Limit Theorems for Additive Functionals of Path-Dependent SDEs *

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Abstract

By using limit theorems of uniform mixing Markov processes and martingale difference sequences, the strong law of large numbers, central limit theorem, and the law of iterated logarithm are established for additive functionals of path-dependent stochastic differential equations.

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1 Introduction and Main Results

Since W. Doeblin [9] in 1938 established the law of large numbers and central limit theorem for denumerable Markov chains, limit theory for additive functionals of Markov processes has been extensively investigated. In general, for an ergodic Markov process $(X_t)_{t\geq 0}$ on a Polish space E, as $t \to \infty$ one describes the convergence of the empirical distribution $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$ to the unique invariant probability measure μ_∞. A standard way is to look at the convergence rate of

$$A_t^f := \frac{1}{t}\int_0^t f(X_s)\mathrm{d}s \to \mu_\infty(f) \quad \text{as} \quad t \to \infty$$

for f in a class of reference functions.This leads to the study of limit theorems for additive functionals of ergodic Markov processes. Classical limit theorems include

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- Strong law of large numbers (SLLN): $\mathbb{P}$-a.s. convergence of A_t^f to $\mu_\infty(f)$;
- Central limit theorem (CLT): The weak convergence of $\frac{1}{\sqrt{t}}\int_0^t\{f(X_s)-\mu_\infty(f)\}\mathrm{d}s$ to a normal random variable;
- Law of iterated logarithm (LIL): the asymptotic range of $\frac{1}{\sqrt{t\log\log t}}\int_0^t f(X_s)\mathrm{d}s$.

Once CLT is established, one may further investigate the large/moderate deviations principles, see for instance [12] and references within.

When the Markov processes are exponentially ergodic in $L^2(\mu_\infty)$ or total variational norm, limit theorems of A_t^f have been established for reference functions $f\in L^2(\mu_\infty)$ or $\mathscr{B}_b(E)$, respectively; see the recent monograph [22] and earlier references [6, 13, 17, 20, 19, 23, 28]. However, these results do not apply to highly degenerate models which are exponentially ergodic merely under a Wasserstein distance; see for instance [15] for 2D Navier-Stokes equations with degenerate stochastic forcing, and [2, 4, 5, 14] for stochastic differential equations (SDEs) with memory.

In this paper, we aim to establish limit theorems for path-dependent SDEs, which were initiated by Itô-Nisio [18]. Due to the path-dependence of the noise term, the corresponding segment solutions are no longer ergodic in the total variational norm (see e.g. [22, Example 5.1.3]). Moreover, the L^2-ergodicity is also unknown because of the lack of Dirichlet form for path-dependent SDEs. So far, there are a few of papers on LLN and CLT for stochastic dynamical systems which are weakly ergodic; see e.g. [21, 22, 24, 27]. In particular, f in [21, 27] is assumed to be (bounded) Lipschitz with respect to a metric and the weak LLN is investigated; In [22], the LLN is established under some additional technical conditions (see [22, Theorem 5.1.10] for more details). In this paper, we will show that limit theorems established in [24] for uniformly mixing Markov processes apply well to the present model for f being Lipchitz continuous with respect to a quasi-metric.

For a fixed number $r_0\in(0,\infty)$, let $\mathscr{C}=C([-r_0,0];\mathbb{R}^d)$ be the collection of all continuous functions $f:[-r_0,0]\to\mathbb{R}^d$ endowed with the uniform norm

$$\|f\|_\infty := \sup_{-r_0\le\theta\le 0}|f(\theta)|.$$

For any continuous path $(\gamma(t))_{t\ge -r_0}$ on $\mathbb{R}^d$, its segment $(\gamma_t)_{t\ge0}$ is a continuous path on $\mathscr{C}$ defined by

$$\gamma_t(\theta):=\gamma(t+\theta),\quad \theta\in[-r_0,0], t\ge 0.$$

Consider the following path-dependent SDE on $\mathbb{R}^d$:

$$\mathrm{d}X(t)=b(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}W(t),\quad t\ge0,\quad X_0=\xi\in\mathscr{C}, \tag{1.1}$$

where $(W(t))_{t\ge0}$ is a d-dimensional Brownian motion on a complete filtration probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\ge0},\mathbb{P})$, and

$$b:\mathscr{C}\to\mathbb{R}^d,\quad \sigma:\mathscr{C}\to\mathbb{R}^d\otimes\mathbb{R}^d$$

are measurable maps satisfying the following assumptions.

(**A1**) (**Continuity**) σ is Lipschitz continuous; b is continuous, and bounded on bounded subsets of $\mathscr{C}$;

(**A2**) (**Dissipativity**) There exist constants $\lambda_1, \lambda_2 > 0$ with $\lambda_1 > \lambda_2 \mathrm{e}^{\lambda_1 r_0}$ such that

$$2\langle \xi(0)-\eta(0), b(\xi)-b(\eta)\rangle \le -\lambda_1|\xi(0)-\eta(0)|^2 + \lambda_2\|\xi-\eta\|_\infty^2, \quad \xi,\eta\in\mathscr{C};$$

(**A3**) (**Invertibility**) σ is invertible with $\sup_{\xi\in\mathscr{C}}\{\|\sigma(\xi)\|+\|\sigma(\xi)^{-1}\|\}<\infty$.

Under (**A1**) and (**A2**), (1.1) admits a unique solution, and the segment (also called functional or window) solution $(X_t)_{t\ge 0}$ is a Markov process on $\mathscr{C}$; see [26, Theorem 2.2] or [4, Proposition 4.1]. Assumption (**A3**) was used in [2, 4, 5, 14] to ensure the exponential ergodicity under the Wasserstein distance induced by a quasi-metric.

Let P_t be the associated Markov process, i.e.,

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi), \ \ f\in\mathscr{B}_b(\mathscr{C}), \ t\ge 0, \ \xi\in\mathscr{C}.$$

For a probability measure μ on $\mathscr{C}$, let μP_t be the law of X_t with initial distribution μ. We then have

$$\int_{\mathscr{C}} f \mathrm{d}(\mu P_t) = \int_{\mathscr{C}} P_t f \mathrm{d}\mu, \ \ t\ge 0, f\in\mathscr{B}_b(\mathscr{C}).$$

To state the main results, we recall the quasi-metric $\rho_{p,\gamma}$, the associated Wasserstein distance $\mathbb{W}_{p,\gamma}$, and the class $C_{p,\gamma}(\mathscr{C})$ of Lipschitz functions, where $p\ge 1$ and $\gamma\in(0,1]$ are constants. Firstly, let

$$\rho_{p,\gamma}(\xi,\eta) = (1\wedge\|\xi-\eta\|_\infty^\gamma)\sqrt{1+\|\xi\|_\infty^p+\|\eta\|_\infty^p}, \quad \xi,\eta\in\mathscr{C}.$$

Note that $(\xi,\eta)\mapsto\rho_{p,\gamma}(\xi,\eta)$ is a quasi-distance, i.e., it is symmetric, lower semi-continuous, and $\rho_{p,\gamma}(\xi,\eta)=0 \Leftrightarrow \xi=\eta$, but the triangle inequality may not hold. Next, let $C_{p,\gamma}(\mathscr{C})$ be the set of all continuous $\mathbb{R}$-valued functions on $\mathscr{C}$ such that

$$\|f\|_{p,\gamma} := \sup_{\xi\in\mathscr{C}} \frac{|f(\xi)|}{1+\|\xi\|_\infty^{p/2}} + \sup_{\xi\neq\eta,\xi,\eta\in\mathscr{C}} \frac{|f(\xi)-f(\eta)|}{\rho_{p,\gamma}(\xi,\eta)} < \infty.$$

Moreover, let $\mathscr{P}_{p,\gamma}(\mathscr{C})$ be the set of probability measures μ on $\mathscr{C}$ with $(\mu\times\mu)(\rho_{p,\gamma})<\infty$. Define

$$\mathbb{W}_{p,\gamma}(\mu,\nu) = \inf_{\pi\in\mathcal{C}(\mu,\nu)} \int_{\mathscr{C}\times\mathscr{C}} \rho_{p,\gamma}(\xi,\eta)\pi(\mathrm{d}\xi,\mathrm{d}\eta), \ \ \mu,\nu\in\mathscr{P}_{p,\gamma}(\mathscr{C}),$$

where $\mathcal{C}(\mu,\nu)$ stands for the set of all couplings of μ and ν; that is, $\pi\in\mathcal{C}(\mu,\nu)$ if and only if it is a probability measure on $\mathscr{C}\times\mathscr{C}$ such that $\pi(\cdot\times\mathscr{C})=\mu(\cdot)$ and $\pi(\mathscr{C}\times\cdot)=\nu(\cdot)$.

The following result concerns with the exponential ergodicity and SLLN for the additive functional $A_t^f(\xi) := \frac{1}{t}\int_0^t f(X_s^\xi)\mathrm{d}s$, where $f\in C_{p,\gamma}(\mathscr{C})$.

