ON 5- AND 10-DISSECTIONS FOR SOME INFINITE PRODUCTS

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Abstract. Quite recently, Xia and Zhao established the 10-dissections for Hirschhorn's two infinite q-series products by two MAPLE packages and modular forms. Utilizing the Jacobi triple product identity, we not only establish the 10-dissections for two infinite q-series products, introduced by Baruah and Kaur, but give an elementary proof of the 10-dissections due to Xia and Zhao. Moreover, we obtain the 5-dissections for four quotients of infinite q-series products related to the Rogers–Ramanujan functions. Using these dissections, the coefficients in these series expansions have periodic sign patterns with some few exceptions.

1. Introduction

Recently, arithmetic properties of coefficients in infinite q-series products have been extensively investigated. These research mainly focus on two aspects. Some results are about vanishing coefficients of the arithmetic progressions in q-series expansions [1, 3, 9, 15, 21, 25, 29]. The other concentrate on periodic sign patterns of coefficients in q-series expansions [2, 6, 7, 10, 14, 18, 20, 22, 23, 27, 28]. In a private communication with Ernest X. W. Xia, Hirschhorn [17] studied vanishing coefficients of the arithmetic progressions in a new type of q-series expansions, the forms of which are different from these of the aforementioned literature. Define

$$\sum_{n=0}^{\infty} a(n)q^n = (-q, -q^4; q^5)_\infty (q, q^9; q^{10})^3_\infty, \tag{1.1}$$

$$\sum_{n=0}^{\infty} b(n)q^n = (-q^2, -q^3; q^5)_\infty (q^3, q^7; q^{10})^3_\infty. \tag{1.2}$$

Here and in what follows, we adopt the following customary q-series notation:

$$(a; q)_n = \prod_{i=0}^{n-1}(1 - aq^i), \qquad n \in \mathbb{N} \cup \{\infty\},$$

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty, \quad \text{for} \quad |q| < 1.$$

Hirschhorn [17] proved that for any $n \geq 0$,

$$a(5n + 2) = a(5n + 4) = 0,$$

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$$b(5n+1)=b(5n+4)=0.$$

The author [24], Baruah and Kaur [4] as well as Dou and Xiao [12] subsequently considered some variants of (1.1) and (1.2) and obtained some comparable results. Baruah and Kaur [4] also established some interlinked identities between $a(n)$ and $b(n)$. Quite recently, the author and Xia [26] gave another proof of these interlinked identities among other things. For instance, for any $n\geq 0$,

$$a(5n+1)=b(5n+3).$$

Motivated by these works, Xia and Zhao [30] further investigated (1.1) and (1.2). Using two MAPLE packages due to Frye and Garvan [13] and modular forms, they obtained the following two 10-dissections.

Theorem 1.1. *We have*

$$(-q,-q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3=\sum_{r=0}^{9}q^rA_r(q^{10}),$$

where

$$A_0(q)=\frac{(q^2;q^2)_\infty^6(q^{20};q^{20})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty^4(q^4;q^4)_\infty^3(q^2,q^3;q^5)_\infty},\tag{1.3}$$

$$A_1(q)=\frac{-2(q^4;q^4)_\infty^2}{(q;q)_\infty^2(q^4,q^6;q^{10})_\infty^2(q^2,q^{18};q^{20})_\infty},\tag{1.4}$$

$$A_3(q)=\frac{2(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q,q^9;q^{10})_\infty^2}{(q;q)_\infty^3(q^2,q^{18};q^{20})_\infty},\tag{1.5}$$

$$A_5(q)=\frac{-2(q^4;q^4)_\infty^2}{(q;q)_\infty^2(q^2,q^8;q^{10})_\infty^2(q^6,q^{14};q^{20})_\infty},\tag{1.6}$$

$$A_6(q)=\frac{(q^2;q^2)_\infty^6(q^{20};q^{20})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^4(q^4;q^4)_\infty^3(q,q^4;q^5)_\infty},\tag{1.7}$$

$$A_8(q)=\frac{-(q^2;q^2)_\infty^6(q^5;q^5)_\infty(q,q^4;q^5)_\infty}{(q;q)_\infty^5(q^4;q^4)_\infty^2(q^8,q^{12};q^{20})_\infty},\tag{1.8}$$

$$A_2(q)=A_4(q)=A_7(q)=A_9(q)=0.\tag{1.9}$$

Theorem 1.2. *We have*

$$(-q^2,-q^3;q^5)_\infty(q^3,q^7;q^{10})_\infty^3=\sum_{r=0}^{9}q^rB_r(q^{10}),$$

where

$$B_0(q)=\frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty(q^2,q^3;q^5)_\infty}{(q;q)_\infty^5(q^4;q^4)_\infty^2(q^4,q^{16};q^{20})_\infty},\tag{1.10}$$

$$B_2(q)=\frac{(q^2;q^2)_\infty^6(q^{20};q^{20})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty^4(q^4;q^4)_\infty^3(q^2,q^3;q^5)_\infty},\tag{1.11}$$

$$B_3(q) = \frac{-2(q^4;q^4)_\infty^2}{(q;q)_\infty^2(q^4,q^6;q^{10})_\infty^2(q^2,q^{18};q^{20})_\infty}, \tag{1.12}$$

$$B_5(q) = \frac{-2(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q^3,q^7;q^{10})_\infty^2}{(q;q)_\infty^3(q^6,q^{14};q^{20})_\infty}, \tag{1.13}$$

$$B_7(q) = \frac{-2(q^4;q^4)_\infty^2}{(q;q)_\infty^2(q^2,q^8;q^{10})_\infty^2(q^6,q^{14};q^{20})_\infty}, \tag{1.14}$$

$$B_8(q) = \frac{(q^2;q^2)_\infty^6(q^{20};q^{20})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^4(q^4;q^4)_\infty^3(q,q^4;q^5)_\infty}, \tag{1.15}$$

$$B_1(q) = B_4(q) = B_6(q) = B_9(q) = 0. \tag{1.16}$$

In [4], Baruah and Kaur considered vanishing coefficients of the arithmetic progressions in following two q-series expansions closely related to (1.1) and (1.2). Define

$$\sum_{n=0}^{\infty} c(n)q^n = (q,q^4;q^5)_\infty(-q,-q^9;q^{10})_\infty^3, \tag{1.17}$$

$$\sum_{n=0}^{\infty} d(n)q^n = (q^2,q^3;q^5)_\infty(-q^3,-q^7;q^{10})_\infty^3. \tag{1.18}$$

The 10-dissections (1.3)–(1.16) prompt us to look for the 10-dissections for q-series expansions (1.17) and (1.18). In the present paper, we not only establish the 10-dissections (see (1.19)–(1.36)) for q-series expansions (1.17) and (1.18), but give an elementary proof of (1.3)–(1.16). Unlike the main techniques used of [26], the main ingredient for proofs of (1.3)–(1.16) and (1.19)–(1.36) is only the Jacobi triple product identity. Moreover, we also establish the 5-dissections for four quotients of infinite q-series products related to the Rogers–Ramanujan functions, two of which were introduced by the author [25]. Define the sequences $\{e(n)\}$, $\{f(n)\}$, $\{g(n)\}$ and $\{h(n)\}$ by

$$\sum_{n=0}^{\infty} e(n)q^n = \frac{(q,q^4;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty^3}, \qquad \sum_{n=0}^{\infty} f(n)q^n = \frac{(q^2,q^3;q^5)_\infty^2}{(q,q^4;q^5)_\infty^3},$$

$$\sum_{n=0}^{\infty} g(n)q^n = \frac{(q^2,q^3;q^5)_\infty^3}{(q,q^4;q^5)_\infty^2}, \qquad \sum_{n=0}^{\infty} h(n)q^n = \frac{(q,q^4;q^5)_\infty^3}{(q^2,q^3;q^5)_\infty^2}.$$

