Functional Inequalities for Weighted Gamma Distribution on the Space of Finite Measures*

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Abstract

Let $\mathbb{M}$ be the space of finite measures on a locally compact Polish space, and let $\mathscr{G}$ be the Gamma distribution on $\mathbb{M}$ with intensity measure $\nu \in \mathbb{M}$. Let ∇^{ext} be the extrinsic derivative with tangent bundle $T\mathbb{M} = \cup_{\eta\in\mathbb{M}} L^2(\eta)$, and let $\mathscr{A} : T\mathbb{M} \to T\mathbb{M}$ be measurable such that $\mathscr{A}_\eta$ is a positive definite linear operator on $L^2(\eta)$ for every $\eta \in \mathbb{M}$. Moreover, for a measurable function V on $\mathbb{M}$, let $\mathrm{d}\mathscr{G}^V = \mathrm{e}^V \mathrm{d}\mathscr{G}$. We investigate the Poincaré, weak Poincaré and super Poincaré inequalities for the Dirichlet form

$$\mathscr{E}_{\mathscr{A},V}(F,G) := \int_{\mathbb{M}} \langle \mathscr{A}_\eta \nabla^{ext} F(\eta), \nabla^{ext} G(\eta)\rangle_{L^2(\eta)}\, \mathrm{d}\mathscr{G}^V(\eta),$$

which characterize various properties of the associated Markov semigroup. The main results are extended to the space of finite signed measures.

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1 Introduction

Let $\mathbb{M}$ be the class of finite measures on a locally compact Polish space E, which is again a Polish space under the weak topology. Recall that a sequence of finite measures $\eta_n \to \eta$

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weakly if $\eta_n(f) \to \eta(f)$ for $f \in C_b(E)$, where and in what follows, for a measure η we denote

*00 (1.1)
$$\eta(f) := \int f \mathrm{d}\eta, \quad f \in L^1(\eta). \tag{1.1}$$

Since M is locally compact, the Borel σ-algebra $\mathscr{B}(\mathbb{M})$ induced by the weak topology coincides with that induced by the vague topology. Let $\nu \in \mathbb{M}$ with $\nu(E) > 0$. The Gamma distribution $\mathscr{G}$ with intensity measure ν is the unique probability measure on $\mathbb{M}$ such that for any finitely many disjoint measurable subsets $\{A_1, \cdots, A_n\}$ of E, $\{\eta(A_i)\}_{1 \le i \le n}$ are independent Gamma random variables with shape parameters $\{\nu(A_i)\}_{1 \le i \le n}$ and scale parameter 1; that is,

*0 (1.2)
$$\int_{\mathbb{M}} f(\eta(A_1), \cdots, \eta(A_n)) \mathscr{G}(\mathrm{d}\eta) = \int_{[0,\infty)^n} f(x_1, \cdots, x_n) \prod_{i=1}^n \gamma_{\nu(A_i)}(\mathrm{d}x_i), \quad f \in \mathscr{B}_b(E), \tag{1.2}$$

where $\mathscr{B}_b(E)$ is the class of bounded measurable functions on E, for a constant $r > 0$

*0' (1.3)
$$\gamma_r(\mathrm{d}s) := 1_{[0,\infty)}(s) \frac{s^{r-1} \mathrm{e}^{-s}}{\Gamma(r)} \mathrm{d}s, \quad \Gamma(r) := \int_0^\infty x^{r-1} \mathrm{e}^{-x} \mathrm{d}x, \tag{1.3}$$

and $\gamma_0 := \delta_0$ is the Dirac measure at point 0. It is well known that $\mathscr{G}$ is concentrated on the class of finite discrete measures

$$\mathbb{M}_{dis} := \Big\{ \sum_{i=1}^\infty s_i \delta_{x_i} : \ s_i \ge 0, x_i \in E, \sum_{i=1}^\infty s_i < \infty \Big\}.$$