Theorem 1.1. *Assume* **(A1)**-**(A3)** *and let* $p\ge 1, \gamma\in(0,1]$. *Then* P_t *has a unique invariant probability measure* $\mu_\infty\in\mathscr{P}_{p,\gamma}(\mathscr{C})$ *such that*

$$\mathbb{W}_{p,\gamma}(\mu P_t,\mu_\infty) \le c\,\mathrm{e}^{-\beta t}\mathbb{W}_{p,\gamma}(\mu,\mu_\infty), \ \ t\ge 0, \mu\in\mathscr{P}_{p,\gamma}(\mathscr{C}) \tag{1.2}$$

holds for some constants $c,\beta>0$. *Moreover, for any* $\xi\in\mathscr{C}$ *and* $f\in C_{p,\gamma}(\mathscr{C})$,

(1) *There exists a constant* $c > 0$ *such that*

$$\mathbb{E}\big|A_t^f(\xi) - \mu_\infty(f)\big|^2 \le c\,(1 + \|\xi\|_\infty^p)\|f\|_{p,\gamma}^2 t^{-1},\ t \ge 1;$$

(2) *For any* $\varepsilon \in (0, \frac{1}{2})$, *there exist a constant* $c_\varepsilon > 0$ *such that* $\mathbb{P}$*-a.s.*

$$\big|A_t^f(\xi) - \mu_\infty(f)\big| \le c_\varepsilon \|f\|_{p,\gamma} t^{-\frac{1}{2}+\varepsilon}, \quad t \ge T_\varepsilon^f(\xi)$$

holds for a family of random variables $\{T_\varepsilon^f(\xi) \ge 1 : f \in C_{p,\gamma}(\mathscr{C}), \xi \in \mathscr{C}\}$ *satisfying*

$$\sup_{f \in C_{p,\gamma}(\mathscr{C})} \frac{\mathbb{E}\,|T_\varepsilon^f(\xi)|^k}{1 + \|f\|_{p,\gamma}^{k(1+k)}} < \infty, \quad k \ge 1.$$

To state the CLT, we introduce the corrector R_f for $f \in C_{p,\gamma}(\mathscr{C})$ defined by

$$R_f(\xi) = \int_0^\infty \big\{P_t f(\xi) - \mu_\infty(f)\big\}\mathrm{d}t, \quad \xi \in \mathscr{C}. \tag{1.3}$$

This function is well-defined since (1.2) and $\mu_\infty \in \mathscr{P}_{p,\gamma}(\mathscr{C})$ imply

$$\begin{aligned}&|P_t f(\xi) - \mu_\infty(f)| \le \|f\|_{p,\gamma}\mathbb{W}_{p,\gamma}(\delta_\xi P_t, \mu_\infty)\\ &\le c_1 \mathrm{e}^{-\beta t}\|f\|_{p,\gamma}\mathbb{W}_{p,\gamma}(\delta_\xi, \mu_\infty) \le c_2 \mathrm{e}^{-\beta t}\|f\|_{p,\gamma}(1 + \|\xi\|_\infty^{p/2}), \quad t \ge 0,\ \xi \in \mathscr{C}\end{aligned} \tag{1.4}$$

for some constants $c_1, c_2 > 0$. Let

$$\varphi_f(\xi) = \mathbb{E}\bigg|\int_0^1 f(X_r^\xi)\mathrm{d}r + R_f(X_1^\xi) - R_f(\xi)\bigg|^2, \quad \xi \in \mathscr{C}. \tag{1.5}$$

For any $D \in [0, \infty)$, let Φ_D be the normal distribution function with zero mean and variance D, where $\Phi_0(z) := 1_{[0,\infty)}(z)$ for $D = 0$. We have the following CLT.

Theorem 1.2. *Assume* **(A1)-(A3)***. For any constants* $p \ge 1$ *and* $\gamma \in (0, 1]$, *let* $f \in C_{p,\gamma}(\mathscr{C})$ *with* $\mu_\infty(f) = 0$. *Then* $D_f := \sqrt{\mu_\infty(\psi_f)} \in [0, \infty)$ *and the following assertion holds:*

(1) *When* $D_f > 0$, *for any* $\varepsilon \in (0, \frac{1}{4})$ *there exists an increasing function* $h_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\sup_{z \in \mathbb{R}} \Big|\mathbb{P}\big(\sqrt{t}A_t^f(\xi) \le z\big) - \Phi_{D_f}(z)\Big| \le h_\varepsilon(\|\xi\|_\infty)t^{-\frac{1}{4}-\varepsilon}, \quad t > 0;$$

(2) *When* $D_f = 0$, *there exists an increasing function* $h_0 : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\sup_{z \in \mathbb{R}} \big(1 \wedge |z|\big)\Big|\mathbb{P}\big(\sqrt{t}A_t^f(\xi) \le z\big) - \Phi_{D_f}(z)\Big| \le h_0(\|\xi\|_\infty)t^{-\frac{1}{4}}, \quad t > 0.$$

Finally, to investigate the LIL, we consider the unit ball in the Camron-Martin space of $C([0,1];\mathbb{R})$:

$$\mathcal{H} := \Big\{ h \in C([0,1];\mathbb{R}) :\ h_t' \text{ exists a.e. } t, \int_0^1 |h_t'|^2 \mathrm{d}t \le 1 \Big\}, \tag{1.6}$$

and the following discrete version of R_f and φ_f for $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$:

$$\widehat{R}_f(\xi) := \sum_{k=0}^{\infty} P_k f(\xi), \quad \widehat{\varphi}_f(\xi) := \mathbb{E}\big| f(\xi) + \widehat{R}^f(X_1^\xi) - \widehat{R}^f(\xi) \big|^2, \quad \xi \in \mathscr{C},$$

which are well defined due to (1.4). For any $n \ge 1$, consider the following random variable on $C([0,1];\mathbb{R})$:

$$\Lambda_n^{f,\xi}(t) := \sum_{k=0}^{n} 1_{[\frac{k}{n}, \frac{k+1}{n})}(t) \frac{\sum_{l=1}^{k-1} f(X_l^\xi) + (nt-k) f(X_k^\xi)}{\widehat{D}_f \sqrt{2n \log\log n}}, \quad t \in [0,1], \tag{1.7}$$

where $\widehat{D}_f := \mu_\infty(\widehat{\varphi}_f)$.

Theorem 1.3. *Assume* **(A1)**-**(A3)**. *Let $p \ge 1, \gamma \in (0,1]$, $\xi \in \mathscr{C}$, and $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$ and $\widehat{D}_f > 0$. Then the sequence $\{\Lambda_n^{f,\xi}(\cdot)\}_{n\ge1}$ is almost surely relatively compact in $C([0,1];\mathbb{R})$, and when $n \to \infty$ the set of limit points coincides with $\mathcal{H}$. Consequently, $\mathbb{P}$-a.s.*

$$\limsup_{n\to\infty} \frac{\sum_{l=1}^{n} f(X_l^\xi)}{\sqrt{2n \log\log n}} = \widehat{D}_f, \quad \liminf_{n\to\infty} \frac{\sum_{l=1}^{n} f(X_l^\xi)}{\sqrt{2n \log\log n}} = -\widehat{D}_f. \tag{1.8}$$

Note that the LIL has been intensively investigated for many different models, see e.g. [3, 7, 8, 10, 13, 20, 25] and references therein. Theorem 1.3 is a supplement in the setting of path-dependent SDEs.

The remainder of this paper is arranged as follows. In Section 2, we recall some known results on SLLN, CLT and LIL for Markov processes, which are then applied to prove the above three results in Sections 3-5 respectively.

2 Some known results

We first state some results presented in [24] for continuous Markov processes on separable Hilbert spaces. Since proofs of these results only use the norm rather than the inner product of the space, they apply also to a Banach space.

Let $\{X_t^x :\ x \in \mathbb{B}, t \ge 0\}$ be a continuous Markov process on a separable Banach space $(\mathbb{B}, \|\cdot\|)$ with respect to a complete filtration probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\ge0}, \mathbb{P})$ such that the associate Markov semigroup

$$P_t f(x) := \mathbb{E} f(X_t^x), \quad t \ge 0, x \in \mathbb{B}, f \in \mathscr{B}_b(\mathbb{B})$$

has a unique invariant probability measure μ_∞. For a constant $\gamma \in (0,1]$ and an increasing function $w \in C([0,\infty);[1,\infty))$, let $C_{w,\gamma}(\mathbb{B})$ be the class of measurable functions on $\mathbb{B}$ such that

$$\|f\|_{w,\gamma} := \sup_{x\in\mathbb{B}} \frac{|f(x)|}{w(\|x\|)} + \sup_{x,y\in\mathbb{B}} \frac{|f(x)-f(y)|}{(1\wedge\|x-y\|^\gamma)(1+w(\|x\|)+w(\|y\|))} < \infty.$$

Note that in [24] $\|f\|_{w,\gamma}$ is defined by using $\|x-y\|^\gamma$ instead of $1\wedge\|x-y\|^\gamma$, but this does not make essential differences since these two definitions are equivalent up to a constant multiplication. We take the present formulation in order to apply the ergodicity result derived in [2]. By [24, Proposition 2.6], we have the following result.

