycki methods. Another approach to solve this problem is to consider the case of small perturbations. An asymptotic result can be found in [17, 19], i.e. more solutions will be obtained as the perturbation shrinks. The advantage of this method is that we do not need to restrict the growth of the nonlinear term $f(u)$. Existence of two solutions can be found in [3]. For the result on two positive solutions to the problem, readers can find them in [9, 18]. In [2], Adachi and Tanaka gave four positive solutions and their convergence result as the perturbation goes to zero. A result on jumping nonlinearities can be found in [11, Section 3.4].

In this note, we focus on the case of coupled nonlinear Schrödinger equations. We show that, unlike for scalar equations with perturbation to even symmetry, for systems with permutation symmetry under small perturbations an infinite number positive solutions survive. In establishing our existence results we will employ the heat flow methods and solutions will be constructed in some suitable invariant sets of the flow having large an permutation symmetry index. In comparison, our results hold for all dimensions $n = 1, 2, 3$, we give infinitely many positive solutions for small perturbations, while by using Bahri-Berestycki methods infinitely many solutions was given in [33] by assuming $|\beta| < \mu$ for $n = 1, 2$ without requiring smallness of perturbations. The results of ours and of [33] are not inclusive with each other, both giving a better understanding of the perturbation problem for coupled systems. In [24], the authors consider the coupled nonlinear Schrödinger systems with non-homogeneous nonlinear term with potentials. Taking the advantages of the certain choice of the potentials, they obtained infinitely many solutions and the convergence results.

1.3. **Organization of this Paper.** In Section 2 we first collect necessary results on the associated parabolic system. Then without the presence of equilibria of the system we construct a domain of attraction for some closed sets. Our two main theorems are proved in Section 3 and Section 4 respectively. Some detailed estimates used in Section 2 are left in the Appendix of Section 5.

2. Preliminaries for Theorem 1.1 and Theorem 1.2

We will prove our main theorems via parabolic flow method. First we consider the following parabolic problem

$$\begin{cases} \partial_t u - \Delta u + u = \mu_1 u^3 + \beta u v^2 + f_1 & \text{in } \Omega, \\ \partial_t v - \Delta v + v = \mu_2 v^3 + \beta v u^2 + f_2 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

2.1. **Basic Properties of the Parabolic Flow.** We use $\eta^t(U)$ to represent the solution to Problem (2.1) with initial value $U = (u_0, v_0)$. First of all, we have the following theorem on existence, uniqueness, regularity and dependence on the initial value from the framework in [14].

Theorem 2.1. *For $s = 1, 2$, let the initial value $U := (u_0, v_0) \in (H^s)^2$. Then there is a unique solution $\eta^t(U) = (u(t), v(t))$ to Problem (2.1) defined on its maximum interval $[0, T(U))$, satisfying*

(I) it holds that

$$\eta^t(U) \in C^1((0, T(U)), (L^2)^2) \cap C([0, T(U)), (H^2)^2);$$

(II) for any $U \in (H^s)^2$ and any $\delta \in [0, T(U))$, there are positive constants r, K such that for any $t \in [0, \delta]$

$$\|U - V\|_{(H^s)^2} < r \quad \Rightarrow \quad \|\eta^t(U) - \eta^t(V)\|_{(H^1)^s} \le K\|U - V\|_{(H^s)^2};$$

Lemma 2.2. *For any solution to Problem (2.1), we have*

$$\partial_t I(\eta^t(U)) = -\int |u_t|^2 + |v_t|^2.$$

Proof. Computing it directly, we have

$$\begin{aligned}
\partial_t I(\eta^t(U)) &= \int \nabla u \cdot \nabla u_t + \nabla v \cdot \nabla v_t + u u_t + v v_t \\
&\quad - \int \mu_1 u^3 u_t + \mu_2 v^3 v_t + \beta u u_t v^2 + \beta u^2 v v_t - \int f_1 u_t + f_2 v_t \\
&= \int \left(\Delta u - u + \mu_1 u^3 + \beta u v^2\right) u_t + \int \left(\Delta v - v + \mu_2 v^3 + \beta v u^2\right) v_t \\
&= -\int |u_t|^2 + |v_t|^2.
\end{aligned}$$

□

Remark 2.3. *It is an immediate consequence that the parabolic flow is a decreasing flow for I. The energy will keep decreasing unless it is on a position of a solution to Problem (1.1), which is nothing but an equilibrium point of the flow $\eta^t(\cdot)$.*

We point out that the method we use in this paper is similar to that in [20]. To achieve our goal, there is still a stability result missing (c.f. [20]). The aim of the stability result is to give a domain of attraction, on which various variational structures will be found. Usually, the domain is the stable set of a trivial solution (c.f. [1, 16, 20, 22, 23]). But in Problem (2.1), there is no obvious trivial equilibrium points, which ceases the traditional method to be effective. To overcome this difficulty, we choose to consider the stable set of a closed subset rather than of one single equilibrium point. We put the detailed computations on energy levels in the Appendix of the present paper. The main result can be summarized as follows.

Lemma 2.4. *There is an $\varepsilon_1 > 0$ small enough such that for any $\varepsilon \in (0, \varepsilon_1)$, if $|f_1|_2 + |f_2|_2 < \varepsilon$, there exist following parameters:*

$$\delta_0 > 0,\ 0 < a < b,\ R > R_2 > 0,\ T_0 > 0$$

such that

(1) $I^a \cap B(0,R)$ and $I^b \cap B(0,R)$ are invariant under the flow $\eta^t(\cdot)$;

(2) for any $U \in I^b \cap B(0,R)$, $\eta^{T_0}(U) \in I^a \cap B(0,R)$;

(3) $(I^a \cap B(0,R))_{\delta_0} \subset B(0,R_2) \subset I^b \cap B(0,R)$.

Here for a given set A, A_{δ_0} denotes the δ_0 neighborhood of the set. The specific selections of parameters are in the Appendix. Then we have the following corollary.

Corollary 2.5. *With the same parameters in the above lemma, we have*

$$\eta^{T_0}\left((I^a \cap B(0,R))_{\delta_0}\right) \subset I^a \cap B(0,R).$$

Due to the stability theory of closed sets in [6, Section 2.12] and the dependence on initial data, we can get the existence of a domain of attraction.

Corollary 2.6. *There is a $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0,\varepsilon_0)$, if $|f_1|_2 + |f_2|_2 \leq \varepsilon$, the set $\mathcal{A}$ defined by*

$$\mathcal{A} = \left\{ U \in (H_0^1(\Omega))^N \middle| \exists T_U \geq 0 \ s.t.\ \eta^{T_U}(U) \in (I^a \cap B(0,R))_{\delta_0} \right\}, \tag{2.2}$$

is an open neighbourhood of the trivial function $(0,0)$ in $H^1 \times H^1$. Moreover, both of $\mathcal{A}$ and $\partial\mathcal{A}$ are invariant under the flow $\eta^t(\cdot)$.