Consider the weighted Gamma distribution $\mathscr{G}^V(\mathrm{d}\eta) := \mathrm{e}^{V(\eta)} \mathscr{G}(\mathrm{d}\eta)$, where V is a measurable function on $\mathbb{M}$. We will investigate functional inequalities for the Dirichlet form induced by $\mathscr{G}^V(\mathrm{d}\eta)$ and a positive definite linear map $\mathscr{A}$ on the tangent space of the extrinsic derivative. See [7] and references therein for Dirichlet forms induced by both extrinsic and intrinsic derivatives, where the intensity measure ν is the Lebesgue measure on $\mathbb{R}^d$ such that the Gamma distribution $\mathscr{G}$ is concentrated on the space of infinite Radon measures on $\mathbb{R}^d$. In this paper, we only consider finite intensity measure ν.

Definition 1.1 ([11])**.** A measurable real function F on $\mathbb{M}$ is called extrinsically differentiable at $\eta \in \mathbb{M}$, if

$$\nabla^{ext} F(\eta)(x) := \frac{\mathrm{d}}{\mathrm{d}s} F(\eta + s\delta_x)\Big|_{s=0} \text{ exists for all } x \in E,$$

such that

$$\|\nabla^{ext} F(\eta)\| := \|\nabla^{ext} F(\eta)(\cdot)\|_{L^2(\eta)} < \infty.$$

If F is extrinsically differentiable at all $\eta \in \mathbb{M}$, we denote $F \in \mathscr{D}(\nabla^{ext})$ and call it extrinsically differentiable on $\mathbb{M}$.

Regarding $L^2(\eta)$ as the extrinsic tangent space at $\eta \in \mathbb{M}$, we define the directional derivatives by

$$\nabla^{ext}_\phi F(\eta) := \langle \nabla^{ext} F(\eta), \phi\rangle_{L^2(\eta)} = \eta\big(\phi \nabla^{ext} F(\eta)\big), \quad \phi \in L^2(\eta).$$

When ϕ is bounded, this coincides with the directional derivative under multiplicative actions:

$$\nabla^{ext}_\phi F(\eta) = \frac{\mathrm{d}}{\mathrm{d}s} F(\mathrm{e}^{s\phi}\eta)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} F((1+s\phi)\eta)\Big|_{s=0}, \quad \phi \in \mathscr{B}_b(E),$$

where $h\eta$ for $h \in \mathscr{B}_b(E)$ is a finite signed measure given by

$$(h\eta)(A) := \eta(1_A h) = \int_A h \,\mathrm{d}\eta, \quad A \in \mathscr{B}(E).$$

To introduce the Dirichlet form induced by the extrinsic derivative and the weighted Gamma distribution $\mathscr{G}^V$, we consider the class $\mathscr{F}C_0^\infty$, which consists of cylindrical functions functions of type

$$F(\eta) := f(\eta(A_1), \cdots, \eta(A_n)), \quad n \geq 1, f \in C_0^\infty(\mathbb{R}^n), \{A_i\}_{1\leq i\leq n} \in \mathscr{I}(E),$$

where $\mathscr{I}(E)$ is the set of all measurable partitions of E. Obviously, such a function F is extrinsically differentiable with

CL (1.4)
$$\nabla^{ext} F(\eta) = \sum_{i=1}^n (\partial_i f)(\eta(A_1), \cdots, \eta(A_n)) \cdot 1_{A_i}.$$

We consider the square field

$$\Gamma_{\mathscr{A}}(F,G) := \langle \mathscr{A}_\eta \nabla^{ext} F(\eta), \nabla^{ext} G(\eta)\rangle_{L^2(\eta)} = \int_E \big[\mathscr{A}_\eta \nabla^{ext} F(\eta)\big] \cdot \big[\nabla^{ext} G(\eta)\big] \,\mathrm{d}\eta,$$

and the pre-Dirichlet form

$$\mathscr{E}_{\mathscr{A},V}(F,G) := \int_{\mathbb{M}} \Gamma_{\mathscr{A}}(F,G) \,\mathrm{d}\mathscr{G}^V, \quad F,G \in \mathscr{F}C_0^\infty,$$

where $\mathscr{A}$ and V satisfy the following assumption.

