Talagrand Inequality on Free Path Space and Application to Stochastic Reaction Diffusion Equations*

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Abstract

By using a split argument due to [1], the transportation cost inequality is established on the free path space of Markov processes. The general result is applied to stochastic reaction diffusion equations with random initial values.

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1 Introduction

Let (E, ρ) be a metric space, and let $\mathscr{P}(E)$ be the class of all probability measures on E. The quadratic Warsserstein distance between $\mu_1, \mu_2 \in \mathscr{P}(E)$ is defined by

$$\mathbb{W}_2(\mu_1, \mu_2) = \inf_{\pi \in \mathscr{C}(\mu_1, \mu_2)} \left\{ \int_{E\times E} \rho^2(x, y)\pi(\mathrm{d}x, \mathrm{d}y) \right\}^{1/2},$$

where $\mathscr{C}(\mu_1, \mu_2)$ is the space of all couplings of μ_1 and μ_2. In the study of Monge-Kontorovich optimal transportation problem, this distance is explained as the minimal cost to transport distribution μ_1 into μ_2 at the cost rate (cost function) ρ. Thus, an inequality involving $\mathbb{W}_2$

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is called a transportation cost inequality (TCI). Since the optimal transportation is usually unknown, in applications it is important to estimate $\mathbb{W}_2$ by easier to calculate quantities, for instance the relative entropy $H(\mu_1|\mu_2) := \int_E \left(\log \frac{d\mu_1}{d\mu_2}\right) d\mu_1$ if μ_1 is absolutely continuous with respect to μ_2, and $H(\mu_1|\mu_2) := \infty$ otherwise.

In 1996, Talagrand [18] established the following beautiful TCI for the standard Gaussian measure μ on $\mathbb{R}^d$ with $\rho(x, y) = |x - y|$:

$$\mathbb{W}_2(\nu, \mu)^2 \leq 2H(\nu|\mu), \quad \nu \in \mathscr{P}(\mathbb{R}^d),$$

where the constant 2 is sharp. Since then, this type TCI has been intensively investigated and applied for various different distributions, and was linked to functional inequalities, concentration phenomena, optimal transport problem, and large deviations, see [2, 3, 8, 5, 11, 13, 19, 23] and references therein. Moreover, Talagrand type TCI has also been established on the path spaces of stochastic processes, see e.g. [4, 26, 27] for diffusion processes on $\mathbb{R}^d$, [14] for multidimensional semi-martingales, [1, 20] for stochastic differential equations (SDEs) with memory, [5, 6, 22, 23, 24] for (reflecting) diffusion processes on Riemannian manifolds, [25] for SDEs driven by pure jump processes, and [12, 17] for SDEs with Lévy or fractional noises.

Recently, by using the Girsanov transformation argument developed from [4], the Talagrand inequality was established on the path space for solutions of stochastic reaction diffusion equations with deterministic initial values, see [10], [15]. In this paper, we aim to extend this result to the case with random initial values. In this case, the distribution of a solution is a probability measure on the free path space, where the initial value is not fixed. Since the Girsanov transformation does not change initial distributions, it does not work for probability measures with different initial distributions. However, two equivalent probability measures on the free path space may have different initial distributions. To overcome this difficulty, we will adopt a split argument used in [1] to reduce the problem to the case with deterministic initial value, to which the Girsanov transformation applies.

The remainder of the paper is organized as follows. In Section 2 we present a general result on the TCI for Markov processes with random initial values, which is then applied in Section 3 to stochastic reaction diffusion equations.

2 A general result

Let (E, ρ) be a Polish space, and let $(P_t)_{t\geq 0}$ be the semigroup of a continuous Markov process on E. For any $T > 0$ and $\mu \in \mathscr{P}(E)$, let P^μ denote the distribution of the Markov process up to time T with initial distribution μ; i.e. letting $P_t(x, \cdot)$ be the associated Markov transition kernel, P^μ is the unique probability measure on the free path space

$$E_T := C([0, T]; E) \text{ equipped with } \rho_T(\xi, \eta) := \sup_{t\in[0,T]} \rho(\xi_t, \eta_t),$$

such that for any $0 = t_0 < t_1 \cdots < t_n = T$ and $\{A_i\}_{0\leq i\leq n} \subset \mathscr{B}(E)$,

$$P^\mu(X_{t_i} \in A_i, 0 \leq i \leq n) = \int_{A_0} \mu(dx_0) \int_{A_1} P_{t_1-t_0}(x_0, dx_1) \cdots \int_{A_n} P_{t_n-t_{n-1}}(x_{n-1}, dx_n),$$

where $X_t, t \geq 0$ denotes the canonical coordinate process on the path space E_T. When $\mu = \delta_x$, the Dirac measure at $x \in E$, we simply denote $P^\mu = P^x$. Then

