Mixed Boundary Value Problems of Semilinear Elliptic PDEs and BSDEs with Singular Coefficients

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Abstract

In this paper, we prove that there exists a unique weak solution to the mixed boundary value problem for a general class of semilinear second order elliptic partial differential equations with singular coefficients. Our approach is probabilistic. The theory of Dirichlet forms and backward stochastic differential equations with singular coefficients and infinite horizon plays a crucial role.

Keywords: Dirichlet forms; Quadratic forms; Fukushima's decomposition; Mixed boundary value problem; Backward stochastic differential equations; Reflecting diffusion processes.

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1 Introduction

In this paper, our aim is to use probabilistic methods to solve the mixed boundary value problem for semilinear second order elliptic partial differential equations (called PDEs for short) of the following form:

$$\begin{cases} Lu(x) = -F(x, u(x), \nabla u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) - \widehat{B} \cdot n(x)u(x) = \Phi(x) & \text{on } \partial D \end{cases} \tag{1.1}$$

The elliptic operator L is given by :

$$\begin{aligned} L &= \frac{1}{2}\nabla \cdot (A\nabla) + B \cdot \nabla - \nabla \cdot (\hat{B}\cdot) + Q \qquad (1.2) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} B_i(x)\frac{\partial}{\partial x_i} - div(\hat{B}\cdot) + Q(x) \end{aligned}$$

on a d-dimensional smooth bounded Euclidean domain D.
$A(x) = (a_{ij})_{1\le i,j\le d}$: $R^d \to R^d \otimes R^d$ is a smooth, symmetric matrix-valued function which is uniformly elliptic. That is, there is a constant $\lambda > 1$ such that

$$\frac{1}{\lambda}I_{d\times d} \le A(\cdot) \le \lambda I_{d\times d}. \tag{1.3}$$

Here $B = (B_1, ..., B_d)$ and $\hat{B} = (\hat{B}_1, ..., \hat{B}_d) : R^d \to R^d$ are Borel measurable functions, which could be singular, and Q is a real-valued Borel measurable function defined on R^d such that, for some $p > \frac{d}{2}$,

$$I_D(|B|^2 + |\hat{B}|^2 + |Q|) \in L^p(D).$$

L is rigorously determined by the following quadratic form:

$$\begin{aligned} \mathcal{Q}(u,v) := (-Lu, v)_{L^2(D)} &= \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_i \int_D B_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ &- \sum_i \int_D \hat{B}_i(x)\frac{\partial v}{\partial x_i}u(x)dx - \int_D Q(x)u(x)v(x)dx. \end{aligned}$$

Details about the operator L can be found in [9], [16] and [20].
The function $F(\cdot,\cdot,\cdot)$ in (1.1) is a nonlinear function defined on $R^d \times R \times R^d$ and $\Phi(x)$ is a bounded measurable function defined on the boundary ∂D and $\gamma = An$, where n denotes the inward normal vector field defined on the boundary ∂D.

To solve the problem (1.1), it turns out that we need to establish the existence and uniqueness of solutions of backward stochastic differential equations (BSDEs) with singular coefficients and infinite horizon, which is of independent interest.

Probabilistic approaches to boundary value problem of second order differential operators have been adopted by many authors and the earliest work went back as early as 1944 in [12].

There has been a lot of study on the Dirichlet boundary problem (see [1], [8], [3],[6], [11] and [22]). However, there are not many articles on the probabilistic approaches to the Neumann boundary problem.

When $A = I$, $B = 0$ and $\hat{B} = 0$, the following Neumann boundary problem

$$\begin{cases} \frac{1}{2}\triangle u(x) + qu(x) = 0, & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial n}(x) = \phi(x) & \text{on } \partial D \end{cases}$$

was solved in [1] and [11], which also gives the solution the following representation:

$$u(x) = E_x[\int_0^\infty e^{\int_0^t q(B_u)du}\phi(B_t)dL_t^0],$$

where $(B_t)_{t>0}$ is the reflecting Brownian motion on the domain D associated with the infinitesimal generator

$$G = \frac{1}{2}\triangle,$$

and L_t^0, $t > 0$ is the boundary local time satisfying $L_t^0 = \int_0^t I_{\partial D}(B_s)dL_s^0$.

But when $\hat{B} \neq 0$, the term $\nabla\cdot(\hat{B}\cdot)$ is just a formal way of writing because the divergence does not exist as $\hat{B}$ is only a measurable vector field. It should be interpreted in the distributional sense. For this reason, the term $\nabla\cdot(\hat{B}\cdot)$ can not be handled by Girsanov transform or Feyman-Kac transform.

The study of the boundary value problems for the general operator L in the PDE literature (see e.g. [9], [20]) was always carried out under the extra condition:

$$-div(\hat{B}) + Q(x) \leq 0$$

in the sense of distribution in order to use the maximum principle.

When $F = 0$, i.e. the linear case, problem (1.1) was studied in [4](see also [3] for the Dirichlet boundary problem). The term $\nabla\cdot(\hat{B}\cdot)$ is tackled using the time-reversal of Girsanov transform of the symmetric reflecting diffusion $(\Omega, P_x^0, X_t^0, t > 0)$ associated with the operator

$$L_0 = \frac{1}{2}\nabla\cdot(A\nabla).$$

The semigroup S_t associated with the operator L has the following representation (see [5]):

$$\begin{aligned} S_t f(x) = \quad & E_x^0[f(X_t^0)\exp(\int_0^t (A^{-1}B)^*(X_s^0)dM_s^0 + (\int_0^t (A^{-1}\hat{B})^*(X_s^0)dM_s^0)\circ\gamma_t^0 \\ & -\frac{1}{2}\int_0^t (B-\hat{B})A^{-1}(B-\hat{B})^*(X_s^0)ds + \int_0^t Q(X_s^0)ds)], \end{aligned}$$

where M^0 is the martingale part of the diffusion X^0 and γ_t^0 is the reverse operator.

The main purpose of this paper is to study the nonlinear equation (1.1)(i.e. $F \neq 0$), which can not be handled by the methods used for the linear case. Our approach is first to solve a backward stochastic differential equation (BSDE) with singular coefficients and infinite horizon to produce a candidate for the solution of the boundary value problem and then to show that the candidate is indeed a solution. The results we obtained for BSDEs with infinite horizon are of independent interest.

We would like to mention that the first results on BSDEs and probabilistic interpretation of solutions of semilinear parabolic PDEs via BSDEs were obtained by Peng and pardoux in [19], [17] and [18]. There the operator L is smooth and the solution is a viscosity solution. We stress that the solutions we considered for PDEs in this paper are Soblev (also called weak) solutions, not viscosity solutions.

In [22], the corresponding Dirichlet problem for the semilinear elliptic PDEs:

$$\begin{cases} Lu(x) = -F(x, u(x), \nabla u(x)), & \text{on } D \\ u(x) = \Phi(x) & \text{on } \partial D \end{cases} \tag{1.4}$$

was solved. The strategy in [3], [22] is to transform the general operator L by a kind of h-transform to an operator of the form: $L_2 = \frac{1}{2}\nabla(A\nabla) + b\cdot\nabla + q$ which does not have the "bad" term such as $\nabla(\hat{B}\cdot)$. This idea is used in current paper too.

The BSDEs we studied are inspired by the ones in [10] where the author gave a probabilistic interpretation of the solution to the following Neumann problem:

$$\begin{cases} (\frac{1}{2}\triangle - \nu)u(x) = 0, & \text{on } D \\ \frac{\partial u}{\partial n} = \phi, & \text{on } \partial D \end{cases}$$

The content of the paper as follows. In Section 2, we study the following BSDEs with infinite horizon:

$$\begin{aligned} &dY(t) = -F(X(t), Y(t), Z(t))dt + e^{\int_0^t q(X(u))dt}\Phi(X(s))dL_t + \langle Z(t), dM(t)\rangle, \\ &\lim_{t\to\infty} e^{\int_0^t d(X(u))du} Y_t = 0 \quad in \quad L^2(\Omega), \end{aligned} \tag{1.5}$$

where $(X(t))_{t>0}$ is the reflecting diffusion associated with an infinitesimal generator of the form: $\mathcal{A} = \frac{1}{2}\nabla(A\nabla) + b\cdot\nabla$, $M(t)$ is the martingale part of $X(t)$, L_t is the boundary local time of X and $d(\cdot)$ is an appropriate measurable function. The existence and uniqueness of an L^2-solution (Y, Z) is obtained.
In Section 3, we solve the linear PDEs of the form:

$$\begin{cases} \frac{1}{2}\nabla(A\nabla u)(x) + b\cdot\nabla u(x) + qu(x) = F(x), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) = \phi(x) & \text{on } \partial D\ . \end{cases} \tag{1.6}$$

under the condition:

$$E_{x_0}[\int_0^\infty e^{\int_0^t q(X(u))du} dL_t] < \infty$$

for some $x_0 \in \bar{D}$. Useful estimates for local time and Girsanov density are proved which will also be used in subsequent sections.
In Section 4, we obtain the solution of the semilinear PDE:

$$\begin{cases} \frac{1}{2}\nabla(A\nabla u)(x) + b\cdot\nabla u(x) + qu(x) = G(x,u(x),\nabla u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial\gamma}(x) = \phi(x) & \text{on } \partial D\ . \end{cases} \tag{1.7}$$

To this end, we first use the solution $(Y_x(t), Z_x(t))$ of the BSDE (1.5) to produce a candidate $u_0(x) = E_x[Y_x(0)]$ and then find a solution u of an equation like (1.6) with a given $F(x) := G(x, u_0(x), v_0(x))$. Finally we identify u with u_0. In Section 5, we consider the general problem:

$$\begin{cases} Lu(x) = -F(x,u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial\gamma}(x) - \widehat{B}\cdot n(x)u(x) = \Phi(x) & \text{on } \partial D \end{cases}\ . \tag{1.8}$$

We apply the transformation introduced in [3] to transform the problem (1.8) to a problem like (1.7). An inverse transformation will yield the solution of the problem (1.8) under the condition that the L^p norm of $\hat{B}$ is sufficiently small.
To remove some of the restrictions imposed on $\hat{B}$ in Section 5, in Section 6, we study the L^1-solutions of the BSDEs (1.5) under appropriate conditions. Our approach is inspired by the one in [2]. The study of L^2-solutions and L^1-solutions of the BSDEs (1.5) are carried out in Section 2 and Section 6 separately because the methods used for these two cases are quite different.

2 BSDEs with Singular Coefficients and Infinity Horizon

Consider the operator

$$L_1 = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

on the domian D equipped with the Neumann boundary condition:

$$\frac{\partial}{\partial\gamma} := \langle An, \nabla\cdot\rangle = 0, \quad on \quad \partial D.$$

By [13], there exists a unique reflecting diffusion process denoted by $(\Omega, \mathcal{F}_t, X_x(t), P_x, \theta_t, x \in D)$ associated with the generator L_1.
Here $\theta : \Omega \to \Omega$ is the shift operator defined as follows:

$$X_x(s)(\theta_t\cdot) = X_x(t+s), \quad s,t \geq 0.$$

Let E_x denote the expectation under the measure P_x.
Set $\tilde{b} = \{\tilde{b}_1, ..., \tilde{b}_d\}$, where $\tilde{b}_i = \frac{1}{2}\sum_j \frac{\partial a_{ij}}{\partial x_j} + b_i$.
Then the process $X_x(t)$ has the following decomposition:

$$X_x(t) = X_x(0) + M_x(t) + \int_0^t \tilde{b}(X_x(s))ds + \int_0^t An(X_x(s))dL_s, \quad P_x - a.s.. \tag{2.1}$$

Here $M_x(t)$ is a $\mathcal{F}_t$ square integrable continuous martingale additive functional. And L_t is a positive increasing continuous additive functional satisfying $L_t = \int_0^t I_{\{X_x(s)\in\partial D\}} dL_s$.
We write $X_x(t)$ as $X(t)$ for short in the following discussion.

In this section, we will study the backward stochastic differential equations with singular coefficients and infinite horizon associated with the martingale part $M_x(t)$ and the local time L_t. A unique L^2 solution of such BSDEs is obtained.

Let $g(\omega, t, y, z) : \Omega \times R^+ \times R \times R^d \to R$ be a progressively measurable function. Consider the following conditions:
(A.1) $(y_1 - y_2)(g(t, y_1, z) - g(t, y_2, z)) \le -a_1(t)|y_1 - y_2|^2$,
(A.2) $|g(t, y, z_1) - g(t, y, z_2)| \le a_2|z_1 - z_2|$,
(A.3) $|g(t, y, z)| \le |g(t, 0, 0)| + a_3(t)(1 + |y|)$.
Here $a_1(t)$ and $a_3(t)$ are two progressively measurable processes and a_2 is a constant.
Set $a(t) = -a_1(t) + \delta a_2^2$, for some constant $\delta > \frac{1}{2\lambda}$, where λ is the constant appeared in (1.3).