(H) For any $\eta \in \mathbb{M}$, let $\mathscr{A}_\eta$ be a bounded linear operator on $L^2(\eta)$ such that

H01 (1.5)
$$\langle \mathscr{A}_\eta h, h\rangle_{L^2(\eta)} \geq 0, \quad h \in L^2(\eta),$$

for any $A \in \mathscr{B}(E)$ and $x \in E$, $\mathscr{A}_\eta 1_A(x)$ is measurable in $(\eta, x) \in \mathbb{M} \times E$ and is extrinsically differentiable in η with

H02 (1.6)
$$\sup_{\eta(E)\leq r} \big\{ \|\mathscr{A}_\eta\|^2_{L^2(\eta)} + \|\nabla^{ext}[\mathscr{A}_\eta 1_A]\|_{L^2(\eta)} \big\} < \infty, \quad r \in (0,\infty),$$

where $\|\cdot\|_{L^2(\eta)}$ is the norm (or the operator norm for linear operators) in $L^2(\eta)$. Moreover, $V \in \mathscr{D}(\nabla^{ext})$ such that

H03 (1.7)
$$\sup_{\eta(E)\leq r} \big\{ |V(\eta)| + \|\nabla^{ext} V(\eta)\|_{L^2(\eta)} \big\} < \infty, \quad r \in (0,\infty).$$

Condition (1.5) is essential for the nonnegativity of $\mathscr{E}_{\mathscr{A},V}$, where conditions (1.6) and (1.7) ensure the boundedness of $\mathscr{A}, V$ and their extrinsic derivatives on the level sets $\{\eta(E) \le r\}$ for $r > 0$. These conditions are standard for establishing functional inequalities by using perturbation argument, see [14, 24] for the study of the finite-dimensional models.

We write $\mathscr{A} = \mathbf{1}$ if $\mathscr{A}_\eta$ is the identity map on $L^2(\eta)$ for every $\eta \in \mathbb{M}$. According to Theorem 3.1 below, the assumption **(H)** implies that $(\mathscr{E}_{\mathscr{A},V}, \mathscr{F}C_0^\infty)$ is closable in $L^2(\mathscr{G}^V)$ and the closure $(\mathscr{E}_{\mathscr{A},V}, \mathscr{D}(\mathscr{E}_{\mathscr{A},V}))$ is a symmetric Dirichlet form. If moreover

*AB2

$$\int_{\mathbb{M}} \Big(1 + \frac{\|\mathscr{A}_\eta\|_{L^2(\eta)}}{1+\eta(E)}\Big) \mathscr{G}^V(\mathrm{d}\eta) < \infty, \tag{1.8}$$

then $1 \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V})$ with $\mathscr{E}_{\mathscr{A},V}(1,1) = 0$. Let $(\mathscr{L}_{\mathscr{A},V}, \mathscr{D}(\mathscr{L}_{\mathscr{A},V}))$ be the associated generator. We aim to investigate functional inequalities for the Dirichlet form $\mathscr{E}_{\mathscr{A},V}$ and the spectral gap of the generator $\mathscr{L}_{\mathscr{A},V}$.

We first consider the Poincaré inequality

PC1

$$\mathscr{G}^V(F^2) \le \frac{1}{\lambda}\mathscr{E}_{\mathscr{A},V}(F,F) + \mathscr{G}^V(F)^2, \quad F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V}), \tag{1.9}$$

where $\lambda > 0$ is a constant. The spectral gap of $\mathscr{L}_{\mathscr{A},V}$, denoted by $\mathrm{gap}(\mathscr{L}_{\mathscr{A},V})$, is the largest constant $\lambda > 0$ such that (1.9) holds. If (1.9) is invalid, i.e. there is no any constant $\lambda > 0$ satisfying the inequality, we write $\mathrm{gap}(\mathscr{L}_{\mathscr{A},V}) = 0$ and say that $\mathscr{L}_{\mathscr{A},V}$ does not have spectral gap. It is well known that (1.9) is equivalent to the exponential convergence of the associated Markov semigroup $P_t^{\mathscr{A},V}$:

$$\|P_t^{\mathscr{A},V} F - \mathscr{G}^V(F)\|_{L^2(\mathscr{G}^V)} \le \mathrm{e}^{-\lambda t}\|F\|_{L^2(\mathscr{G}^V)}, \quad t \ge 0, F \in L^2(\mathscr{G}^V).$$