MK
$$P^\mu = \int_E P^x \mu(\mathrm{d}x), \quad \mu \in \mathscr{P}(E). \tag{2.1}$$

Let $\mathbb{W}_2$ and $\mathbb{W}_{2,T}$ be the Wasserstein distances induced by ρ on $\mathscr{P}(E)$ and ρ_T on $\mathscr{P}(E_T)$ respectively. We aim to establish the TCI for P^μ by using those for $\{P^x : x \in E\}$ and μ.

T1 **Theorem 2.1.** *Assume that for some constants $c_1, c_2 \in (0, \infty)$ one has*

TM2
$$\mathbb{W}_{2,T}(Q, P^x)^2 \leq c_1 H(Q|P^x), \quad x \in E, Q \in \mathscr{P}(E_T), \tag{2.2}$$

TM3
$$\mathbb{W}_{2,T}(P^x, P^y)^2 \leq c_2 \rho(x, y)^2, \quad x, y \in E. \tag{2.3}$$

If $\mu \in \mathscr{P}(E)$ satisfies

TM1
$$\mathbb{W}_2(\nu, \mu)^2 \leq c_0 H(\nu|\mu), \quad \nu \in \mathscr{P}(E) \tag{2.4}$$

for some constant $c_0 \in (0, \infty)$, then

TM0
$$\mathbb{W}_{2,T}(Q, P^\mu)^2 \leq C H(Q|P^\mu), \quad Q \in \mathscr{P}(E_T) \tag{2.5}$$

holds for $C = \big(\sqrt{c_1} + \sqrt{c_0 c_2}\big)^2$. On the other hand, (2.5) *implies* (2.4) *for $c_0 = C$.*

Proof. (1) We first deduce (2.5) from (2.4). Let $Q = F P^\mu \in \mathscr{P}(E_T)$ and $u_0 : E_T \to E$ with $u_0(\xi) = \xi_0$. Then

QQ
$$\{Q \circ u_0^{-1}\}(\mathrm{d}x) = p(x)\mu(\mathrm{d}x) =: \nu(\mathrm{d}x) \tag{2.6}$$

holds for

$$p(x) := \int_{E_T} F(\xi) P^x(\mathrm{d}\xi), \quad x \in E.$$

By the triangle inequality,

TRA
$$\mathbb{W}_{2,T}(Q, P^\mu) \leq \mathbb{W}_{2,T}(Q, P^\nu) + \mathbb{W}_{2,T}(P^\nu, P^\mu). \tag{2.7}$$

Below we estimate these two terms respectively.

To estimate $\mathbb{W}_{2,T}(Q, P^\nu)$, we note that (2.1) implies

$$\begin{aligned}
&\int_{E_T} f(\xi_0) F(\xi) P^\mu(\mathrm{d}\xi) = \int_E f(x)\mu(\mathrm{d}x) \int_{E_T} F(\xi) P^x(\mathrm{d}\xi) \\
&= \int_E f(x) p(x) \mu(\mathrm{d}x) = \int_{E_T} (fp)(\xi_0) P^\mu(\mathrm{d}\xi), \quad f \in \mathscr{B}_b(E).
\end{aligned}$$

Therefore, letting $\mathbb{E}^\mu$ be the expectation with respect to P^μ, we have

QQ2 (2.8)
$$p \circ u_0 = \mathbb{E}^\mu(F|u_0).$$

Now, let

$$F_x(\xi) = 1_{\{p(x)>0\}} \frac{F(\xi)}{p(x)}, \quad x \in E, \xi \in E_T.$$

By (2.2), if $p(x) > 0$ then

$$\mathbb{W}_{2,T}(F_x P^x, P^x)^2 \le c_1 P^x(F_x \log F_x).$$

So, for any $G, H \in \mathscr{C}$, where

$$\mathscr{C} := \big\{(G,H) : G, H \in C_b(E_T), G(\xi) \le H(\eta) + \rho_T(\xi,\eta)^2 \text{ for } \xi, \eta \in E_T\big\},$$

we have

$$\int_{E_T} F_x G \mathrm{d}P^x - \int_{E_T} H \mathrm{d}P^x \le c_1 \int_{E_T} (F_x \log F_x) \mathrm{d}P^x, \quad p(x) > 0.$$