Lemma 2.1. *If there exist $\varphi,\psi \in C(\mathbb{R}_+;\mathbb{R}_+)$ with $\int_0^\infty \varphi(t)\mathrm{d}t < \infty$ such that*

$$|P_t f(x) - \mu_\infty(f)| \le \varphi(t)\psi(\|x\|)\|f\|_{w,\gamma}, \quad f \in C_{w,\gamma}(\mathbb{B}), \quad t \ge 0, \quad x \in \mathbb{B}, \tag{2.1}$$

and for some $k \in \mathbb{N}$,

$$\mathbb{E}\psi(\|X_t^x\|)^{2k} < \infty, \quad t \ge 0,\ x \in \mathbb{B}, \tag{2.2}$$

then for any $f \in C_{w,\gamma}(\mathbb{B})$,

$$\begin{aligned} &\mathbb{E}\Big|\frac{1}{t}\int_0^t f(X_s^x)\mathrm{d}s - \mu_\infty(f)\Big|^{2\kappa} \\ &\le t^{-k}\Big(2k(2k-1)\varphi(0)\int_0^\infty \varphi(s)\mathrm{d}s\Big)\|f\|_{p,\gamma}^{2k}\,\mathbb{E}\psi(\|X_t^x\|)^{2k}, \quad t \ge 1. \end{aligned} \tag{2.3}$$

Next, [24, Corollary 2.4] gives the following result on SLLN.

Lemma 2.2. *Under conditions of Lemma 2.1, if there exist a constant $q \in (0,1/2)$, a function $\tau \in C(\mathbb{R}_+;\mathbb{R}_+)$ and random variables $\{M_x \ge 1 : x \in \mathbb{B}\}$ such that*

$$\mathbb{E}M_x^{\frac{1}{q}} \le \tau(\|x\|), \quad x \in \mathbb{B}, \tag{2.4}$$

$$\mathbb{P}\Big(\|X_t^x\| \le w^{-1}(t^q) \ \ \text{for } t \ge M_x\Big) = 1, \quad x \in \mathbb{B}, \tag{2.5}$$

where w^{-1} is the inverse of w. Then for any $\varepsilon \in (0,\frac{1}{2})$, there exist a constant $c_\varepsilon > 0$ and a family of random variables $\{T_{\varepsilon,x}^f \ge 1 : x \in \mathbb{B}, f \in C_{w,\gamma}(\mathbb{B})\}$ such that $\mathbb{P}$-a.s.

$$\Big|\frac{1}{t}\int_0^t f(X_s^x)\mathrm{d}s - \mu_\infty(f)\Big| \le c_\varepsilon\|f\|_{w,\gamma}t^{-\frac{1}{2}+\varepsilon}, \quad t \ge T_{\varepsilon,x}^f,\ x \in \mathbb{B},\ f \in C_{w,\gamma}(\mathbb{B}), \tag{2.6}$$

and

$$\sup_{f\in C_{w,\gamma}(\mathbb{B})} \frac{\mathbb{E}|T_{\varepsilon,x}^f|^k}{1+\|f\|_{w,\gamma}^{k(1+k)}} < \infty, \quad k \in \mathbb{N}. \tag{2.7}$$

Let $f \in C_{w,\gamma}(\mathbb{B})$ and $x \in \mathbb{B}$, assume that

$$M_t^{f,x} := \int_0^t \big\{f(X_s^x) - P_s f(x)\big\}\mathrm{d}s + \int_t^\infty \big\{P_{s-t} f(X_t^x) - P_s f(x)\big\}\mathrm{d}s, \quad t \geq 0 \tag{2.8}$$

is a well-defined square integrable martingale. Consider its discrete time quadratic variation process

$$\langle M^{f,x}\rangle_k := \sum_{i=1}^k \mathbb{E}\big((M_i^{f,x} - M_{i-1}^{f,x})^2 \big| \mathscr{F}_{i-1}\big), \quad k \in \mathbb{N}.$$

Let $\lfloor t \rfloor = \sup\{k \in \mathbb{Z}_+ : k \leq t\}$ be the integer part of $t \geq 0$. The following CLT is due to [24, Theorem 2.8].

Lemma 2.3. *Let $f \in C_{w,\gamma}(\mathbb{B})$ and $x \in \mathbb{B}$ such that $M_t^{f,x}$ in (2.8) is a well-defined square integrable martingale. Assume that*

$$\mathbb{E}\Big(\sup_{t\in[k,k+1]} \mathrm{e}^{|\psi(\|X_t^x\|)|^\alpha}\Big) \leq \kappa(\|x\|), \qquad k \geq 0, \quad x \in \mathbb{B} \tag{2.9}$$

holds for some constant $\alpha > 0$ and continuous function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$. Then

(1) *For any constants $D, q > 0$ and $\varepsilon \in (0, 1/4)$, there exists an increasing function $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for any $x \in \mathbb{B}$ and $f \in C_{w,\gamma}(\mathbb{B})$,*

$$\begin{aligned}&\sup_{z\in\mathbb{R}} \Big|\mathbb{P}\Big(\frac{1}{\sqrt{t}}\int_0^t f(X_s^x)\mathrm{d}s \leq z\Big) - \Phi_D(z)\Big| \\ &\leq t^{-\frac{1}{4}+\varepsilon} h(\|x\|, \|f\|_{w,\gamma}) + D^{-4q}\lfloor t\rfloor^{q(1-4\varepsilon)}\mathbb{E}\Big|\lfloor t\rfloor^{-1}\langle M^{f,x}\rangle_{\lfloor t\rfloor} - D^2\Big|^{2q}, \quad t \geq 1;\end{aligned}$$

(2) *There exists an increasing function $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for any $x \in \mathbb{B}$ and $f \in C_{w,\gamma}(\mathbb{B})$,*

$$\begin{aligned}&\sup_{z\in\mathbb{R}} \Big((1 \wedge |z|)\Big|\mathbb{P}\Big(\frac{1}{\sqrt{t}}\int_0^t f(X_s^x)\mathrm{d}s \leq z\Big) - \Phi_0(z)\Big|\Big) \\ &\leq t^{-\frac{1}{4}} h(\|x\|, \|f\|_{w,\gamma}) + \lfloor t\rfloor^{-\frac{1}{2}}(\mathbb{E}\langle M^{f,x}\rangle_{\lfloor t\rfloor})^{1/2}, \quad t \geq 1.\end{aligned}$$

Finally, let $(M_n)_{n\geq 0}$ be a square integrable martingale and let $Z_n = M_n - M_{n-1}$ be the martingale difference. The following result is taken from [16, Theorem 1].

Lemma 2.4. *Assume that $S_n := \mathbb{E}M_n \to \infty$ as $n \to \infty$, and there exists a constant $\delta > 0$ such that*

$$\sum_{n=1}^\infty S_n^{-4}\mathbb{E}(Z_n^4 \mathbf{1}_{\{|Z_n|\leq\delta S_n\}}) < \infty, \qquad \sum_{n=1}^\infty S_n^{-1}\mathbb{E}(Z_n \mathbf{1}_{\{|Z_n|\leq\delta S_n\}}) < \infty, \tag{2.10}$$

and $\mathbb{P}$*-a.s.*

$$\lim_{n\to\infty} S_n^{-2}\sum_{k=1}^{n} Z_k^2 = 1. \tag{2.11}$$

Then the sequence $(\Lambda_n)_{n\geq 1}$ *of random variables on* $C([0,1];\mathbb{R})$ *defined by*

$$\Lambda_n(t) = \sum_{k=0}^{n-1} 1_{\{S_k^2\leq tS_n^2\leq S_{k+1}^2\}} \frac{M_k + (S_n^2 t - S_k^2)(S_{k+1}^2 - S_k^2)Z_{k+1}}{\sqrt{2S_n^2\log\log S_n^2}}, \quad t\in[0,1]$$

is almost surely relatively compact, and the set of its limits points coincides with $\mathcal{H}$ *in* (1.6)*.*

3 Proof of Theorem 1.1

It suffices to verify conditions in Lemmas 2.1 and 2.2 for the present model, where $\mathbb{B} = \mathscr{C}, w(r) = 1 + r^{p/2}, r\geq 0$. To this end, we present the following lemma.