Theorem 1.3. *We have*

$$(q,q^4;q^5)_\infty(-q,-q^9;q^{10})_\infty^3 = \sum_{r=0}^{9} q^r C_r(q^{10}),$$

where

$$C_0(q) = \frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty(q^{20};q^{20})_\infty(q^2,q^8;q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3(q,q^4;q^5)_\infty(q^2,q^{18};q^{20})_\infty}, \tag{1.19}$$

$$C_1(q) = \frac{2(q^4;q^4)_\infty^3(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty(q,q^4;q^5)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^4,q^{16};q^{20})_\infty}, \tag{1.20}$$

$$C_3(q) = \frac{-2(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty^3(q^4,q^6;q^{10})_\infty^2}, \tag{1.21}$$

$$C_4(q) = \frac{-2(q^2;q^2)_\infty^6(q^{10};q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q^4,q^{16};q^{20})_\infty}, \tag{1.22}$$

$$C_5(q) = \frac{-2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty(q,q^4;q^5)_\infty^2}, \tag{1.23}$$

$$C_6(q) = \frac{-(q^2;q^2)_\infty^7(q^5;q^5)_\infty^2(q,q^4;q^5)_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty^6(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^2,q^8;q^{10})_\infty^2}, \tag{1.24}$$

$$C_8(q) = \frac{(q^2;q^2)_\infty^7(q^5;q^5)_\infty^2(q^{20};q^{20})_\infty(q^2,q^3;q^5)_\infty}{(q;q)_\infty^6(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^6,q^{14};q^{20})_\infty}, \tag{1.25}$$

$$C_9(q) = \frac{4(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^4(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty^2(q^5;q^5)_\infty^2(q,q^9;q^{10})_\infty}, \tag{1.26}$$

$$C_2(q) = C_7(q) = 0. \tag{1.27}$$

Theorem 1.4. *We have*

$$(q^2,q^3;q^5)_\infty(-q^3,-q^7;q^{10})_\infty^3 = \sum_{r=0}^{9} q^r D_r(q^{10}),$$

where

$$D_0(q) = \frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty^2(q,q^4;q^5)_\infty(q^6,q^{14};q^{20})_\infty}{(q;q)_\infty^6(q^4;q^4)_\infty^2}, \tag{1.28}$$

$$D_2(q) = \frac{-(q^2;q^2)_\infty^6(q^5;q^5)_\infty(q^{20};q^{20})_\infty(q^2,q^8;q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3(q,q^4;q^5)_\infty(q^2,q^{18};q^{20})_\infty}, \tag{1.29}$$

$$D_3(q) = \frac{2(q^4;q^4)_\infty^3(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty(q,q^4;q^5)_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty(q^{20};q^{20})_\infty(q^4,q^{16};q^{20})_\infty}, \tag{1.30}$$

$$D_4(q) = \frac{2q(q^2;q^2)_\infty^6(q^{10};q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q^8,q^{12};q^{20})_\infty}, \tag{1.31}$$

$$D_5(q) = \frac{-2(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^3(q^2,q^8;q^{10})_\infty^2}, \tag{1.32}$$

$$D_7(q) = \frac{2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty(q,q^4;q^5)_\infty^2}, \tag{1.33}$$

$$D_8(q) = \frac{-(q^2;q^2)_\infty^7(q^5;q^5)_\infty^2(q,q^4;q^5)_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty^6(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^2,q^8;q^{10})_\infty^2}, \tag{1.34}$$

$$D_9(q) = \frac{-4(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^4(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty^2(q^5;q^5)_\infty^2(q^3,q^7;q^{10})_\infty}, \tag{1.35}$$

$$D_1(q) = D_6(q) = 0. \tag{1.36}$$

The identities (1.27) and (1.36) were established by Baruah and Kaur [4, Theorem 1.12].

Theorem 1.5. *We have*

$$\frac{(q,q^4;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty^3}=\sum_{r=0}^{4} q^r E_r(q^5),$$

where

$$E_0(q)=\frac{1}{(q,q^4;q^5)_\infty^4(q^2,q^3;q^5)_\infty}, \tag{1.37}$$

$$E_1(q)=\frac{-2}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^2}, \tag{1.38}$$

$$E_2(q)=\frac{4}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^3}, \tag{1.39}$$

$$E_3(q)=\frac{-3}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^4}, \tag{1.40}$$

$$E_4(q)=\frac{1}{(q^2,q^3;q^5)_\infty^5}. \tag{1.41}$$

Theorem 1.6. *We have*

$$\frac{(q^2,q^3;q^5)_\infty^2}{(q,q^4;q^5)_\infty^3}=\sum_{r=0}^{4} q^r F_r(q^5),$$

where

$$F_0(q)=\frac{1}{(q,q^4;q^5)_\infty^5}, \tag{1.42}$$

$$F_1(q)=\frac{3}{(q,q^4;q^5)_\infty^4(q^2,q^3;q^5)_\infty}, \tag{1.43}$$

$$F_2(q)=\frac{4}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^2}, \tag{1.44}$$

$$F_3(q)=\frac{2}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^3}, \tag{1.45}$$

$$F_4(q)=\frac{1}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^4}. \tag{1.46}$$

Theorem 1.7. *We have*

$$\frac{(q^2,q^3;q^5)_\infty^3}{(q,q^4;q^5)_\infty^2}=\sum_{r=0}^{4} q^r G_r(q^5),$$

where

$$G_0(q)=\frac{-3q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^5}+\frac{1}{(q,q^4;q^5)_\infty^7}, \tag{1.47}$$

$$G_1(q)=\frac{-q}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^6}+\frac{2}{(q,q^4;q^5)_\infty^6(q^2,q^3;q^5)_\infty}, \tag{1.48}$$

$$G_3(q) = \frac{-5}{(q,q^4;q^5)_\infty^4(q^2,q^3;q^5)_\infty^3}, \tag{1.49}$$

$$G_4(q) = \frac{-5}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^4}, \tag{1.50}$$

$$G_2(q) = 0. \tag{1.51}$$

Theorem 1.8. *We have*

$$\frac{(q,q^4;q^5)_\infty^3}{(q^2,q^3;q^5)_\infty^2} = \sum_{r=0}^{4} q^r H_r(q^5),$$

where

$$H_0(q) = \frac{1}{(q,q^4;q^5)_\infty^6(q^2,q^3;q^5)_\infty} + \frac{2q}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^6}, \tag{1.52}$$

$$H_1(q) = \frac{-3}{(q,q^4;q^5)_\infty^5(q^2,q^3;q^5)_\infty^2} - \frac{q}{(q^2,q^3;q^5)_\infty^7}, \tag{1.53}$$

$$H_2(q) = \frac{5}{(q,q^4;q^5)_\infty^4(q^2,q^3;q^5)_\infty^3}, \tag{1.54}$$

$$H_3(q) = \frac{-5}{(q,q^4;q^5)_\infty^3(q^2,q^3;q^5)_\infty^4}, \tag{1.55}$$

$$H_4(q) = 0. \tag{1.56}$$

The identities (1.51) and (1.56) were proved in [25, Eqs. (1.7) and (1.8)].

According to Theorem 1.1 and Theorem 1.2, Xia and Zhao [30] obtained that the sequences $\{a(n)\}$ and $\{b(n)\}$ have periodic sign patterns for any $n \geq 0$.

Corollary 1.9. *For any $n \geq 0$, we have*

$$a(n) \begin{cases} > 0, & \text{if } n \equiv 0,3,6 \pmod{10}, \\ < 0, & \text{if } n \equiv 1,5,8 \pmod{10}, \\ = 0, & \text{if } n \equiv 2,4 \pmod{5}, \end{cases} \tag{1.57}$$

$$b(n) \begin{cases} > 0, & \text{if } n \equiv 0,2,8 \pmod{10}, \\ < 0, & \text{if } n \equiv 3,5,7 \pmod{10}, \\ = 0, & \text{if } n \equiv 1,4 \pmod{5}. \end{cases} \tag{1.58}$$

In view of (1.19)–(1.56), we obtain the following analogous results.

Corollary 1.10. *For any $n \geq 0$, we have*

$$c(n) \begin{cases} > 0, & \text{if } n \equiv 0,1,8,9 \pmod{10}, \\ < 0, & \text{if } n \equiv 3,4,5,6 \pmod{10}, \\ = 0, & \text{if } n \equiv 2,7 \pmod{10}, \end{cases} \tag{1.59}$$

$$d(n)\begin{cases} >0, & \textit{if } \ n\equiv 0,3,4,7 \pmod{10},\\ <0, & \textit{if } \ n\equiv 2,5,8,9 \pmod{10},\\ =0, & \textit{if } \ n\equiv 1,6 \pmod{10},\end{cases} \tag{1.60}$$

except for $d(4)=0$.

Corollary 1.11. *For any* $n\geq 0$*, we have*

$$e(n)\begin{cases} >0, & \textit{if } \ n\equiv 0,2,4 \pmod{5},\\ <0, & \textit{if } \ n\equiv 1,3 \pmod{5},\end{cases} \tag{1.61}$$

$$f(n)>0, \tag{1.62}$$

except for $e(9)=0$.

Corollary 1.12. *For any* $n\geq 0$*, we have*

$$g(n)\begin{cases} >0, & \textit{if } \ n\equiv 0,1 \pmod{5},\\ <0, & \textit{if } \ n\equiv 3,4 \pmod{5},\\ =0, & \textit{if } \ n\equiv 2 \pmod{5},\end{cases} \tag{1.63}$$

$$h(n)\begin{cases} >0, & \textit{if } \ n\equiv 0,2 \pmod{5},\\ <0, & \textit{if } \ n\equiv 1,3 \pmod{5},\\ =0, & \textit{if } \ n\equiv 4 \pmod{5}.\end{cases} \tag{1.64}$$

The remainder of this paper is organized as follows. In Sect. 2, we collect some necessary notation and some identities related to the Rogers–Ramanujan functions. In Sect. 3, we first prove Theorem 1.3 and Theorem 1.4, then sketch the elementary proofs of Theorem 1.1 and Theorem 1.2. The proofs of Theorems 1.5–1.8 and Corollaries 1.10–1.12 are presented in Sect. 4. In the last section, we conclude with two remarks.