Lemma 2.1 *Assume the conditions (A.1)-(A.3) and*

$$E_x[\int_0^\infty e^{2\int_0^t a(u)du}|g(t, 0, 0)|^2 dt] < \infty.$$

Then there exists a unique solution $(Y_x(t), Z_x(t))$ to the following backward stochastic differential equation:

$$\begin{aligned} &Y_x(t) = Y_x(T) + \int_t^T g(s, Y_x(s), Z_x(s))ds - \int_t^T < Z_x(s), dM_x(s) >, \quad t < T; \\ &\lim_{t\to\infty} e^{\int_0^t a(u)du} Y_x(t) = 0, \quad in \quad L^2(\Omega). \end{aligned} \tag{2.2}$$

Moreover,

$$E_x[\sup_t e^{2\int_0^t a(u)du}|Y_x(t)|^2] < \infty \quad and \quad E_x[\int_0^\infty e^{2\int_0^s a(u)du}|Z_x(s)|^2 ds] < \infty. \tag{2.3}$$

Proof.
Existence:
The proof of this lemma is similar to that of Theorem 3.2 in [22], but the terminal conditions here are different. By Theorem 3.1 in [22], the following BSDE has a unique solution $(Y_x^n(t), Z_x^n(t))$:

$$Y_x^n(t) = \int_t^n g(s, Y_x^n(s), Z_x^n(s))ds - \int_t^n < Z_x^n(s), dM_x(s) >, \quad t \le n; \tag{2.4}$$

and moreover,

$$Y_x^n(t) = 0, \quad Z_x^n(t) = 0, \quad t > n.$$

Fix $t > 0$ and $n > m > t$. It follows that

$$
\begin{aligned}
& e^{2\int_0^t a(u)du}|Y_x^n(t) - Y_x^m(t)|^2 + \int_t^\infty e^{2\int_0^s a(u)du}\langle A(X(s))(Z_x^n(s) - Z_x^m(s)), (Z_x^n(s) - Z_x^m(s))\rangle ds \\
= \quad & -2\int_t^n a(s)e^{2\int_0^s a(u)du}|Y_x^n(s) - Y_x^m(s)|^2 ds \\
& +2\int_t^n e^{2\int_0^s a(u)du}(Y_x^n(s) - Y_x^m(s))(g(s, Y_x^n(s), Z_x^n(s)) - g(s, Y_x^m(s), Z_x^m(s)))ds \\
& +2\int_m^n e^{2\int_0^s a(u)du}(Y_x^n(s) - Y_x^m(s))g(s, 0, 0)ds \\
& -2\int_t^n e^{2\int_0^s a(u)du}(Y_x^n(s) - Y_x^m(s)) < Z_x^n(s) - Z_x^m(s), dM_x(t) >
\end{aligned}
$$

Choose two positive numbers δ_1 and δ_2 such that $\delta_1 > \frac{1}{2\lambda}$ and $\delta_1 + \delta_2 < \delta$. Then from

$$
\begin{aligned}
& 2\int_t^n e^{2\int_0^s a(u)du}(Y_x^n(s) - Y_x^m(s))(g(s, Y_x^n(s), Z_x^n(s)) - g(s, Y_x^m(s), Z_x^m(s)))ds \\
\leq \quad & -2\int_t^n a_1(s)e^{2\int_0^s a(u)du}|Y_x^n(s) - Y_x^m(s)|^2 ds \\
& +2\delta_1 a_2^2\int_t^n e^{2\int_0^s a(u)du}|Y_x^n(s) - Y_x^m(s)|^2 ds \\
& +\frac{1}{2\lambda\delta_1}\int_t^n e^{2\int_0^s a(u)du}\langle A(X(s))(Z_x^n(s) - Z_x^m(s)), (Z_x^n(s) - Z_x^m(s))\rangle ds
\end{aligned}
$$

and

$$
\begin{aligned}
& 2\int_m^n e^{2\int_0^s a(u)du}(Y_x^n(s) - Y_x^m(s))g(s, 0, 0)ds \\
\leq \quad & 2\delta_2 a_2^2\int_m^n e^{2\int_0^s a(u)du}|Y_x^n(s) - Y_x^m(s)|^2 ds + \frac{1}{2\delta_2 a_2^2}\int_m^n e^{2\int_0^s a(u)du}|g(s, 0, 0)|^2 ds,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
& E_x[e^{2\int_0^t a(u)du}|Y_x^n(t) - Y_x^m(t)|^2] + \frac{1}{\lambda}(1 - \frac{1}{2\lambda\delta_1})E_x[\int_t^\infty e^{2\int_0^s a(u)du}|Z_x^n(s) - Z_x^m(s)|^2 ds] \\
& \leq \frac{1}{2\delta_2 a_2^2}E_x[\int_m^n e^{2\int_0^s a(u)du}|g(s, 0, 0)|^2 ds].
\end{aligned}
$$

This implies that

$$
E_x[\int_0^\infty e^{2\int_0^s a(u)du}|Z_x^n(s) - Z_x^m(s)|^2 ds] \to 0, \quad as \quad m, n \to \infty.
$$

Hence there exists $\tilde{Z}_x$ such that

$$
\tilde{Z}_x = \lim_{n\to\infty} e^{\int_0^\cdot a(u)du} Z_x^n \quad in \quad L^2([0, \infty) \times \Omega).
$$

At the same time, we also obtain the following estimates:

$$\begin{aligned}
&\sup_t e^{2\int_0^t a(u)du}|Y_x^n(t)-Y_x^m(t)|^2\\
\leq\ & \frac{1}{2\delta_2 a_2^2}\int_m^n e^{2\int_0^s a(u)du}|g(s,0,0)|^2 ds\\
&+2\sup_t|\int_t^n e^{2\int_0^s a(u)du}(Y_x^n(s)-Y_x^m(s)) < Z_x^n(s)-Z_x^m(s), dM_x(t)>|.
\end{aligned}$$

Taking expectation on both sides of the above inequality, by BDG inequality, we obtain

$$\begin{aligned}
&E_x[\sup_t e^{2\int_0^t a(u)du}|Y_x^n(t)-Y_x^m(t)|^2]\\
\leq\ & \frac{1}{2\delta_2 a_2^2}E_x[\int_m^n e^{2\int_0^s a(u)du}|g(s,0,0)|^2 ds]\\
&+C_1E_x[\{\int_t^n e^{4\int_0^s a(u)du}|Y_x^n(s)-Y_x^m(s)|^2|Z_x^n(s)-Z_x^m(s)|^2 ds\}^{\frac{1}{2}}]\\
\leq\ & \frac{1}{2\delta_2 a_2^2}E_x[\int_m^n e^{2\int_0^s a(u)du}|g(s,0,0)|^2 ds]+\frac{1}{2}E_x[\sup_t e^{2\int_0^t a(u)du}|Y_x^n(t)-Y_x^m(t)|^2]\\
&+C_2E_x[\int_0^\infty e^{2\int_0^s a(u)du}|Z_x^n(s)-Z_x^m(s)|^2 ds]
\end{aligned}$$

Thus

$$\begin{aligned}
&E_x[\sup_t e^{2\int_0^t a(u)du}|Y_x^n(t)-Y_x^m(t)|^2]\\
\leq\ & \frac{1}{\delta_2 a_2^2}E_x[\int_m^n e^{2\int_0^s a(u)du}|g(s,0,0)|^2 ds]+2C_2E_x[\int_0^\infty e^{2\int_0^s a(u)du}|Z_x^n(s)-Z_x^m(s)|^2 ds]\\
\to\ & 0, \quad as \quad m,n\to\infty.
\end{aligned}$$

So, there exists $\{\tilde{Y}_x(t)\}$ such that

$$\lim_{n\to\infty}E_x[\sup_t|\tilde{Y}_x(t)-e^{\int_0^t a(u)du}Y_x^n(t)|^2]=0.$$

For any $\varepsilon>0$, there exist a positive number N such that for any $n\geq N$,

$$E_x[\sup_t|\tilde{Y}_x(t)-e^{\int_0^t a(u)du}Y_x^n(t)|^2]<\frac{\varepsilon}{2}.$$

For $t>N$, noticing $Y_x^N(t)=0$, it follows that

$$\begin{aligned}
E_x[|\tilde{Y}_x(t)|^2] &\leq 2E_x[|\tilde{Y}_x(t)-e^{\int_0^t a(u)du}|Y_x^N(t)|^2]+2E_x[e^{2\int_0^t a(u)du}|Y_x^N(t)|^2]\\
&\leq 2E_x[\sup_t|\tilde{Y}_x(t)-e^{\int_0^t a(u)du}|Y_x^N(t)|^2]+2E_x[e^{2\int_0^t a(u)du}|Y_x^N(t)|^2]\\
&< \varepsilon.
\end{aligned}$$

Thus we have $\lim_{t\to 0} E_x[|\tilde{Y}_x(t)|^2] = 0$.
By chain rule, it is easy to see from (2.4) that

$$Y_x(t) = e^{-\int_0^t a(u)du}\tilde{Y}_x(t) \quad and \quad Z_x(t) = e^{-\int_0^t a(u)du}\tilde{Z}_x(t)$$

satisfy the equation (2.2) and

$$\lim_{t\to\infty} E_x[e^{2\int_0^t a(u)du}|Y_x(t)|^2] = \lim_{t\to\infty} E_x[|\tilde{Y}_x(t)|^2] = 0.$$

From the above proof, we also see that (2.3) holds.
Uniqueness:
Suppose that (Y_x^1, Z_x^1) and (Y_x^2, Z_x^2) are two solutions of the equation (2.2).
Set $\bar{Y}_x(t) = Y_x^1(t) - Y_x^2(t)$ and $\bar{Z}_x(t) = Z_x^1(t) - Z_x^2(t)$. Then

$$\begin{aligned} d(e^{\int_0^t a(u)du}\bar{Y}_x(t)) &= -e^{\int_0^t a(u)du}(g(t, Y_x^1(t), Z_x^1(t)) - g(t, Y_x^2(t), Z_x^2(t)))dt \\ &+ a(t)e^{\int_0^t a(u)du}\bar{Y}_x(t)dt \\ &+ e^{\int_0^t a(u)du}\langle \bar{Z}_x(t), dM_x(t)\rangle. \end{aligned} \tag{2.5}$$

By Ito's formula, we get, for any $t < T$,

$$\begin{aligned} & e^{2\int_0^t a(u)du}|\bar{Y}_x(t)|^2 + \int_t^T e^{2\int_0^t a(u)du}\langle A(X(s))\bar{Z}_x(s), \bar{Z}_x(s)\rangle ds \\ = \; & e^{2\int_0^T a(u)du}|\bar{Y}_x(T)|^2 + 2\int_t^T e^{2\int_0^s a(u)du}\bar{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))ds \\ & -2\int_t^T a(s)e^{2\int_0^s a(u)du}|\bar{Y}_x(s)|^2 ds \\ & -2\int_t^T a(s)e^{2\int_0^s a(u)du}\bar{Y}_x(s)\langle \bar{Z}_x(s), dM_x(s)\rangle \end{aligned} \tag{2.6}$$

By condition (A.1) and (A.2), we have

$$\begin{aligned} & 2\int_t^T e^{2\int_0^s a(u)du}\bar{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))ds \\ = \; & 2\int_t^T e^{2\int_0^s a(u)du}\bar{Y}_x(s)(g(s, Y_x^1(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^1(s)))ds \\ + \; & 2\int_t^T e^{2\int_0^s a(u)du}\bar{Y}_x(s)(g(s, Y_x^2(s), Z_x^1(s)) - g(s, Y_x^2(s), Z_x^2(s)))ds \\ \le \; & -2\int_t^T a_1(s)e^{2\int_0^s a(u)du}|\bar{Y}_x(s)|^2 ds + a_2\int_t^T e^{2\int_0^s a(u)du}\bar{Y}_x(s)|\bar{Z}_x(s)|ds \\ \le \; & -2\int_t^T a_1(s)e^{2\int_0^s a(u)du}|\bar{Y}_x(s)|^2 ds + c'a_2\int_t^T e^{2\int_0^s a(u)du}|\bar{Y}_x(s)|^2 ds \\ & +a_2\frac{1}{c'\lambda}\int_t^T e^{2\int_0^s a(u)du}|\bar{Z}_x(s)|^2 ds. \end{aligned} \tag{2.7}$$

Choosing $c' = 2\delta a_2$, we obtain

$$
\begin{aligned}
& |e^{\int_0^t a(u)du}\bar{Y}_x(t)|^2 + (1 - \frac{1}{2\delta\lambda})\int_t^T e^{2\int_0^t a(u)du}\langle A(X(s))\bar{Z}_x(s), \bar{Z}_x(s)\rangle ds \\
\leq \quad & e^{2\int_0^T a(u)du}|\bar{Y}_x(T)|^2 - 2\int_t^T a(s)e^{2\int_0^s a(u)du}\bar{Y}_x(s)\langle \bar{Z}_x(s), dM_x(s)\rangle
\end{aligned}
\tag{2.8}
$$

Taking expectation on both sides of the above inequality, we get that, for any $t < T$,

$$E_x[e^{2\int_0^t a(u)du}|\bar{Y}_x(t)|^2] \leq E_x[e^{2\int_0^T a(u)du}|\bar{Y}_x(T)|^2].$$

For both Y^1 and Y^2 satisfy the terminal condition in (2.2), so that

$$\lim_{T\to\infty} E_x[e^{2\int_0^T a(u)du}|\bar{Y}_x(T)|^2] = 0,$$

which leads to $E_x[e^{2\int_0^t a(u)du}|\bar{Y}_x(t)|^2] = 0$.
We conclude that $Y_x^1(t) = Y_x^2(t)$ and $Z_x^1(t) = Z_x^2(t)$. $\square$

We now want to apply Lemma 2.1 to a particular situation.
Let $F(x, y, z) : R^d \times R \times R^d \to R$ be a Borel measurable function. Consider the following conditions:
(D.1) $(y_1 - y_2)(F(x, y_1, z) - F(x, y_2, z)) \leq -d_1(x)|y_1 - y_2|^2$,
(D.2) $|F(x, y, z_1) - F(x, y, z_2)| \leq d_2|z_1 - z_2|$,
(D.3) $|F(x, y, z)| \leq |F(x, 0, z)| + K(x)(1 + |y|)$.
Set $d(x) = -d_1(x) + \delta d_2^2$ for some constant $\delta > \frac{1}{2\lambda}$.
The follows result follows from Lemma 2.1.