When $\mathrm{gap}(\mathscr{L}_{\mathscr{A},V}) = 0$, the following weak Poincaré inequality was introduced in [13]:

WPC

$$\mathscr{G}^V(F^2) \le \alpha(r)\mathscr{E}_{\mathscr{A},V}(F,F) + r\|F\|_\infty^2, \quad F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V}),\ \mathscr{G}^V(F) = 0, r > 0, \tag{1.10}$$

where $\alpha : (0,\infty) \to (0,\infty)$ corresponds to a non-exponential convergence rate of $P_t^{\mathscr{A},V}$ as $t \to \infty$, see [13, Theorems 2.1 and 2.3]. In particular, (1.10) implies

$$\|P_t^{\mathscr{A},V} - \mathscr{G}^V\|_{L^\infty(\mathscr{G}^V)\to L^2(\mathscr{G}^V)} \le \inf\big\{r > 0 : \alpha(r)\log r^{-1} \le 2t\big\} \downarrow 0 \text{ as } t \uparrow \infty.$$

We also consider the super Poincaré inequality

SUP

$$\mathscr{G}^V(F^2) \le r\mathscr{E}_{\mathscr{A},V}(F,F) + \beta(r)\mathscr{G}^V(|F|)^2, \quad r > 0, F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V}), \tag{1.11}$$

where $\beta : (0,\infty) \to (0,\infty)$ is a decreasing function. The existence of super Poincaré inequality is equivalent to the uniform integrability of $P_t^{\mathscr{A},V}$ for $t > 0$, and, when $P_t^{\mathscr{A},V}$ has an asymptotic density with respect to $\mathscr{G}^V$, it is also equivalent to the compactness of $P_t^{\mathscr{A},V}$ in $L^2(\mathscr{G}^V)$, see [24, Theorem 3.2.1] for details. According to [24, Definition 3.1.2], $P_t^{\mathscr{A},V}$ is said to have an asymptotic density, if $\|P_t^{\mathscr{A},V} - P_n\|_{L^2(\mathscr{G}^V)} \to 0$ for a sequence of bounded linear operators $\{P_n\}_{n\ge1}$ having densities with respect to $\mathscr{G}^V$. We say that $\mathscr{E}_{\mathscr{A},V}$ does not satisfy the super Poincaré inequality, if there is no $\beta : (0,\infty) \to (0,\infty)$ satisfying (1.11). In particular, (1.11) holds with $\beta(r) = \mathrm{e}^{cr^{-1}}$ for some constant $c > 0$ if and only if the log-Sobolev inequality

LSI0 (1.12)
$$\mathscr{G}^V(F^2 \log F^2) \le C\mathscr{E}_{\mathscr{A},V}(F,F), \quad F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V}), \mathscr{G}^V(F^2) = 1 \tag{1.12}$$

holds for some constant $C > 0$. It is well known (see [2, 6]) that (1.12) is equivalent to the hypercontractivity of $P_t^{\mathscr{A},V}$:

$$\|P_t^{\mathscr{A},V}\|_{L^2(\mathscr{G}^V)\to L^4(\mathscr{G}^V)} = 1 \ \text{ for large } t > 0,$$

as well as the exponential convergence in entropy:

$$\mathscr{G}^V((P_t^{\mathscr{A},V}F)\log P_t^{\mathscr{A},V}F) \le \mathrm{e}^{-2t/C}\mathscr{G}^V(F\log F), \quad t \ge 0, F \ge 0, \mathscr{G}^V(F) = 1.$$

See [21, 22, 23] or [24] for more results on the super Poincaré inequalities, for instance, estimates on the semigroup $P_t^{\mathscr{A},V}$ and higher order eigenvalues of the generator $\mathscr{L}_{\mathscr{A},V}$ using the function β in (1.11).