Due to the specific construction of $\mathcal{A}$, the following corollary is obvious.

Corollary 2.7. $\inf_{U \in \partial\mathcal{A}} I(U) =: c_0 > 0$

In the process of obtaining equilibrium points via heat flow, global existence of the trajectory plays an important role. With this result, we can find a (PS) sequence along the flow line. Therefore, a solution to Problem (1.1) can be found (c.f. [13]). We apply the method in [20, Section 2.2] to this problem. One can also find them in [10, 12, 25].

Lemma 2.8. *Assume for a solution $\eta^t(U)$ to Problem (2.1), there is $C > 0$ such that $\lim_{t \to T(U)} |I(U) - I(\eta^t(U))| \leq C$. Then $T(U) = +\infty$.*

2.2. **Monotonicity of the Parabolic Flow.** Usually, we describe the monotonicity by showing the mapping preserves some kind of positivity. However, inspirited by [21], we find it will be easier if we show that the flow has a shrinking error term in variational methods. For example, we show the flow preserves the positivity by proving that the small negative term keeps shrinking. We apply this idea to the flow $\eta^t(\cdot)$ via a related technique in [13, 20].

Lemma 2.9. *For Problem (2.1), if $\beta \leq 0$, then there is a $\rho_0 > 0$, such that if for any $\rho \in (0,\rho_0)$ and a global solution $U(t) = (u(t), v(t))$, $|u_-(t_0)|_4 \leq \rho$ with some $t_0 \geq 0$, then for any $t > t_0$, $|u_-(t)|_4 < \rho$. For $v(t)$, the same result holds. Moreover, the constant $\rho > 0$ is independent of f_j.*

Proof.

$$\partial_t \int |u_-|^4 = 4\int u_-^3 \partial_t u \le 4\int u_-^3\left(\Delta u - u + \mu_1 u^3 + \beta u v^2\right)$$
$$\le -3\int |\nabla(u^2)|^2 - 4\int |u_-|^4 + 4\mu_1 \int |u_-|^6.$$

Denote $W = u_-^2$. The last line of above computation can be dominated by

$$-C\|W\|^2 + C|W|_3^3.$$

Noticing $\frac{1}{3} = \frac{\frac{1}{2}}{6} + \frac{1-\frac{1}{2}}{2}$ and using the Sobolev embedding, we conclude that

$$|W|_3^3 \le C\|W\|^{\frac{3}{2}} |W|_2^{\frac{3}{2}}.$$

Hence,

$$\partial_t \int |u_-|^4 \le -C\|W\|^2 + C\|W\|^{\frac{3}{2}}|W|_2^{\frac{3}{2}}$$
$$\le -C\|W\|^{\frac{3}{2}}|W|_2^{\frac{3}{2}} + C\|W\|^{\frac{3}{2}}|W|_2^{\frac{3}{2}}$$
$$= -C\|W\|^{\frac{3}{2}}|W|_2^{\frac{3}{2}}\left(1 - C|u_-|_4^2\right)$$

is negative when $|u_-|_4$ is small.

$\square$

Corollary 2.10. *Consider Problem (2.1). If the initial value (u_0, v_0) of a trajectory $(u(t), v(t))$ satisfies $u_0, v_0 \ge 0$ in Ω and β satisfies the conditions in above lemmas, then for any $t \in [0, T(u_0, v_0))$, $u(t), v(t) \ge 0$ in Ω.*

Proof. We argue it by contradiction. Suppose that the result is violated for some time t_0 by u. Then, due to Theorem 2.1, there must be a time $t_1 \in (0, t_0]$ such that

- $|u(t_1)|_4 \le \frac{\rho_0}{2}$;
- $\partial_t |u_-|_4^4\big|_{t=t_1} > 0$.

Here the ρ_0 is the positive number given in Lemma 2.9. Then the two entries lead us to a contradiction with the same lemma.

$\square$

2.3. **Verification of (PS) Condition.** In this subsection, we verify the (PS) condition of the energy I of Problem (1.1). Recall

$$I(u,v) = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{\mu}{4}\int u^4 + v^4 - \frac{\beta}{2}\int u^2 v^2 - \int f\cdot(u+v).$$

For any sequence $(u_n, v_n) \in H^2$ with $|I(u_n, v_n)| \leq C$ and $\|I'(u_n, v_n)\| = o(1)$, as a routine in [32], it is easy to see that

$$\begin{aligned}
C + 1 + \|u_n\| + \|v_n\| &\geq I(u_n, v_n) - \frac{1}{4} I'(u_n, v_n)(u_n, v_n) \\
&= \frac{1}{2}\left(\|u_n\|^2 + \|v_n\|^2\right) - \frac{3}{4}\int (f_1 u_n + f_2 v_n) \\
&\geq \frac{1}{2}\left(\|u_n\|^2 + \|v_n\|^2\right) - C\left(\|u_n\| + \|v_n\|\right).
\end{aligned}$$

This gives that the sequence $((u_n, v_n))_n \subset H^1 \times H^1$ is bounded. With the help of Sobolev embedding,

$$\begin{aligned}
(u_n, v_n) &\rightharpoonup (u, v) \qquad \text{in } H^1 \times H^1 \\
(u_n, v_n) &\to (u, v) \qquad \text{in } L^r \times L^r
\end{aligned}$$

for some $(u, v) \in H^2$ and for any $r \in (2, 6)$. On the other hand, based on an elementary calculation,

$$\begin{aligned}
\|u_n - u\|^2 + \|v_n - v\|^2 &= \left(I'(u_n, v_n) - I'(u, v)\right)\left((u_n, v_n) - (u, v)\right) \\
&\quad + \mu_1 \int (u_n - u)\big(u_n^3 - u^3\big) \\
&\quad + \mu_2 \int (v_n - v)\big(v_n^3 - v^3\big) \\
&\quad - \beta \int (u_n - u)\big(uv^2 - u_n v_n^2\big) + (v_n - v)\big(u^2 v - u_n^2 v_n\big) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Estimate the integrals respectively.