Lemma 2.2 *Assume the conditions (D.1)-(D.3) and*

$$E_x[\int_0^\infty e^{2\int_0^t d(X(u))du}|F(X(t), 0, 0)|^2 dt] < \infty.$$

Then there exists a unique solution $(Y_x(t), Z_x(t))$ *to the following equation:*

$$
\begin{aligned}
& Y_x(t) = Y_x(T) + \int_t^T F(X(s), Y_x(s), Z_x(s))ds - \int_t^T < Z_x(s), dM_x(s) >, \quad t < T; \\
& \lim_{t\to\infty} e^{\int_0^t d(X(u))du}Y_x(t) = 0, \quad in \quad L^2(\Omega).
\end{aligned}
\tag{2.9}
$$

Consider the following condition instead of $(D.3)$.
(D.3)$'$ $|F(X(t), y, z)| \leq K(t)$, for any $y \in R$ and $z \in R^d$.

Let Φ be a bounded measurable function defined on ∂D, and function $\tilde{q} \in L^p(D)$, for $p > \frac{d}{2}$.
The following theorem is the main result in this section.

Theorem 2.1 *Assume the conditions (D.1), (D.2) and (D.3)',*

$$E_{x_0}[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))du} dL_s] < \infty$$

for some $x_0 \in D$ and for $x \in D$,

$$E_x[\int_0^\infty e^{2\int_0^t d(X(u))du}\{e^{2\int_0^t \tilde{q}(X(u))du} + |K(t)|^2\}dt] < \infty. \quad (2.10)$$

Then there exists a unique solution (Y_x, Z_x) to the following BSDE:

$$\begin{aligned} Y_x(t) &= Y_x(T) + \int_t^T F(X(s), Y_x(s), Z_x(s))ds - \int_t^T e^{\int_0^s \tilde{q}(X(u))dt}\Phi(X(s))dL_s \\ &\quad - \int_t^T \langle Z_x(s), dM_x(s)\rangle, \quad for \quad t < T, \end{aligned} \quad (2.11)$$

and

$$\lim_{t\to\infty} e^{\int_0^t d(X(u))du} Y_t = 0 \quad in \quad L^2(\Omega). \quad (2.12)$$

Proof.
Uniqueness:
Suppose that (Y_x^1, Z_x^1) and (Y_x^2, Z_x^2) are two solutions of the equation (2.11) satisfying (2.12). Set $\bar{Y}_x(t) = Y_x^1(t) - Y_x^2(t)$ and $\bar{Z}_x(t) = Z_x^1(t) - Z_x^2(t)$. Then

$$\begin{aligned} d(e^{\int_0^t d(X(u))du}\bar{Y}_x(t)) &= -e^{\int_0^t d(X(u))du}(F(X(t), Y_x^1(t), Z_x^1(t)) - F(X(t), Y_x^2(t), Z_x^2(t)))dt \\ &+ d(X(t))e^{\int_0^t d(X(u))du}\bar{Y}_x(t)dt \\ &+ e^{\int_0^t d(X(u))du}\langle \bar{Z}_x(t), dM_x(t)\rangle. \end{aligned} \quad (2.13)$$

By Ito's formula, we get, for any $t < T$,

$$\begin{aligned} & e^{2\int_0^t d(X(u))du}|\bar{Y}_x(t)|^2 + \int_t^T e^{2\int_0^t d(X(u))du}\langle A(X(s))\bar{Z}_x(s), \bar{Z}_x(s)\rangle ds \\ = \;& e^{2\int_0^T d(X(u))du}|\bar{Y}_x(T)|^2 \\ + \;& 2\int_t^T e^{2\int_0^s d(X(u))du}\bar{Y}_x(s)(F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s)))ds \\ - \;& 2\int_t^T d(X(s))e^{2\int_0^s d(X(u))du}|\bar{Y}_x(s)|^2 ds \\ - \;& 2\int_t^T d(X(s))e^{2\int_0^s d(X(u))du}\bar{Y}_x(s)\langle \bar{Z}_x(s), dM_x(s)\rangle \end{aligned} \quad (2.14)$$

By (D.1) and (D.2), we have

$$2\int_t^T e^{2\int_0^s d(X(u))du}\bar{Y}_x(s)(F(X(s), Y_x^1(s), Z_x^1(s)) - F(X(s), Y_x^2(s), Z_x^2(s)))ds$$

$$
\begin{aligned}
&= 2\int_t^T e^{2\int_0^s d(X(u))du}\bar{Y}_x(s)(F(X(s),Y_x^1(s),Z_x^1(s)) - F(X(s),Y_x^2(s),Z_x^1(s)))ds \\
&+ 2\int_t^T e^{2\int_0^s d(X(u))du}\bar{Y}_x(s)(F(X(s),Y_x^2(s),Z_x^1(s)) - F(X(s),Y_x^2(s),Z_x^2(s)))ds \\
&\leq -2\int_t^T d_1(X(s))e^{2\int_0^s d(X(u))du}|\bar{Y}_x(s)|^2 ds + d_2\int_t^\infty e^{2\int_0^s d(X(u))du}\bar{Y}_x(s)|\bar{Z}_x(s)|ds \\
&\leq -2\int_t^T d_1(X(s))e^{2\int_0^s d(X(u))du}|\bar{Y}_x(s)|^2 ds + cd_2\int_t^T e^{2\int_0^s d(X(u))du}|\bar{Y}_x(s)|^2 ds \\
&\quad + d_2\frac{1}{c\lambda}\int_t^T e^{2\int_0^s d(X(u))du}|\bar{Z}_x(s)|^2 ds. \qquad (2.15)
\end{aligned}
$$

Choosing $c = 2\delta d_2$, we obtain from (2.15)

$$
\begin{aligned}
&|e^{-\int_0^t d(X(u))du}\bar{Y}_x(t)|^2 + (1-\frac{1}{2\delta\lambda})\int_t^T e^{-2\int_0^t d(X(u))du}\langle A(X(s))\bar{Z}_x(s),\bar{Z}_x(s)\rangle ds \\
&\leq e^{2\int_0^T d(X(u))du}|\bar{Y}_x(T)|^2 - 2\int_t^T d(X(s))e^{-2\int_0^s d(X(u))du}\bar{Y}_x(s)\langle\bar{Z}_x(s),dM_x(s)\rangle \qquad (2.16)
\end{aligned}
$$

Taking expectation on both sides of the above inequality and letting T tend to infinity, we obtain that

$$E_x[e^{2\int_0^t d(X(u))du}|\bar{Y}_x(t)|^2] = 0$$

We conclude that $Y_x^1(t) = Y_x^2(t)$ and hence from (2.16), $Z_x^1(t) = Z_x^2(t)$.

Existence:
First of all, the assumption (2.10) implies (see [**?**])

$$\sup_x E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))du}dL_s] < \infty.$$

1°: There exists $(p_x(t), q_x(t))$ such that

$$dp_x(t) = e^{\int_0^t \tilde{q}(X(u))du}\Phi(X(t))dL_t + < q_x(t), dM_x(t) >, \qquad (2.17)$$

and $e^{\int_0^t d(X(u))du}p_x(t) \to 0$ as $t \to \infty$ in $L^2(\Omega)$.
In fact, let

$$
\begin{aligned}
p_x(t) &:= -E_x[\int_t^\infty e^{\int_0^s \tilde{q}(X(u))du}\Phi(X(s))dL_s|\mathcal{F}_t] \\
&= \int_0^t e^{\int_0^s \tilde{q}(X(u))du}\Phi(X(s))L_s - E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))du}\Phi(X(s))dL_s|\mathcal{F}_t]. \qquad (2.18)
\end{aligned}
$$

By the martingale representation theorem in [22], there exists a process $q_x(t)$, such that

$$
\begin{aligned}
-E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))du}\Phi(X(s))dL_s|\mathcal{F}_t] &= -E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u))du}\Phi(X(s))dL_s] \\
&+ \int_0^t < q_x(s), dM_x(s) > . \qquad (2.19)
\end{aligned}
$$

Then (p_x, q_x) satisfies the equation (2.17).
Moreover,

$$\begin{aligned} p_x(t) &:= -E_x[\int_t^\infty e^{\int_0^s \tilde{q}(X(u))du}\Phi(X(s))dL_s|\mathcal{F}_t] \\ &= -e^{\int_0^t \tilde{q}(X(u))du}E_x[\int_t^\infty e^{\int_t^s \tilde{q}(X(u))du}\Phi(X(s))dL_s|\mathcal{F}_t] \\ &= -e^{\int_0^t \tilde{q}(X(u))du}E_x[\int_0^\infty e^{\int_t^{s+t} \tilde{q}(X(u))du}\Phi(X(s+t))dL_{s+t}|\mathcal{F}_t] \\ &= -e^{\int_0^t \tilde{q}(X(u))du}E_x[\int_0^\infty e^{\int_0^s \tilde{q}(X(u+t))du}\Phi(X(s+t))dL_{s+t}|\mathcal{F}_t] \\ &= -e^{\int_0^t \tilde{q}(X(u))du}E_{X(t)}[\int_0^\infty e^{\int_0^l \tilde{q}(X(u))du}\Phi(X(l))dL_l] \end{aligned} \tag{2.20}$$

The last equality follows from the fact that $L_{t+s} = L_t + L_s \circ \theta_t$. Therefore,

$$\sup_x |p_x(t)| \le e^{\int_0^t \tilde{q}(X(u))du} \sup_{x\in D}|\Phi(x)| \cdot \sup_{x\in\bar{D}} E_x[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))du}dL_t].$$

Set $M = \sup_{x\in D}|\Phi(x)| \cdot \sup_{x\in\bar{D}} E_x[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))du}dL_t]$.
In view of (2.10), we have $\lim_{t\to\infty} e^{\int_0^t (d+\tilde{q})(X(u))du} = 0$ in $L^2(\Omega)$.
Hence,

$$e^{\int_0^t d(X(u))du}p_x(t) \le Me^{\int_0^t (d+\tilde{q})(X(u))du} \to 0 \quad as \quad t\to\infty, \quad in \quad L^2(\Omega). \tag{2.21}$$

2° : Set $g(t,y,z) = F(X(t), p_x(t)+y, q_x+z)$. Then

$$\begin{aligned} & (y_1-y_2)(g(t,y_1,z)-g(t,y_2,z)) \\ = \ & (y_1-y_2)(F(X(t),p_x(t)+y_1,q_x+z)-F(X(t),p_x(t)+y_2,q_x+z)) \\ \le \ & -d_1(X(t))|y_1-y_2|^2. \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} |g(t,y,z_1)-g(t,y,z_2)| &= |F(X(t),p_x(t)+y,q_x+z_1)-F(X(t),p_x(t)+y,q_x+z_1)| \\ &\le d_2|z_1-z_2|. \end{aligned} \tag{2.23}$$

Moreover,

$$\begin{aligned} & E_x[\int_0^\infty e^{2\int_0^t d(X(u))du}|g(X(t),0,0)|^2dt] \\ \le \ & E_x[\int_0^\infty e^{2\int_0^t d(X(u))du}|F(X(t),p_x(t),q_x(t))|^2dt] \\ \le \ & E_x[\int_0^\infty e^{2\int_0^t d(X(u))du}|K(t)|^2dt] \\ < \ & \infty. \end{aligned} \tag{2.24}$$

g satisfies all the conditions of the Lemma 2.2. Hence, there exist processes (k_x, l_x) such that

$$dk_x(t) = -g(t, k_x(t), l_x(t))dt + < l_x(t), dM_x(t) >,$$

and

$$e^{\int_0^t d(X(u))du} k_x(t) \to 0,$$

as $t \to \infty$.
Putting $Y_x(t) = p_x(t) + k_x(t)$ and $Z_x(t) = q_x(t) + l_x(t)$, we find that $(Y_x(t), Z_x(t))$ satisfies the following equation

$$dY_x(t) = e^{\int_0^t \tilde{q}(X(u))du} \phi(X(t))dL_t - F(t, Y_x(t), Z_x(t))dt + < Z_x(t), dM_x > .$$

and

$$\lim_{t\to\infty} e^{\int_0^t d(X(u))du} Y_t = 0.$$

Corollary 2.1 *Suppose all the assumptions in Theorem 2.1 hold. If, in addition,*

$$\sup_x E_x[\int_0^\infty e^{\int_0^t d(X(u))du} |K(t)|^2 dt] < \infty,$$

then it follows that

$$\sup_{x\in D} |Y_x(0)| < \infty.$$

Proof.
As shown in the proof of Theorem 2.1, $Y_x(t)$ has the decomposition: $Y_x(t) = p_x(t) + k_x(t)$. Setting $t = 0$ in (2.20), it follows that

$$\begin{aligned} |p_x(0)| &\leq E_{X(t)}[|\int_0^\infty e^{\int_0^l \tilde{q}(X(u))du} \Phi(X(l))dL_l|] \\ &\leq \|\Phi\|_\infty \sup_x E_x[\int_0^\infty e^{\int_0^l \tilde{q}(X(u))du} dL_l] \\ &< \infty. \end{aligned} \tag{2.25}$$

By Ito's formula, we obtain

$$\begin{aligned} de^{2\int_0^t d(X(u))du} |k_x(t)|^2 &= -2e^{2\int_0^t d(X(u))du} k_x(t) g(t, k_x(t), l_x(t))dt \\ &+ 2e^{2\int_0^t d(X(u))du} k_x(t) d(X(t))dt + 2e^{2\int_0^t d(X(u))du} k_x(t) < l_x(t), dM_x(t) > \\ &+ e^{2\int_0^t d(X(u))du} \langle A(X(t)) l_x(t), l_x(t) \rangle dt \end{aligned}$$

Choosing two positive numbers δ_1 and δ_2 such that $\delta_1 > \frac{1}{2\lambda}$ and $\delta_1 + \delta_2 < \delta$, similar calculations as in the proof of Theorem 2.1 yield that, for any $t < T$,

$$\begin{aligned} & E_x[e^{2\int_0^t d(X(u))du} |k_x(t)|^2] + \frac{1}{\lambda}(1 - \frac{1}{2\lambda\delta_1}) E_x[\int_t^T e^{2\int_0^s d(X(u))du} |l_x(s)|^2 ds] \\ \leq\ & E_x[e^{2\int_t^T d(X(u))du} |k_x(T)|^2] + \frac{1}{2\delta_2 d_2^2} E_x[\int_t^T e^{2\int_0^s d(X(u))du} |g(s, 0, 0)|^2 ds]. \end{aligned}$$

Setting $t = 0$, we have

$$\begin{aligned} |k_x(0)|^2 = E_x[|k_x(0)|^2] &\leq E_x[e^{2\int_0^T d(X(u))du}|k_x(T)|^2] \\ &+ \frac{1}{2\delta_2 d_2^2} E_x[\int_0^T e^{2\int_0^s d(X(u))du}|g(s,0,0)|^2 ds]. \end{aligned}$$

Let $T \to \infty$ to obtain that

$$\sup_x |k_x(0)| \leq \left(\frac{1}{2\delta_2 d_2^2} \sup_x E_x[\int_0^\infty e^{2\int_0^s d(X(u))du}|g(s,0,0)|^2 ds] \right)^{\frac{1}{2}} < \infty,$$

where the fact that $e^{2\int_0^T d(X(u))du} k_x(T) \to 0$ as $T \to \infty$, has been used. Hence, we have $\sup_x |Y_x(0)| \leq \sup_x |p_x(0)| + \sup_x |k_x(0)| < \infty$.