Lemma 3.1. *Under assumptions of Theorem* 1.1*, for any* $p\geq 1$ *and* $\gamma\in(0,1]$*, there exist constants* $c,\beta>0$ *such that*

$$\mathbb{E}\|X_t^\xi\|_\infty^p \leq c\,(1 + \mathrm{e}^{-\beta t}\|\xi\|_\infty^p), \quad \xi\in\mathscr{C}, t\geq 0, \tag{3.1}$$

and

$$\mathbb{W}_{p,\gamma}(\mu P_t, \nu P_t) \leq c\,\mathrm{e}^{-\beta t}\mathbb{W}_{p,\gamma}(\mu,\nu), \quad \mu,\nu\in\mathscr{P}_{p,\gamma}(\mathscr{C}), t\geq 0. \tag{3.2}$$

Consequently, P_t *has a unique invariant probability measure* μ_∞ *and* $\mu_\infty(\|\cdot\|_\infty^p)<\infty$ *for all* $p\geq 1$*.*

Proof. (1) By Jensen's inequality, concerning (3.1) we only need to consider $p\geq 2$. Since $\lambda_1 - \lambda_2\mathrm{e}^{\lambda_1 r_0} > 0$, there exists a constant $\varepsilon\in(0,\lambda_1)$ such that

$$\lambda_\varepsilon := \lambda_1 - \lambda_2\mathrm{e}^{(\lambda_1-\varepsilon)r_0} - \varepsilon > 0. \tag{3.3}$$

According to (**A1**) and (**A3**), we may find a constant $c_0>0$ such that

$$2\,\langle\xi(0), b(\xi)\rangle + \|\sigma(\xi)\|_{\mathrm{HS}}^2 \leq c_0 - (\lambda_1-\varepsilon)|\xi(0)|^2 + \lambda_2\|\xi\|_\infty^2, \quad \xi\in\mathscr{C}.$$

So, by Itô's formula,

$$\begin{aligned} \mathrm{e}^{(\lambda_1-\varepsilon)t}|X^\xi(t)|^2 &= |\xi(0)|^2 + M^\xi(t) + \int_0^t \mathrm{e}^{(\lambda_1-\varepsilon)s}\Big((\lambda_1-\varepsilon)|X^\xi(s)|^2 \\ &\quad + 2\langle X^\xi(s), b(X_s^\xi)\rangle + \|\sigma(X_s^\xi)\|_{\mathrm{HS}}^2\Big)\mathrm{d}s \\ &\leq |\xi(0)|^2 + M^\xi(t) + c_1\,\mathrm{e}^{(\lambda_1-\varepsilon)t} + \lambda_2\int_0^t \mathrm{e}^{(\lambda_1-\varepsilon)s}\|X_s^\xi\|_\infty^2\mathrm{d}s \end{aligned} \tag{3.4}$$

holds for some constant $c_1 > 0$ and the martingale

$$M^\xi(t) := 2\int_0^t \mathrm{e}^{(\lambda_1-\varepsilon)s}\langle \sigma^*(X_s^\xi)X^\xi(s), \mathrm{d}W(s)\rangle, \quad t \geq 0.$$

Noting that

$$\mathrm{e}^{(\lambda_1-\varepsilon)t}\|X_t^\xi\|_\infty^2 \leq \mathrm{e}^{(\lambda_1-\varepsilon)r_0}\Big(\|\xi\|_\infty^2 \vee \sup_{0\leq s\leq t}(\mathrm{e}^{(\lambda_1-\varepsilon)s}|X^\xi(s)|^2)\Big),$$

we deduce from (3.4) that

$$\begin{aligned}\|X_t^\xi\|_\infty^2 \leq \mathrm{e}^{(\lambda_1-\varepsilon)r_0}\Big\{&c_1 + \mathrm{e}^{-(\lambda_1-\varepsilon)t}\|\xi\|_\infty^2 + \mathrm{e}^{-(\lambda_1-\varepsilon)t}N^\xi(t)\\ &+ \lambda_2\int_0^t \mathrm{e}^{-(\lambda_1-\varepsilon)(t-s)}\|X_s^\xi\|_\infty^2\mathrm{d}s\Big\}, \quad t \geq 0,\end{aligned}$$

where $N^\xi(t) := \sup_{0\leq s\leq t} M^\xi(s)$. By invoking Gronwall's inequality (see e.g. [11, Theorem 11]), this implies

$$\begin{aligned}\|X_t^\xi\|_\infty^2 \leq{}& \mathrm{e}^{(\lambda_1-\varepsilon)r_0}\Big\{c_1 + \mathrm{e}^{-(\lambda_1-\varepsilon)t}\|\xi\|_\infty^2 + \mathrm{e}^{-(\lambda_1-\varepsilon)t}N^\xi(t)\Big\}\\ &+ \lambda_2\mathrm{e}^{2(\lambda_1-\varepsilon)r_0}\int_0^t \Big\{c_1 + \mathrm{e}^{-(\lambda_1-\varepsilon)s}\|\xi\|_\infty^2 + \mathrm{e}^{-(\lambda_1-\varepsilon)s}N^\xi(s)\Big\}\mathrm{e}^{-\lambda_\varepsilon(t-s)}\mathrm{d}s, \quad t \geq 0.\end{aligned}$$

Combining this with Hölder's inequality, for fixed $p \geq 2$ we may find constants $c_2, c_3 > 0$ such that

$$\begin{aligned}\mathbb{E}\|X_t^\xi\|_\infty^p \leq{}& c_2 + c_2\mathrm{e}^{-\frac{p}{2}\lambda_\varepsilon t}\|\xi\|_\infty^p + c_2\mathrm{e}^{-\frac{p}{2}(\lambda_1-\varepsilon)t}(N^\xi(t))^{p/2}\\ &+ c_2\mathbb{E}\Big|\int_0^t \mathrm{e}^{-(\lambda_1-\varepsilon)s}\mathrm{e}^{-\lambda_\varepsilon(t-s)}N^\xi(s)\mathrm{d}s\Big|^{p/2}\\ \leq{}& c_3 + c_3\mathrm{e}^{-\frac{p}{2}\lambda_\varepsilon t}\|\xi\|_\infty^p + c_3\mathrm{e}^{-\frac{p}{2}(\lambda_1-\varepsilon)t}\mathbb{E}(N^\xi(t))^{p/2}\\ &+ c_3\int_0^t \mathrm{e}^{-\frac{p}{2}(\lambda_1-\varepsilon)s-\lambda_\varepsilon(t-s)}\mathbb{E}(N^\xi(s))^{p/2}\mathrm{d}s.\end{aligned} \tag{3.5}$$

On the other hand, by means of **(A3)** and using BDG's and Hölder's inequalities, there exist constants $c_4, c_5 > 0$ such that

$$\begin{aligned}&\mathrm{e}^{-\frac{p}{2}(\lambda_1-\varepsilon)t}\mathbb{E}(N^\xi(t))^{p/2} \leq c_4\mathbb{E}\Big(\int_0^t \mathrm{e}^{-2(\lambda_1-\varepsilon)(t-s)}|X^\xi(s)|^2\mathrm{d}s\Big)^{p/4}\\ &\leq c_4\mathbb{E}\Big[\Big(\int_0^t \mathrm{e}^{-2(\lambda_1-\varepsilon)(t-s)}|X^\xi(s)|^p\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_0^t \mathrm{e}^{-2(\lambda_1-\varepsilon)(t-s)}\mathrm{d}s\Big)^{\frac{p-2}{4}}\Big]\\ &\leq c_5 + \frac{(1\wedge\lambda_\varepsilon)^2}{4c_3}\int_0^t \mathrm{e}^{-2(\lambda_1-\varepsilon)(t-s)}\mathbb{E}|X^\xi(s)|^p\mathrm{d}s, \quad t \geq 0.\end{aligned}$$

Substituting this into (3.5), and noting that due to $\lambda_1 - \varepsilon > \lambda_\varepsilon > 0$ we have

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\lambda_\varepsilon(t-s)}\mathrm{d}s \int_0^s \mathrm{e}^{-2(\lambda_1-\varepsilon)(s-r)}\mathbb{E}|X^\xi(r)|^p\mathrm{d}r\\&= \int_0^t \mathrm{e}^{2(\lambda_1-\varepsilon)r-\lambda_\varepsilon t}\mathbb{E}|X^\xi(r)|^p\mathrm{d}r \int_r^t \mathrm{e}^{-(2(\lambda_1-\varepsilon)-\lambda_\varepsilon)s}\mathrm{d}s\\&\le \frac{1}{2(\lambda_1-\varepsilon)-\lambda_\varepsilon}\int_0^t \mathrm{e}^{-\lambda_\varepsilon(t-r)}\mathbb{E}|X^\xi(r)|^p\mathrm{d}r\\&\le \frac{1}{\lambda_\varepsilon}\int_0^t \mathrm{e}^{-\lambda_\varepsilon(t-r)}\mathbb{E}|X^\xi(r)|^p\mathrm{d}r,\end{aligned}$$

we may find a constant $C > 0$ such that

$$\mathbb{E}\|X_t^\xi\|_\infty^p \le C(1+\|\xi\|_\infty^p) + \frac{\lambda_\varepsilon}{2}\int_0^t \mathrm{e}^{-\lambda_\varepsilon(t-s)}\mathbb{E}\|X_s^\xi\|_\infty^p\mathrm{d}s, \quad t \ge 0.$$

By a truncation argument with stopping times, we may and do assume that $\mathbb{E}\|X_t^\xi\|_\infty^p < \infty$, so that by Gronwall's inequality, this implies the desired estimate (3.1) for some constants $c, \beta > 0$.