$$\begin{aligned}
|I_1| &= \left|\left(I'(u_n, v_n) - I'(u, v)\right)\left((u_n, v_n) - (u, v)\right)\right| \\
&\leq \|I'(u_n, v_n)\| \cdot \left(\sup_n \|(u_n, v_n)\| + \|(u, v)\|\right) + \left|I'(u, v)\left((u_n, v_n) - (u, v)\right)\right| \to 0,
\end{aligned}$$

since $\|I'(u_n, v_n)\| = o(1)$ and $(u_n, v_n) \rightharpoonup (u, v)$ in $H^1 \times H^1$. For the second term,

$$\begin{aligned}
|I_2| &\leq \mu_1 \int \big|(u_n - u)\big(u_n^3 - u^3\big)\big| = \mu \int (u_n - u)^2 (u_n^2 + u_n u + u^2) \\
&\leq |u_n - u|_4^2 |u_n + u_n u + u^2|_2 \leq C |u_n - u|_4^2 \to 0.
\end{aligned}$$

This is valid due to $(u_n, v_n) \to (u, v)$ in $L^r \times L^r$ for any $r \in (2, 6)$. The same with I_3.

$$\begin{aligned}
|I_4| &\leq |\beta| \int \big|(u_n - u)\big(uv^2 - u_n v_n^2\big)\big| + \big|(v_n - v)\big(u^2 v - u_n^2 v_n\big)\big| \\
&\leq C\left(|u_n - u|_4 + |v_n - v|_4\right) \to 0.
\end{aligned}$$

This implies that $(u_n, v_n) \to (u, v)$ in $H^1 \times H^1$ and the (PS) condition holds for I.

3. Proof of Theorem 1.1

3.1. **Existence of a solution with negative energy.** Due to Lemma 2.4, we know that $I^a \cap B(0,R)$ is invariant under the flow $\eta^t(\cdot)$. We can find the specific definitions on a and R in appendix. Moreover, $\emptyset \neq I^0 \cap B(0,R) \subset I^a \cap B(0,R)$ is also invariant. Due to the form of energy I, it has a negative lower bound on $I^0 \cap B(0,r)$. We select a smooth function $\delta(\widetilde{f_1}, \widetilde{f_2}) \in I^0 \cap B(0,R)$, where $\widetilde{f_1}$ and $\widetilde{f_2}$ are nonnegative C_0^∞ functions such that $\int f_1\widetilde{f_1} + f_2\widetilde{f_2} > 0$ and $\delta > 0$ is small enough. Compute that

$$I(\delta(\widetilde{f_1}, \widetilde{f_2})) = \frac{\delta^2}{2}\left(\|\widetilde{f_1}\|^2 + \|\widetilde{f_2}\|^2\right) - \frac{\delta^4}{4}\int \mu_1\widetilde{f_1}^4 + \mu_2\widetilde{f_2}^4 + 2\beta\widetilde{f_1}^2\widetilde{f_1}^2$$
$$- \delta\int f_1\widetilde{f_1} + f_2\widetilde{f_2}.$$

Then $I(\delta(\widetilde{f_1}, \widetilde{f_2})) < 0$ when $\delta > 0$ is small. This implies that $\delta(\widetilde{f_1}, \widetilde{f_2}) \in I^0 \cap B(0,r)$ if $\delta > 0$ is smaller. Consider the trajectory $(u(t), v(t))$ of Problem (2.1) with $u(0) = \delta\widetilde{f_1}$ and $v(0) = \delta\widetilde{f_2}$. Now we apply a method in [13] to find a (PS) sequence. We firstly notice that the trajectory $(u(t), v(t))$ is of global existence since it is included in $I^0 \cap B(0,R)$, which is a $H^1 \times H^1$ bounded invariant set. The same argument also give that $\lim_{t\to\infty} I(u(t), v(t)) = -c < 0$ for some constant c. Using Lemma 2.2,

$$\int_0^\infty \int |u_t|^2 + |v_t|^2 dxdt = I(u(0), v(0)) + c < \infty.$$

Therefore, we can find a sequence of time $t_k \to \infty$ with $\int |u_t|^2(t_k) + |v_t|^2(t_k)dx \to 0$. Then we have $\|I'(u(t_k), v(t_k))\|_{H^{-1}\times H^{-1}} \to 0$. Indeed, if we select two functions $\phi, \psi \in H^1$,

$$\begin{aligned} I'(u(t_k), v(t_k))(\phi, \psi) &\leq |\Delta u - u + \mu_1 u^3 + \beta uv^2 + f_1|_2|\phi|_2 \\ &\quad + |\Delta v - v + \mu_2 v^3 + \beta vu^2 + f_2|_2|\psi|_2 \\ &\leq o(1)\left(\|\phi\| + \|\psi\|\right). \end{aligned}$$

Therefore we have a sequence $(u_k, v_k) \in H^1 \times H^1$ satisfies

- $I(u_k, v_k) \to -c$;
- $\|I'(u_k, v_k)\|_{H^1\times H^1} \to 0$;
- $u_k, v_k \geq 0$ in Ω.

The third entry follows from the monotonicity. Using the (PS) condition, the sequence tends to its limit (u_1, v_1), which is a solution to Problem (1.1) with negative energy and $u_1, v_1 \geq 0$ in Ω. The regularity and maximum principle of elliptic equations imply that $u_1, v_1 > 0$ in Ω.

3.2. **Existence of a solution with positive energy.** To obtain the second solution, we focus on the boundary $\partial\mathcal{A}$ of the domain of attraction $\mathcal{A}$. Select two smooth positive functions with disjoint supports, say ϕ and ψ. Notice that

$$I(t(\phi,\psi)) = \frac{t^2}{2}\left(\|\phi\|^2+\|\psi\|^2\right) - \frac{t^4}{4}\int \mu_1\phi^4+\mu_2\psi^4 - t\int f_1\phi+f_2\psi.$$

The assumption that $|f_1|_2+|f_2|_2$ is small ensure that there is a t_0 such that $I(t_0(\phi,\psi)) > \frac{S_4^{-2}}{16(\mu_1+\mu_1+2|\beta|+1)}$. Here the constant $S_p>0$ satisfies that $|u|_p \le S_p\|u\|$ for any $u\in H_0^1(\Omega)$. A detailed computation is in the Appendix. Therefore we can find a $t_1>0$ such that $t_1(\phi,\psi)\in\partial\mathcal{A}$. With a similar method, we can select a non-negative (PS) sequence from the flow $\eta^t(t_1(\phi,\psi))$, which will lead us to a positive solution $(u_2,v_2)\in\partial\mathcal{A}$. Finally, noticing that $\inf_{U\in\partial\mathcal{A}} I(U)>0$, we have $I(u_2,v_2)>0$.