3 Linear PDEs

Set

$$L_2 = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q$$

where $b = (b_1, ..., b_d)$ is a R^d-valued Borel measurable function, and q is a Borel measurable function on R^d such that:

$$I_D(|b|^2 + |q|) \in L^p(D), \quad p > \frac{d}{2}.$$

In this section, we solve the following linear boundary value problem:

$$\begin{cases} \frac{1}{2}\nabla \cdot (A\nabla u)(x) + b \cdot \nabla u(x) + q(x)u(x) = F(x), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) = \phi & \text{on } \partial D \ , \end{cases} \tag{3.1}$$

where F and ϕ are bounded measurable functions on D.

It is well known that operator L_2 defined on a bounded domain D with Neumann boundary condition $\frac{\partial u}{\partial \gamma}(x) = 0$ is associated with the quadratic form:

$$\begin{aligned} \mathcal{E}(f,g) : &= -\int_D L_2 f(x)g(x)dx \\ &= \frac{1}{2}\int_D \langle A\nabla f, \nabla g\rangle dx - \int_D b \cdot \nabla f(x)g(x)dx - \int_D q(x)f(x)g(x)dx \end{aligned}$$

Definition 3.1 *A bounded continuous function $u(x)$ defined on D is a weak solution of the problem* (3.1) *if $u \in W^{1,2}(D)$, and for any $g \in C^\infty(\overline{D})$,*

$$\mathcal{E}(u,g) = \int_{\partial D} \phi(x)g(x)\sigma(dx) - \int_D F(x)g(x)dx,$$

where σ denotes the $d-1$ dimensional Lebesgue measure on ∂D.

Consider the operator

$$L_0 = \frac{1}{2}\nabla \cdot (A\nabla u) \tag{3.2}$$

on domain D with boundary condition $\frac{\partial u}{\partial \gamma} = 0$ on ∂D.
L_0 is associated with a reflecting diffusion process (X^0, P_x^0). By [13], X^0 has the following decomposition:

$$\begin{aligned} dX_t^0 &= \sigma(X_t^0)dW_t + \frac{1}{2}\nabla A(X_t^0)dt + \gamma(X_t^0)dL_t^0, \\ L_t^0 &= \int_0^t I_{\{X_s^0 \in \partial D\}} dL_s^0, \end{aligned} \tag{3.3}$$

where the matrix $\sigma(x)$ is the positive definite symmetric square root of the matrix $A(x)$ and $\{W_t\}_{t>0}$ is a d-dimensional standard Brownian motion.
It is well known that operator L_0 is associated with the regular Dirichlet form:

$$\mathcal{E}^0(u,v) = \frac{1}{2}\int_D a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx$$

and the domain of $\mathcal{E}^0$ is $W^{1,2}(D) := \{u \in L^2(D) : \frac{\partial u}{\partial x_i} \in L^2(D)\}$.

The following lemma can be proved similarly as the Corollary 3.8 in [11] using the heat kernel estimates in [21].

Lemma 3.1 *There exists a constant $K > 0$, such that*

$$\sup_{x\in\bar{D}} E_x^0[L_t^0] \le K\sqrt{t} \quad \text{and} \quad \inf_{x\in\bar{D}} E_x^0[L_t^0] > 0.$$

Moreover, we have $\sup_{x\in\bar{D}} E_x^0[(L_t^0)^n] \le K_n t^{\frac{n}{2}}$, for some constant $K_n > 0$.

Set $M_t^0 = \int_0^t \sigma(X_s^0)dW_s$ and

$$Z_t = e^{\int_0^t <A^{-1}b(X_s^0), dM_s^0> - \frac{1}{2}\int_0^t bA^{-1}b^*(X_s^0)ds + \int_0^t q(X_s^0)ds}, \tag{3.4}$$

where b^* is the transpose of the row vector b.

The proof of the following two lemmas are inspired by that of the Lemma 2.1 and Theorem 2.2 in [11].

Lemma 3.2 *For $t > 0$, there are two strictly positive functions $M_1(t)$ and $M_2(t)$ such that, for any $x \in \overline{D}$, $M_1(t) \le E_x^0[\int_0^t Z_s dL_s^0] \le M_2(t)$. Furthermore, $M_2(t) \to 0$ as $t \to 0$.*

PROOF.
1°: Put

$$\tilde{M}(t) = e^{\int_0^t <A^{-1}b(X_s^0), dM_s^0> - \frac{1}{2}\int_0^t bA^{-1}b^*(X_s^0)ds}, \tag{3.5}$$
$$e_q(t) = e^{\int_0^t q(X_s^0)ds},$$
$$M_q(t) = \sup_{x\in\overline{D}} E_x^0[\int_0^t |q(X_s^0)|ds].$$

Then we have

$$\begin{aligned}
\sup_{x\in\overline{D}} E_x^0[\int_0^t Z_s dL_s^0] &= \sup_{x\in\overline{D}} E_x^0[\int_0^t \tilde{M}(s)e_q(s)dL_s^0] \\
&\leq \sup_{x\in\overline{D}} E_x^0[\max_{0\leq s\leq t} |\tilde{M}(s)|^2]^{\frac{1}{2}} \cdot \sup_{x\in\overline{D}} E_x^0[e_{2|q|}(t)(L_t^0)^2]^{\frac{1}{2}} \\
&\leq \underbrace{\sup_{x\in\overline{D}} E_x^0[|\tilde{M}(t)|^2]^{\frac{1}{2}}}_{(I)} \cdot \underbrace{\sup_{x\in\overline{D}} E_x^0[e_{4|q|}(t)]^{\frac{1}{4}}}_{(II)} \cdot \underbrace{\sup_{x\in\overline{D}} E_x^0[(L_t^0)^4]^{\frac{1}{4}}}_{(III)}
\end{aligned} \tag{3.6}$$

By Khash'Minskii's lemma and Theorem 2.1 in [15], (I) and (II) are bounded if t belongs to a bounded interval. Because of $E_x^0[(L_t^0)^n] \leq K_n t^{\frac{n}{2}}$, we see that $M_2(t) := K(I)(II)\sqrt{t}$ is the required upper bound.
2°: Since

$$E_x^0[L_t^0]^2 \leq E_x^0[\int_0^t \tilde{M}^{-1}(s)e_{-q}(s)dL_s^0] \cdot E_x^0[\int_0^t \tilde{M}(s)e_q(s)dL_s^0], \tag{3.7}$$

we obtain

$$E_x^0[\int_0^t \tilde{M}(s)e_q(s)dL_s^0] \geq \frac{E_x^0[L_t^0]^2}{E_x^0[\int_0^t \tilde{M}^{-1}(s)e_{-q}(s)dL_s^0]}. \tag{3.8}$$

Here

$$\begin{aligned}
\tilde{M}^{-1}(t) &= e^{-\int_0^t <A^{-1}b(X_s^0), dM_s^0> + \frac{1}{2}\int_0^t bA^{-1}b^*(X_s^0)ds} \\
&= e^{-\int_0^t <A^{-1}b(X_s), dM_s^0> - \frac{1}{2}\int_0^t bA^{-1}b^*(X_s^0)ds} \cdot e^{\int_0^t bA^{-1}b^*(X_s^0)ds} \\
&:= N(t) \cdot e^{\int_0^t bA^{-1}b^*(X_s^0)ds}
\end{aligned} \tag{3.9}$$

By the proof of the first part, replacing $\tilde{M}_t$, q by N_t and $bA^{-1}b^* - q$ respectively, it is seen that there exists $K(t) > 0$ such that $\sup_{x\in\overline{D}} E_x^0[\int_0^t \tilde{M}^{-1}(s)e_{-q}(s)dL_s^0] \leq K(t)$.
As $\inf_{x\in\overline{D}} E_x^0[L_t^0] > 0$, we complete the proof of the lemma by setting $M_1(t) = \frac{\inf_{x\in\overline{D}} E_x^0[L_t^0]^2}{K(t)}$.□

Set $G(x) := E_x^0[\int_0^\infty Z_s dL_s^0]$.

Lemma 3.3 *If there is a point $x_0 \in \overline{D}$, such that $G(x_0) < \infty$, then there are two positive constants K and β such that $\sup_{x\in\overline{D}} E_x^0[Z_t] \leq Ke^{-\beta t}$.*

Proof.
By Girsanov Theorem and Feymann-Kac formula, $L_2 = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q$ is associated with the semigroup $\{T_t\}_{t>0}$, where $T_t f(x) = E^0_x[Z_t f(X^0_t)]$ for $f \in L^2(D)$.
By the upper and lower bound estimates of the heat kernel $p_2(t,x,y)$ associated with T_t in [21], the following inequality holds,

$$c^{-1}\int_D f(x)dx \leq E^0_x[Z_1 f(X^0_1)] \leq c\int_D f(x)dx, \tag{3.10}$$

where c is a positive constant. Since

$$G(x) = \sum_{n=0}^{\infty} E^0_x[Z_n E^0_{X^0_n}[\int_0^1 Z_s L^0(ds)]] \geq M_1(1)\sum_{n=0}^{\infty} E^0_x[Z_n]$$

and $G(x_0) < \infty$, there is a positive integer number N such that

$$\frac{1}{2c^2} \geq E^0_{x_0}[Z_N] = E^0_{x_0}[Z_1 E^0_{X_1}[Z_{N-1}]] \geq c^{-1}\int_D E^0_x[Z_{N-1}]m(dx).$$

This implies

$$\int_D E^0_x[Z_{N-1}]m(dx) \leq \frac{1}{2c}.$$

Thus

$$\sup_{x\in\overline{D}} E^0_x[Z_N] = \sup_{x\in\overline{D}} E^0_x[Z_1 E^0_{X_1}[Z_{N-1}]] \leq c\int_D E^0_x[Z_{N-1}]m(dx) \leq \frac{1}{2}. \tag{3.11}$$

For any $t > 0$, there exists a positive number n such that $\frac{t}{N} \in [n-1, n)$. Then by (3.11), it follows that

$$\begin{aligned} E^0_x[Z_t] \leq \frac{1}{2^{n-1}} E^0_x[Z_{t-N(n-1)}] &\leq \left(\sup_{x\in D, 0\leq t\leq N} E^0_x[Z_t]\right)\frac{1}{2^{n-1}} \\ &\leq 2\sup_{x\in D, 0\leq t\leq N} E^0_x[Z_t]e^{-\frac{ln2}{N}t}.\Box \end{aligned} \tag{3.12}$$

Theorem 3.1 *If there exists $x_0 \in \overline{D}$ such that $G(x_0) < \infty$, then there exists a unique bounded continuous weak solution of the problem* (3.1)*:*

Proof.
Existence :
Due to Theorem 3.2 in [4], there exists a unique, bounded, continuous weak solution u_2 of the following problem:

$$\begin{cases} L_2 u_2(x) = 0, & \text{on } D \\ \frac{1}{2}\frac{\partial u_2}{\partial \gamma}(x) = \phi & \text{on } \partial D\ . \end{cases} \tag{3.13}$$

Thus by the linearity of the problem (3.1), we only need to show that the following problem has a bounded continuous weak solution:

$$\begin{cases} L_2 u_1(x) = F(x), & \text{on } D \\ \frac{\partial u_1}{\partial \gamma}(x) = 0 & \text{on } \partial D \end{cases} \tag{3.14}$$

The semigroup associated with operator L_2 is $\{T_t, t > 0\}$. By Lemma 3.3, we have

$$\sup_{x \in D} |T_t F(x)| = \sup_{x \in D} |E_x^0[Z_t F(X_t^0)]| \leq K e^{-\beta t} \|F\|_\infty.$$