2. Preliminary results

We first collect some necessary notation and identities which are needed to prove the main results of this paper.

Let $k>0, l\geq 0$ be integers and let

$$W(q)=\sum_{n=0}^{\infty} w(n)q^n$$

be a formal power series. Define an operator $H_{k,l}$ by

$$H_{k,l}\left(W(q)\right)=\sum_{n=0}^{\infty} w(kn+l)q^{kn+l}.$$

For two given formal power series

$$W_1(q)=\sum_{n=0}^{\infty} w_1(n)q^n \qquad \text{and} \qquad W_2(q)=\sum_{n=0}^{\infty} w_2(n)q^n,$$

if $w_1(n) \geq w_2(n)$ (resp. $w_1(n) > w_2(n)$) holds for all n, then we denote $W_1(q) \succeq W_2(q)$ (resp. $W_1(q) \succ W_2(q)$).

Ramanujan's general theta function is defined by [11, p. 8]

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where $|ab| < 1$. The function $f(a,b)$ enjoys the well-known Jacobi triple product identity [5, p. 35, Entry 19]:

$$f(a,b) = (-a, -b, ab; ab)_\infty. \tag{2.1}$$

The two important special cases of (2.1) are [5, p. 36, Entry 22 (i) and (ii)]

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \tag{2.2}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{2.3}$$

It follows easily from (2.3) that

$$\psi(-q) = \frac{(q;q)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty}. \tag{2.4}$$

The identities (2.1)–(2.3) will be frequently used without explicit mention in the sequel.

The celebrated Rogers–Ramanujan identities [16, Eqs. (17.4.2) and (17.4.3)] state that

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \tag{2.5}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}. \tag{2.6}$$

Here $G(q)$ and $H(q)$ are known as the Rogers–Ramanujan functions. The Rogers–Ramanujan continued fraction can be interpreted as [16, Eq. (8.1.2)]

$$R(q) = \frac{H(q)}{G(q)} = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}. \tag{2.7}$$

Finally, we require the following identities related to (2.5)–(2.7).

Corollary 2.1. *We have*

$$\begin{aligned}\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q;q)_\infty^6} \bigg(&\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \\ &- 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4 \bigg),\end{aligned} \tag{2.8}$$

$$\frac{R(q)^5}{R(q^5)} = \frac{1-2qR(q^5)+4q^2R(q^5)^2-3q^3R(q^5)^3+R(q^5)^4}{1+3qR(q^5)+4q^2R(q^5)^2+2q^3R(q^5)^3+R(q^5)^4}, \tag{2.9}$$

$$G(q)^{11}H(q)-q^2G(q)H(q)^{11} = 1+11q\left(G(q)H(q)\right)^6. \tag{2.10}$$

Proof. The identity (2.8) appears in [16, Eq. (8.4.4)], (2.9) comes from [16, Eq. (9.2.13)] and (2.10) is from [16, Eq. (17.4.5)]. □

3. Proofs of Theorems 1.1–1.4

We are ready to prove Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3. We first prove (1.19), (1.22), (1.24) and (1.25).

We start with

$$\begin{aligned}
&(q,q^4;q^5)_\infty(-q,-q^9;q^{10})_\infty^3 \\
&\quad = (q,q^4,q^6,q^9;q^{10})_\infty(-q,-q^9;q^{10})_\infty^3 \\
&\quad = (q^4,q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty(-q,-q^9;q^{10})_\infty^2 \\
&\quad = \frac{(q^4,q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q^{10};q^{10})_\infty^2}\left(\sum_{m,n=-\infty}^{\infty} q^{5m^2+4m+5n^2+4n}\right).
\end{aligned}$$

If m and n have the same parity, take $(m,n)=(r+s,r-s)$. If m and n have the opposite parity, let $(m,n)=(r+s-1,r-s)$, where r and s are integers. Therefore, we obtain

$$\begin{aligned}
&(q,q^4;q^5)_\infty(-q,-q^9;q^{10})_\infty^3 \\
&\quad = \frac{(q^4,q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q^{10};q^{10})_\infty^2} \\
&\qquad \times\left(\sum_{r,s=-\infty}^{\infty} q^{5(r+s)^2+4(r+s)+5(r-s)^2+4(r-s)}\right. \\
&\qquad \left. + \sum_{r,s=-\infty}^{\infty} q^{5(r+s-1)^2+4(r+s-1)+5(r-s)^2+4(r-s)}\right) \\
&\quad = \frac{(q^4,q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q^{10};q^{10})_\infty^2}\big(\varphi(q^{10})(-q^2,-q^{18},q^{20};q^{20})_\infty \\
&\qquad + 2q\psi(q^{20})(-q^8,-q^{12},q^{20};q^{20})_\infty\big),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} c(2n)q^n &= \frac{\varphi(q^5)(q^{10};q^{10})_\infty}{(q^5;q^5)^2}(q^2,q^3;q^5)_\infty(q,q^9;q^{10})_\infty(-q,-q^9;q^{10})_\infty \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty^2}(q^2,q^3;q^5)_\infty(q^2,q^{18};q^{20})_\infty
\end{aligned} \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} c(2n+1)q^n = \frac{2\psi(q^{10})(q^{10};q^{10})_\infty}{(q^5;q^5)^2}(q^2,q^3;q^5)_\infty(q,q^9;q^{10})_\infty(-q^4,-q^6;q^{10})_\infty$$
$$= \frac{2(q^{20};q^{20})_\infty^2}{(q^5;q^5)^2}(q^2,q^3;q^5)_\infty(q,q^9;q^{10})_\infty(-q^4,-q^6;q^{10})_\infty. \tag{3.2}$$

Notice that

$$(q^2,q^3;q^5)_\infty = \frac{1}{(q^5;q^5)_\infty}\sum_{m=-\infty}^{\infty}(-1)^m q^{(5m^2+m)/2}$$
$$= \frac{1}{(q^5;q^5)_\infty}\left(\sum_{m=-\infty}^{\infty} q^{10m^2+m} - q^2\sum_{m=-\infty}^{\infty} q^{10m^2+9m}\right). \tag{3.3}$$

Combining (3.1) and (3.3), we find that

$$\sum_{n=0}^{\infty} c(2n)q^n = \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^3}(q^2,q^{18},q^{20};q^{20})_\infty$$
$$\times\left(\sum_{m=-\infty}^{\infty} q^{10m^2+m} - q^2\sum_{m=-\infty}^{\infty} q^{10m^2+9m}\right)$$
$$= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^3}\left(\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+8m+10n^2+n}\right.$$
$$\left. - q^2\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+8m+10n^2+9n}\right)$$
$$= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^3}(S_1-S_2), \tag{3.4}$$

where

$$S_1 = \sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+8m+10n^2+n}, \tag{3.5}$$

$$S_2 = q^2\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+8m+10n^2+9n}. \tag{3.6}$$

In S_1, suppose that $8m+n\equiv 0 \pmod 5$, then $-2m+n\equiv 0 \pmod 5$. Equivalently, $m+2n\equiv 0 \pmod 5$. Assume $-2m+n=5r$ and $m+2n=-5s$, it follows that $m=-2r-s$ and $n=r-2s$. Therefore, we obtain

$$H_{5,0}(S_1) = \sum_{r,s=-\infty}^{\infty}(-1)^s q^{10(-2r-s)^2+8(-2r-s)+10(r-2s)^2+(r-2s)}$$

$$\begin{aligned}&=\sum_{r,s=-\infty}^{\infty}(-1)^s q^{50r^2+15r+50s^2+10s}\\&=(-q^{35},-q^{65},q^{100};q^{100})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty.\end{aligned}\tag{3.7}$$

In the same vein,

$$H_{5,0}(S_2)=-q^5(-q^{15},-q^{85},q^{100};q^{100})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty.\tag{3.8}$$

According to (3.4), (3.7) and (3.8), we arrive at

$$\begin{aligned}C_0(q)&=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5(q^4;q^4)_\infty^3}(q^8,q^{12},q^{20};q^{20})_\infty\\&\quad\times\left((-q^7,-q^{13},q^{20};q^{20})_\infty+q(-q^3,-q^{17},q^{20};q^{20})_\infty\right)\\&=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5(q^4;q^4)_\infty^3}(q^8,q^{12},q^{20};q^{20})_\infty\\&\quad\times\left(\sum_{m=-\infty}^{\infty}q^{10m^2+3m}+q\sum_{m=-\infty}^{\infty}q^{10m^2+7m}\right)\\&=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5(q^4;q^4)_\infty^3}(q^8,q^{12},q^{20};q^{20})_\infty\\&\quad\times\left(\sum_{m=-\infty}^{\infty}q^{\left(5(2m)^2+3(2m)\right)/2}+\sum_{m=-\infty}^{\infty}q^{\left(5(2m-1)^2+3(2m-1)\right)/2}\right)\\&=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5(q^4;q^4)_\infty^3}(q^8,q^{12},q^{20};q^{20})_\infty\left(\sum_{m=-\infty}^{\infty}q^{(5m^2+3m)/2}\right)\\&=\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5(q^4;q^4)_\infty^3}(q^8,q^{12},q^{20};q^{20})_\infty(-q,-q^4,q^5;q^5)_\infty\\&=\frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty(q^{20};q^{20})_\infty(q^2,q^8;q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3(q,q^4;q^5)_\infty(q^2,q^{18};q^{20})_\infty},\end{aligned}$$

as desired.