Integrating with respect to $\nu(\mathrm{d}x) := p(x)\mu(\mathrm{d}x)$ and using (2.1), we obtain

$$\begin{aligned}
Q(G) - P^\nu(H) &= \int_{E_T} G \mathrm{d}Q - \int_{E_T} H \mathrm{d}P^\nu \\
&= \int_E \bigg\{ \int_{E_T} F_x G \mathrm{d}P^x - \int_{E_T} H \mathrm{d}P^x \bigg\} p(x)\mu(\mathrm{d}x) \\
&\le c_1 \int_E \bigg\{ \int_{E_T} (F_x \log F_x) \mathrm{d}P^x \bigg\} p(x)\mu(\mathrm{d}x) \\
&= c_1 \int_{E_T} \big\{F \log F - F \log \mathbb{E}^\mu(F|u_0)\big\} \mathrm{d}P^\mu \\
&= c_1 H(Q|P^\mu) - c_1 \mathbb{E}^\mu[F \log \mathbb{E}^\mu(F|u_0)] \le c_1 H(Q|P^\mu),
\end{aligned}$$

where the last step is due to the fact that

$$\begin{aligned}
&\mathbb{E}^\mu[F \log \mathbb{E}^\mu(F|u_0)] = \mathbb{E}^\mu[\mathbb{E}^\mu(F|u_0) \log \mathbb{E}^\mu(F|u_0)] \\
&\ge \mathbb{E}^\mu[\mathbb{E}^\mu(F|u_0)] \log \mathbb{E}^\mu[\mathbb{E}^\mu(F|u_0)] = \mathbb{E}^\mu[F] \log \mathbb{E}^\mu[F] = 0.
\end{aligned}$$

Therefore, by the Kontorovich dual formula, we arrive at

EE1 (2.9)
$$\mathbb{W}_{2,T}(Q, P^\nu)^2 = \sup_{(G,H)\in\mathscr{C}} \big\{Q(G) - P^\nu(H)\big\} \le c_1 H(Q|P^\mu).$$

On the other hand, by (2.3), for any $(G,H) \in \mathscr{C}$ we have

EE' (2.10)
$$\int_{E_T} G \mathrm{d}P^x - \int_{E_T} H \mathrm{d}P^y \le c_2 \rho(x,y)^2, \quad x, y \in E.$$

Let $\pi \in \mathscr{C}(\nu,\mu)$ be the optimal coupling such that

$$\mathbb{W}_2(\nu,\mu)^2 = \int_{E\times E} \rho(x,y)^2 \pi(\mathrm{d}x,\mathrm{d}y).$$

Integrating (2.10) with respect to $\pi(\mathrm{d}x,\mathrm{d}y)$, and applying (2.1), we obtain

$$\int_{E_T} G\mathrm{d}P^\nu - \int_{E_T} H\mathrm{d}P^\mu = \int_{E\times E}\bigg\{\int_{E_T} G\mathrm{d}P^x - \int_{E_T} H\mathrm{d}P^y\bigg\}\pi(\mathrm{d}x,\mathrm{d}y) \le c_2\mathbb{W}_2(\nu,\mu)^2.$$

Combining this with the Kontorovich dual formula, and applying (2.4), we arrive at

TM5 (2.11) $$\mathbb{W}_{2,T}(P^\nu,P^\mu)^2 \le c_2\mathbb{W}_2(\nu,\mu)^2 \le c_0c_2\mu(p\log p).$$

Since (2.1), (2.8) and Jensen's inequality imply

$$\begin{aligned}\mu(p\log p) &= \int_{E_T} \big\{(p\circ u_0)\log p\circ u_0\big\}\mathrm{d}P^\mu \\ &= \mathbb{E}^\mu[\mathbb{E}^\mu(F|u_0)\log\mathbb{E}^\mu(F|u_0)] \le \mathbb{E}^\mu[\mathbb{E}^\mu(F\log F|u_0)] = H(Q|P^\mu),\end{aligned}$$

it follows from (2.11) that

$$\mathbb{W}_{2,T}(P^\nu,P^\mu)^2 \le c_0c_2H(Q|P^\mu).$$

Combining this with (2.7) and (2.9), we prove (2.5)