Then

$$u_1(x) := \int_0^\infty T_t F(x) dt$$

is well defined and has the following bound:

$$\sup_{x \in D} |u_1(x)| \leq \frac{K}{\beta} \|F\|_\infty.$$

The function $u_1(x)$ is also continuous on D.
In fact, fixing any $x \in D$ and $\epsilon > 0$, we can firstly choose a constant $t_0 > 0$, such that $\sup_{z \in D} | \int_0^{t_0} T_s F(z) ds| < \frac{\epsilon}{3}$. And because $T_{t_0} u_1(x)$ is continuous, there exists a constant $\delta > 0$, such that for any y with $|y - x| < \delta$, $|T_{t_0} u_1(x) - T_{t_0} u_1(y)| \leq \frac{\epsilon}{3}$.
We find that

$$\begin{aligned} T_t u_1(x) = E_x^0[Z_t u_1(X_t^0)] &= E_x^0[Z_t \int_0^\infty E_{X_t^0}[Z_s F(X_s^0)] ds] \\ &= \int_0^t E_x^0[Z_{t+s} u_1(X_{t+s}^0)] ds \\ &= \int_t^\infty T_s F(x) ds \\ &= u_1(x) - \int_0^t T_s F(x) ds. \end{aligned} \tag{3.15}$$

For any y satisfying $|y - x| < \delta$, it follows that

$$|u_1(x) - u_1(y)| \leq |T_{t_0} u_1(x) - T_{t_0} u_1(y)| + |\int_0^{t_0} T_s F(x) ds| + |\int_0^{t_0} T_s F(y) ds| \leq \epsilon. \tag{3.16}$$

This implies that the function u_1 is continuous on domain D.
Denote the resolvents associated with operator L_2 by $\{G_\beta, \beta > 0\}$. Note that

$$\begin{aligned} G_\beta u_1(x) &= \int_0^\infty e^{-\beta t} T_t u_1(x) dt \\ &= \int_0^\infty e^{-\beta t} u_1(x) dt - \int_0^\infty e^{-\beta t} \int_0^t T_s F(x) ds dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\beta}u_1(x) - \int_0^\infty \int_0^t e^{-\beta t} T_s F(x) ds dt \\
&= \frac{1}{\beta}u_1(x) - \int_0^\infty T_s F(x) (\int_s^\infty e^{-\beta t} dt) ds \\
&= \frac{1}{\beta}u_1(x) - \frac{1}{\beta} G_\beta F(x).
\end{aligned} \tag{3.17}
$$

We have

$$
\beta(u_1(x) - \beta G_\beta u_1(x)) = \beta G_\beta F(x).
$$

Therefore,

$$
\lim_{\beta\to\infty} \int_D \beta(u_1(x) - \beta G_\beta u_1(x)) u_1(x) dx = \lim_{\beta\to\infty} \int_D \beta G_\beta F(x) u(x) dx = \int_D F(x) u_1(x) dx < \infty.
$$

This implies that $u_1 \in D(\mathcal{E})$ (see [16]) and u_1 is a weak solution of equation (3.14). By the linearity, $u = u_1 + u_2$ is a bounded continuous weak solution of equation (3.1).
$Uniqueness$:
Let v_1 and v_2 be two bounded continuous weak solutions of the equation (3.1). Then $v_1 - v_2$ is the solution of equation (3.13) with $\phi = 0$. Then by the uniqueness of the equation (3.13) proved in [4], we know that $v_1 = v_2$. $\square$

4 Semilinear PDEs

Recall that

$$
L_1 = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}
$$

and $L_2 = L_1 + q$ are two operators both defined on the domain D and equipped with the Neumann boundary condition $\frac{\partial}{\partial\gamma} = 0$ on ∂D.
$(\Omega, \mathfrak{F}_t, X(t), P_x, x \in D)$ is the reflecting diffusion process associated with the operator L_1 with the decomposition introduced in (2.1).

In this section, we solve the following semilinear boundary value problem:

$$
\begin{cases} L_2 u(x) = -G(x, u(x), \nabla u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial\gamma}(x) = \phi(x) & \text{on } \partial D \end{cases} \tag{4.1}
$$

Let $\mathcal{E}(\cdot, \cdot)$ be the quadratic form associated with the operator L_2:

$$
\mathcal{E}(u, v) = \frac{1}{2} \int_D < A\nabla u, \nabla v > dx - \int_D < b, \nabla u > v dx - \int_D q u v dx.
$$

Definition 4.1 *A bounded continuous function $u(x)$ defined on D is called a weak solution of the equation (4.1) if $u \in W^{1,2}(D)$, and for any $g \in C^\infty(\overline{D})$,*

$$\mathcal{E}(u,g) = \int_{\partial D} \phi(x)g(x)\sigma(dx) + \int_D G(x,u(x),\nabla u(x))g(x)dx.$$

Recall that L_t is the boundary local time of $X(t)$ defined in (2.1) and L_t^0 is the boundary local time of X_t^0 in (3.3).
As a consequence of the Girsanov theorem, we have:

Lemma 4.1 *Suppose that the function f satisfies $E_x[\int_0^T e^{\int_0^t f(X(u))du} dL_t] < \infty$. Then it holds that*

$$E_x[\int_0^T e^{\int_0^t f(X(u))du} dL_t] = E_x^0[\int_0^T \tilde{M}_t e^{\int_0^t f(X_u^0)du} dL_t^0],$$

where $\tilde{M}_t$ was defined in (3.5).

The following lemma is deduced from Theorem 3.2 in [4].

Lemma 4.2 *Suppose that the function $\tilde{q} \in L^p(D)$ and $p > \frac{d}{2}$. If there exists some point $x_0 \in D$, such that*

$$E_{x_0}[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))du} dL_t] < \infty, \tag{4.2}$$

then it holds that

$$\sup_x E_x[\int_0^\infty e^{\int_0^t \tilde{q}(X(u))du} dL_t] < \infty.$$

Let $G(x,y,z) : R^d \times R \times R^d \to R$ be a bounded Borel measurable function. Introduce the following conditions:
(H.1) $(y_1 - y_2)(G(x,y_1,z) - G(x,y_2,z)) \le -h_1(x)|y_1 - y_2|^2$,
(H.2) $|G(x,y,z_1) - G(x,y,z_2)| \le h_2|z_1 - z_2|$.
Set $h(t) = -h_1(X(t)) + \delta h_2^2 + q(X(t))$ and $\tilde{h}(t) = -h_1(X(t)) + \delta h_2^2$ for some constant $\delta > \frac{1}{2\lambda}$.

Theorem 4.1 *Suppose that the conditions (H.1) and (H.2) are satisfied. Assume*

$$E_{x_1}[\int_0^\infty e^{2\int_0^t (q(X(u)) + \tilde{h}(u))du} dt] < \infty, \quad for \quad some \quad x_1 \in D, \tag{4.3}$$

and there exists some point $x_0 \in D$, such that

$$E_{x_0}[\int_0^\infty e^{\int_0^t q(X(u))du} dL_t] < \infty. \tag{4.4}$$

Then the semilinear Neumann boundary value problem (4.1) has a unique continuous weak solution.

PROOF.
Set
$$\tilde{G}(X(t),y,z) := e^{\int_0^t q(X(u))dt} G(x, e^{-\int_0^t q(X(u))dt} y, e^{-\int_0^t q(X(u))dt} z).$$
Then
$$(y_1 - y_2)(\tilde{G}(X(t),y_1,z) - \tilde{G}(X(t),y_2,z)) \le -h_1(x)|y_1-y_2|^2 \tag{4.5}$$
and
$$|\tilde{G}(X(t),y,z_1) - \tilde{G}(X(t),y,z_2)| \le h_2|z_1-z_2|. \tag{4.6}$$
Note that
$$\tilde{G}(X(t),y,z) \le e^{\int_0^t q(X(u))dt}\|G\|_\infty.$$
By Theorem 2.1 there exists a unique process $(\hat{Y}_x, \hat{Z}_x)$ satisfying
$$\begin{aligned}&d\hat{Y}_x(t) = -\tilde{G}(X(t),\hat{Y}_x(t),\hat{Z}_x(t))dt + e^{\int_0^t q(X(u))du}\phi(X(t))dL(t) + \langle \hat{Z}_x(t), dM_x(t)\rangle\\ &e^{\int_0^t \tilde{h}(u)du}\hat{Y}_x(t) \to 0 \quad as \quad t\to\infty.\end{aligned}$$
Furthermore, Corollary 2.1 implies that $\sup_x \hat{Y}_x(0) < \infty$.
From Ito's formula, it follows that
$$\begin{aligned}& d(e^{-\int_0^t q(X(u))dt}\hat{Y}_x(t))\\ =\;& -q(X(t))e^{-\int_0^t q(X(u))dt}\hat{Y}_x(t)dt - e^{-\int_0^t q(X(u))dt}\tilde{G}(X(t),\hat{Y}_x(t),\hat{Z}_x(t))dt\\ & +\phi(X(t))dL_t + < e^{-\int_0^t q(X(u))dt}\hat{Z}_x(t), dM_x(t) > .\end{aligned}$$
Setting $Y_x(t) := e^{-\int_0^t q(X(u))dt}\hat{Y}_x(t)$ and $Z_x(t) := e^{-\int_0^t q(X(u))dt}\hat{Z}_x(t)$, we obtain
$$dY_x(t) = -(q(X(t))Y_x(t) + G(X(t),Y_x(t),Z_x(t)))dt + \phi(X(t))dL_t + < Z_x(t), dM_x(t) > .$$
Moreover,
$$e^{\int_0^t h(u)dt}Y_x(t) = e^{\int_0^t h(u)dt}e^{-\int_0^t q(X(u))dt}\hat{Y}_x(t) = e^{\int_0^t \tilde{h}(X(u))dt}\hat{Y}_x(t) \to 0 \quad as \quad t\to\infty. \tag{4.7}$$
So by Ito's formula, we have that, for any $t<T$,
$$\begin{aligned}& e^{\int_0^t h(u)du}Y_x(t)\\ =\;& e^{\int_0^T h(u)du}Y_x(T) + \int_t^T e^{\int_0^s h(u)du}\left(G(X_x(s),Y_x(s),Z_x(t)) + q(X_x(s))Y_x(s)\right)ds\\ -\;& \int_t^T e^{\int_0^s h(u)du}\phi(X(s))dL_s - \int_t^T h(s)e^{\int_0^s h(u)du}Y_x(s)ds\\ -\;& \int_t^T e^{\int_0^s h(u)du}\langle Z_x(t), dM_x(t)\rangle.\end{aligned} \tag{4.8}$$

Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$.
Since $Y_x(0) = \hat{Y}_x(0)$, we know that u_0 is a bounded function on domain D. By the Markov property of X and the uniqueness of (Y_x, Z_x) , it is easy to see that

$$Y_x(t) = u_0(X(t)), \quad Z_x(t) = v_0(X(t)).$$

So that $\sup_{x\in D, t>0} |Y_x(t)| \leq \|u_0\|_\infty < \infty$.
Now consider the following problem:

$$\begin{cases} L_2 u(x) = -G(x, u_0(x), v_0(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) = \phi(x) & \text{on } \partial D \end{cases} \tag{4.9}$$

By Theorem 3.1, problem (4.9) has a unique continuous weak solution $u(x)$. Next we will show that $u = u_0$.
Since u belongs to the domain of the Dirichlet form associated with the process $X(t)$, it follows from the Fukushima's decomposition that:

$$\begin{aligned} & du(X(t)) \\ = & -[G(X(t), u_0(X(t)), v_0(X(t))) + q(X(t))u(X(t))]dt + \phi(X(t))dL(t) + \langle \nabla u(X(t)), dM_x(t)\rangle \\ = & -[G(X(t), Y_x(t), Z_x(t)) + q(X(t))u(X(t))] + \phi(X(t))dL(t) + \langle \nabla u(X(t)), dM_x(t)\rangle \end{aligned}$$

From the condition (4.3) and the boundedness of $u(x)$, it follows that

$$\lim_{t\to\infty} E_x[e^{2\int_0^t h(u)du} u^2(X(t))] \leq \|u\|_\infty^2 \lim_{t\to\infty} E_x[e^{2\int_0^t (\tilde{h}+q)(u)du}] = 0.$$

By Ito's formula, it follows that, for any $t < T$,

$$\begin{aligned} & e^{\int_0^t h(u)du} u(X(t)) \\ = & e^{\int_0^T h(u)du} u(X(T)) + \int_t^T e^{\int_0^s h(u)du}[G(X(s), Y_x(s), Z_x(s)) + q(X(s))u(X(s))]ds \\ - & \int_t^T e^{\int_0^s h(u)du} \phi(X(s))dL(s) - \int_t^T h(s)e^{\int_0^s h(u)du} u(X(s))ds \\ - & \int_t^T e^{\int_0^s h(u)du} \langle \nabla u(X(t)), dM_x(t)\rangle. \end{aligned} \tag{4.10}$$

Set

$$v_x(t) = u(X(t)) - Y_x(t) \quad \textit{and} \quad R_x(t) = \nabla u(X(t)) - Z_x(t).$$

Subtracting the equations (4.8) from (4.10), we obtain the following equation: for any $t < T$,

$$\begin{aligned} & e^{\int_0^t h(u)du} v(X(t)) \\ = & e^{\int_0^T h(u)du} v(X(T)) + \int_t^\infty (q(X(u)) - h(u))e^{\int_0^s h(u)du} v(X(s))ds \\ & - \int_t^\infty e^{\int_0^s h(u)du} < R_x(t), , dM_x(t) > \end{aligned}$$

$$
\begin{aligned}
&= e^{\int_0^T h(u)du} v(X(T)) - \int_t^T \tilde{h}(s) e^{\int_0^s h(u)du} v(X(s))ds \\
&\quad - \int_t^T e^{\int_0^s h(u)du} < R_x(t), dM_x(t) > .
\end{aligned}
$$