Since

$$\begin{aligned}H_{5,2}(S_1)&=-q^2(-q^{25},-q^{75},q^{100};q^{100})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty,\\H_{5,2}(S_2)&=q^2(-q^{25},-q^{75},q^{100};q^{100})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty.\end{aligned}$$

Thus we obtain

$$\begin{aligned}C_4(q)&=\frac{-2(q^2;q^2)_\infty^6}{(q;q)_\infty^5(q^4;q^4)_\infty^3}(-q^5,-q^{15},q^{20};q^{20})_\infty(q^8,q^{12},q^{20};q^{20})_\infty\\&=\frac{-2(q^2;q^2)_\infty^6(q^{10};q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^2(q^5;q^5)_\infty(q^4,q^{16};q^{20})_\infty},\end{aligned}$$

as desired.

Similarly,

$$H_{5,3}(S_1) = -q^{13}(-q^5, -q^{95}, q^{100}; q^{100})_\infty (q^{20}, q^{80}, q^{100}; q^{100})_\infty,$$
$$H_{5,3}(S_2) = q^{3}(-q^{45}, -q^{55}, q^{100}; q^{100})_\infty (q^{20}, q^{80}, q^{100}; q^{100})_\infty.$$

We obtain

$$\begin{aligned}
C_6(q) &= \frac{-(q^2;q^2)_\infty^6}{(q;q)_\infty^5 (q^4;q^4)_\infty^3}(q^4,q^{16},q^{20};q^{20})_\infty \\
&\quad \times \left((-q^9,-q^{11},q^{20};q^{20})_\infty + q^2(-q,-q^{19},q^{20};q^{20})_\infty\right) \\
&= \frac{-(q^2;q^2)_\infty^6}{(q;q)_\infty^5 (q^4;q^4)_\infty^3}(q^4,q^{16},q^{20};q^{20})_\infty \\
&\quad \times \left(\sum_{m=-\infty}^{\infty} q^{10m^2+m} + q^2 \sum_{m=-\infty}^{\infty} q^{10m^2+9m}\right) \\
&= \frac{-(q^2;q^2)_\infty^6}{(q;q)_\infty^5 (q^4;q^4)_\infty^3}(q^4,q^{16},q^{20};q^{20})_\infty \left(\sum_{m=-\infty}^{\infty} q^{(5m^2+m)/2}\right) \\
&= \frac{-(q^2;q^2)_\infty^6}{(q;q)_\infty^5 (q^4;q^4)_\infty^3}(q^4,q^{16},q^{20};q^{20})_\infty (-q^2,-q^3;q^5;q^5)_\infty \\
&= \frac{-(q^2;q^2)_\infty^7 (q^5;q^5)_\infty^2 (q,q^4;q^5)_\infty (q^2,q^{18};q^{20})_\infty}{(q;q)_\infty^6 (q^4;q^4)_\infty^2 (q^{10};q^{10})_\infty (q^2,q^8;q^{10})_\infty^2}.
\end{aligned}$$

This proves (1.24).

Following the similar technique, we get

$$H_{5,4}(S_1) = q^{9}(-q^{15}, -q^{85}, q^{100}; q^{100})_\infty (q^{20}, q^{80}, q^{100}; q^{100})_\infty,$$
$$H_{5,4}(S_2) = -q^{4}(-q^{35}, -q^{65}, q^{100}; q^{100})_\infty (q^{20}, q^{80}, q^{100}; q^{100})_\infty.$$

Therefore,

$$\begin{aligned}
C_8(q) &= \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5 (q^4;q^4)_\infty^3}(q^4,q^{16},q^{20};q^{20})_\infty \\
&\quad \times \left((-q^7,-q^{13},q^{20};q^{20})_\infty + q(-q^3,-q^{17},q^{20};q^{20})_\infty\right) \\
&= \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^5 (q^4;q^4)_\infty^3}(q^4,q^{16},q^{20};q^{20})_\infty (-q,-q^4,q^5;q^5)_\infty \\
&= \frac{(q^2;q^2)_\infty^7 (q^5;q^5)_\infty^2 (q^2,q^3;q^5)_\infty (q^{20};q^{20})_\infty}{(q;q)_\infty^6 (q^4;q^4)_\infty^3 (q^{10};q^{10})_\infty (q^6,q^{14};q^{20})_\infty}.
\end{aligned}$$

This proves (1.25).

The proofs of (1.20), (1.21), (1.23) and (1.26) are a little trickier.

According to (2.4) and (3.2), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c(2n+1)q^n &= \frac{2(q^{20};q^{20})_\infty^2}{(q^5;q^5)^2(q^{10};q^{10})_\infty^2}(q^2,q^3;q^5)_\infty\left(\sum_{m,n=-\infty}^{\infty}(-1)^m q^{5m^2+4m+5n^2+n}\right)\\ &= \frac{2(q^{20};q^{20})_\infty^2}{(q^5;q^5)^2(q^{10};q^{10})_\infty^2}(q^2,q^3;q^5)_\infty\\ &\quad\times\left(\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}q^{5(r+s)^2+4(r+s)+5(r-s)^2+(r-s)}\right.\\ &\quad\left.+\sum_{r,s=-\infty}^{\infty}(-1)^{r+s-1}q^{5(r+s-1)^2+4(r+s-1)+5(r-s)^2+(r-s)}\right)\\ &= \frac{2(q^{20};q^{20})_\infty^3}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty^3}(q^2,q^3;q^5)_\infty\\ &\quad\times\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+3m}-q\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+7m}\right).\end{aligned} \tag{3.9}$$

Combining (3.3) and (3.9),

$$\begin{aligned}\sum_{n=0}^{\infty} c(2n+1)q^n &= \frac{2(q^{20};q^{20})_\infty^3}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^3}\\ &\quad\times\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+3m}-q\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+7m}\right)\\ &\quad\times\left(\sum_{n=-\infty}^{\infty}q^{10n^2+n}-q^2\sum_{n=-\infty}^{\infty}q^{10n^2+9n}\right)\\ &= \frac{2(q^{20};q^{20})_\infty^3}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^3}\left(T_1-T_2-T_3+T_4\right),\end{aligned}$$

where

$$T_1=\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+3m+10n^2+n}, \tag{3.10}$$

$$T_2=q^2\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+3m+10n^2+9n}, \tag{3.11}$$

$$T_3=q\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+7m+10n^2+n}, \tag{3.12}$$

$$T_4 = q^3 \sum_{m,n=-\infty}^{\infty} (-1)^m q^{10m^2+7m+10n^2+9n}. \tag{3.13}$$

Since

$$\begin{aligned}
H_{5,0}(T_1) &= (-q^{45}, -q^{55}, q^{100}; q^{100})_\infty (q^{45}, q^{55}, q^{100}; q^{100})_\infty, \\
H_{5,0}(T_2) &= -q^{10}(-q^{5}, -q^{95}, q^{100}; q^{100})_\infty (q^{45}, q^{55}, q^{100}; q^{100})_\infty, \\
H_{5,0}(T_3) &= q^{10}(-q^{45}, -q^{55}, q^{100}; q^{100})_\infty (q^{5}, q^{95}, q^{100}; q^{100})_\infty, \\
H_{5,0}(T_4) &= -q^{20}(-q^{5}, -q^{95}, q^{100}; q^{100})_\infty (q^{5}, q^{95}, q^{100}; q^{100})_\infty.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
C_1(q) &= \frac{2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} \Big((q^9, q^{11}, q^{20}; q^{20})_\infty \\
&\quad \times \big((-q^9, -q^{11}, q^{20}; q^{20})_\infty + q^2(-q, -q^{19}, q^{20}; q^{20})_\infty \big) \\
&\quad - q^2 (q, q^{19}, q^{20}; q^{20})_\infty \\
&\quad \times \big((-q^9, -q^{11}, q^{20}; q^{20})_\infty + q^2(-q, -q^{19}, q^{20}; q^{20})_\infty \big) \Big) \\
&= \frac{2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} \Big((-q^2, -q^3, q^5; q^5)_\infty \\
&\quad \times \big((q^9, q^{11}, q^{20}; q^{20})_\infty - q^2 (q, q^{19}, q^{20}; q^{20})_\infty \big) \Big) \\
&= \frac{2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} (-q^2, -q^3, q^5; q^5)_\infty \\
&\quad \times \left(\sum_{m=-\infty}^{\infty} (-1)^m q^{10m^2+m} - q^2 \sum_{m=-\infty}^{\infty} (-1)^m q^{10m^2+9m} \right) \\
&= \frac{2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} (-q^2, -q^3, q^5; q^5)_\infty \left(\sum_{m=-\infty}^{\infty} (-1)^{m(m-1)/2} q^{(5m^2+m)/2} \right) \\
&= \frac{2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} (-q^2, -q^3, q^5; q^5)_\infty (q^2, -q^3, -q^5; -q^5)_\infty \\
&= \frac{2(q^4;q^4)_\infty^3 (q^5;q^5)_\infty^2 (q, q^4; q^5)_\infty^2 (q^{10};q^{10})_\infty}{(q;q)_\infty^4 (q^2;q^2)_\infty (q^{20};q^{20})_\infty (q^4, q^{16}; q^{20})_\infty},
\end{aligned}$$

as desired.