(2) To deduce (2.4) from (2.5), for $\nu = p\mu$ we take $Q = (p\circ u_0)P^\mu$. Let $\Pi \in \mathscr{C}(Q,P^\mu)$ be the optimal coupling such that

$$\mathbb{W}_{2,T}(Q,P^\mu)^2 = \int_{E_T\times E_T} \rho_T^2\,\mathrm{d}\Pi.$$

We have $\pi := \Pi\circ(u_0,u_0)^{-1} \in \mathscr{C}(\nu,\mu)$, so that

$$\begin{aligned}\mathbb{W}_2(\nu,\mu)^2 &\le \int_{E\times E}\rho^2\mathrm{d}\pi = \int_{E_T\times E_T}\rho^2(\xi_0,\eta_0)\Pi(\mathrm{d}\xi,\mathrm{d}\eta) \\ &\le \int_{E_T\times E_T}\rho_T^2(\xi,\eta)\Pi(\mathrm{d}\xi,\mathrm{d}\eta) = \mathbb{W}_{2,T}(Q,P^\mu)^2.\end{aligned}$$

Combining this with (2.5) and noting that (2.1) implies

$$H(Q|P^\mu) = \int_{E_T}\big\{(p\circ u_0)\log p\circ u_0\big\}\mathrm{d}P^\mu = \int_E (p\log p)\mathrm{d}\mu = H(\nu|\mu),$$

we derive (2.4) for $c_0 = C$.

□

3 TCI for stochastic reaction diffusion equations with random initial values

Let $C_0([0,1]) = \{u \in C([0,1]) : u(0) = u(1) = 0\}$. Consider the following SPDE on $C_0([0,1])$:

3.1
$$\begin{cases} \mathrm{d}u_t(x) = \frac{1}{2}u_t''(x)dt + b(u_t(x))dt + \sigma(u_t(x))W(\mathrm{d}t,\mathrm{d}x), \quad x \in (0,1), \\ u_t \in C_0([0,1]), \qquad t \geq 0, \end{cases} \tag{3.1}$$

where $W(\mathrm{d}t, \mathrm{d}x)$ is a space-time white noise on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with natural filtration $\mathscr{F}_t$ generated by the Brownian sheet $\{W(t,x) : (t,x) \in [0,\infty) \times [0,1]\}$, u_0 is a $C_0([0,1])$-valued random variable independent of W, and $b, \sigma : \mathbb{R} \to \mathbb{R}$ are locally bounded measurable functions. We say that an adapted, continuous process $\{u_t\}_{t\geq 0}$ on $C_0([0,1])$ is a solution to (3.1), if $\mathbb{P}$-a.s.

3.2
$$\begin{aligned} \int_0^1 u_t(x)\phi(x)\mathrm{d}x &= \int_0^1 u_0(x)\phi(x)\mathrm{d}x + \frac{1}{2}\int_0^t \mathrm{d}s \int_0^1 u_s(x)\phi''(x)\mathrm{d}x \\ &+ \int_0^t \mathrm{d}s \int_0^1 b(u_s(x))\phi(x)\mathrm{d}x + \int_0^t \int_0^1 \sigma(u_s(x))\phi(x)W(\mathrm{d}s,\mathrm{d}x), \quad t \geq 0, \phi \in C_0^2([0,1]), \end{aligned} \tag{3.2}$$

where $C_0^2([0,1]) := \{\phi \in C^2([0,1]) : \phi(0) = \phi(1) = 0\}$. According to [21], u_t is a solution to (3.1) if and only if $\mathbb{P}$-a.s.

3.3
$$u_t(x) = P_t u_0(x) + \int_0^t P_{t-s}\{b(u_s)\}(x)\mathrm{d}s + \int_0^t \int_0^1 p_{t-s}(x,y)\sigma(u_s(y))W(\mathrm{d}s,\mathrm{d}y), \quad t \geq 0, \tag{3.3}$$

where P_t and $p_t(x,y)$ are the Dirichlet heat semigroup and heat kernel generated by $\frac{1}{2}\Delta$ on $[0,1]$.

We will apply Theorem 2.1 to

$$E := C_0([0,1]), \quad E_T := C([0,T]; E) = C([0,T]; C_0([0,1])),$$

and P^μ being the distribution of the solution $(u_t)_{t\in[0,T]}$ with initial distribution $\mu \in \mathscr{P}(E)$. To this end, we need the following assumption.

(H) σ is bounded, b and σ are Lipschitz continuous.

According to [21], when b and σ are Lipschitz continuous, (3.1) admits a unique solution for any (random) initial value u_0 on E. The boundedness of σ was used in [15] to establish the TCI for solutions of (3.1) with deterministic initial values.