(b) By (3.1), the Lyapunov condition **(A3)** in [2, Theorem 1.1] holds for $V(\xi) := \|\xi\|_\infty^p, \xi \in \mathscr{C}$ and $\gamma = \beta$. In terms of [2, Theorem 1.1], this together with **(A1)** and **(A2)** implies (3.2) for possibly different constants $c, \beta > 0$, which then implies the existence and uniqueness of the invariant probability measure $\mu_\infty \in \mathscr{P}_{p,\gamma}(\mathscr{C})$. Since $p \ge 1$ is arbitrary, we conclude that $\mu_\infty(\|\cdot\|_\infty^p) < \infty$ holds for all $p \ge 1$. □

Proof of Theorem 1.1. From (1.4) and (3.1) we see that assumptions in Lemma 2.1 holds for $\mathbb{B} = \mathscr{C}, w(r) = 1 + r^{p/2}, k = 1, \varphi(t) = c\,\mathrm{e}^{-\beta t}$, and $\psi(r) = 1 + r^{p/2}$. Then (1) follows from Lemma 2.1 .

Next, to prove (2), we only need to verify conditions (2.4) and (2.5) in Lemma 2.2. For $q \in (0, 1/2)$, consider the following $[0,\infty]$-valued random variables:

$$\begin{aligned}M &:= \inf\Big\{T \ge 0 : 16^{\frac{1}{p}}\|X_t^\xi\|_\infty^2 \le t^{\frac{4q}{p}} \ \text{ for } \ t \ge T\Big\},\\M' &:= \inf\Big\{m \in \mathbb{N} : 16^{\frac{1}{p}}\sup_{t\in[k,k+1]}\|X_t^\xi\|_\infty^2 \le k^{\frac{4q}{p}} \ \text{ for } \ \mathbb{N} \ni k \ge m+1\Big\}.\end{aligned}$$

Obviously, $M \le M'$. Since

$$\sup_{t\in[k,k+1]}\|X_t^\xi\|_\infty \le \max_{i\in\{0,1,\cdots,\lfloor 1/r_0\rfloor+1\}}\|X_{k+ir_0}^\xi\|_\infty, \tag{3.6}$$

by (3.1) and applying Chebyshev's inequality, we may find a constant $C(\xi) > 0$ such that

$$\begin{aligned}&\sum_{k=1}^\infty \mathbb{P}\Big(\sup_{t\in[k,k+1]}\|X_t^\xi\|_\infty^2 \ge \frac{k^{\frac{4q}{p}}}{16^{\frac{1}{p}}}\Big)\\&\le 2^{1+\frac{1}{q}}\sum_{k=1}^\infty \frac{\mathbb{E}(\sup_{t\in[k,k+1]}\|X_t^\xi\|_\infty^{\frac{p}{2}(1+\frac{1}{q})})}{k^{1+q}} \le C(\xi)\sum_{k=1}^\infty \frac{1}{k^{1+q}} < \infty.\end{aligned}$$

So, by Borel-Cantelli's lemma, there exists an $\mathbb{N}$-valued random variable K such that

$$\mathbb{P}\Big(\sup_{t\in[k,k+1]}\|X_t^\xi\|_\infty^2 \le \frac{k^{\frac{4q}{p}}}{16^{\frac{1}{p}}} \ \text{ for } \ k\ge K\Big)=1.$$

Therefore, $\mathbb{P}$-a.s. $M \le M' < \infty$ and (2.5) holds true. Moreover, (3.1) and Chebyshev's inequality also imply

$$\begin{aligned}\mathbb{E}\,|M'|^{\frac{1}{q}} &= \sum_{k=0}^\infty k^{\frac{1}{q}}\mathbb{P}(M'=k) \le \sum_{k=1}^\infty k^{\frac{1}{q}}\mathbb{P}\Big(\sup_{t\in[k,k+1]}\|X_t^\xi\|_\infty^2 > \frac{k^{\frac{4q}{p}}}{16^{\frac{1}{p}}}\Big)\\ &\le 16^{\frac{\alpha}{p}}\sum_{k=1}^\infty \frac{\mathbb{E}(\sup_{t\in[k,k+1]}\|X_t\|_\infty^{2\alpha})}{k^{1+q}}\\ &\le c\,(1+\|\xi\|_\infty^{2\alpha}),\quad \alpha := \frac{p}{4\,q}(1+q+1/q)\end{aligned}$$

for some constant $c>0$. This, together with $M\le M'$, leads to

$$\mathbb{E}\,M^{\frac{1}{q}} \le c\,(1+\|\xi\|_\infty^{2\alpha}),\quad q\in(0,1/2), \tag{3.7}$$

which ensures condition (2.4). Therefore, the proof is finished by Lemma 2.2. □

4 Proof of Theorem 1.2

To apply Lemma 2.3, for fixed $f\in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f)=0$, consider

$$M_t^{f,\xi} := \int_0^t \big\{f(X_u^\xi)-P_uf(\xi)\big\}\mathrm{d}u + \int_t^\infty \big\{P_{u-t}f(X_t^\xi)-P_uf(\xi)\big\}\mathrm{d}u,\quad t\ge 0,\, \xi\in\mathscr{C}. \tag{4.1}$$

Since $\mu_\infty(f)=0$, (1.4) implies

$$|P_tf(\xi)| \le c\,\mathrm{e}^{-\beta t}\|f\|_{p,\gamma}(1+\|\xi\|_\infty^{p/2}),\quad t\ge 0,\, \xi\in\mathscr{C}$$

for some constants $c,\beta>0$. So, there exists an increasing function $c:\mathbb{R}_+\to\mathbb{R}_+$ such that (3.1) yields

$$\begin{aligned}&\mathbb{E}\bigg|\int_0^t \big\{f(X_u^\xi)-P_uf(\xi)\big\}\mathrm{d}u + \int_t^\infty \big|P_{u-t}f(X_t^\xi)-P_uf(\xi)\big|\mathrm{d}u\bigg|^k\\ &\le c(t)\|f\|_{p,\gamma}^k\Big(1+\mathbb{E}\|X_t^\xi\|_\infty^{pk/2}+\|\xi\|_\infty^{pk/2}+\int_0^t(1+\|\xi\|_\infty^{pk/2}+\mathbb{E}\|X_u^\xi\|_\infty^{pk/2})\mathrm{d}u\Big)\\ &<\infty,\quad t\ge 0.\end{aligned}$$

Hence, $M_t^{f,\xi}$ is a well-defined martingale with $\mathbb{E}|M_t^{f,\xi}|^k<\infty$ for all $k\ge 1$.

Next, consider

$$\langle M^{f,\xi}\rangle_k := \sum_{i=1}^{k} \mathbb{E}\Big((M_i^{f,\xi} - M_{i-1}^{f,\xi})^2\Big|\mathscr{F}_{i-1}\Big), \quad k \in \mathbb{N}.$$

Let R_f and φ_f be defined as in (1.3) and (1.5), respectively. By the Markov property of $(X_t^\xi)_{t\ge 0}$, we have

$$M_i^{f,\xi} = M_{i-1}^{f,\xi} + \int_{i-1}^{i} f(X_u^\xi)\mathrm{d}u + R_f(X_i^\xi) - R_f(X_{i-1}^\xi)$$

so that

$$\mathbb{E}((M_i^{f,\xi} - M_{i-1}^{f,\xi})^2|\mathscr{F}_{i-1}) = \varphi_f(X_{i-1}^\xi).$$

Consequently, we arrive at

$$\langle M^{f,\xi}\rangle_k = \sum_{i=0}^{k-1} \varphi_f(X_i^\xi), \quad k \in \mathbb{N}. \tag{4.2}$$

Lemma 4.1. *Under assumptions of Theorem 1.1, for any $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$,*

$$0 \le \mu_\infty(\varphi_f) = 2\,\mu_\infty(fR_f) < \infty. \tag{4.3}$$

Proof. Firstly, by Lemma 3.1 and (1.4), we have $\mu_\infty(\|\cdot\|_\infty^p) < \infty$ for all $p \ge 1$ so that $\mu_\infty(\varphi_f) < \infty$ for any $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$.

Next, by the Markov property of $(X_t^\xi)_{t\ge 0}$ and noting that (1.3) implies

$$P_tR_f(\xi) = R_f(\xi) - \int_0^t P_sf(\xi)\mathrm{d}s, \ t \ge 0,$$

we have

$$\mathbb{E}\big[f(X_s^\xi)R_f(X_1^\xi)\big] = P_s(fP_{1-s}R_f)(\xi) = P_s(fR_f)(\xi) - \int_0^{1-s} P_s(fP_rf)(\xi)\mathrm{d}r, \quad s \in [0,1],$$

and

$$\begin{aligned}
&\mathbb{E}\Big(\int_0^1 f(X_s^\xi)\mathrm{d}s\Big)^2 = 2\mathbb{E}\int_0^1 f(X_s^\xi)\mathrm{d}s \int_s^1 f(X_r^\xi)\mathrm{d}r \\
&= 2\int_0^1 \mathrm{d}s \int_s^1 P_s(fP_{r-s}f)(\xi)\mathrm{d}r = 2\int_0^1 \mathrm{d}s \int_0^{1-s} P_s(fP_rf)(\xi)\mathrm{d}r.
\end{aligned}$$