3.3. **Uniqueness of the solution** (u_1,v_1)**.** To give the uniqueness of the small solution (v_1,v_2), we firstly give a L^∞ estimate for it. Notice that, due to the computation in Appendix, $\|u_1\|+\|v_1\|\to 0$ as $|f_1|_2+|f_2|_2\to 0$. Using the explicit form of Problem (1.1), we can estimate the H^2 norm of (u_1,v_1) as $|f_1|_2+|f_2|_2\to 0$:

$$|\Delta u_1|_2+|\Delta v_1|_2 \le C\left(|u_1|_2+|v_1|_2+\|u_1\|^3+\|v_1\|^3+|f_1|_2+|f_2|_2\right)\to 0,$$

which gives the fact that $|u_1|_\infty+|v_1|_\infty\to 0$ as $|f_1|_2+|f_2|_2\to 0$ due to the Sobolev embeddings in $\mathbb{R}^3$. Suppose there is another solution (u_1',v_1') with a small H^1-norm, then we have

$$\begin{aligned}
-\Delta(u_1-u_1')+(u_1-u_1') &= \mu_1(u_1-u_1')({u_1}^2+u_1u_1'+{u_1'}^2)\\
&\quad+\beta(u_1-u_1'){v_1}^2+\beta u_1'(v_1+v_1')(v_1-v_1')\\
-\Delta(v_1-v_1')+(v_1-v_1') &= \mu_2(v_1-v_1')({v_1}^2+v_1v_1'+{v_1'}^2)\\
&\quad+\beta(v_1-v_1'){u_1}^2+\beta v_1'(u_1+u_1')(u_1-u_1').
\end{aligned}$$

Multiplying them by u_1-u_1' and v_1-v_1' respectively, integrating and adding them up. We can do the following estimating

$$\begin{aligned}
\|u_1-u_1'\|^2+\|v_1-v_1'\|^2 \le & C\left(|u_1|_\infty+|v_1|_\infty+|u_1'|_\infty+|v_1'|_\infty\right)^2\\
&\cdot\left(\|u_1-u_1'\|^2+\|v_1-v_1'\|^2\right).
\end{aligned}$$

Notice that $C\left(|u_1|_\infty+|v_1|_\infty+|u_1'|_\infty+|v_1'|_\infty\right)\to 0$ as $|f_1|_2+|f_2|_2\to 0$, we conclude that $u_1=u_1'$ and $v_1=v_1'$.

3.4. **A remark.** If we do not require the positivity of solution, i.e. we only want to prove the existence, we may skip the step of monotonicity. Then we can conclude the following theorem with the rest of the process the same:

Theorem 3.1. *There is ε_0 such that if the non-zero functions f_1 and f_2 satisfy $0 < |f_1|_2 + |f_2|_2 \leq \varepsilon_0$, Problem (1.1) possesses two nontrivial solutions (u_1, v_1) and (u_2, v_2) with $I(u_1, v_1) > 0$ and $I(u_2, v_2) < 0$.*

4. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. A symmetry under permutation is the key to the multiplicity result. Readers can find more results about permutation symmetry in [15, 20, 30, 31]. We denote the permutation by $\sigma(u, v) = (v, u)$. Obviously, the Problem (1.1) and (2.1) is invariant under the permutation if we set $\mu_1 = \mu_2 =: \mu$ and $f_1 = f_2 =: f$, i.e. the problem are in the form of

$$\begin{cases} -\Delta u + u = \mu u^3 + \beta u v^2 + f & \text{in } \Omega, \\ -\Delta v + v = \mu v^3 + \beta v u^2 + f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

and of

$$\begin{cases} \partial_t u - \Delta u + u = \mu u^3 + \beta u v^2 + f & \text{in } \Omega, \\ \partial_t v - \Delta v + v = \mu v^3 + \beta v u^2 + f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{4.2}$$

We proceed the usual program of symmetric mountain pass theorem to this theorem in the symmetry of permutation. To do this, firstly, we need to define the genus generated by the permutation σ. The following definition and properties can be found in [15, 20, 30].

Definition 4.1. *Let E be a Banach space and a $\mathbb{Z}_2$ action on E given by σ. Denote the fixed point set by F_σ. For any closed subset $A \subset E \backslash F_\sigma$, the index $\gamma(A)$ is defined as the smallest number $m \in \mathbb{N}$ such that there exists a continuous mapping $h : A \to \mathbb{C}^m \backslash \{0\}$ with*

$$h(\sigma U) = e^{i\pi} h(U).$$

If there is no such a mapping, we set $\gamma(A) = \infty$.

Proposition 4.2. *The index γ satisfies the following properties:*

- *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$;*
- *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;*
- *if a closed set $\emptyset \neq A \subset E \backslash F_\sigma$ satisfies $A \cap \sigma(A) = \emptyset$, then $\gamma(A \cup \sigma(A)) = 1$;*
- *if $g : A \to E \backslash F_\sigma$ is continuous and satisfies $g(\sigma(U)) = \sigma g(U)$ for all $U \in A$, then*
$$\gamma(A) \leq \gamma(\overline{g(A)});$$
- *if $A \cap F_\sigma = \emptyset$ and $\gamma(A) > 1$, A is an infinite set;*
- *if A is compact and $\gamma(A) < \infty$, then there exist an open σ-invariant neighbourhood $\mathcal{N}$ of A such that $\gamma(A) = \gamma(\overline{N})$;*

- *if S is the boundary of a bounded neighbourhood of the origin in a m-dimensional complex linear space such that $e^{i\pi}U \in S$, and $\Psi : S \to E\backslash F_\sigma$ is continuous and satisfies for any $U \in S$, $\Psi(e^{i\pi}U) = \sigma\left(\Psi(U)\right)$, then $\gamma\left(\Psi(S)\right) \geq m$.*

We can find the proofs of these properties in [15, 20, 30], which are variants of the ones in [27, Section 7].

4.1. **Proof of the first Theorem 1.2.** Due to the routine of symmetric mountain pass theorem, we need to prove a certain invariant set, where we preform the minimax procedure, possesses infinitely large genus (c.f. [21, 22, 23, 20, 30]). [30] shows a way to solve this problem. In [30], the authors gave the explicit formula of the subset of invariant set using the advantage of Nehari manifolds, which is very hard to be done in our problem since it involves solving cubic equations if we want to process the same deduction. To overcome this difficulty, we replace the Nehari manifold (c.f. [20]) with the boundary of a domain of attraction. It is convenient to do so because in the case of $f_1 = f_2 \neq 0$, we do not need to consider about the semi-trivial solutions.

Before doing that, we need to check that there is no non-negative fixed point of the permutation σ on $\partial\mathcal{A}$. First let us check that there is no equilibrium point (u, v) with $u = v \geq 0$. Indeed, let $u = v$ in the energy I, we have

$$\begin{aligned} I(u,u) &= \|u\|^2 - \frac{\mu+\beta}{2}\int u^4 - 2\int fu \\ &= \frac{\mu+\beta}{2}\int u^4 - \int fu \leq 0. \end{aligned}$$

Combining the fact that $\inf_{U\in\partial\mathcal{A}} I(U) \geq c > 0$, we can conclude that $\partial\mathcal{A}$ contains no positive equilibrium point which is a fixed point of permutation σ. Then, if there is a non-negative fixed point $(u', u') \in \partial\mathcal{A}$, we consider the flow line $\eta^t(u', u')$. Using the same process in the last section, we can find an equilibrium point $(u'', u'') \in \partial\mathcal{A}$ and $u'' > 0$. This contradicts the deduction above. Therefore, there is no non-negative fixed point on $\partial\mathcal{A}$.