Set $g(t) = e^{\int_0^t h(u)du} v(t)$. Taking conditional expectation on both sides of (4.11), we find that

$$
\begin{aligned}
g(t) &= E_x[g(T) - \int_t^T \tilde{h}(s) g(s) ds | \mathcal{F}_t] \\
&= E_x[g(T)(1 - \int_t^T \tilde{h}(s)ds) + \int_t^T \int_s^T \tilde{h}(s)\tilde{h}(s_1) g(s_1) ds_1 ds | \mathcal{F}_t] \\
&= E_x[g(T)(1 - \int_t^T \tilde{h}(s)ds + \frac{1}{2}(\int_t^T \tilde{h}(s)ds)^2) \\
&\quad +(-1)^3 \int_t^T \int_s^T \int_{s_1}^T \tilde{h}(s)\tilde{h}(s_1)\tilde{h}(s_2) g(s_2) ds_2 ds_1 ds | \mathcal{F}_t].
\end{aligned}
$$

Keeping iterating, we obtain

$$
\begin{aligned}
g(t) &= E_x[g(T)(\sum_{k=0}^n \frac{(-\int_t^T \tilde{h}(s)ds)^n}{n!}) \\
&\quad +(-1)^{n+1} \int_t^T \int_s^T \int_{s_1}^T ... \int_{s_{n-1}}^T \tilde{h}(s)\tilde{h}(s_1)...\tilde{h}(s_n) g(s_n) ds_n ... ds_1 ds | \mathcal{F}_t]
\end{aligned}
$$

Since $E_x[|g(T)| e^{\int_t^T |\tilde{h}|(s)ds}] < \infty$, letting $n \to \infty$, by dominated convergence theorem, it follows that

$$
g(t) = E_x[g(T) e^{-\int_t^T \tilde{h}(s)ds} | \mathcal{F}_t].
$$

Then

$$
v(t) = E_x[v(T) e^{\int_t^T (h(s) - \tilde{h}(s))ds} | \mathcal{F}_t] \leq (\|u_0\|_\infty + \|u\|_\infty) E_x[e^{\int_t^T q(X(s))ds} | \mathcal{F}_t]. \tag{4.11}
$$

Hence, it follows that

$$
0 \leq e^{\int_0^t q(X(s))ds} |v(t)| \leq (\|u_0\|_\infty + \|u\|_\infty) \lim_{T\to\infty} E_x[e^{\int_0^T q(X(s))ds} | \mathcal{F}_t]. \tag{4.12}
$$

Since the condition (4.4) implies

$$
\lim_{T\to\infty} E_x[e^{\int_0^T q(X(s))ds}] = 0,
$$

we deduce that $E_x[e^{\int_0^t q(X(s))ds} |v(t)|] = 0$ and hence $v(t) = 0$, $P_x - a.s.$.
Therefore, for any $t > 0$, we have $u(X(t)) = Y_x(t)$ and $\nabla u(X(t)) = Z_x(t)$ by the uniqueness of the Doob-Meyer decomposition of semimartingales. In particular, $u(x) = E_x[u(X_x(0))] =$

$E_x[Y_x(0)] = u_0(x)$. This shows that $u(x)$ is a weak solution of the equation (4.1).
If $\tilde{u}$ is another solution of the problem (4.1). Then the processes $\tilde{Y}_x(t) := \tilde{u}(X(t))$ and $\tilde{Z}_x(t) := \nabla\tilde{u}(X(t))$ satisfy the following equation

$$d\tilde{Y}_x(t) = -G(X(t), \tilde{Y}_x(t), \tilde{Z}_x(t))dt - \phi(X(t))dL_t + < \tilde{Z}_x(t), dM_x(t) > . \tag{4.13}$$

Set $\bar{Y}_x(t) = e^{\int_0^t q(X(u))du}\tilde{Y}_x(t)$ and $\bar{Z}_x(t) = e^{\int_0^t q(X(u))du}\tilde{Z}_x(t)$.
By chain rule, it follows that

$$d\bar{Y}_x(t) = -\tilde{G}(X(t), \bar{Y}_x(t), \bar{Z}_x(t))dt + e^{\int_0^t q(X(u))du}\phi(X(t))dL(t) + \langle \bar{Z}_x(t), dM_x(t)\rangle$$

Moreover, because $\tilde{u}$ is bounded, we have

$$\lim_{t\to\infty} e^{\int_0^t \tilde{h}(u)du}\bar{Y}_x(t) = \lim_{t\to\infty} e^{\int_0^t h(u)du}\tilde{u}(X(t)) = 0.$$

Therefore, from the uniqueness of the solution of the BSDE in Theorem 2.1, we have

$$\tilde{Y}_x(t) = Y_x(t) \quad \tilde{Z}_x(t) = Z_x(t).$$

In particular,

$$\tilde{u}(x) = E_x[\tilde{Y}_x(t)] = E_x[Y_x(t)] = u(x).$$

5 Semilinear Elliptic PDEs with Singular Coefficients

Recall the operator

$$L = \frac{1}{2}\nabla\cdot(A\nabla) + B\cdot\nabla - \nabla\cdot(\hat{B}\cdot) + Q$$

on the domain D equipped with the mixed boundary condition on ∂D:

$$\frac{1}{2}\frac{\partial u}{\partial\gamma} - \langle\hat{B}, n\rangle u(x) = 0.$$

The quadratic form associated with L is given by:

$$\begin{aligned}\mathcal{Q}(u,v) := (-Lu, v) = \quad & \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx - \sum_i\int_D B_i(x)\frac{\partial u}{\partial x_i}v(x)dx \\ - \quad & \sum_i\int_D \hat{B}_i(x)\frac{\partial v}{\partial x_i}u(x)dx - \int_D Q(x)u(x)v(x)dx,\end{aligned}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(D)$.
The domain of the quadratic form is

$$\mathcal{D}(\mathcal{Q}) = W^{1,2}(D) := \{u : u \in L^2(D), \frac{\partial u}{\partial x_i} \in L^2(D), i = 1, ..., d\}.$$

Let $\{S_t,\, t \geq 0\}$ denote the semigroup generated by L.

In this section, our aim is to solve the following equation:

$$\begin{cases} Lf(x) = -F(x, f(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial f}{\partial \gamma}(x) - < \widehat{B}, n > (x) f(x) = \Phi(x) & \text{on } \partial D \end{cases} \tag{5.1}$$

Definition 5.1 *A bounded continuous function $f(x)$ defined on D is called a weak solution of the equation* (5.1) *if $f \in W^{1,2}$, and for any $g \in C^\infty(\bar{D})$,*

$$\mathcal{Q}(u, g) = \int_{\partial D} \Phi(x) g(x) \sigma(dx) + \int_D F(x, u(x)) g(x) dx.$$

Here the function $F : R^d \times R \to R$ is a bounded measurable function and satisfies the following condition:
(E.1) $(y_1 - y_2)(F(x, y_1) - F(x, y_2)) \leq -r_1(x)|y_1 - y_2|^2$.

Recall the following regular Dirichlet form

$$\begin{cases} \mathcal{E}^0(u, v) = \frac{1}{2}\sum_{i,j}\int_D a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx, \\ D(\mathcal{E}^0) = W^{1,2}(D) \end{cases} \tag{5.2}$$

associated with the operator $L_0 = \frac{1}{2}\nabla(A\nabla)$ equipped with the Neumann boundary condition $\frac{\partial}{\partial \gamma} = 0$ on ∂D.
The associated reflecting diffusion process is denoted by $\{\Omega, \mathfrak{F}_t, X_t^0, \theta_t^0, \gamma_t^0, P_x^0\}$. Here θ_t^0 and γ_t^0 are the shift and reverse operators defined by

$$\begin{aligned} X_s^0(\theta_t^0(\omega)) &= X_{t+s}^0(\omega), s, t \geq 0 \\ X_s^0(\gamma_t^0(\omega)) &= X_{t-s}^0(\omega), s \leq t. \end{aligned}$$

The process $(X_t^0)_{t\geq 0}$ has the decomposition in (3.3). The martingale part of X_t^0 is $M_t^0 = \int_0^t \sigma(X_s^0) dW_s$.
The following probabilistic representation of semigroup S_t was proved in [5]

$$\begin{aligned} S_t f(x) = \quad & E_x^0[f(X_t^0) \exp(\int_0^t (A^{-1}B)^*(X_s^0) dM_s^0 + (\int_0^t (A^{-1}\hat{B})^*(X_s^0) dM_s^0) \circ \gamma_t^0 \\ & - \frac{1}{2}\int_0^t (B - \hat{B})A^{-1}(B - \hat{B})^*(X_s^0) ds + \int_0^t Q(X_s^0) ds)] \end{aligned} \tag{5.3}$$

E_x^0 denotes the expectation under P_x^0.
Set

$$\begin{aligned} \hat{Z}_t \;=\; & \exp(\int_0^t (A^{-1}B)^*(X_s^0) dM_s^0 + (\int_0^t (A^{-1}\hat{B})^*(X_s^0) dM_s^0) \circ \gamma_t^0 \\ & - \frac{1}{2}\int_0^t (B - \hat{B})A^{-1}(B - \hat{B})^*(X_s^0) ds + \int_0^t Q(X_s^0) ds). \end{aligned} \tag{5.4}$$

By [3] and [21], there exists a bounded, continuous functions $v \in W^{1,p}(D)$ satisfying that

$$\begin{aligned}
& \left(\int_0^t (A^{-1}\hat{B})^*(X_s^0)dM_s^0\right) \circ \gamma_t^0 \\
= & -\int_0^t \nabla v(X_s^0)dM_s + v(X_t^0) - v(X_0^0) - \int_0^t (A^{-1}\hat{B})^*(X_s^0)dM_s
\end{aligned} \tag{5.5}$$

Moreover, v satisfies the following equations: for $g \in C^1(\bar{D})$,

$$\int_D < A\nabla v, \nabla g > (x)dx = \int_D < \hat{B}, \nabla g > (x)dx. \tag{5.6}$$

Thus the representation of S_t becomes:

$$\begin{aligned}
S_t f(x) &= e^{-v(x)} E_x^0[f(X_t^0)e^{v(X_t^0)} \exp(\int_0^t (A^{-1}(B - \hat{B} - A\nabla v))^* dM_s^0 \\
&\quad -\frac{1}{2}\int_0^t (B - \hat{B} - A\nabla v)^* A^{-1}(B - \hat{B} - A\nabla v)(X_s^0)ds \\
&\quad + \int_0^t (Q + \frac{1}{2}(\nabla v)A(\nabla v)^* - \langle B - \hat{B}, \nabla v\rangle)(X_s^0)ds)] \\
&= e^{-v(x)}\tilde{S}_t[fe^v](x).
\end{aligned} \tag{5.7}$$

Here, setting $b := B - \hat{B} - (A\nabla v)$ and $q := Q + \frac{1}{2}(\nabla v)A(\nabla v)^* - \langle B - \hat{B}, \nabla v\rangle$, we see that $\tilde{S}_t$ is the semigroup generated by the following operator:

$$\begin{aligned}
L_2 &= \frac{1}{2}\nabla \cdot (A\nabla) + (B - \hat{B} - (A\nabla v)) \cdot \nabla + (Q + \frac{1}{2}(\nabla v)A(\nabla v)^* - \langle B - \hat{B}, \nabla v\rangle) \\
&= \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla + q
\end{aligned}$$

equipped with the boundary condition $\frac{\partial}{\partial \gamma} = 0$.

In this section, we will stick to this particular choice of b and q.

Recall that

$$\tilde{M}(t) = e^{\int_0^t A^{-1}b(X_s^0)dM_s^0 - \frac{1}{2}\int_0^t bA^{-1}b^*(X_s^0)ds}$$

and set $Z_t = \tilde{M}(t)e^{\int_0^t q(X_s^0)ds}$.
Then from (5.5), it follows that $\hat{Z}(t) = Z_t e^{v(X_t^0) - v(X_0^0)}$.
Recall the operator $L_1 = \frac{1}{2}\nabla \cdot (A\nabla) + b \cdot \nabla$ with Neumann boundary condition, which is associated with the reflecting diffusion $(X(t), P_x)$. It is known from [14] that

$$dP_x|_{\mathcal{F}_t} = \tilde{M}_t dP_x^0|_{\mathcal{F}_t},$$

and

$$X(t) = x + \int_0^t \sigma(X(s))dW_s + \int_0^t (\frac{1}{2}\nabla A + b)(X(s))ds + \int_0^t \gamma(X(s))dL_s, \quad P_x - a.s.$$

where $\{W_t\}$ is a d-dimensional Brownian motion and L_t is the local time satisfying that $L_t = \int_0^t I_{\partial D}(X(s))dL_s$.