In the same vein, we obtain

$$\begin{aligned}
H_{5,1}(T_1) &= q^{11}(-q^{35}, -q^{65}, q^{100}; q^{100})_\infty (q^{5}, q^{95}, q^{100}; q^{100})_\infty, \\
H_{5,1}(T_2) &= -q^{16}(-q^{15}, -q^{85}, q^{100}; q^{100})_\infty (q^{5}, q^{95}, q^{100}; q^{100})_\infty, \\
H_{5,1}(T_3) &= q(-q^{35}, -q^{65}, q^{100}; q^{100})_\infty (q^{45}, q^{55}, q^{100}; q^{100})_\infty,
\end{aligned}$$

$$H_{5,1}(T_4) = -q^6(-q^{15}, -q^{85}, q^{100}; q^{100})_\infty (q^{45}, q^{55}, q^{100}; q^{100})_\infty.$$

Then we get

$$\begin{aligned}
C_3(q) &= \frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} \Big((q^9, q^{11}, q^{20}; q^{20})_\infty \\
&\quad \times \big((-q^7, -q^{13}, q^{20}; q^{20})_\infty + q(-q^3, -q^{17}, q^{20}; q^{20})_\infty \big) \\
&\quad - q^2 (q, q^{19}, q^{20}; q^{20})_\infty \\
&\quad \times \big((-q^7, -q^{13}, q^{20}; q^{20})_\infty + q(-q^3, -q^{17}, q^{20}; q^{20})_\infty \big) \Big) \\
&= \frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} \Big((-q, -q^4, q^5; q^5)_\infty \\
&\quad \times \big((q^9, q^{11}, q^{20}; q^{20})_\infty - q^2 (q, q^{19}, q^{20}; q^{20})_\infty \big) \Big) \\
&= \frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} (-q, -q^4, q^5; q^5)_\infty (q^2, -q^3, -q^5; -q^5)_\infty \\
&= \frac{-2(q^4;q^4)_\infty^2 (q^5;q^5)_\infty (q^8, q^{12}; q^{20})_\infty}{(q;q)_\infty^3 (q^4, q^6; q^{10})_\infty^2}.
\end{aligned}$$

This establishes (1.21).

Similarly, we obtain

$$\begin{aligned}
H_{5,2}(T_1) &= -q^7(-q^{15}, -q^{85}, q^{100}; q^{100})_\infty (q^{35}, q^{65}, q^{100}; q^{100})_\infty, \\
H_{5,2}(T_2) &= q^2(-q^{35}, -q^{65}, q^{100}; q^{100})_\infty (q^{35}, q^{65}, q^{100}; q^{100})_\infty, \\
H_{5,2}(T_3) &= q^{12}(-q^{15}, -q^{85}, q^{100}; q^{100})_\infty (q^{15}, q^{85}, q^{100}; q^{100})_\infty, \\
H_{5,2}(T_4) &= -q^7(-q^{35}, -q^{65}, q^{100}; q^{100})_\infty (q^{15}, q^{85}, q^{100}; q^{100})_\infty
\end{aligned}$$

and

$$\begin{aligned}
C_5(q) &= \frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} \Big((q^7, q^{13}, q^{20}; q^{20})_\infty \\
&\quad \times \big((-q^7, -q^{13}, q^{20}; q^{20})_\infty + q(-q^3, -q^{17}, q^{20}; q^{20})_\infty \big) \\
&\quad + q (q^3, q^{17}, q^{20}; q^{20})_\infty \\
&\quad \times \big((-q^7, -q^{13}, q^{20}; q^{20})_\infty + q(-q^3, -q^{17}, q^{20}; q^{20})_\infty \big) \Big) \\
&= \frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} \Big((-q, -q^4, q^5; q^5)_\infty \\
&\quad \times \big((q^7, q^{13}, q^{20}; q^{20})_\infty + q (q^3, q^{17}, q^{20}; q^{20})_\infty \big) \Big) \\
&= \frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2 (q^2;q^2)_\infty^3} (-q, -q^4, q^5; q^5)_\infty
\end{aligned}$$

$$\times\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+3m}+q\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+7m}\right)$$
$$=\frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2(q^2;q^2)_\infty^3}(-q,-q^4,q^5;q^5)_\infty\left(\sum_{m=-\infty}^{\infty}(-1)^{m(m+1)/2}q^{(5m^2+3m)/2}\right)$$
$$=\frac{-2(q^4;q^4)_\infty^3}{(q;q)_\infty^2(q^2;q^2)_\infty^3}(-q,-q^4,q^5;q^5)_\infty(-q,q^4,-q^5;-q^5)_\infty$$
$$=\frac{-2(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty(q,q^4;q^5)_\infty^2}.$$

This establishes (1.23).

Since

$$H_{5,4}(T_1)=q^9(-q^{25},-q^{75},q^{100};q^{100})_\infty(q^{15},q^{85},q^{100};q^{100})_\infty,$$
$$H_{5,4}(T_2)=-q^9(-q^{25},-q^{75},q^{100};q^{100})_\infty(q^{15},q^{85},q^{100};q^{100})_\infty,$$
$$H_{5,4}(T_3)=-q^4(-q^{25},-q^{75},q^{100};q^{100})_\infty(q^{35},q^{65},q^{100};q^{100})_\infty,$$
$$H_{5,4}(T_4)=q^4(-q^{25},-q^{75},q^{100};q^{100})_\infty(q^{35},q^{65},q^{100};q^{100})_\infty.$$

Therefore, we obtain

$$C_9(q)=\frac{4(q^4;q^4)_\infty^3}{(q;q)_\infty^2(q^2;q^2)_\infty^3}(-q^5,-q^{15},q^{20};q^{20})_\infty$$
$$\times\big((q^7,q^{13},q^{20};q^{20})_\infty+q(q^3,q^{17},q^{20};q^{20})_\infty\big)$$
$$=\frac{4(q^4;q^4)_\infty^3}{(q;q)_\infty^2(q^2;q^2)_\infty^3}(-q^5,-q^{15},q^{20};q^{20})_\infty(-q,q^4,-q^5;q^5)_\infty$$
$$=\frac{4(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^4(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty^2(q^2;q^2)_\infty^2(q^5;q^5)_\infty^2(q,q^9;q^{10})_\infty}.$$

Finally,

$$H_{5,1}(S_1)=q^{11}(-q^{45},-q^{55},q^{100},q^{100})_\infty(1,q^{100},q^{100};q^{100})_\infty=0,$$
$$H_{5,1}(S_2)=q^{21}(-q^{5},-q^{95},q^{100},q^{100})_\infty(1,q^{100},q^{100};q^{100})_\infty=0,$$
$$H_{5,3}(T_1)=-q^{13}(-q^{5},-q^{95},q^{100};q^{100})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty,$$
$$H_{5,3}(T_2)=q^{3}(-q^{45},-q^{55},q^{100};q^{100})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty,$$
$$H_{5,3}(T_3)=-q^{13}(-q^{5},-q^{95},q^{100};q^{100})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty,$$
$$H_{5,3}(T_4)=q^{3}(-q^{45},-q^{55},q^{100};q^{100})_\infty(q^{25},q^{75},q^{100};q^{100})_\infty,$$

from which we obtain (1.27). This completes the proof. □

The proof of Theorem 1.4 is similar to that of Theorem 1.3.