Then it follows from (1.5) that

$$
\begin{aligned}
\varphi_f(\xi) &= R_f(\xi)^2 + P_1(R_f)^2(\xi) + \mathbb{E}\bigg(\int_0^1 f(X_s^\xi)\mathrm{d}s\bigg)^2 + 2\int_0^1 \mathbb{E}\big[f(X_s^\xi)R_f(X_1^\xi)\big]\mathrm{d}s \\
&\quad - 2R_f(\xi)\int_0^1 P_r f(\xi)\mathrm{d}r - 2R_f(\xi)P_1R_f(\xi) \\
&= R_f(\xi)^2 + P_1(R_f)^2(\xi) + 2\int_0^1 \mathrm{d}s\int_0^{1-s} P_s(fP_rf)(\xi)\mathrm{d}r + 2\int_0^1 P_s(fR_f)(\xi)\mathrm{d}s \\
&\quad - 2\int_0^1 \mathrm{d}s\int_0^{1-s} P_s(fP_rf)(\xi)\mathrm{d}r - 2R_f(\xi)\int_0^1 P_rf(\xi)\mathrm{d}r - 2R_f(\xi)^2 \\
&\qquad + 2R_f(\xi)\int_0^1 P_sf(\xi)\mathrm{d}s \\
&= P_1(R_f^2)(\xi) - R_f(\xi)^2 + 2\int_0^1 P_s(fR_f)(\xi)\mathrm{d}s.
\end{aligned}
\tag{4.4}
$$

Since μ_∞ is P_t-invariant, integrating with respect to $\mu_\infty(\mathrm{d}\xi)$ on both sides of (4.4) gives $\mu_\infty(\varphi_f) = 2\mu_\infty(fR_f)$.

□

Lemma 4.2. *Under assumptions of Theorem* 1.1*, there exists a constant* $C > 0$ *such that*

$$\|\varphi_f\|_{2p,\gamma} \le C\,\|f\|_{p,\gamma}^2, \quad f \in C_{p,\gamma}(\mathscr{C}), \mu_\infty(f) = 0. \tag{4.5}$$

Proof. By (1.3) and (1.4), in addition to $\mu_\infty(\|\cdot\|_\infty^p) < \infty$, there exists a constant $c_1 > 0$ such that

$$|R_f(\xi)| \le c_1\|f\|_{p,\gamma}(1 + \|\xi\|_\infty^p), \quad f \in C_{p,\gamma}(\mathscr{C}), \xi \in \mathscr{C}. \tag{4.6}$$

Next, applying (3.2) to $\mu = \delta_\xi$ and $\nu = \delta_\eta$, we obtain

$$|P_tf(\xi) - P_tf(\eta)| \le c\,\mathrm{e}^{-\beta t}\|f\|_{p,\gamma}\rho_{p,\gamma}(\xi,\eta). \tag{4.7}$$

This and (1.3) imply

$$|R_f(\xi) - R_f(\eta)| \le \int_0^\infty |P_tf(\xi) - P_tf(\eta)|\mathrm{d}t \le \frac{c}{\beta}\|f\|_{p,\gamma}\rho_{p,\gamma}(\xi,\eta). \tag{4.8}$$

Moreover, it follows from (4.6) and (4.8) that

$$|R_f(\xi)^2 - R_f(\eta)^2| = |R_f(\xi) + R_f(\eta)|\cdot|R_f(\xi) - R_f(\eta)| \le c'\,\|f\|_{p,\gamma}^2\,\rho_{2p,\gamma}(\xi,\eta) \tag{4.9}$$

for some constant $c' > 0$. Combining (4.7)-(4.9) with (4.4), we finish the proof. □

Lemma 4.3. *Under assumptions of Theorem 1.1, there exist constants* $\delta, c > 0$ *such that*

$$\mathbb{E}\Big(\sup_{t\in[k,k+1]} \mathrm{e}^{\delta\,\|X_t^\xi\|_\infty^2}\Big) \le \mathrm{e}^{c\,(1+\|\xi\|_\infty^2)}, \quad k \ge 0,\ \xi \in \mathscr{C}, \quad t \ge 0. \tag{4.10}$$

Proof. In terms of [1, Lemma 2.1], there exist constants $c_0, \varepsilon_0 > 0$ such that

$$\sup_{t\geq 0} \mathbb{E}\mathrm{e}^{\varepsilon_0 \|X_t^\xi\|_\infty^2} \leq \mathrm{e}^{c_0(1+\|\xi\|_\infty^2)}, \quad \xi \in \mathscr{C}. \tag{4.11}$$

On the other hand, (3.6) implies

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[k,k+1]} \mathrm{e}^{\varepsilon_0 \|X_t^\xi\|_\infty^2}\Big) &\leq \mathbb{E}\Big(\max_{i\in\{0,1,\cdots,\lfloor 1/r_0\rfloor+1\}} \mathrm{e}^{\varepsilon_0 \|X_{k+ir_0}^\xi\|_\infty^2}\Big)\\ &\leq (\lfloor 1/r_0\rfloor + 2) \max_{i\in\{0,1,\cdots,\lfloor 1/r_0\rfloor+1\}} \mathbb{E}\,\mathrm{e}^{\varepsilon_0 \|X_{k+ir_0}^\xi\|_\infty^2}.\end{aligned}$$

Combining this with (4.11), we prove (4.10). □

Proof of Theorem 1.2. Let $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$. By Lemmas 3.1 and 4.3, the results in Lemma 2.3 applies to $D = D_f$. Below we consider $D_f > 0$ and $D_f = 0$, respectively.

(a) Let $D_f > 0$. By Lemma 2.3(1), for any $\varepsilon, q > 0$, there exists an increasing function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned}&\sup_{z\in\mathbb{R}} \Big|\mathbb{P}\Big(\sqrt{t} A_t^f(\xi) \leq z\Big) - \Phi_{D_f}(z)\Big|\\ &\leq h_1(\|\xi\|_\infty, \|f\|_{p,\gamma}) t^{-\frac{1}{4}+\varepsilon} + D_f^{-4q} \lfloor t\rfloor^{q(1-4\varepsilon)} \mathbb{E}\Big|\frac{1}{\lfloor t\rfloor}\langle M^{f,\xi}\rangle_{\lfloor t\rfloor} - D_f^2\Big|^{2q}, \quad t \geq 1.\end{aligned} \tag{4.12}$$

So, if we can find an increasing function $\widehat{h} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathbb{E}\Big|\frac{1}{\lfloor t\rfloor}\langle M^{f,\xi}\rangle_{\lfloor t\rfloor} - D_f^2\Big|^{2q} \leq \widehat{h}(\|\xi\|_\infty, \|f\|_{p,\gamma}) \lfloor t\rfloor^{-q}, \ \xi \in \mathscr{C}, \quad t \geq 1, \tag{4.13}$$

then the desired estimate in Theorem 1.2(1) follows from (4.13) with large enough $q > 0$, say, $q > \frac{1}{16\varepsilon}$. By (1.4) for $2p$ instead of p,

$$|P_t\varphi_f(\xi) - D_f^2| \leq c\,\|\varphi_f\|_{2p,\gamma}\,\mathrm{e}^{-\beta t}(1 + \|\xi\|_\infty^p)$$

holds for some constants $c, \beta > 0$. Combining this with (3.1), (4.2) and (4.5), we prove (4.13).

(b) Let $D_f = 0$. With $q = 1$ the estimate (4.13) reduces to

$$\mathbb{E}\Big|\frac{1}{\lfloor t\rfloor}\langle M^{f,\xi}\rangle_{\lfloor t\rfloor}\Big|^2 \leq \widehat{h}(\|\xi\|_\infty, \|f\|_{p,\gamma}) \lfloor t\rfloor^{-1}, \ \xi \in \mathscr{C}, \quad t \geq 1. \tag{4.14}$$

Combining this with Lemma 2.3(2), we prove Theorem 1.2(2). □

5 Proof of Theorem 1.3

Let us fix $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$. To apply Lemma 2.4, for any $\xi \in \mathscr{C}$, we consider

$$M_n^\xi := \sum_{k=0}^n \big\{f(X_k^\xi) - P_k f(\xi)\big\} + \sum_{k=n+1}^\infty \big\{P_{k-n} f(X_n^\xi) - P_k f(\xi)\big\}, \quad n \geq 0.$$

The argument after (4.1) implies that $(M_n^\xi)_{n\geq 0}$ is a well-defined square integrable martingale. Let

$$S_n^\xi = \sqrt{\mathbb{E}\big|M_n^\xi\big|^2}, \quad Z_n^\xi = M_n^\xi - M_{n-1}^\xi, \quad n \geq 1,$$

and let $\widehat{R}_f$ and $\widehat{\varphi}_f$ be given before Theorem 1.3. Following the arguments of Lemmas 4.1 and 4.2, we have

$$0 \leq \widehat{D}_f^2 := \mu_\infty(\widehat{\varphi}_f) = 2\mu_\infty(f\widehat{R}_f) < \infty, \tag{5.1}$$

and for some constant $c > 0$,

$$\|\widehat{\varphi}_f\|_{2p,\gamma} \leq c\,\|f\|_{p,\gamma}^2, \quad f \in C_{p,\gamma}(\mathscr{C}). \tag{5.2}$$