Lemma 5.1 *Assume that there exists $x_0 \in D$, such that*

$$E^0_{x_0}[\int_0^\infty |\hat{Z}_t|^2 e^{\int_0^t (2Q-4r_1)(X^0_u)du} dL^0_t] < \infty. \tag{5.8}$$

Then there exists a positive number $\varepsilon > 0$, if $\|\hat{B}\|_{L^p} \le \varepsilon$, the following inequality holds:

$$\sup_{x\in D} E_x[\int_0^\infty e^{2\int_0^t (-r_1+q)(X(u))du} dt] < \infty. \tag{5.9}$$

PROOF.

$$\begin{aligned} E_x[e^{2\int_0^t(-r_1+q)(X(u))du}] &= E^0_x[\tilde{M}(t)e^{2\int_0^t(-r_1+q)(X^0_u)du}] \\ &= E^0_x[Z(t)e^{\int_0^t(-2r_1+q)(X(u))du}] \\ &\le C_1E^0_x[\hat{Z}(t)e^{-2\int_0^t(r_1(X(u))du}e^{\int_0^t(Q+\frac{1}{2}<A\nabla v-2(B-\hat{B}),\nabla v>)(X^0_u)du}] \\ &\le C_1E^0_x[\hat{Z}^2(t)e^{2\int_0^t(Q-2r_1)(X^0_u)du}]^{\frac{1}{2}} \cdot E^0_x[e^{\int_0^t<A\nabla v-2(B-\hat{B}),\nabla v>(X^0_u)du}]^{\frac{1}{2}} \end{aligned}$$

By Lemma 3.3 and condition (5.8), there exists two constant $c_2, \beta > 0$ such that

$$\sup_{x\in D} E_x[\hat{Z}^2(t)e^{2\int_0^t(Q-r_1)(X^0_u)du}] < c_2e^{-\beta t}.$$

Moreover, for $p > d$, by the Theorem 2.1 in [15], there exist two positive constants c_3 and c_4 such that

$$E^0_x[e^{\int_0^t<A\nabla v-2(B-\hat{B}),\nabla v>(X^0_u)du}] \le c_3e^{c_4t},$$

where $c_4 = c\| < A\nabla v - 2(B-\hat{B}), \nabla v > \|_{L^{p/2}}$.
Since $|\nabla v|_{L^p} \le C|\hat{B}|_{L^p(D)}$ (see [21]), there exists $\varepsilon > 0$, such that $|\hat{B}|_{L^p(D)} \le \varepsilon$ implies $c_4 < \beta$. Thus (5.9) holds. □

Theorem 5.1 *Assume* (5.8) *and for some point $x_0 \in D$*

$$E^0_{x_0}[\int_0^\infty \hat{Z}_s dL^0_t] < \infty \tag{5.10}$$

Then there exists $\varepsilon > 0$ such that if $\|\hat{B}\|_{L^p} \le \varepsilon$, the problem (5.1) *has a unique, bounded, continuous weak solution $u(x)$.*

PROOF.
Existence: Set $\tilde{F}(x,y) = e^{v(x)}F(x, e^{-v(x)}y)$ and $\phi(x) = e^{v(x)}\Phi(x)$.
From the boundedness of v , $\tilde{F}$ is also bounded.
And $\tilde{F}$ satisfies

$$(y_1-y_2)(\tilde{F}(x,y_1) - \tilde{F}(x,y_2)) \le -r_1(x)|y_1-y_2|^2.$$

Moreover, there is a constant $c > 0$, such that

$$\begin{aligned}\infty > E^0_{x_0}[\int_0^\infty \hat{Z}_s dL^0_s] &= E^0_{x_0}[\int_0^\infty Z_s e^{v(X^0_s)-v(X^0_0)} dL^0_s] \\ &\geq cE^0_{x_0}[\int_0^\infty Z_s dL^0_s] = cE^0_{x_0}[\int_0^\infty \tilde{M}_s e^{\int_0^s q(X^0_u)du} dL^0_s] \qquad (5.11)\end{aligned}$$

By Lemma 4.1, we know that, at $x_0 \in D$,

$$E_{x_0}[\int_0^\infty e^{\int_0^s q(X_u)du} dL_s] < \infty. \qquad (5.12)$$

Furthermore, by Lemma 4.2, it follows that

$$\sup_x E_x[\int_0^\infty e^{\int_0^t q(X(u))du} dL_t] < \infty. \qquad (5.13)$$

By Lemma 5.1, the following condition is satisfied :

$$E_x[\int_0^\infty e^{2\int_0^t (q-r_1)(X(u))du} dt] < \infty, \qquad (5.14)$$

So $\tilde{F}$ satisfies all of the conditions in Theorem 4.1 replacing G by $\tilde{F}$. Thus the following problem

$$\begin{cases} L_2 u(x) = -\tilde{F}(x, u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) = \phi & \text{on } \partial D \end{cases} \qquad (5.15)$$

has a unique bounded continuous weak solution $u(x)$.
Set $f(x) = e^{-v(x)}u(x)$. Then we claim the function $f(x)$ is the weak solution of the equation (5.1).
Because function v is continuous and bounded, $f(x)$ is also continuous. From the fact that function u is the weak solution of the problem (5.15), we obtain, for any function $\psi \in C^\infty(D)$,

$$\begin{aligned}\mathcal{E}(u, e^{-v}\psi) &= \frac{1}{2}\int_D < A\nabla u, \nabla(e^{-v}\psi) > - < b, \nabla u > e^{-v}\psi - e^{-v}qu\psi dx \\ &= \int_{\partial D} e^{-v}\phi\psi d\sigma + \int_D \tilde{F}(x, u(x))e^{-v}\psi dx. \qquad (5.16)\end{aligned}$$

As in the proof of Theorem 5.1 in [22], we can show that the left side of the equation (5.16) equals to

$$\mathcal{Q}(f, \psi) = \frac{1}{2}\int_D [< A\nabla f, \nabla\psi > - < B, \nabla u > \psi - < \hat{B}, \nabla\psi > f - Qf\psi]dx.$$

At the same time, by the definition of the function ϕ and $\tilde{F}$, the right side of the equation (5.16) equals to

$$\int_{\partial D} \Phi\psi d\sigma + \int_D F(x, f(x))\psi dx.$$

Thus it follows that, for any $\psi \in C^{\infty}(D)$,

$$\mathcal{Q}(f,\psi) = \int_{\partial D} \Phi\psi d\sigma + \int_D F(x, f(x))\psi dx.$$

which proves that function f is a weak solution of the problem (5.1).
Uniqueness:
If $\bar{f}$ is another solution of the problem (5.1), then $\bar{u} := e^v f$ can be shown to be the solution of the equation (5.15). Then by the uniqueness of the problem (5.15) proved in the Theorem 4.1, we find $\bar{u} = u$. Therefore, $f = \bar{f}$.□

6 L^1 solutions of the BSDE and Semilinear PDEs

Recall the operator

$$L_1 = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}$$

on the domian D equipped with the Neumann boundary condition $\frac{\partial}{\partial \gamma} = 0$, on ∂D.
And $(\Omega, \mathcal{F}_t, X(t), P_x, x \in D)$ is the reflecting diffusion process associated with the generator L_1.
Then the process $X(t)$ has the following decomposition:

$$X(t) = X(0) + M(t) + \int_0^t \tilde{b}(X(s))ds + \int_0^t An(X(s))dL_s, \quad P_x - a.s..$$

Here $\tilde{b} = \{\tilde{b}_1, ..., \tilde{b}_d\}$ with $\tilde{b}_i = \frac{1}{2}\sum_j \frac{\partial a_{ij}}{\partial x_j} + b_i$. $M(t)$ is the $\mathcal{F}_t$ square integrable continuous martingale additive functional.

In this section, we will consider the L^1 solutions of the BSDEs in Section 2 and use this result to solve the nonlinear elliptic partial differential equation with the mixed boundary condition.

Let $f : \Omega \times R^+ \times R \to R$ be progressively measurable. Consider the following conditions:
(I.1) $(y - y')(f(t, y) - f(t, y')) \le d(t)|y - y'|^2$, where $d(t)$ is a progressively measurable process;
(I.2)$E[\int_0^\infty e^{\int_0^s d(u)du}|f(s, 0)|ds] < \infty$;
(I.3) $P_x - a.s.$, for any $t > 0$, $y \to f(t, y)$ is continuous;
(I.4) $\forall r > 0, \quad T > 0$, $\psi_r(t) := \sup_{|y|\le r} |f(t, y) - f(t, 0)| \in L^1([0, T] \times \Omega, dt \times dP_x)$.

The following lemma is deduced from Corollary 2.3 in [2].

Lemma 6.1 *Suppose a pair of progressively measurable processes (Y, Z) with values in $R \times R^d$ such that $t \to Z_t$ belongs to $L^2([0, T])$ and $t \to f(t, Y_t)$ belongs to $L^1([0, T])$, $P_x - a.s.$.*
If

$$Y_t = \xi + \int_t^T f(r, Y_r)dr - \int_t^T < Z_r, dM_r >, \tag{6.1}$$

then the following inequality holds, for $0 \le t < u \le T$,

$$|Y_t| \le |Y_u| + \int_t^u \hat{Y}_s f(s, Y_s) ds - \int_t^u \hat{Y}_s \langle Z_r, dM_r \rangle.$$

where $\hat{y} = \frac{y}{|y|} I_{\{y \ne 0\}}$.

The following lemma can be proved by modifying the proof of Proposition 6.4 in [2].

Lemma 6.2 *Assume that conditions (I.1)-(I.4) with* $d(t) \equiv 0$. *Then there exists a unique solution* (Y, Z) *of the BSDE*

$$Y_t = \int_t^T f(r, Y_r) dr - \int_t^T \langle Z_r, dM_r \rangle, \quad for \quad t \le T. \tag{6.2}$$

Moreover, for each $\beta \in (0, 1)$, $E[\sup_{t \le T} |Y_t|^\beta] + E[(\int_0^T |Z_r|^2 dr)^{\frac{\beta}{2}}] < \infty$.

Suppose $\beta \in (0, 1)$.
$\mathcal{S}^\beta$ denotes the set of real-valued, adapted and continuous process $\{Y_t\}_{t \ge 0}$ such that

$$\|Y\|^\beta := E[\sup_{t>0} |Y_t|^\beta] < \infty.$$

It is known that $\| \cdot \|^\beta$ deduces a complete metric on $\mathcal{S}^\beta$.
M^β denotes the set of R^d-valued predictable processes $\{Z_t\}$ such that

$$\|Z\|_{M^\beta} := E[(\int_0^\infty |Z_t|^2 dt)^{\frac{\beta}{2}}] < \infty.$$

M^β is also a complete metric space with the distance deduced by $\| \cdot \|_{M^\beta}$.

Lemma 6.3 *Under the same assumption as the Lemma* 6.2, *there exists a unique solution* (Y, Z) *of the BSDE*

$$\begin{aligned} &Y_t = Y_T + \int_t^T f(r, Y_r) dr - \int_t^T \langle Z_r, dM_r \rangle, \quad any \quad t \le T; \\ &\lim_{t \to \infty} Y_t = 0, \quad P-a.s.. \end{aligned} \tag{6.3}$$

Proof. Existence:
By the Lemma 6.2 above, there exists (Y^n, Z^n) such that, for $0 \le t \le n$,

$$Y_t^n = \int_t^n f(r, Y_r^n) dr - \int_t^n \langle Z_r^n, dM_r \rangle,$$

and $Y_t^n = Z_t^n = 0$, for $t \ge n$.
Fix $t > 0$ and $t < n < n+i$, then

$$Y_t^{n+i} - Y_t^n = \int_t^{n+i} (f(r, Y_r^{n+i}) - f(r, Y_r^n)) dr - \int_t^{n+i} \langle (Z_r^{n+i} - Z_r^n), dM_r \rangle + \int_n^{n+i} f(r, 0) dr$$

Set $F^n(r,y) = f(r, y+Y_r^n) - f(r,Y_r^n) + f(r,0)I_{\{r>n\}}$, $y_t^n = Y_t^{n+i} - Y_t^n$ and $z_t^n = Z_t^{n+i} - Z_t^n$. Then (y_t^n, z_t^n) is the solution of the following BSDE:

$$y_t^n = \int_t^{n+i} F(r, y_r^n)dr - \int_t^{n+i} \langle z_r^n, dM_r\rangle. \tag{6.4}$$

So that by the condition (I.1) with $d(t) \equiv 0$, it follows from Lemma 6.1 that

$$\begin{aligned} |y_t^n| &\leq \int_t^{n+i} \langle \hat{y}_r^n, F^n(r, y_r^n)\rangle dr - \int_t^{n+i} \langle \hat{y}_r^n, z_r^n dM_r\rangle \\ &\leq \int_t^{n+i} \frac{I_{\{y_r^n \neq 0\}}}{|y_r^n|}\langle y_r^n, f(r, y_r^n + Y_r^n) - f(r, Y_r^n)\rangle dr + \int_n^{n+i} |f(s,0)|ds \\ &\quad - \int_t^{n+i} \langle \hat{y}_r^n, z_r^n dM_r\rangle \\ &\leq \int_n^{n+i} |f(s,0)|ds - \int_t^{n+i} \langle \hat{y}_r^n, z_r^n dM_r\rangle. \end{aligned} \tag{6.5}$$

Taking conditional expectation on both side of the inequality, we got

$$|y_t^n| \leq E[\int_n^{n+i} |f(s,0)|ds|\mathcal{F}_t] := M_t^n,$$

where M_t^n is a martingale. Then by Doob's inequality and condition (I.2), it follows that, for $\beta \in (0,1)$,

$$\begin{aligned} E[\sup_t |y_t^n|^\beta] \leq E[\sup_t (M_t^n)^\beta] &\leq \frac{1}{1-\beta} E[\int_n^{n+i} |f(s,0)|ds]^\beta \\ &\to 0, \quad as \quad n \to \infty. \end{aligned} \tag{6.6}$$