Now we turn to sketch the elementary proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Similarly, we start with

$$\begin{aligned}
&(-q,-q^4;q^5)_\infty(q,q^9;q^{10})_\infty^3 \\
&= (-q,-q^4,-q^6,-q^9;q^{10})_\infty(q,q^9;q^{10})_\infty^3 \\
&= (-q^4,-q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty(-q,-q^9;q^{10})_\infty^2 \\
&= \frac{(-q^4,-q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q^{10};q^{10})_\infty^2}\left(\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}q^{5m^2+4m+5n^2+4n}\right) \\
&= \frac{(-q^4,-q^6;q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q^{10};q^{10})_\infty^2}\big(\varphi(q^{10})(-q^2,-q^{18},q^{20};q^{20})_\infty \\
&\quad - 2q\psi(q^{20})(-q^8,-q^{12},q^{20};q^{20})_\infty\big),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}a(2n)q^n &= \frac{\varphi(q^5)(q^{10};q^{10})_\infty}{(q^5;q^5)^2}(-q^2,-q^3;q^5)_\infty(q,q^9;q^{10})_\infty(-q,-q^9;q^{10})_\infty \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^4(q^{20};q^{20})_\infty^2}(-q^2,-q^3;q^5)_\infty(q^2,q^{18};q^{20})_\infty \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^3}\left(\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+8m+10n^2+n}\right. \\
&\quad \left. + q^2\sum_{m,n=-\infty}^{\infty}(-1)^m q^{10m^2+8m+10n^2+9n}\right) \\
&= \frac{(q^{10};q^{10})_\infty^6}{(q^5;q^5)_\infty^5(q^{20};q^{20})_\infty^3}(S_1+S_2)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}a(2n+1)q^n &= \frac{-2\psi(q^{10})(q^{10};q^{10})_\infty}{(q^5;q^5)^2}(-q^2,-q^3;q^5)_\infty(q,q^9;q^{10})_\infty(-q^4,-q^6;q^{10})_\infty \\
&= \frac{-2(q^{20};q^{20})_\infty^2}{(q^5;q^5)^2}(-q^2,-q^3;q^5)_\infty(q,q^9;q^{10})_\infty(-q^4,-q^6;q^{10})_\infty \\
&= \frac{-2(q^{20};q^{20})_\infty^3}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^3} \\
&\quad \times\left(\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+3m} - q\sum_{m=-\infty}^{\infty}(-1)^m q^{10m^2+7m}\right) \\
&\quad \times\left(\sum_{n=-\infty}^{\infty}q^{10n^2+n} + q^2\sum_{n=-\infty}^{\infty}q^{10n^2+9n}\right)
\end{aligned}$$

$$= \frac{-2(q^{20};q^{20})_\infty^3}{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^3}\left(T_1+T_2-T_3-T_4\right),$$

where S_i $(i=1,2)$ and T_j $(j=1,2,3,4)$ are defined as (3.5), (3.6) and (3.10)–(3.13). The proofs of (1.3)–(1.9) are quite analogous to (1.19)–(1.27), thus we omit the details. □

The proof of Theorem 1.2 is similar to that of Theorem 1.1.

Remark 3.1. It is worth mentioning that our proofs of (1.9) and (1.16) are different from Hirschhorn's proofs [17], because the summation forms of (3.5), (3.6) and (3.10)–(3.13) are different from the summation forms in Hirschhorn's proofs. In the same vein, the proofs of (1.27) and (1.36) are different from those of Baruah and Kaur [4].

4. Proofs of Theorems 1.5–1.8 and Corollaries 1.10–1.12

Proof of Theorem 1.5. In view of (2.5) and (2.6), we rewrite (2.8) as

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\left(\frac{G(q^5)^4}{H(q^5)^4}+q\frac{G(q^5)^3}{H(q^5)^3}+2q^2\frac{G(q^5)^2}{H(q^5)^2}+3q^3\frac{G(q^5)}{H(q^5)}\right.$$
$$\left.+5q^4-3q^5\frac{H(q^5)}{G(q^5)}+2q^6\frac{H(q^5)^2}{G(q^5)^2}-q^7\frac{H(q^5)^3}{G(q^5)^3}+q^8\frac{H(q^5)^4}{G(q^5)^4}\right). \tag{4.1}$$

By (4.1), we obtain

$$\begin{aligned}G(q)H(q) &= \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty} = \frac{(q^5;q^5)_\infty}{(q;q)_\infty}\\
&= (q^5;q^5)_\infty\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\left(\frac{G(q^5)^4}{H(q^5)^4}+q\frac{G(q^5)^3}{H(q^5)^3}+2q^2\frac{G(q^5)^2}{H(q^5)^2}+3q^3\frac{G(q^5)}{H(q^5)}\right.\\
&\quad\left.+5q^4-3q^5\frac{H(q^5)}{G(q^5)}+2q^6\frac{H(q^5)^2}{G(q^5)^2}-q^7\frac{H(q^5)^3}{G(q^5)^3}+q^8\frac{H(q^5)^4}{G(q^5)^4}\right)\\
&= G(q^5)^5H(q^5)^5\left(\frac{G(q^5)^2}{H(q^5)^2}-2q\frac{G(q^5)}{H(q^5)}+4q^2-3q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}\right)\\
&\quad\times\left(\frac{G(q^5)^2}{H(q^5)^2}+3q\frac{G(q^5)}{H(q^5)}+4q^2+2q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}\right).\end{aligned} \tag{4.2}$$

Based on (2.7), we rewrite (2.9) as

$$\frac{H(q)^5}{G(q)^5} = \frac{H(q^5)}{G(q^5)}\frac{\frac{G(q^5)^2}{H(q^5)^2}-2q\frac{G(q^5)}{H(q^5)}+4q^2-3q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}}{\frac{G(q^5)^2}{H(q^5)^2}+3q\frac{G(q^5)}{H(q^5)}+4q^2+2q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}}. \tag{4.3}$$

Multiplying (4.2) and (4.3) yields

$$\frac{H(q)^6}{G(q)^4} = G(q^5)^4 H(q^5)^6 \left(\frac{G(q^5)^2}{H(q^5)^2} - 2q\frac{G(q^5)}{H(q^5)} + 4q^2 - 3q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2} \right)^2,$$

from which we obtain

$$\begin{aligned} \frac{(q,q^4;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty^3} &= \frac{H(q)^3}{G(q)^2} \\ &= G(q^5)^2 H(q^5)^3 \left(\frac{G(q^5)^2}{H(q^5)^2} - 2q\frac{G(q^5)}{H(q^5)} + 4q^2 - 3q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2} \right) \\ &= G(q^5)^4 H(q^5) - 2qG(q^5)^3 H(q^5)^2 + 4q^2 G(q^5)^2 H(q^5)^3 \\ &\quad - 3q^3 G(q^5) H(q^5)^4 + q^4 H(q^5)^5. \end{aligned} \tag{4.4}$$

The identities (1.37)–(1.41) follow from (4.4). □

Proof of Theorem 1.6. From (4.3), we find that

$$\frac{G(q)^5}{H(q)^5} = \frac{G(q^5)}{H(q^5)} \frac{\frac{G(q^5)^2}{H(q^5)^2} + 3q\frac{G(q^5)}{H(q^5)} + 4q^2 + 2q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2}}{\frac{G(q^5)^2}{H(q^5)^2} - 2q\frac{G(q^5)}{H(q^5)} + 4q^2 - 3q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2}}. \tag{4.5}$$

Multiplying (4.2) and (4.5) gives

$$\frac{G(q)^6}{H(q)^4} = G(q^5)^6 H(q^5)^4 \left(\frac{G(q^5)^2}{H(q^5)^2} + 3q\frac{G(q^5)}{H(q^5)} + 4q^2 + 2q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2} \right)^2,$$

from which we obtain

$$\begin{aligned} \frac{(q^2,q^3;q^5)_\infty^2}{(q,q^4;q^5)_\infty^3} &= \frac{G(q)^3}{H(q)^2} \\ &= G(q^5)^3 H(q^5)^2 \left(\frac{G(q^5)^2}{H(q^5)^2} + 3q\frac{G(q^5)}{H(q^5)} + 4q^2 + 2q^3\frac{H(q^5)}{G(q^5)} + q^4\frac{H(q^5)^2}{G(q^5)^2} \right) \\ &= G(q^5)^5 + 3qG(q^5)^4 H(q^5) + 4q^2 G(q^5)^3 H(q^5)^2 \\ &\quad + 2q^3 G(q^5)^2 H(q^5)^3 + q^4 G(q^5) H(q^5)^4. \end{aligned} \tag{4.6}$$

In light of (4.6), we obtain (1.42)–(1.46). The proof is completed. □

Proof of Theorem 1.7. Replacing q by q^5 in (2.10), we see that

$$G(q^5)^{11} H(q^5) - 11q^5 \left(G(q^5) H(q^5) \right)^6 - q^{10} G(q^5) H(q^5)^{11} = 1. \tag{4.7}$$

With the help of (4.4) and (4.7), we obtain

$$\frac{(q^2,q^3;q^5)_\infty^3}{(q,q^4;q^5)_\infty^2} = \frac{G(q)^2}{H(q)^3} = \frac{1}{H(q)^3/G(q)^2}$$

$$= \frac{G(q^5)^{11}H(q^5) - 11q^5\left(G(q^5)H(q^5)\right)^6 - q^{10}G(q^5)H(q^5)^{11}}{G(q^5)^4H(q^5) - 2qG(q^5)^3H(q^5)^2 + 4q^2G(q^5)^2H(q^5)^3 - 3q^3G(q^5)H(q^5)^4 + q^4H(q^5)^5}$$
$$= G(q^5)^7 + 2qG(q^5)^6H(q^5) - 5q^3G(q^5)^4H(q^5)^3 - 5q^4G(q^5)^3H(q^5)^4 - 3q^5G(q^5)^2H(q^5)^5 - q^6G(q^5)H(q^5)^6. \tag{4.8}$$