Lemma 5.1. *Under assumptions of Theorem* 1.1, $\mathbb{P}$*-a.s.*

$$\frac{1}{n}\sum_{k=1}^n (Z_k^\xi)^2 \to \widehat{D}_f^2. \tag{5.3}$$

Proof. According to the proof of [3, Lemma 3.2], it suffices to show that the maps

$$\mathscr{C} \ni \xi \mapsto \Lambda_1(\xi) := \mathbb{E}\Big(\Big|\limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^n (Z_k^\xi)^2\Big) - \widehat{D}_f^2\Big| \wedge 1\Big)$$
$$\mathscr{C} \ni \xi \mapsto \Lambda_2(\xi) := \mathbb{E}\Big(\Big|\liminf_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^n (Z_k^\xi)^2\Big) - \widehat{D}_f^2\Big| \wedge 1\Big)$$

are continuous. For simplicity, we only prove the continuity of Λ_1 as that of the other is completely similar. By definition it is easy to see that

$$Z_n^\xi = f(X_n^\xi) + \sum_{k=n}^\infty \big\{P_{k-n} f(X_n^\xi) - P_{k+1-n} f(X_{n-1}^\xi)\big\}, \quad n \geq 1. \tag{5.4}$$

Combining this with (1.4), we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned} \big|Z_n^\xi - Z_n^\eta\big| &\leq \big|f(X_n^\xi) - f(X_n^\eta)\big| \\ &\quad + \sum_{k=n}^\infty \big(|P_{k-n} f(X_n^\xi) - P_{k-n} f(X_n^\eta)| + |P_{k+1-n} f(X_{n-1}^\xi) - P_{k+1-n} f(X_{n-1}^\eta)|\big) \\ &\leq c_1 \rho_{p,\gamma}(X_n^\xi, X_n^\eta) + c_1 \sum_{k=n}^\infty \mathrm{e}^{-\beta(k-n)}\big\{\rho_{p,\gamma}(X_n^\xi, X_n^\eta) + \rho_{p,\gamma}(X_{n-1}^\xi, X_{n-1}^\eta)\big\} \\ &\leq c_2\big\{\rho_{p,\gamma}(X_n^\xi, X_n^\eta) + \rho_{p,\gamma}(X_{n-1}^\xi, X_{n-1}^\eta)\big\}. \end{aligned} \tag{5.5}$$

Similarly, (1.4) with $\mu_\infty(f) = 0$ and (5.4) also imply

$$|Z_n^\xi| \le c_3(1 + \|X_n^\xi\|_\infty + \|X_{n-1}^\xi\|_\infty)^{p/2}, \quad n \ge 1, \xi \in \mathscr{C}$$

for some constant $c_3 > 0$. Combining this with (5.5) and setting

$$A_k^{\xi,\eta} := 1 + \|X_k^\xi\|_\infty + \|X_k^\eta\|_\infty + \|X_{k-1}^\xi\|_\infty + \|X_{k-1}^\eta\|_\infty, \quad k \ge 1,$$

we may find a constant $c_4 > 0$ such that

$$\begin{aligned}
&|\Lambda_1(\xi) - \Lambda_1(\eta)| \\
&\le \left|\mathbb{E}\Big|\limsup_{l\to\infty}\sup_{n\ge l}\frac{1}{n}\sum_{k=1}^n\big(|Z_k^\xi|^2 - \widehat{D}_f^2\big)\Big| - \mathbb{E}\Big|\limsup_{l\to\infty}\sup_{n\ge l}\frac{1}{n}\sum_{k=1}^n\big(|Z_k^\eta|^2 - \widehat{D}_f^2\big)\Big|\right| \\
&\le \mathbb{E}\bigg[\limsup_{l\to\infty}\sup_{n\ge l}\frac{1}{n}\sum_{k=1}^n\big|Z_k^\xi - Z_k^\eta\big|\big(|Z_k^\xi| + |Z_k^\eta|\big)\bigg] \\
&\le c_4\mathbb{E}\bigg[\limsup_{l\to\infty}\sup_{n\ge l}\frac{1}{n}\sum_{k=1}^n\big\{\rho_{p,\gamma}(X_k^\xi, X_k^\eta) + \rho_{p,\gamma}(X_{k-1}^\xi, X_{k-1}^\eta)\big\}|A_k^{\xi,\eta}|^{\frac{p}{2}}\bigg].
\end{aligned} \tag{5.6}$$

Since $\rho_{p,\gamma}(\xi,\eta) \le (1 + \|\xi\|_\infty + \|\eta\|_\infty)^{\frac{p}{2}}$ for all $\xi, \eta \in \mathscr{C}$, for any $m \ge 1$ and $l \ge m$ we have

$$\begin{aligned}
&\sup_{n\ge l}\frac{1}{n}\sum_{k=1}^n\big\{\rho_{p,\gamma}(X_k^\xi, X_k^\eta) + \rho_{p,\gamma}(X_{k-1}^\xi, X_{k-1}^\eta)\big\}|A_k^{\xi,\eta}|^{\frac{p}{2}} \\
&\le \frac{1}{l}\sum_{k=1}^m\big\{\rho_{p,\gamma}(X_k^\xi, X_k^\eta) + \rho_{p,\gamma}(X_{k-1}^\xi, X_{k-1}^\eta)\big\}|A_k^{\xi,\eta}|^{\frac{p}{2}} \\
&\quad + \sum_{k=m}^\infty |A_k^{\xi,\eta}|^{\frac{3p}{4}}\sqrt{\rho_{p,\gamma}(X_k^\xi, X_k^\eta) + \rho_{p,\gamma}(X_{k-1}^\xi, X_{k-1}^\eta)}.
\end{aligned}$$

Combining this with (5.6), (3.1), (3.2), and applying the Schwarz inequality, we may find constants $c_5, c_6 > 0$ such that

$$\begin{aligned}
&\limsup_{\eta\to\xi}|\Lambda_1(\xi) - \Lambda_1(\eta)| \le c_4\sum_{k=m}^\infty\mathbb{E}\Big[|A_k^{\xi,\eta}|^{\frac{3p}{4}}\sqrt{\rho_{p,\gamma}(X_k^\xi, X_k^\eta) + \rho_{p,\gamma}(X_{k-1}^\xi, X_{k-1}^\eta)}\Big] \\
&\le c_5\sum_{k=m}^\infty\big(\mathbb{E}\{\rho_{p,\gamma}(X_k^\xi, X_k^\eta) + \rho_{p,\gamma}(X_{k-1}^\xi, X_{k-1}^\eta)\}\big)^{\frac{1}{2}}\big(\mathbb{E}|Z_k^{\xi,\eta}|^{\frac{3p}{2}}\big)^{\frac{1}{2}} \\
&\le c_6(1 + \|\xi\|_\infty + \|\eta\|_\infty)^{\frac{3p}{4}}\sum_{k=m}^\infty \mathrm{e}^{-\beta k/2}, \quad k \ge 1.
\end{aligned}$$

Letting $k \to \infty$, we consequently prove $\limsup_{\eta\to\xi}|\Lambda_1(\xi) - \Lambda_1(\eta)| = 0$. □

Proof of Theorem 1.3. Let $f \in C_{p,\gamma}(\mathscr{C})$ with $\mu_\infty(f) = 0$ and $\widehat{D}_f > 0$, and let $\xi \in \mathscr{C}$. Below we prove assertions (1) and (2), respectively.

(1) By Lemma 2.4, for the first assertion we only need to verify conditions (2.10) and (2.11) for $(S_n, Z_n) = (S_n^\xi, Z_n^\xi)$.

Firstly, by (1.4) and (5.2), there exist constants $c = c(f,\xi)$ and $\beta > 0$ such that

$$|P_k\widehat{\varphi}_f(\xi) - \widehat{D}_f^2| = |P_k\widehat{\varphi}_f(\xi) - \mu_\infty(\widehat{\varphi}_f)| \le c\,\mathrm{e}^{-\beta k}, \quad k \ge 0. \tag{5.7}$$

Consequently,

$$\lim_{n\to\infty} \frac{(S_n^\xi)^2}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P_k\widehat{\varphi}_f(\xi) = \widehat{D}_f^2 > 0, \tag{5.8}$$

so that $S_n^\xi \to \infty$ as $n \to \infty$. Next, by following the argument to derive (4.8), there exists a constant $c_1 = c_1(f) > 0$ such that

$$|\widehat{R}_f(\xi_1) - \widehat{R}_f(\xi_2)| \le c_1\rho_{p,\gamma}(\xi_1,\xi_2), \quad \xi_1,\xi_2 \in \mathscr{C}.$$

Combining this with (3.1), we may find constants $c_2 = c_2(f), c_3 = c_3(f,\xi) > 0$ such that

$$\begin{aligned}
\mathbb{E}|Z_n^\xi|^4 &\le 8\,\mathbb{E}|f(X_{n-1}^\xi)|^4 + 8\,\mathbb{E}|\widehat{R}_f(X_n^\xi) - \widehat{R}_f(X_{n-1}^\xi)|^4 \\
&\le c_2\Big\{1 + \mathbb{E}\|X_n^\xi\|_\infty^{2p} + \mathbb{E}\|X_{n-1}^\xi\|_\infty^{2p}\Big\} \\
&\le c_3, \quad n \ge 1.
\end{aligned}$$

This together with (5.8) yields

$$\sum_{n=1}^\infty (S_n^\xi)^{-4}\mathbb{E}\Big((Z_n^\xi)^4\mathbf{1}_{\{|Z_n^\xi|<S_n^\xi\}}\Big) \le \sum_{n=1}^\infty (S_n^\xi)^{-4}\mathbb{E}(Z_n^\xi)^4 < \infty. \tag{5.9}$$

Combining this with Chebyshev's inequality, we obtain

$$\sum_{n=1}^\infty (S_n^\xi)^{-1}\mathbb{E}\Big(|Z_n^\xi|\mathbf{1}_{\{|Z_n^\xi|\ge S_n^\xi\}}\Big) \le \sum_{n=1}^\infty (S_n^\xi)^{-4}\mathbb{E}(Z_n^\xi)^4 < \infty. \tag{5.10}$$

Therefore, (2.10) holds true for $(S_n, Z_n) = (S_n^\xi, Z_n^\xi)$.