The remainder of the paper is organised as follows. In section 2, we state the main results of the paper, and illustrate these results by a typical example with specific interactions. In Section 3, we establish the integration by parts formula which implies the closability of $(\mathscr{E}_{\mathscr{A},V}, \mathscr{F}C_0^\infty)$. Then the main results are proved in Section 4, and extended in Section 5 to the space $\mathbb{M}_s$ of finite signed measures.

2 Main results and an example

We first consider $\mathscr{E}_{\mathbf{1},0}$ in $L^2(\mathscr{G})$ whose restriction on $\mathbb{M}_1 := \{\mu \in \mathbb{M} : \mu(E) = 1\}$ gives rise to the Dirichlet form of the Fleming−Viot process. Corresponding to results of [16, 17] for the Fleming−Viot process, we have the following result. See also [12, 26] for functional inequalities of different type measure-valued processes.

T1.2 **Theorem 2.1.** *Let $V = 0$ and $\mathscr{A} = \mathbf{1}$.*

(1) $\mathrm{gap}(\mathscr{L}_{\mathbf{1},0}) = 1$, *i.e.* $\lambda = 1$ *is the largest constant such that* (1.9) *holds for $V = 0$ and $\mathscr{A} = \mathbf{1}$.*

(2) *If* $\mathrm{supp}\,\nu$ *contains infinitely many points, then* $\mathscr{E}_{\mathbf{1},0}$ *does not satisfy the super Poincaré inequality.*

(3) *There exists a constant $c_0 > 0$ such that when* $\mathrm{supp}\,\nu$ *is a finite set, the log-Sobolev inequality*

LSI0' (2.1)
$$\mathscr{G}(F^2 \log F^2) \le \frac{c_0}{1 \wedge \delta}\mathscr{E}_{\mathbf{1},0}(F,F), \quad F \in \mathscr{D}(\mathscr{E}_{\mathbf{1},0}), \mathscr{G}(F^2) = 1 \tag{2.1}$$

holds, where $\delta := \min\{\nu(\{x\}) : x \in \mathrm{supp}\,\nu\}$.

To extend this result to $\mathscr{E}_{\mathscr{A},V}$, we will adopt a split argument by making perturbations to $\mathscr{E}_{\mathbf{1},0}$ on bounded sets and estimating the principal eigenvalue of $\mathscr{L}_{\mathscr{A},V}$ outside. To this end, we take

$$\rho(\eta) = 2\sqrt{\eta(E)}, \quad \eta \in \mathbb{M}$$

and let $\mathbf{B}_N = \{\eta \in \mathbb{M} : \rho(\eta) \le N\}$ for $N > 0$. Since (1.4) implies

RRD
$$\nabla^{ext}\rho(\eta) = \frac{1}{\sqrt{\eta(E)}}, \quad \eta \in \mathbb{M} \setminus \{0\}, \tag{2.2}$$

we have

*OW
$$\Gamma_{\mathbf{1}}(\rho, \rho) := \eta\big(|\nabla^{ext}\rho(\eta)|^2\big) = \frac{\eta(E)}{\eta(E)} = 1. \tag{2.3}$$

According to (3.1) below, we set

*TP
$$\mathscr{L}_{\mathscr{A},V}\rho(\eta) = \frac{2}{\rho(\eta)}\big[(\nu - \eta)(\mathscr{A}_\eta 1) + \eta\big(\nabla^{ext}[\mathscr{A}_\eta 1(\cdot)](\cdot)\big) + \nabla^{ext}_{\mathscr{A}_\eta 1}V(\eta)\big] - \frac{4}{\rho(\eta)^2}\eta(\mathscr{A}_\eta 1), \tag{2.4}$$

where

$$\eta\big(\nabla^{ext}[\mathscr{A}_\eta 1(\cdot)](\cdot)\big) := \int_E \nabla^{ext}[\mathscr{A}_\eta 1(x)](x)\,\eta(\mathrm{d}x).$$