Next, we define the following functions. For any positive integer K, we do the following steps.

- First of all, we select an annulus $\Omega' \subset \Omega$ (without loss of generality, we can assume that the center of the annulus is at the origin of $\mathbb{R}^3$) and divide it into K disjoint sub-domains Ω_k with $\cup_{k=1}^K \overline{\Omega_k} = \overline{\Omega'}$;
- Secondly, for a fixed $k = 1, \ldots, K$, we define a smooth positive radial function $w_k(x,t) = w_k(r,t)$ with $r = |x|$ on $\Omega_k \times \mathbb{S}^1$ such that $w_k(x,t) \neq 0$ for any $t \in \mathbb{S}^1$ and $\mathrm{supp} w_k(x,t) \cap \mathrm{supp} w_k(x, t+\pi) = \emptyset$.

Now consider the K-dimensional complex Euclidean space $\mathbb{C}^K$ with its elements denoted by $z = (z_1, \ldots, z_K) = (\alpha_1 e^{i\theta_1}, \ldots, \alpha_K e^{i\theta_K})$. Define a mapping

$$\Phi(z) = \left(\sum_{k=1}^{K} \alpha_k w_k(x, \theta_k), \sum_{k=1}^{K} \alpha_k w_k(x, \theta_k + \pi) \right).$$

It is easy to see that the mapping Φ satisfies $\Phi(-z) = \sigma \circ \Phi(z)$ and Φ is a homeomorphism. Furthermore, any $U \in \Phi(\mathbb{C}^K)$ satisfies $\{tU | t \geq 0\} \cap \partial\mathcal{A} \neq \emptyset$. Using Borsuk's theorem, we have $\gamma\left(\Phi(\mathbb{C}^K) \cap \partial\mathcal{A}\right) \geq K$. Notice that any $U \in \Phi(\mathbb{C}^K)$ has two non-negative components. Denote

$$c_k = \inf_{A \in \Gamma_k} \sup_{U \in A} I(U),$$

where

$$\Gamma_k = \left\{ A \subset X \cap \partial\mathcal{A} \middle| \gamma(A) \geq k \right\}, X = \left\{ (u, v) \in H_0^1 \times H_0^1 \middle| u, v \geq 0 \right\}.$$

Now we claim that c_k's are critical values having positive critical points on these critical levels. Then the first part of the result of Theorem 1.2 follows. The proof of the above claim is a straightforward routine in variational arguments. We only sketch it here while refering [20, 27] for more details. Assume that the critical set K_{c_k} at the level c_k contains no positive critical points. Then for any $\varepsilon > 0$, we can find a positive number $T_1 > 0$ such that when we denote $\eta_0(U) := \eta^{T_1}(U)$ the mapping η_0 satisfies the following properties:

- $\eta_0 : X \cap \partial\mathcal{A} \to X \cap \partial\mathcal{A}$ is continuous;
- $\eta_0\left(X \cap \partial\mathcal{A} \cap I^{c_k+\varepsilon}\right) \subset X \cap \partial\mathcal{A} \cap I^{c_k-\varepsilon}$.

The constructing of deformation mapping is a traditional technique in the topics on modern variational methods. We refer [11, 27, 32] for a detailed computation. Readers can also find a variant with parabolic flow in [20], which is similar with the condition here. With the same $\varepsilon > 0$, we can find a set $A \in \Gamma_k$ such that

- $\sup_A I(U) < c_k + \varepsilon$;
- $\sup_{\eta_0(A)} I(U) < c_k - \varepsilon$;
- $\eta_0(A) \in \Gamma_k$.

These lead us to a contradiction with the definition of the value c_k. Therefore, the critical set K_{c_k} contains at least one positive critical point. If for $k, p > 0$, $c_k = \cdots = c_{k+p}$, using a routine in [27], we can address that $\gamma(K_{c_k} \cap X) \geq p + 1$. This requires the following property of η_0: for small $\varepsilon > 0$ an a neighbourhood $\mathcal{N}$ of K_c, we have

$$\eta_0(X \cap \partial\mathcal{A} \cap I^{c+\varepsilon} \backslash \mathcal{N}) \subset X \cap \partial\mathcal{A} \cap I^{c-\varepsilon}.$$

The proof of Theorem 1.2 completed.

□

4.2. **Proof of the second part of Theorem 1.2.** Here we only need to check that the simplex contains no fixed point of the permutation σ. The rest part of the proof ia the same with the one of the first part.

Now we check the following claim:

Claim 4.3. *For any $R > 0$, there is a $\delta_R > 0$ such that there is no positive fixed point in $\partial\mathcal{A}_\beta \cap I^R$ for Problem (2.1) with $\beta \in (-\mu, -\mu + \delta_R)$.*

Proof. In this part, we denote the energy and the boundary of attracting domain by I_β and $\mathcal{A}_\beta$ respectively, to show the influence of the parameter β on them. As in the last section, we firstly check that there is no fixed point which is also a critical point in $\partial\mathcal{A}_\beta \cap I^R$. We argue it by contradiction. Suppose there is a $R > 0$ such that for any $\beta \in (-\mu, -\mu + \delta)$, there is always a fixed critical point $u_\beta \in \partial\mathcal{A}_\beta \cap I^R$. Then

$$\begin{aligned} R \geq & \|u_\beta\|^2 - \frac{\mu+\beta}{2}\int u_\beta^4 - 2\int f u_\beta \\ = & \frac{1}{2}\|u_\beta\|^2 - \frac{5}{2}\int f u_\beta. \end{aligned}$$

This implies the sequence $(u_\beta)_\beta \subset H^1$ is bounded. On the other hand, since $\beta \to -\mu$, we have

$$|-\Delta u_\beta + u_\beta - f|_2 = \frac{\mu+\beta}{2}|u_\beta^3|_2 \leq C(\mu+\beta)\|u_\beta\|^3 \to 0.$$