On the other hand, (5.3) and (5.8) imply $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{1}{(S_n^\xi)^2}\sum_{k=1}^n (Z_k^\xi)^2 = \lim_{n\to\infty} \frac{n}{(S_n^\xi)^2}\left(\frac{1}{n}\sum_{k=1}^n (Z_k^\xi)^2\right) = 1. \ \ \mathbb{P}\text{-a.s.} \tag{5.11}$$

So, (2.11) holds for $(S_n, Z_n) = (S_n^\xi, Z_n^\xi)$ as well, and hence the assertion in (1) follows from Lemma 2.4.

(2) It remains to prove (1.8). By the first assertion, $\Lambda_n^{f,\xi}(t)$ is almost surely relatively compact in $C([0,1];\mathbb{R})$ and the set of its limits points coincides with $\mathcal{H}$. Since $\|h\|_{\mathcal{H}} \le 1$ for any $h \in \mathcal{H}$, this implies $\mathbb{P}$-a.s.

$$\limsup_{n\to\infty} \sup_{t\in[0,1]} |\Lambda_n^{f,\xi}(t)| \le 1. \tag{5.12}$$

Observing that (1.7) implies

$$\Lambda_n^{f,\xi}(1) = \frac{\sum_{l=1}^{n-1} f(X_l^\xi)}{\widehat{D}_f\sqrt{2n\log\log n}}, \quad n \ge 1, \tag{5.13}$$

it follows from (5.12) that

$$\limsup_{n\to\infty} \frac{\sum_{l=1}^{n} f(X_l^\xi)}{\sqrt{2n\log\log n}} = \widehat{D}_f \limsup_{n\to\infty} \Lambda_n^{f,\xi}(1) \le \widehat{D}_f, \quad \mathbb{P}\text{-a.s.} \tag{5.14}$$

On the other hand, since the limits points of $(\Lambda_n^{f,\xi}(t))$ coincides with $\mathcal{H}$ and $h \in \mathcal{H}$ with $h(t) = t, t \in [0,1]$, there exists a subsequence $n_k \uparrow \infty$ as $k \to \infty$ such that $\mathbb{P}$-a.s.

$$\lim_{k\to\infty} \sup_{t\in[0,1]} |\Lambda_{n_k}^{f,\xi}(t) - h(t)| = 0.$$

In particular, combining this with (1.7) for $k = n-1$ and $t = \frac{k}{n}$, we deduce $\mathbb{P}$-a.s.

$$\lim_{k\to\infty} \frac{\sum_{l=1}^{n_k} f(X_l^\xi)}{\sqrt{2n_k\log\log n_k}} = \lim_{k\to\infty} \widehat{D}_f \Lambda_{n_k}^{f,\xi}(1) = \widehat{D}_f,$$

which together with (5.14) yields

$$\limsup_{n\to\infty} \frac{\sum_{l=1}^{n} f(X_l^\xi)}{\sqrt{2n\log\log n}} = \widehat{D}_f \quad \mathbb{P}\text{-a.s.}$$

Replacing f by $-f$, this formula reduces to

$$\liminf_{n\to\infty} \frac{\sum_{l=1}^{n} f(X_l^\xi)}{\sqrt{2n\log\log n}} = -\widehat{D}_f \quad \mathbb{P}\text{-a.s.}$$

Therefore, (1.8) holds. □

References

[1] Bao, J., Wang, F.-Y., Yuan, C., Hypercontractivity for functional stochastic differential equations, *Stochastic Process. Appl.*, **125** (2015), 3636–3656.

[2] Bao, J., Wang, F.-Y., Yuan, C., Ergodicity for Neutral Type SDEs with Infinite Length of Memory, arXiv:1805.03431.

[3] Bolt, W., Majewski, A. A., Szarek, T., An invariance principle for the law of the iterated logarithm for some Markov chains, *Studia Math.*, **212** (2012), 41–53.

[4] Butkovsky, O., Scheutzow, M., Invariant measures for stochastic functional differential equations, *Electron. J. Probab.*, **22** (2017), Paper No. 98, 23 pp.

[5] Butkovsky, O., Subgeometric rates of convergence of Markov processes in the Wasserstein metric, *Ann. Appl. Probab.*, **24** (2014), 526–552.

[6] Cattiaux, P., Chafa¨i, D., Guillin, A., Central limit theorems for additive functionals of ergodic Markov diffusions processes, *ALEA Lat. Am. J. Probab. Math. Stat.*, **9** (2012), 337–382.

[7] Chen, X., The law of the iterated logarithm for functionals of Harris recurrent Markov chains: self-normalization, *J. Theoret. Probab.*, **12** (1999), 421–445.

[8] Derriennic, Y., Lin, M., The central limit theorem for Markov chains started at a point, *Probab. Theory Related Fields*, **125** (2003), 73–76.

[9] Doeblin, W., Sur deux problemes de M. Kolmogoroff concernant les chanes d énombrables, *Bull. Soc. Math. France*, **66** (1938), 210–220.

[10] Dos Reis, G., Salkeld, W., Tugaut, J., Freidlin-Wentzell LDP in path space for McKean-Vlasov equations and the Functional Iterated Logarithm Law, arXiv:1708.04961v3.

[11] Dragomir, S. S., *Some Gronwall type inequalities and applications*, Nova Science Publishers, Inc., Hauppauge, NY, 2003.

[12] Gao, F., Long time asymptotics of unbounded additive functionals of Markov processes, *Electron. J. Probab.*, **22** (2017), 1–21.

[13] Hall, P., Heyde, C. C., *Martingale limit theory and its applications*, Academic Press, New York-London, 1980.

[14] Hairer, M., Mattingly, J. C., Scheutzow, M., Asymptotic coupling and a general form of Harris' theorem with applications to stochastic delay equations, *Probab. Theory Related Fields*, **149** (2011), 223–259.

[15] Hairer, M., Mattingly, J. C., Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing, *Ann. Math.*, **164** (2006), 993–1032.

[16] Heyde, C. C., Scott, D. J., Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments, *Ann. Probab.*, **1** (1973), 428–436.

[17] Ibragimov, I. A., Linnik, Yu. V., *Independent and stationary sequences of random variables*, Wolters-Noordhoff Publishing, Groningen, 1971.

[18] Itô, K., Nisio, M., On stationary solutions of a stochastic differential equation, *J. Math. Kyoto Univ.*, **4** (1964), 1–75.

[19] Jacod, J., Shiryaev, A. N., Limit theorems for stochastic processes, Springer-Verlag, Berlin, 1987.

[20] Kipnis, C., Varadhan, S. R. S., Central limit theorem for additive functionals of reversible Markov process and applications to simple exclusions, *Comm. Math. Phys.*, **104** (1986), 1–19.

[21] Komorowski, T., Walczuk, A., Central limit theorem for Markov processes with spectral gap in the Wasserstein metric, *Stochastic Process. Appl.*, **122** (2012), 2155–2184.

[22] Kulik, A., *Ergodic behavior of Markov processes with applications to limit theorems*, De Gruyter Studies in Mathematics, 67, De Gruyter, Berlin, 2018.

[23] Meyn, S. P., Tweedie, R. L., *Markov Chains and Stochastic Stability*, Springer-Verlag, Berlin, 1993.

[24] Shirikyan, A., Law of large numbers and central limit theorem for randomly forced PDE's, *Probab. Theory Related Fields*, **134** (2006), 215–247.

[25] Strassen, V., An invariance principle for the law of the iterated logarithm, *Probab. Theory Related Fields*, **3** (1964), 211–226.

[26] von Renesse, M.-K., Scheutzow, M., Existence and uniqueness of solutions of stochastic functional differential equations, *Random Oper. Stoch. Equ.*, **18** (2010), 267–284.

[27] Walczuk, A., Central limit theorem for an additive functional of a Markov process, stable in the Wesserstein metric, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **62** (2008), 149–159.

[28] Wu, L., Forward-backward martingale decomposition and compactness results for additive functionals of stationary ergodic Markov processes, *Ann. Inst. H. Poincaré Probab. Statist.*, **35** (1999), 121–141.