The identities (1.47)–(1.51) follow immediately. □

Proof of Theorem 1.8. Combining (4.6) and (4.7) yields

$$\frac{(q,q^4;q^5)_\infty^3}{(q^2,q^3;q^5)_\infty^2} = \frac{H(q)^2}{G(q)^3} = \frac{1}{G(q)^3/H(q)^2}$$
$$= \frac{G(q^5)^{11}H(q^5) - 11q^5\left(G(q^5)H(q^5)\right)^6 - q^{10}G(q^5)H(q^5)^{11}}{G(q^5)^5 + 3qG(q^5)^4H(q^5) + 4q^2G(q^5)^3H(q^5)^2 + 2q^3G(q^5)^2H(q^5)^3 + q^4G(q^5)H(q^5)^4}$$
$$= G(q^5)^6H(q^5) - 3qG(q^5)^5H(q^5)^2 + 5q^2G(q^5)^4H(q^5)^3 - 5q^3G(q^5)^3H(q^5)^4 + 2q^5G(q^5)H(q^5)^6 - q^6H(q^5)^7, \tag{4.9}$$

from which we obtain (1.52)–(1.56). □

Proof of Corollary 1.10. We only prove the case $n \equiv 0 \pmod{10}$ in (1.59) here, the other can be proved similarly.

In light of (2.3), we find that

$$C_0(q) = \frac{(q^2;q^2)_\infty^6(q^5;q^5)_\infty(q^{20};q^{20})_\infty(q^2,q^8;q^{10})_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty^3(q,q^4;q^5)_\infty(q^2,q^{18};q^{20})_\infty}$$
$$\succeq \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^3}\cdot\frac{(q^5;q^5)_\infty(q^2,q^8;q^{10})_\infty^2}{(q;q)_\infty^2}\cdot\frac{1}{(q,q^4;q^5)_\infty}$$
$$\succeq \frac{1}{(q,q^4;q^5)_\infty^3}\left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)^3 \succ 0,$$

as required. □

The inequalities in (1.61) and (1.62) follow immediately from (1.37)–(1.46).

Proof of Corollary 1.12. The equality and inequalities in (1.64) follow from (1.52)–(1.56), thus we only need to prove (1.63).

The cases $n \equiv 3, 4, 2 \pmod 5$ in (1.63) are immediate consequences of (1.49)–(1.51), respectively. Therefore we need to consider the following two cases:

1) $n \equiv 1 \pmod 5$. According to (2.5) and (2.6),

$$\frac{1}{(q,q^4;q^5)_\infty} - \frac{1}{(q^2,q^3;q^5)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} - \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}$$
$$= \sum_{n=0}^{\infty}\frac{q^{n^2}(1-q^n)}{(q;q)_n} = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{n-1}} \succeq 0.$$

It follows immediately that

$$\begin{aligned} G_1(q) &= \frac{-q}{(q,q^4;q^5)_\infty(q^2,q^3;q^5)_\infty^6} + \frac{2}{(q,q^4;q^5)_\infty^6(q^2,q^3;q^5)_\infty} \\ &\succeq \frac{1}{(q^2,q^3;q^5)_\infty^6}\left(\frac{2}{(q,q^4;q^5)_\infty} - \frac{q}{(q,q^4;q^5)_\infty}\right) \\ &\succeq \frac{1}{(q,q^4;q^5)_\infty} + \left(\frac{1}{(q,q^4;q^5)_\infty} - \frac{q}{(q,q^4;q^5)_\infty}\right) \\ &= \frac{1}{(q,q^4;q^5)_\infty} + \frac{1}{(q^4,q^6;q^5)_\infty} \succ 0. \end{aligned}$$

This implies that $g(5n+1) > 0$ holds for all $n \geq 0$.

2) $n \equiv 0 \pmod 5$. Firstly, notice that

$$\begin{aligned} G_0(q) &= \frac{-3q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^5} + \frac{1}{(q,q^4;q^5)_\infty^7} \\ &\succeq \frac{1}{(q,q^4;q^5)_\infty^5(q^2,q^3;q^5)_\infty^2} - \frac{3q}{(q,q^4;q^5)_\infty^2(q^2,q^3;q^5)_\infty^5} \\ &= \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty^2}\left(\frac{1}{(q,q^4;q^5)_\infty^3} - \frac{3q}{(q^2,q^3;q^5)_\infty^3}\right). \end{aligned}$$

If

$$\frac{1}{(q,q^4;q^5)_\infty^3} \succeq \frac{3q}{(q^2,q^3;q^5)_\infty^3},$$

then we further have

$$\begin{aligned} G_0(q) &\succeq \frac{1}{(q,q^2,q^3,q^4;q^5)_\infty^2}\left(\frac{1}{(q,q^4;q^5)_\infty^3} - \frac{3q}{(q^2,q^3;q^5)_\infty^3}\right) \\ &\succeq \frac{1}{(q,q^4;q^5)_\infty^3} - \frac{3q}{(q^2,q^3;q^5)_\infty^3}. \end{aligned} \tag{4.10}$$

Now we denote

$$\sum_{n=0}^{\infty} \alpha(n)q^n = \frac{1}{(q,q^4;q^5)_\infty^3}, \qquad \sum_{n=0}^{\infty} \beta(n)q^n = \frac{1}{(q^2,q^3;q^5)_\infty^3}.$$

Partition-theoretically, $\alpha(n)$ (resp. $\beta(n)$) can be interpreted as the number of partition triples of n with all parts congruent to ± 1 (resp. ± 2) modulo 5. To obtain $g(5n) > 0$, we first need to prove $3\beta(n-1) \leq \alpha(n)$. Furthermore,

$$3\beta(n-1) < \alpha(n) \qquad \text{if} \quad n \geq 5. \tag{4.11}$$

Define two sets $\mathcal{S}_n$ and $\mathcal{E}_n$ by

$$\mathcal{S}_n = \{\pi = (\pi^1, \pi^2, \pi^3) |\ \text{all parts in partitions } \pi^1, \pi^2, \pi^3 \ \equiv \pm 1 \pmod 5,$$

$$s(\pi^1)+s(\pi^2)+s(\pi^3)=n\},$$
$$\mathcal{E}_n=\{\pi=(\pi^1,\pi^2,\pi^3)|\text{ all parts in partitions } \pi^1,\pi^2,\pi^3 \equiv \pm 2 \pmod 5),$$
$$s(\pi^1)+s(\pi^2)+s(\pi^3)=n\},$$

where $s(\pi^i)$ denotes the sum of all parts in partition π^i.

According to [19], there exists an injection τ: $\mathcal{RR}_2(n) \hookrightarrow \mathcal{RR}_1(n)$, where $\mathcal{RR}_1(n)$ (resp. $\mathcal{RR}_2(n)$) denotes the set of partitions of n with all parts congruent to ± 1 (resp. ± 2) modulo 5. For a given $\pi=(\pi^1,\pi^2,\pi^3)\in\mathcal{E}_n$, we define an injection $\widetilde{\tau}$: $\mathcal{E}_n \hookrightarrow \mathcal{S}_n$ by $\lambda=\widetilde{\tau}(\pi)=(\tau(\pi^1),\tau(\pi^2),\tau(\pi^3))$. If $\pi\in\mathcal{E}_{n-1}$, then $\lambda\in\mathcal{S}_{n-1}$. Finally we add a part of size one to λ and denote this new partition triples by $\widehat{\lambda}$. Of course, we have $\widehat{\lambda}\in\mathcal{S}_n$. For the partition triple λ, there are three choices to append this new part. Thus we obtain $3\beta(n-1)\le\alpha(n)$. Furthermore, by (2.4) in [19], τ is an injection but not a surjection if $n\ge 4$. By the definition of $\widetilde{\tau}$, we obtain (4.11) immediately.

On the other hand, it is easy to compute that

$$g(0)=1,\quad g(5)=4,\quad g(10)=22,\quad g(15)=60,\quad g(20)=160. \tag{4.12}$$

According to (4.10)–(4.12), we obtain that $g(5n)>0$ holds for all $n\ge 0$.

This completes the proof. □

5. Final remarks

We conclude this paper with two remarks.