This shows that $(u_\beta)_\beta \subset H^1$ is a (PS) sequence of functional $I_0(u) := \frac{1}{2}\|u\|^2 - \int fu$. Therefore we have a subsequence of $(u_\beta)_\beta$ (we still denote it by $(u_\beta)_\beta$) converges in H^1. Denote the limit by u. Notice that u is the only solution of $-\Delta u + u = f$ and $f \neq 0$. This gives that $u \neq 0$. Then, on one hand, due to the fact that $u_\beta \in \partial\mathcal{A}_\beta$,

$$I_\beta(u_\beta, u_\beta) = \|u_\beta\|^2 - \frac{\mu+\beta}{2}\int u_\beta^4 - 2\int f u_\beta \geq c > 0$$

for some fixed $c > 0$ independence on β. Using the convergence of $(u_\beta)_\beta$ in H^1, we know that $\|u\|^2 - 2\int fu \geq \frac{c}{2}$. On the other hand, since u is the only solution of $-\Delta u + u = f$ and $f \neq 0$, we have

$$\|u\|^2 - 2\int fu = -\|u\|^2 < 0.$$

We have a contradiction. Therefore, there is no positive fixed critical point in $\partial\mathcal{A}_\beta \cap I^R$. If there is a positive fixed point in $\partial\mathcal{A}_\beta \cap I^R$. Using the fact that all the flow line start on $\partial\mathcal{A}_\beta$ exists globally, the dissipation of the system and the maximum principle, we can find a positive fixed point in $\partial\mathcal{A}_\beta \cap I^R$, which contradicts the conclusion above.

$\square$

Now we have shown that there is no positive fixed point in $\partial\mathcal{A}_\beta \cap I^R$. Notice that for a fixed simplex $\Phi(\mathbb{C}^K)$, $\sup_{U\in\Phi(\mathbb{C}^K)\cap\partial\mathcal{A}_\beta} I(U)$ is independence on β. Then we can complete the proof with the same method as that in the last section.

5. Appendix: Proof of Lemma 2.4

In this part, we prove Lemma 2.4, which gives the existence of domain of attraction, under the condition of small perturbation for the following general case for Problem :

$$\begin{cases} \partial_t u_j - \Delta u_j + u_j = \mu_j u_j^3 + \sum_{i\neq j} \beta_{ij} u_j u_i^2 + f_j & in\ \Omega, \\ u_j = 0 & on\ \partial\Omega, \\ u_j(x,0) \in H_0^1(\Omega) \qquad \text{for } j = 1, \ldots, N. \end{cases} \tag{5.1}$$

In the following paragraph, we denote $F = (f_1, \cdots, f_N)$ and $U = (u_1, \cdots, u_N)$. The energy is a C^2 functional defined as

$$\begin{aligned} I(U) = \frac{1}{2}\sum_{j=1}^N \int |\nabla u_j|^2 + u_j^2 - \frac{1}{4}\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 - \sum_{j=1}^N \int f_j u_j \\ : \left(H_0^1(\Omega)\right)^N \to \mathbb{R}. \end{aligned}$$

Before we start computing in detail, we outline the strategy briefly. Usually, when we consider a domain of attraction for a system with homogeneous nonlinear term, we always give the existence of the domain of attraction by verifying the stability of the trivial solution (c.f. [1, 16, 20, 22, 23]) or at least of a stable solution (c.f. [25]). In this problem, there is no trivial solution, which causes an obstruction in the traditional method. To overcome this difficulty, we choose to look for a "stable domain" rather than one stable equilibrium point. Related stability theory can be found in [6, Section 2.12].

Denote

$$\begin{aligned} I(tU) = \frac{t^2}{2}\sum_{j=1}^N \int |\nabla u_j|^2 + u_j^2 - \frac{t^4}{4}\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 - t\sum_{j=1}^N \int f_j u_j \\ := g_U(t). \end{aligned}$$

We begin our construction by steps.

In the following computation, **Step.1-3** are estimates on energy levels. In **Step.4**, an estimate on lower bound of derivative is obtained. At last, we summary the result as a proof of Lemma 2.4 in **Step.5**.

Step.1. A Mountain-pass Level

In the following computation, we always assume that S_p is a constant that satisfies $|u|_p \leq S_p\|u\|$ for any $u \in H_0^1(\Omega)$ and $S_p > 1$. We note that this constant is not necessarily the best. Denote $B := \sum_{j=1}^N \mu_j + \sum_{i\neq j} |\beta_{ij}| + 1 > 0$. Then,

$$\begin{aligned} I\Big|_{\|U\|=\frac{S_4^{-2}}{\sqrt{NB}}} &\geq \frac{1}{2}\|U\|^2 - \frac{NB}{4}\sum_{j=1}^N \int u_j^4 - |F|_2 \cdot |U|_2\Big|_{\|U\|=\frac{S_4^{-2}}{\sqrt{NB}}} \\ &\geq \frac{1}{2}\|U\|^2 - \frac{BNS_4^4}{4}\|U\|^4 - |F|_2 \cdot \|U\|\Big|_{\|U\|=\frac{S_4^{-2}}{\sqrt{NB}}} \\ &\geq \frac{S_4^{-4}}{8BN}, \end{aligned}$$

where we assume that $|F|_2 \leq \frac{S_4^{-2}}{8\sqrt{BN}}$.

Step.2. Estimate on Monotonicity

In this step, we estimate the monotonicity of the energy I along different directions. In a word, we try to verify the following claim.

Claim 5.1. *There is a $\varepsilon_0 > 0$, when $|F|_2 \leq \varepsilon_0$, for any $\|U\| = 1$, $I(tU)$ is strictly increasing for $t \in \left[0, \frac{S_4^{-2}}{\sqrt{BN}}\right] \cap \{t > 0 : g_U(t) > 0\} \neq \emptyset$.*

In the following computation of this step, we always assume that $\|U\| = 1$.

According to the sign of $\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2$ and $\sum_{j=1}^N \int f_j u_j$, we discuss the monotonicity by dividing it into several cases.

(I). $\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 > 0$.

(i). $\sum_{j=1}^N \int f_j u_j > 0$.