Therefore, $\{Y^n\}$ is a Cauchy sequence under the norm $\|\cdot\|_\infty^\beta$. So that there is a process Y such that $E[\sup_t |Y_t - Y_t^n|^\beta] \to 0$.
This also implies that $Y_t \to 0$, as $t \to \infty$, $P_x - a.s.$.
Moreover, by the equation (6.4), Ito's formula and the condition (I.1), it follows that

$$\begin{aligned} &|y_t^n|^2 + \int_t^{n+i} \langle A(X(r)) z_r^n, z_r^n\rangle dr \\ = \;& 2\int_t^{n+i} \langle y_r^n, F^n(r, y_r^n)\rangle dr - 2\int_t^{n+i} \langle y_r^n, z_r^n dM_r\rangle \\ \leq \;& 2\int_n^{n+i} \langle y_r^n, f^n(r,0)\rangle dr + 2|\int_t^{n+i} \langle y_r^n, z_r^n dM_r\rangle| \\ \leq \;& \sup_r |y_r^n|^2 + (\int_n^{n+i} |f(r,0)|dr)^2 + 2|\int_t^{\infty} \langle y_r^n, z_r^n dM_r\rangle|, \end{aligned}$$

and thus that

$$(\int_t^{n+i} |z_r^n|^2 dr)^{\frac{\beta}{2}} \leq c_1[\sup_r |y_r^n|^\beta + (\int_n^{n+i} |f(r,0)|dr)^\beta + |\int_t^{n+i} < y_r^n, z_r^n dM_r > |^{\frac{\beta}{2}}].$$

Taking expectation on both sides of the inequality and applying the BDG inequality, we obtain

$$
\begin{aligned}
& E[(\int_t^{n+i} |z_r^n|^2 dr)^{\frac{\beta}{2}}] \\
\leq\ & c_1(E[\sup_r |y_r^n|^\beta] + E[(\int_t^{n+i} |f(r,0)|dr)^\beta]) + c_2 E[(\int_t^{n+i} |y_r^n|^2 |z_r^n|^2 dr)^{\frac{\beta}{4}}] \\
\leq\ & c_1(E[\sup_r |y_r^n|^\beta] + E[(\int_n^{n+i} |f(r,0)|dr)^\beta]) + c_2 E[(\sup_r |y_r^n|^{\frac{\beta}{2}} \int_t^{n+i} |z_r^n|^2 dr)^{\frac{\beta}{4}}] \\
\leq\ & (c_1 + \frac{c_2}{2})(E[\sup_r |y_r^n|^\beta] + E[(\int_n^{n+i} |f(r,0)|dr)^\beta]) + \frac{1}{2} E[(\int_t^{n+i} |z_r^n|^2 dr)^{\frac{\beta}{2}}].
\end{aligned}
$$

Therefore, we know that there is a constant $C > 0$, such that

$$
\begin{aligned}
E[(\int_0^\infty |z_s^n|^2 ds)^{\frac{\beta}{2}}] &\leq CE[\sup_t |y_t^n|^\beta + (\int_n^{n+i} |f(s,0)|ds)^\beta] \\
&\leq CE[\sup_t |y_t^n|^\beta] + CE[\int_n^{n+i} |f(s,0)|ds]^\beta \\
&\to 0 \quad as \quad n \to \infty.
\end{aligned}
$$

So that $\{Z_t^n\}$ is a Cauchy sequence in M^β. Let Z denote the limit of $\{Z^n\}$.
At last, by the condition (I.3), we find that

$$
\int_0^T f(t, Y_t^n)dt \to \int_0^T f(t, Y_t)dt, \quad P_x - a.s.. \tag{6.7}
$$

Therefore, (Y,Z) is the solution satisfies the BSDE (6.3).
Uniqueness:
Consider (Y, Z) and (Y', Z') are two solutions to (6.3). Then by the same method as in the proof of Lemma 2.1, we can show that,

$$
\forall t > 0, \quad |Y_t - Y_t'| = 0, \quad P - a.s.. \qquad \square
$$

(I.5) The process $d(t)$ is a progressively measurable process satisfying

$$
d(\cdot) \in L^1[[0,T] \times \Omega, dt \otimes P], \quad for \quad any \quad T > 0.
$$

Theorem 6.1 *Assume the conditions (I.1)-(I.4). Then there exists a unique process* (Y, Z) *such that,*

$$
\begin{aligned}
& Y_t = Y_T + \int_t^T f(r, Y_r)dr - \int_t^T < Z_r, dM_r >, \quad for \quad any \quad t < T; \\
& \lim_{t\to\infty} e^{\int_0^t d(u)du} Y_t = 0, \quad P - a.s.
\end{aligned} \tag{6.8}
$$

PROOF.
Existence:
Set $\hat{f}(t,y) = e^{\int_0^t d(u)du} f(t, e^{-\int_0^t d(u)du} y) - d(t)y$. Then
(1) $(y-y')(\hat{f}(t,y) - \hat{f}(t,y')) \leq 0$;
(2)$\hat{f}(t,0) = e^{\int_0^t d(u)du} f(t,0)$. So $E[\int_0^\infty |\hat{f}(s,0)|ds] = E[\int_0^\infty e^{\int_0^t d(u)du}|f(t,0)|ds] < \infty$.
(3) $\sup_{|y|\leq r} |\hat{f}(t,y) - \hat{f}(t,0)| \leq \psi_r(t) + |d(t)|r$, where the process $\psi_r(t) + |d(t)|r \in L^1([0,T]\times\Omega, dt\otimes P)$, for $T > 0$.
Therefore, $\hat{f}$ satisfies all the conditions of the Lemma 6.3. So there exists a pair of processes $(\hat{Y}, \hat{Z})$ satisfying the equation:

$$\hat{Y}_t = \hat{Y}_T + \int_t^T \hat{f}(r, \hat{Y}_r)dr - \int_t^T \langle \hat{Z}_r, dM_r \rangle,$$

and obviously $\lim_{t\to\infty} \hat{Y}_t = 0$.
By the chain rule and the definition of the function $\hat{f}$, it follows that

$$de^{-\int_0^t d(u)du} \hat{Y}_t = -f(t, e^{-\int_0^t d(u)du} \hat{Y}_t)dt + \langle e^{-\int_0^t d(u)du} \hat{Z}_t, dM_t \rangle.$$

Set $Y_t = e^{-\int_0^t d(u)du} \hat{Y}_t$ and $Z_t = e^{-\int_0^t d(u)du} \hat{Z}_t$. Then the process (Y, Z) is the solution to the equation (6.8).
Uniqueness:
The uniqueness of the solution to (6.8) follows from the uniqueness of the solution to equation (6.3). □

Let $G(x,y) : R^d \times R \to R$ be a bounded Borel measurable function. Consider the following conditions:
(H.1)$'$ $(y_1 - y_2)(G(x, y_1, z) - G(x, y_2, z)) \leq -h_1(x)|y_1 - y_2|^2$, where $h_1 \in L^p(D)$ for $p > \frac{d}{2}$.
(H.2)$'$ $y \to G(x,y)$ is continuous.

Theorem 6.2 *Assume the Conditions* $(H.1)'$ *and* $(H.2)'$ *and that there is some point* $x_0 \in D$*, such that*

$$E_{x_0}[\int_0^\infty e^{\int_0^s q(X(u))du} dL_s] < \infty. \tag{6.9}$$

Then the semilinear Neumann boundary value problem

$$\begin{cases} L_2 u(x) = -G(x, u(x)), & \text{on } D \\ \frac{\partial u}{\partial \gamma}(x) = \phi(x) & \text{on } \partial D \end{cases} \tag{6.10}$$

has a unique continuous weak solution.

Proof.
Step 1
Set $\tilde{G}(X(t),y)=e^{\int_0^t q(X(u))dt}G(x,e^{-\int_0^t q(X(u))dt}y)$. Then there exists a unique solution $(\hat{Y}_x,\hat{Z}_x)$ to the following BSDE:
for any $T>0$ and $0<t<T$,

$$\begin{aligned}\hat{Y}_x(t) &= \hat{Y}_x(T)+\int_t^T \tilde{G}(X(t),\hat{Y}_x(s))ds-\int_t^T e^{\int_0^s q(X(u))dt}\phi(X(s))dL_s\\ &\quad -\int_t^T \langle \hat{Z}_x(s),dM_x(s)\rangle\end{aligned}$$

and

$$\lim_{t\to\infty} e^{-\int_0^t h_1(X(u))du}\hat{Y}_t=0 \quad P_x-a.s.$$

The uniqueness follows from the uniqueness proved in Theorem 6.1. Only the existence of solution $(\hat{Y}_x,\hat{Z}_x)$ needs to be proved:
(a) Similarly as the proof of Theorem 2.1, we can show that there exists $(p_x(t),q_x(t))$ such that

$$\begin{aligned}&dp_x(t)=e^{\int_0^t q(X(u))du}\phi(X(t))dL_t+<q_x(t),dM_x(t)>,\\ &e^{-\int_0^t h_1(X(u))du}p_x(t)\to 0,\quad as\quad t\to\infty,\quad P_x-a.s..\end{aligned} \tag{6.11}$$

(b) Set $g(x,y)=\tilde{G}(x,y+p_x(t))$. Then it follows that

$$(y-y')(g(x,y)-g(x,y'))\le -h_1(x)|y-y'|^2.$$

The condition (6.9) and Lemma 3.3 imply, for $x\in D$,

$$E_x[\int_0^\infty e^{\int_0^s(-h_1+q)(X(u))du}ds]<\infty.$$

Furthermore, as the function G is bounded, we see that condition $(I.2)$ is satisfied:

$$\begin{aligned}&E_x[\int_0^\infty e^{-\int_0^s h_1(X(u))du}|g(X(s),0)|ds]\\ =\ &E_x[\int_0^\infty e^{-\int_0^s h_1(X(u))du}|\tilde{G}(X(s),p_x(s))|ds]\\ =\ &E_x[\int_0^\infty e^{\int_0^s(-h_1+q)(X(u))du}|G(X(s),e^{-\int_0^s q(X(u))du}p_x(s))|ds]\\ \le\ &\|G\|_\infty E_x[\int_0^\infty e^{\int_0^t(-h_1+q)(X(u))du}dt]\\ <\ &\infty.\end{aligned} \tag{6.12}$$

Obviously condition (I.3) is satisfied, i.e., $y\to g(x,y)$ is continuous.
Moreover, the condition (I.4) is also satisfied. In fact, for any $r>0$,

$$\psi_r(t)=\sup_r|\tilde{G}(X(t),y)-\tilde{G}(X(t),0)|\le 2\|G\|_\infty e^{\int_0^t q(X_t)dt},$$

and for any $T > 0$, by the fact that $q \in L^p(D)$ with $p > \frac{d}{2}$ and Theorem 2.1 in [15], $E_x[\int_0^T e^{\int_0^t q(X_u)du} dt] < \infty$.
Therefore, the function $g(x, y)$ satisfies all of the conditions of Theorem 6.1 . There exists a pair of processes $(y_x(t), z_x(t))$ such that for any $T > 0$ and $0 < t < T$,

$$y_x(t) = y_x(T) + \int_t^T g(X(s), y_x(s))ds - \int_t^T \langle z_x(s), dM_x(s)\rangle \tag{6.13}$$

and

$$\lim_{t\to\infty} e^{-\int_0^t h_1(X(u))du} y_x(t) = 0 \quad P_x - a.s. \tag{6.14}$$

Put $\hat{Y}_x(t) = p_x(t) + y_x(t)$ and $\hat{Z}_x(t) = q_x(t) + z_x(t)$. It follows that $(\hat{Y}_x(t), \hat{Z}_x(t))$ satisfies the following equation

$$d\hat{Y}_x(t) = e^{\int_0^t q(X(u))du} \phi(X(t))dL_t - \tilde{G}(t, \hat{Y}_x(t))dt + < \hat{Z}_x(t), dM_x >,$$

$$\lim_{t\to\infty} e^{-\int_0^t h_1(X(u))du} \hat{Y}_t = 0 \quad P_x - a.s..$$

Step 2.
Put $Y_x(t) := e^{-\int_0^t q(X(u))dt} \hat{Y}_x(t)$ and $Z_x(t) := e^{-\int_0^t q(X(u))dt} \hat{Z}_x(t)$, we have

$$dY_x(t) = -F(X(t), Y_x(t)) + \phi(X(t))dL_t + < Z_x(t), dM_x(t) >,$$

where $F(x, y) = q(x)y + G(x, y)$. Moreover,

$$\begin{aligned} e^{\int_0^t (-h_1+q)(X(u))du} Y_x(t) &= e^{\int_0^t (-h_1+q)(X(u))(u)dt} e^{-\int_0^t q(X(u))dt} \hat{Y}_x(t) \\ &= e^{-\int_0^t h_1(X(u))dt} \hat{Y}_x(t) \to 0 \quad as \quad t \to \infty. \end{aligned}$$

Put $u_0(x) = Y_x(0)$ and $v_0(x) = Z_x(0)$.
Now as in the proof of Theorem, 4.1, we can solve the following equation

$$\begin{cases} L_2 u(x) = -G(x, u_0(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) = \phi(x) & \text{on } \partial D \end{cases} \tag{6.15}$$

and prove that the solution u coincides with $u_0(x)$. This completes the proof of the theorem. □

Suppose that $F : R^d \times R \to R$ is a bounded measurable function and $r_1 \in L^p(D)$. Consider the following conditions :
(E.1) $(y_1 - y_2)(G(x, y_1, z) - G(x, y_2, z)) \le -r_1(x)|y_1 - y_2|^2$;
(E.3) $y \to F(x, y)$ is continuous;

Now, after establishing Theorem 6.2, following the same proof as that of Theorem 5.1, we finally have the following main result.

Theorem 6.3 *Suppose that the function F satisfies the condition (E.1) and (E.2), and there exists $x_0 \in D$ such that*

$$E^0_{x_0}[\int_0^\infty \hat{Z}_s dL^0_t] < \infty. \tag{6.16}$$

Then the following problem

$$\begin{cases} Lu(x) = -F(x, u(x)), & \text{on } D \\ \frac{1}{2}\frac{\partial u}{\partial \gamma}(x) - < \widehat{B}, n > (x)u(x) = \Phi(x) & \text{on } \partial D \end{cases} \tag{6.17}$$

has a unique, bounded, continuous weak solution.

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