T2 **Theorem 3.1.** *Assume* **(H)** *and let* $\mu \in \mathscr{P}(E)$. *Then*

TMM
$$W_2(Q, P^\mu) \leq CH(Q|P^\mu), \quad Q \in \mathscr{P}(E_T) \tag{3.4}$$

holds for some constant $C > 0$ *if and only if*

TMM2
$$W_2(\nu, \mu) \leq cH(\nu|\mu), \quad \nu \in \mathscr{P}(E) \tag{3.5}$$

holds for some constant $c > 0$.

Proof. In the present case, we have

$$\rho(f,g) = \sup_{x\in[0,1]} |f(x)-g(x)|, \quad f,g \in E := C_0([0,1]),$$
$$\rho_T(\xi,\eta) = \sup_{(t,x)\in[0,T]\times[0,1]} |\xi_t(x)-\eta_t(x)|, \quad \xi,\eta \in E_T := C([0,T]];E).$$

According to [15], (2.2) holds for some constant $c_1>0$. So, by Theorem 2.1, it suffices to verify (2.3). Letting u_t^f be the unique solution of (3.1) with $u_0 = f \in E := C_0([0,1])$, we only need to prove

5.1 (3.6)
$$\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_t^f(x)-u_t^g(x)|^2\right] \le c_2 \sup_{x\in[0,1]} |f(x)-g(x)|^2, \quad f,g \in C_0([0,1])$$

for some constant $c_2>0$. Indeed, since the law of $(u_t^f, u_t^g)_{t\in[0,T]}$ is a coupling of P^f and P^g, we have

$$\mathbb{W}_{2,T}(P^f,P^g)^2 \le \mathbb{E}[\rho_T(u^f,u^g)^2] = \mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_t^f(x)-u_t^g(x)|^2\right].$$

Below we prove the estimate (3.6).

By (3.3) we have

add 0302.1 (3.7)
$$\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_t^f(x)-u_t^g(x)|^2\right] \le 3\rho(f,g)^2 + 3(I_1+I_2),$$

where

$$I_1 := \mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left|\int_0^t\int_0^1 p_{t-s}(x,y)\big[b(u_s^f(y)) - b(u_s^g(y))\big]\,\mathrm{d}s\mathrm{d}y\right|^2\right],$$
$$I_2 := \mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} \left|\int_0^t\int_0^1 p_{t-s}(x,y)\big[\sigma(u_s^f(y)) - \sigma(u_s^g(y))\big]W(\mathrm{d}s,\mathrm{d}y)\right|^2\right].$$

Noting that the Dirichlet heat kernel satisfies

$$\sup_{x\in[0,1]} \int_0^t \mathrm{d}s \int_0^1 p_{t-s}(x,y)^2 \mathrm{d}y \le \frac{\sqrt{2t}}{\sqrt{\pi}}, \quad t>0,$$

and due to **(H)** we have

LL (3.8)
$$|b(x)-b(y)| \vee |\sigma(x)-\sigma(y)| \le K|x-y|, \quad x,y \in [0,1]$$

for some constant $K>0$, by Hölder's inequality we obtain

I

$$\begin{aligned} I_1 \le{}& K^2\mathbb{E}\Bigg\{\sup_{(t,x)\in[0,T]\times[0,1]}\Bigg[\bigg(\int_0^t\int_0^1 p_{t-s}(x,y)^2\,\mathrm{d}s\mathrm{d}y\bigg) \\ &\times\bigg(\int_0^t\int_0^1 |u_s^f(y)-u_s^g(y)|^2\,dsdy\bigg)\Bigg]\Bigg\} \\ \le{}&\sqrt{\frac{2T}{\pi}}K^2\int_0^T\mathbb{E}\left[\sup_{(r,y)\in[0,s]\times[0,1]}|u_r^f(y)-u_r^g(y)|^2\right]\mathrm{d}s. \end{aligned}\tag{3.9}$$

To estimate the term I_2, we recall the following inequality due to [15]: for any $T,\varepsilon>0$, there exists a constant $C_{T,\varepsilon}>0$ such that for any adapted random field $\gamma(t,x)$ with $\mathbb{E}[\sup_{(s,x)\in[0,t]\times[0,1]}|\gamma(s,x)|^2]<\infty, t\ge 0$, we have