Note that

$$g_U(t) = \frac{t^2}{2} - \frac{t^4}{4}\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 - t\sum_{j=1}^N \int f_j u_j$$

and

$$g_U'(t) = t - t^3 \sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 - \sum_{j=1}^N \int f_j u_j$$

are both C^1 functions. With an elementary computation, we find that g_U' achieves its maximum

$$g'_{MAX} = \frac{2^{\frac{3}{2}}}{3\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2\right)^{\frac{1}{2}}} - \sum_{j=1}^N \int f_j u_j$$

at $t_0 = \frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{\frac{1}{2}}}$ for $t \geq 0$. Furthermore, if we assume that $|F|_2 \leq \frac{S_4^{-2}}{\sqrt{3BN}}$, we have

$$\begin{aligned}\sum_{j=1}^N \int f_j u_j &\leq |F|_2|U|_2 \leq |F|_2\|U\| = |F|_2\\ &\leq \frac{S_4^{-2}}{\sqrt{3BN}} \leq \frac{1}{\left(3BN\sum_{j=1}^N S_4^4\|u_j\|^4\right)^{\frac{1}{2}}}\\ &\leq \frac{1}{\left(3BN\sum_{j=1}^N \int u_j^4\right)^{\frac{1}{2}}} \leq \frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{\frac{1}{2}}}.\end{aligned}$$

This gives that

$$\begin{aligned}g'_{MAX} = g'_U &\left(\frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{\frac{1}{2}}}\right)\\ &\geq \frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{\frac{1}{2}}} > 0.\end{aligned}$$

Notice that

$$\begin{aligned}I(tU)\Big|_{t=t_0} &= g_U\left(\frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{\frac{1}{2}}}\right)\\ &= \frac{1}{2}\cdot\frac{1}{3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2}\\ &\quad - \frac{1}{4}\cdot\frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{2}}\cdot\sum_{j=1}^N\int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\\ &\quad - \frac{1}{\left(3\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j}\beta_{ij}u_i^2u_j^2\right)^{\frac{1}{2}}}\cdot\sum_{j=1}^N f_j u_j\\ &\geq \frac{5}{36}T^2 - \frac{1}{\sqrt{3}}|F|_2 T,\end{aligned}$$

where

$$T = \frac{1}{\left(\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2\right)^{\frac{1}{2}}} \geq \frac{\sum_{j=1}^N \int |\nabla u_j|^2 + u_j^2}{\left(NB \sum_{j=1}^N \int u_j^4\right)^{\frac{1}{2}}}$$
$$\geq \frac{S_4^{-2}}{\sqrt{NB}}.$$

If we assume that $|F|_2 \leq \frac{5\sqrt{3}S_4^{-2}}{72\sqrt{NB}}$, we have

$$I(tU)\Big|_{t=t_0} \geq \frac{5}{36}T^2 - \frac{1}{\sqrt{3}}|F|_2 T$$
$$\geq \frac{5}{36}\left(\frac{S_4^{-2}}{\sqrt{NB}}\right)^2 - \frac{1}{\sqrt{3}}|F|_2 \frac{S_4^{-2}}{\sqrt{NB}} \geq \frac{5S_4^{-4}}{72BN} > 0.$$

Recall that we have $g_U\left(\frac{S_4^{-2}}{\sqrt{BN}}\right) > 0$ due to **Step.1**. In this part, we proved that $g_U'\left(\frac{S_4^{-2}}{\sqrt{BN}}\right) > 0$. Notice that the conditions $g_U(t) \geq 0$ and $t \leq t_0$ imply that $g_U'(t) > 0$. Using the basic knowledge of polynomial, we can conclude that $g_U(t)$ is strictly increasing for $t \in \left[0, \frac{S_4^{-2}}{\sqrt{BN}}\right] \cap \{t > 0 : g_U(t) > 0\} \neq \emptyset$.

(ii). $\sum_{j=1}^N \int f_j u_j = 0$.

In this case,

$$g_U(t) = \frac{t^2}{2} - \frac{t^4}{4}\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2.$$

Clearly, we have that $g_U(t)$ is strictly increasing for $t \in \left[0, \frac{S_4^{-2}}{\sqrt{BN}}\right] \cap \{t > 0 : g_U(t) > 0\} \neq \emptyset$.

(iii). $\sum_{j=1}^N \int f_j u_j < 0$.

Since $g_U'(t)$ has only one zero point $t_1 > t_0$ on positive half axis and $g_U'(t) > 0$ for $t \in [0, t_0]$, $g_U(t)$ is strictly increasing for $t \in \left[0, \frac{S_4^{-2}}{\sqrt{BN}}\right] \cap \{t > 0 : g_U(t) > 0\} \neq \emptyset$.

(II). $\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 = 0$.

(i). $\sum_{j=1}^N \int f_j u_j > 0$.

In this case,

$$g_U(t) = \frac{t^2}{2} - t\sum_{j=1}^N \int f_j u_j.$$

If we assume that $|F|_2 \le \frac{S_4^{-2}}{4\sqrt{BN}}$,

$$
\begin{aligned}
I(tU)\Big|_{t=\frac{S_4^{-2}}{\sqrt{BN}}} &= g_U\left(\frac{S_4^{-2}}{\sqrt{BN}}\right) \\
&\ge \frac{1}{2}\frac{S_4^{-4}}{BN} - \frac{S_4^{-2}}{\sqrt{BN}}|F|_2 \ge \frac{S_4^{-4}}{4BN}.
\end{aligned}
$$

Due to the elementary properties of polynomial, we have that $g_U(t)$ is strictly increasing for $t \in \left[0, \frac{S_4^{-2}}{\sqrt{BN}}\right] \cap \{t > 0 : g_U(t) > 0\} \neq \emptyset$.

(ii). $\sum_{j=1}^N \int f_j u_j \le 0$.

The conclusion holds clearly.

(III). $\sum_{j=1}^N \int \mu_j u_j^4 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j^2 < 0$.

(i). $\sum_{j=1}^N \int f_j u_j > 0$.

In this case, the function $g_U'(t)$ has only one zero point on the positive half axis. Check that, with a similar computation in **(i)** of **(II)**, we have that when $|F|_2 \le \frac{S_4^{-2}}{4\sqrt{BN}}$ $g_U(t)$ is strictly increasing for $t \in \left[0, \frac{S_4^{-2}}{\sqrt{BN}}\right] \cap \{t > 0 : g_U(t) > 0\} \neq \emptyset$.

(ii). $\sum_{j=1}^N \int f_j u_j \le 0$.

The conclusion holds obviously in this case.

Step.3. Decreasing Levels

Firstly, we give several computations.

(1). On the sphere of radius $\|U\| = \frac{S_4^{-2}}{\sqrt{BN}}$.

$$
\begin{aligned}
I\Big|_{\|U\|=\frac{S_4^{-2}}{\sqrt{BN}}} &\ge \frac{1}{2}\cdot\frac{S_4^{-4}}{BN} - \frac{1}{4}\cdot BN\cdot S_4^{-4}\frac{S_4^{-8}}{B^2N^2} - |F|_2\cdot\frac{S_4^{-2}}{\sqrt{BN}} \\
&\ge \frac{S_4^{-4}}{8BN},
\end{aligned}
$$

with $|F|_2 \le \frac{S_4^{-2}}{8\sqrt{BN}}$.