1) The identities (4.8) and (4.9) imply that if one of Theorem 1.5 or Theorem 1.7 (resp. Theorem 1.6 or Theorem 1.8) is proved, the other can be proved immediately. In Sect. 4, we first prove Theorem 1.5 and Theorem 1.6. Now we choose to prove Theorem 1.7 and Theorem 1.8 first. At this time, we need to use the following 5-dissections for Euler's product [16, Eq. (8.1.1)]:

$$(q;q)_\infty=(q^{25};q^{25})_\infty\left(\frac{G(q^5)}{H(q^5)}-q-q^2\frac{H(q^5)}{G(q^5)}\right). \tag{5.1}$$

By (5.1), we find that

$$\begin{aligned}\frac{1}{G(q)H(q)}&=(q,q^2,q^3,q^4;q^5)_\infty=\frac{(q;q)_\infty}{(q^5;q^5)_\infty}\\&=\frac{(q^{25};q^{25})_\infty}{(q^5;q^5)_\infty}\left(\frac{G(q^5)}{H(q^5)}-q-q^2\frac{H(q^5)}{G(q^5)}\right)\\&=G(q^5)H(q^5)\left(\frac{G(q^5)}{H(q^5)}-q-q^2\frac{H(q^5)}{G(q^5)}\right).\end{aligned} \tag{5.2}$$

By means of (4.5), (4.7) and (5.2), we obtain

$$\begin{aligned}\frac{G(q)^4}{H(q)^6} &= \frac{G(q)^5}{H(q)^5}\cdot\frac{1}{G(q)H(q)}\cdot 1\\ &= G(q^5)^2\left(G(q^5)^{11}H(q^5) - 11q^5\left(G(q^5)H(q^5)\right)^6 - q^{10}G(q^5)H(q^5)^{11}\right)\\ &\quad\times\left(\frac{\frac{G(q^5)^2}{H(q^5)^2}+3q\frac{G(q^5)}{H(q^5)}+4q^2+2q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}}{\frac{G(q^5)^2}{H(q^5)^2}-2q\frac{G(q^5)}{H(q^5)}+4q^2-3q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}}\right)\\ &\quad\times\left(\frac{G(q^5)}{H(q^5)}-q-q^2\frac{H(q^5)}{G(q^5)}\right)\\ &= \Big(G(q^5)^7+2qG(q^5)^6H(q^5)-5q^3G(q^5)^4H(q^5)^3-5q^4G(q^5)^3H(q^5)^4\\ &\qquad -3q^5G(q^5)^2H(q^5)^5-q^6G(q^5)H(q^5)^6\Big)^2.\end{aligned}$$

The identities (1.47)–(1.51) follow immediately.

In the same vein, it follows from (4.3), (4.7) and (5.2) that

$$\begin{aligned}\frac{H(q)^4}{G(q)^6} &= H(q^5)^2\Big(G(q^5)^{11}H(q^5) - q^{10}G(q^5)H(q^5)^{11} - 11q^5\left(G(q^5)H(q^5)\right)^6\Big)\\ &\quad\times\left(\frac{\frac{G(q^5)^2}{H(q^5)^2}-2q\frac{G(q^5)}{H(q^5)}+4q^2-3q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}}{\frac{G(q^5)^2}{H(q^5)^2}+3q\frac{G(q^5)}{H(q^5)}+4q^2+2q^3\frac{H(q^5)}{G(q^5)}+q^4\frac{H(q^5)^2}{G(q^5)^2}}\right)\\ &\quad\times\left(\frac{G(q^5)}{H(q^5)}-q-q^2\frac{H(q^5)}{G(q^5)}\right)\\ &= \Big(G(q^5)^6H(q^5)-3qG(q^5)^5H(q^5)^2+5q^2G(q^5)^4H(q^5)^3-5q^3G(q^5)^3H(q^5)^4\\ &\qquad +2q^5G(q^5)H(q^5)^6-q^6H(q^5)^7\Big)^2,\end{aligned}$$

from which we obtain (1.52)–(1.56).

2) On the other hand, Chern [8] investigated the asymptotic behavior of the coefficients of a general family of infinite q-series products. All equalities and inequalities in (1.57)–(1.64) can be proved for sufficiently large n by Chern's asymptotic formula.

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References

[1] K. Alladi and B. Gordon, Vanishing coefficients in the expansion of products of Rogers–Ramanujan type, Proc. Rademacher Centenary Conference, (G. E. Andrews and D. Bressoud, Eds.), *Contemp. Math.* **166** (1994), 129–139. 1

[2] G. E. Andrews, Ramanujan's "lost" notebook III. The Rogers–Ramanujan continued fraction, *Adv. Math.* **41** (1981), 186–208. 1

[3] G. E. Andrews and D. M. Bressoud, Vanishing coefficients in infinite product expansions, *J. Austral. Math. Soc. Ser. A* **27** (1979), no. 2, 199–202. 1

[4] N. D. Baruah and M. Kaur, Some results on vanishing coefficients in infinite product expansions, *Ramanujan J.* (2019), in press. doi: 10.1007/s11139-019-00172-x. 2, 3, 4, 18

[5] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991. 8

[6] S. H. Chan and H. Yesilyurt, The periodicity of the signs of the coefficients of certain infinite products, *Pacific J. Math.* **225** (2006), no. 1, 13–32. 1

[7] S.-D. Chen and S.-S. Huang, On the series expansion of the Göllnitz-Gordon continued fraction, Int. J. Number Theory **1** (2005), no. 1, 53–63. 1

[8] S. Chern, Asymptotics for the Taylor coefficients of certain infinite products, *Ramanujan J.* (2020) in press. doi: 10.1007/s11139-020-00273-y. 23

[9] S. Chern and D. Tang, Vanishing coefficients in quotients of theta functions of modulus five, *Bull. Aust. Math. Soc.* (2020), in press. doi: 10.1017/S0004972720000271. 1

[10] S. Chern and D. Tang, 5-Dissections and sign patterns of Ramanujan's parameter and its companion, to appear in *Czechoslovak Math. J.* 1

[11] S. Cooper, *Ramanujan's Theta Functions*, Springer International Publishing, Cham, 2017. 8

[12] D. Q. J. Dou and J. Xiao, The 5-dissections of two infinite product expansions, *Ramanujan J.* (2020), in press. doi: 10.1007/s11139-019-00200-w. 2

[13] J. Frye and F. Garvan, Automatic proof of theta-function identities, in: *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory (Springer, Cham, 2019)*, pp. 195–258. 2

[14] M. D. Hirschhorn, On the expansion of Ramanujan's continued fraction, *Ramanujan J.* **2** (4) (1998), 521–527. 1

[15] M. D. Hirschhorn, On the expansion of a continued fraction of Gordon, *Ramanujan J.* **5** (4) (2011), 369–375. 1

[16] M. D. Hirschhorn, *The Power of q*, Developments in Mathematics Vol. 49, Springer, Cham, 2017. 8, 9, 22

[17] M. D. Hirschhorn, Two remarkable q-series expansions, *Ramanujan J.* **49** (2) (2018), 451–463. 1, 18

[18] M. D. Hirschhorn and Roselin, On the 2-, 3-, 4- and 6-dissections of Ramanujan's cubic continued fraction and its reciprocal, Ramanujan rediscovered: Proceedings of a conference on elliptic functions, partitions, and q-Series, in memory of K. Venkatachaliengar, Bangalore, June 1–5, 2009, India: Ramanujan Mathematical Society, 2010, pp. 125–138 1

[19] K. W. J. Kadell, An injection for the Ehrenpreis Rogers–Ramanujan problem, *J. Combin. Theory Ser. A* **86** (2) (1999) 390–394. 22

[20] B. L. S. Lin, On the expansion of a continued fraction of order twelve, *Int. J. Number Theory* 9 (2013), no. 8, 2019–2031. 1

[21] J. Mc Laughlin, Further results on vanishing coefficients in infinite product expansions, *J. Austral. Math. Soc. Ser. A* **98** (1) (2015), 69–77. 1

[22] K. G. Ramanathan, Generalisations of some theorems of Ramanujan, *J. Number Theory* **29** (2) (1988), 118–137. 1

[23] B. Richmond and G. Szekeres, The Taylor coefficients of certain infinite products, *Acta Sci. Math. (Szeged)* **40** (1978), no. 3–4, 347–369. 1

[24] D. Tang, Vanishing coefficients in some q-series expansions, *Int. J. Number Theory* **15** (2019), no. 4, 763–773. 2

[25] D. Tang, Vanishing coefficiets in four quotients of infinite product expansions, *Bull. Aust. Math. Soc.* **100** (2) (2019), 216–224. 1, 3, 6

[26] D. Tang and E. X. W. Xia, Several q-series related to Ramnujan's theta functions, *Ramanujan J.* (2019), in press. doi: 10.1007/s11139-019-00187-4. 2, 3

[27] A. Vanitha, On the expansion of Ramanujan's continued fraction of order sixteen, *Tamsui Oxf. J. Inf. Math. Sci.* **30** (1) (2016), 81–99. 1

[28] E. X. W. Xia and O. X. M. Yao, On 2- and 4-dissections for some infinite products, *Rocky Mountain J. Math.* **43** (6) (2013), 2033–2047. 1

[29] E. X. W. Xia and O. X. M. Yao, 4-dissections and 8-dissections for some infinite products, *Rocky Mpuntain J. Math.* **44** (5) (2014), 1685–1696. 1

[30] E. X. W. Xia and A. X. H. Zhao, Generalizations of Hirschhorn's results on two remarkable q-series expansions, *Exp. Math.* (2020), in press. doi: 10.1080/10586458.2020.1712565. 2, 6

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