Let

*TP0
$$\begin{aligned} &\xi(r) = \sup_{\rho(\eta)=r} \mathscr{L}_{\mathscr{A},V}\rho(\eta), \quad \underline{a}(r) = \inf_{\rho(\eta)=r}\inf_{\|\phi\|_{L^2(\eta)}=1} \langle \mathscr{A}_\eta\phi, \phi\rangle_{L^2(\eta)}, \\ &\bar{a}(r) = \sup_{\rho(\eta)=r}\sup_{\|\phi\|_{L^2(\eta)}=1} \langle \mathscr{A}_\eta\phi, \phi\rangle_{L^2(\eta)}, \quad r > 0. \end{aligned} \tag{2.5}$$

Under **(H)**, $|V(\eta)| + \|\mathscr{A}_\eta\|_{L^2(\eta)}$ is bounded on $\mathbf{B}_r := \{\rho \le r\}$ for $r \in (0, \infty)$. So, these functions are bounded on $[k, K]$ for any constants $K > k > 0$. Moreover, define

*TP1
$$\sigma_k := \sup_{t \ge k} \int_t^\infty \mathrm{e}^{\int_k^r \frac{\xi(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r \int_k^t \frac{1}{\underline{a}(r)}\mathrm{e}^{-\int_k^r \frac{\xi(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r, \quad k > 0. \tag{2.6}$$

Obviously, σ_k is non-increasing in k and might be infinite. We will see in Theorem 2.2(1) that under certain conditions $\sigma_k < \infty$ implies the validity of Poincaré inequality.

We have the following extension of Theorem 2.1 to $\mathscr{E}_{\mathscr{A},V}$. When $\operatorname{supp}\nu$ is finite the model reduces to finite-dimensional diffusions, for which one may derive super Poincaré inequalities by making perturbations to (2.1). As the present study mainly focusses on the infinite-dimensional model, we exclude this case in the following result.

TSP **Theorem 2.2.** *Assume* **(H)** *and* (1.8). *Suppose that* $\underline{a}(r)^{-1}$ *is locally bounded in* $r \in [0, \infty)$ *and*

0*0
$$\psi(s) := \int_0^s [\bar{a}(r)]^{-1/2}\,\mathrm{d}r \uparrow \infty \text{ as } s \uparrow \infty. \tag{2.7}$$

Then the following assertions hold.

(1) *If* $\lim_{k\to\infty}\sigma_k<\infty$ *(equivalently,* $\sigma_k<\infty$ *for all* $k>0$*), then*

$$\mathrm{gap}(\mathscr{L}_{\mathscr{A},V})\ge\sup\left\{\frac{1}{2\Phi\big(\psi^{-1}(\psi(k)+32\sigma_k+1)\big)+32\sigma_k}:\ k>0\right\}>0,$$

where

$$\Phi(N):=\Big(1\vee\frac{N^2}{4\nu(E)}\Big)\exp\Big[\sup_{\rho\le N}V-\inf_{\rho\le N}V\Big]\sup_{r\le N}\underline{a}(r)^{-1},\quad N>0.$$

(2) *If* $\mathrm{supp}\,\nu$ *contains infinitely many points, then* $\mathscr{E}_{\mathscr{A},V}$ *does not satisfy the super Poincaré inequality.*

(3) *The weak Poincaré inequality* (1.10) *holds for*

$$\alpha(r):=\inf\Big\{2\Phi(N):\ \mathscr{G}^V(\rho>N)\le\frac{r}{1+r}\Big\},\quad r>0.$$

The following result shows that the condition in Theorem 2.2(1) is sharp when $\mathscr{A}_\eta$ and $V(\eta)$ depend only on $\rho(\eta)$.