(2). On the sphere of radius $\|U\| = \frac{S_4^{-2}}{4\sqrt{BN}}$.

$$
\begin{aligned}
I\Big|_{\|U\|=\frac{S_4^{-2}}{4\sqrt{BN}}} &\le \frac{1}{2}\left(\frac{S_4^{-2}}{4\sqrt{BN}}\right)^2 + \frac{1}{4}\cdot BNS_4^4\left(\frac{S_4^{-2}}{4\sqrt{BN}}\right)^4 + |F|_2\cdot\frac{S_4^{-2}}{4\sqrt{BN}} \\
&\le \frac{S_4^{-4}}{16BN}
\end{aligned}
$$

if we set $|F|_2 \leq \frac{S_4^{-2}}{64\sqrt{BN}}$. Under the same condition, we have $I\Big|_{\|U\|=\frac{S_4^{-2}}{4\sqrt{BN}}} \geq \frac{S_4^{-4}}{128BN}$.

(3). On the sphere of radius $\|U\| = \frac{S_4^{-2}}{16\sqrt{BN}}$.

$$\begin{aligned}
I\Big|_{\|U\|=\frac{S_4^{-2}}{16\sqrt{BN}}} &\leq \frac{1}{2}\left(\frac{S_4^{-2}}{16\sqrt{BN}}\right)^2 + \frac{1}{4}\cdot BNS_4^4\left(\frac{S_4^{-2}}{16\sqrt{BN}}\right)^4 + |F|_2\cdot\frac{S_4^{-2}}{16\sqrt{BN}}\\
&\leq \frac{S_4^{-4}}{2^8BN} < \frac{S_4^{-4}}{128BN}
\end{aligned}$$

if we set $|F|_2 \leq \frac{S_4^{-2}}{2^{13}\sqrt{BN}}$. Under the same condition, we have $I\Big|_{\|U\|=\frac{S_4^{-2}}{16\sqrt{BN}}} \geq \frac{2^7-1}{2^{16}}\cdot\frac{S_4^{-4}}{BN}$.

(4). On the sphere of radius $\|U\| = \frac{S_4^{-2}}{2^6\sqrt{BN}}$.

$$\begin{aligned}
I\Big|_{\|U\|=\frac{S_4^{-2}}{2^6\sqrt{BN}}} &\leq \frac{1}{2}\left(\frac{S_4^{-2}}{2^6\sqrt{BN}}\right)^2 + \frac{1}{4}\cdot BNS_4^4\left(\frac{S_4^{-2}}{2^6\sqrt{BN}}\right)^4 + |F|_2\cdot\frac{S_4^{-2}}{2^6\sqrt{BN}}\\
&\leq \frac{S_4^{-4}}{2^{10}BN} < \frac{2^7-1}{2^{16}}\cdot\frac{S_4^{-4}}{BN}
\end{aligned}$$

if we set $|F|_2 \leq \frac{S_4^{-2}}{2^{20}\sqrt{BN}}$. Under the same condition, we have $I\Big|_{\|U\|=\frac{S_4^{-2}}{2^6\sqrt{BN}}} \geq \frac{2^{12}-1}{2^{25}}\cdot\frac{S_4^{-4}}{BN}$.

(5). On the sphere of radius $\|U\| = \frac{S_4^{-2}}{2^8\sqrt{BN}}$.

$$\begin{aligned}
I\Big|_{\|U\|=\frac{S_4^{-2}}{2^8\sqrt{BN}}} &\leq \frac{1}{2}\left(\frac{S_4^{-2}}{2^8\sqrt{BN}}\right)^2 + \frac{1}{4}\cdot BNS_4^4\left(\frac{S_4^{-2}}{2^8\sqrt{BN}}\right)^4 + |F|_2\cdot\frac{S_4^{-2}}{2^8\sqrt{BN}}\\
&\leq \frac{S_4^{-4}}{2^{15}BN} < \frac{2^{12}-1}{2^{25}}\cdot\frac{S_4^{-4}}{BN}
\end{aligned}$$

if we set $|F|_2 \leq \frac{S_4^{-2}}{2^{24}\sqrt{BN}}$.

(6). Conclusion.

Denote $a = \frac{2^{12}-1}{2^{25}}\cdot\frac{S_4^{-4}}{BN}$, $b = \frac{S_4^{-4}}{128BN}$ and $R_0 = \frac{S_4^{-2}}{2^8\sqrt{BN}}$, $R_1 = \frac{S_4^{-2}}{2^6\sqrt{BN}}$, $R_2 = \frac{S_4^{-2}}{16\sqrt{BN}}$, $R_3 = \frac{S_4^{-2}}{4\sqrt{BN}}$, $R = \frac{S_4^{-2}}{\sqrt{BN}}$. Using Claim 5.1, we have the following inclusion:

$$B(0,R_0) \subset I^a \cap B(0,R) \subset B(0,R_1) \subset B(0,R_2) \subset I^b \cap B(0,R) \subset B(0,R_3). \tag{5.2}$$

Notice that since $I^a \cap B(0,R)$ is a connected component of I^a due to the shape of I^a, it is flow invariant. The same argument holds for $I^b \cap B(0,R)$.

Step.4. Estimate on the Derivatives

Inclusion (5.2) implies that

$$I_a^b \cap B(0,R) \subset \overline{B(0,R_3)} \backslash B(0,R_0).$$

Denote

$$G(U) = \sum_{j=1}^{N} \left| \Delta u_j - u_j + \sum_{j=1}^{N} \mu_j u_j^3 + \sum_{i\neq j} \beta_{ij} u_i^2 u_j + f_j \right|_2 .$$

In order to prove that there is a $\delta > 0$ such that $G(U) \geq \delta$ on $I_a^b \cap B(0,R_0)$, we prove the same inequality on $\overline{B(0,R_3)} \backslash B(0,R_0)$. For any $U \in \overline{B(0,R_3)} \backslash B(0,R_0)$,

$$\begin{aligned} G(U) \geq \frac{I'(U)U}{|U|_2} &\geq \frac{1}{\|U\|} \left(\sum_{j=1}^{N} \int |\nabla u_j|^2 + u_j^2 - BN \sum_{j=1}^{N} \int |u_j|^4 - |F|_2 \|U\| \right) \\ &= \|U\| - BNS_4^4 \|U\|^3 - |F|_2 \geq \frac{S_4^{-2}}{2^9 \sqrt{BN}} =: \delta > 0 \end{aligned}$$

if $|F|_2 \leq \frac{S_4^{-2}}{2^{10}\sqrt{BN}}$. This gives Lemma 2.4 immediately.

Step.5. Proof of Lemma 2.4

Entry (1) of Lemma 2.4 follows from (6) of **Step.3**. Estimate on the positive lower bound in **Step.4** gives (2) of Lemma 2.4. To prove (3) of Lemma 2.4, we notice that it is sufficient to let $\delta_0 = \frac{R_2 - R_1}{100} > 0$ in (6) of **Step.3**.

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