4.4-1

$$\begin{aligned} &\mathbb{E}\left[\sup_{(s,x)\in[0,t]\times[0,1]}\left|\int_0^s\int_0^1 p_{s-r}(x,y)\gamma(r,y)W(\mathrm{d}r,\mathrm{d}y)\right|^2\right] \\ &\le\varepsilon\mathbb{E}\Big[\sup_{(s,x)\in[0,t]\times[0,1]}|\gamma(s,x)|^2\Big]+C_{T,\varepsilon}\int_0^t\mathbb{E}\Big[\sup_{(r,x)\in[0,s]\times[0,1]}|\gamma(r,x)|^2\Big]\mathrm{d}r,\quad t\in[0,T]. \end{aligned}\tag{3.10}$$

Applying this to $\gamma(s,x)=\sigma(u_s^f(x))-\sigma(u_s^g(x))$ and using (3.8), we obtain that for any $\epsilon>0$,

term II

$$\begin{aligned} I_2\le{}&\epsilon\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|\sigma(u_t^f(x))-\sigma(u_t^g(x))|^2\right] \\ &+C_{T,\varepsilon}\mathbb{E}\int_0^T\sup_{y\in[0,1]}\left|\sigma(u_s^f(y))-\sigma(u_s^g(y))\right|^2\,\mathrm{d}s \\ \le{}&\epsilon K^2\mathbb{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|u_t^f(x)-u_t^g(x)|^2\right] \\ &+C_{T,\varepsilon}K^2\int_0^T\mathbb{E}\left[\sup_{(r,y)\in[0,s]\times[0,1]}\left|u_r^f(y)-u_r^g(y)\right|^2\right]\mathrm{d}s,\ t\in[0,T]. \end{aligned}\tag{3.11}$$

So, setting

$$Y(t):=\mathbb{E}\left[\sup_{(s,x)\in[0,t]\times[0,1]}|u_s^f(x)-u_s^g(x)|^2\right],$$

which is finite for all $t\in[0,\infty)$ due to assumption **(H)**, by combining (3.7)-(3.11) together we obtain

$$Y(t)\le 3\rho(f,g)^2+3\sqrt{\frac{2T}{\pi}}K^2\int_0^t Y(s)\,\mathrm{d}s+3\epsilon K^2Y(t)+3C_{T,\epsilon}K^2\int_0^t Y(s)\,\mathrm{d}s,\quad t\in[0,T].$$

Choosing $\varepsilon=\frac{1}{6K^2}$, we find a constant $c(T)>0$ such that

$$Y(t)\le 6\rho(f,g)^2+c(T)\int_0^t Y(s)\,\mathrm{d}s,\quad t\in[0,T].$$

By Gronwall's inequality and $Y(t) < \infty$ for $t \geq 0$, this implies (3.6) for $c_2 = 6e^{c(T)T}$.

$\square$

To illustrate Theorem 3.1, we present examples of μ satisfying (3.5), such that (3.4) holds true. By [5, Theorem 3.1], the heat measure on the loop space $C_0([0,1])$ satisfies (3.5). Next, by Gross [9], the log-Sobolev inequality holds for the Brownian bridge measure μ_0 on $C_0([0,1])$:

$$\mu_0(F^2 \log F^2) \leq 2T\mu_0(\|DF\|_H^2)^2, \quad F \in \mathscr{D}(D), \mu_0(F^2) = 1,$$

where $(D, \mathscr{D}(D))$ is the Malliavin gradient operator and $\|h\|_H := (\int_0^T |h_t'|^2 \mathrm{d}t)^{\frac{1}{2}}$ is the Cameron-Martin norm. So, by a standard perturbation argument, the log-Sobolev inequality

$$\mu(F^2 \log F^2) \leq 2T e^{\mathrm{osc}(V)} \mu(\|DF\|_H^2)^2, \quad F \in \mathscr{D}(D), \mu(F^2) = 1,$$

holds for any probability measure $\mathrm{d}\mu = e^V \mathrm{d}\mu_0$ with $V \in \mathscr{B}_b(C_0([0,1]))$, where $\mathrm{osc}(V) := \sup V - \inf V$. According to [16, Theorem 1.10], this implies

$$\tilde{\mathbb{W}}_2(\nu, \mu)^2 \leq 2T e^{\mathrm{osc}(V)} H(\nu, \mu), \quad \nu \in \mathscr{P}(C_0([0,1])),$$

where $\tilde{\mathbb{W}}_2$ is the Wasserstein distance induced by the Cameron-Martin distance on E. Since the Cameron-Martin distance is larger than the uniform distance ρ, (3.5) holds for this class of measures μ.

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