C1.2 **Corollary 2.3.** *Assume* **(H)** *and* (1.8). *Let* $V(\eta)=v(\rho(\eta))$ *and* $\mathscr{A}_\eta=a(\rho(\eta))\mathbf{1}$ *for large* $\rho(\eta)$ *and some* $a,v\in C^1([0,\infty))$ *with* $\underline{a}(r)>0$ *for* $r\ge0$*. Then*

$$\xi(r):=\sup_{\rho(\eta)=r}\mathscr{L}_{\mathscr{A},V}\rho(\eta)=a(r)\Big(\frac{1}{r}+v'(r)-\frac{r}{2}\Big)+\frac{r}{2}a(r),\quad \textit{for large } r>0,$$

and $\mathrm{gap}(\mathscr{L}_{\mathscr{A},V})>0$ *if and only if* $\lim_{k\to\infty}\sigma_k<\infty$.

As in the proof of [14, Corollary 1.3] using [14, Theorem 1.1], it is easy to see that Theorem 2.2(2) implies the following result.

C2.4 **Corollary 2.4.** *Assume* **(H)** *and* (1.8). *If* $\inf_{r\ge0}\underline{a}(r)>0$ *and* $\limsup_{r\to\infty}\frac{\xi(r)}{\underline{a}(r)}<0$*, then* $\mathrm{gap}(\mathscr{L}_{\mathscr{A},V})>0$.

The above two corollaries are concerned with the validity of Poincaré inequality. On the other hand, according to Theorem 2.2(3), the weak Poincaré inequality always holds under **(H)**, (1.8) and (2.7). We will see in the proof that the rate function α is derived by comparing $\mathscr{E}_{\mathscr{A},V}$ with $\mathscr{E}_{\mathbf{1},0}$ on bounded sets $\mathbf{B}_N$, $N>0$. However, when these two Dirichlet forms are far away, this α is less sharp. As a principle, to derive a sharper weak Poincaré inequality, one should compare $\mathscr{E}_{\mathscr{A},V}$ with a closer Dirichlet form which satisfies the Poincaré inequality. In this spirit, we present below an alternative result on the weak Poincaré inequality. To state the result, we introduce the class $\mathscr{H}$ as follows.

Class $\mathscr{H}$: We denote $h \in \mathscr{H}$, if $0 \le h \in C^1([0,\infty))$ with $h'(r) > 0$ for $r > 0$, such that

0*1

$$\xi_h(r) := \xi(r) - \frac{2}{r} h(r) \inf_{\rho(\eta)=r} \eta(\mathscr{A}_\eta 1), \quad r > 0 \tag{2.8}$$

satisfies

0*2

$$\sigma_{1,h} := \sup_{t\ge 1} \int_t^\infty \mathrm{e}^{\int_1^r \frac{\xi_h(s)}{\underline{a}(s)} \mathrm{d}s} \mathrm{d}r \int_1^t \frac{1}{\underline{a}(r)} \mathrm{e}^{-\int_1^r \frac{\xi_h(s)}{\underline{a}(s)} \mathrm{d}s} \mathrm{d}r < \infty. \tag{2.9}$$

It is easy to see that $\mathscr{H} \ne \emptyset$ under the conditions of Theorem 2.2 and $\inf \underline{a} > 0$. For any $h \in \mathscr{H}$, let $V_h = V - h(\rho) + c(h)$, where $c(h) \in \mathbb{R}$ is such that $\mathscr{G}^{V_h}$ is a probability measure on $\mathbb{M}$. By Theorem 2.2(1) with $k = 1$, for any $h \in \mathscr{H}$, the Poincaré inequality

PCW

$$\mathscr{G}^{V_h}(F^2) \le C(h) \mathscr{E}_{\mathscr{A},V_h} + \mathscr{G}^{V_h}(F)^2, \quad F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V_h}) \tag{2.10}$$

holds for

CH

$$C(h) := 2\Phi_{1,h}\big(\psi^{-1}(\psi(1) + 32\sigma_{1,h} + 1)\big) + 32\sigma_{1,h}, \quad h \in \mathscr{H}. \tag{2.11}$$

NT **Theorem 2.5.** *Assume* **(H)**, (1.8) *and* (2.7). *If $\mathscr{H} \ne \emptyset$, then* (1.10) *holds for*

$$\alpha(r) := \inf\Big\{ C(h) \mathrm{e}^{h(N)} :\ h \in \mathscr{H},\ N > 0 \text{ with } \mathscr{G}^V(\rho > N) \le \frac{r}{1+r} \Big\}, \quad r > 0,$$

where $C(h)$ is given by (2.9) *and* (2.11).

To conclude this section, we present below a simple example to illustrate the main results. For simplicity, we only consider $\mathscr{A}_\eta = \mathbf{1}$. But by a simple comparison argument, the assertions apply also to $\mathscr{A}_\eta$ with $\langle \mathscr{A}_\eta \phi, \phi \rangle_{L^2(\eta)} \ge c \|\phi\|^2_{L^2(\eta)}$ for some constant $c > 0$ and all $\eta \in \mathbb{M}$, $\phi \in L^2(\eta)$.

Example 2.6. *Consider the following potential V_0 with interactions given by $\psi_i \in \mathscr{B}_b(E \times E), i = 1, 2, 3$:*

$$V_0(\eta) = \frac{2(\eta\times\eta)(\psi_1)}{3\eta(E)^{3/2}} + \frac{(\eta\times\eta)(\psi_2)}{\eta(E)} + (\eta\times\eta)(\psi_3) - p\log(1+\eta(E)),$$

where $p \in \mathbb{R}$ is a constant. Let $\theta_i = \sup \psi_i, 1 \le i \le 3$. Assume that one of the following conditions hold:

(1) $\min\big\{\theta_3, \theta_2 - 1, \theta_1 \cdot 1_{\{\theta_2 = 1\}}\big\} < 0$;

(2) $\theta_1 = \theta_2 - 1 = \theta_3 = 0$ *and* $p > \nu(E)$.

Then $Z := \mathscr{G}(\mathrm{e}^{V_0}) \le \frac{1}{\Gamma(\nu(E))} \int_0^\infty (1+s)^{-p} s^{\nu(E)-1} \mathrm{e}^{\theta_1 s^{1/2} - (1-\theta_2)s + \theta_3 s^2} \mathrm{d}s < \infty$, *so that $\mathscr{G}^V$ for $V := V_0 - \log Z$ is a probability measure on $\mathbb{M}$, and the following assertions hold:*

(*a*) *Condition (1) implies* $\mathrm{gap}(\mathscr{L}_{\mathbf{1},V}) > 0$*;*

(*b*) *Under condition* (2)*, let*

$$\theta = \max\big\{12 \times 1_{\{\|\psi_3\|_\infty > 0\}},\ 8 \times 1_{\{\|\psi_2 - 1\|_\infty > 0\}},\ 6 \times 1_{\{\|\psi_1\|_\infty > 0\}},\ 5\big\}.$$

Then there exists a constant $c > 0$ *such that the weak Poincaré inequality* (1.10) *holds for*

$$\alpha(r) = cr^{-\frac{\theta}{2(p-\nu(E))}}, \quad r > 0.$$

Proof. Obviously, the assumptions in Theorem 2.2 hold for V and $\mathscr{A}_\eta = \mathbf{1}$. By definition it is easy to see that

$$\begin{aligned}\nabla^{ext} V(\eta)(x) = & \frac{\eta(E)\eta(\psi_1(x,\cdot) + \psi_1(\cdot,x)) - (\eta\times\eta)(\psi_1)}{\eta(E)^{5/2}} + \eta(\psi_3(x,\cdot) + \psi_3(\cdot,x)) \\ & + \frac{\eta(E)\eta(\psi_2(x,\cdot) + \psi_2(\cdot,x)) - (\eta\times\eta)(\psi_2)}{\eta(E)^2} + \frac{p}{1+\eta(E)}.\end{aligned}$$

Then

$$\begin{aligned}\nabla_1^{ext} V(\eta) := \eta\big(\nabla^{ext} V(\eta)\big) &\le \theta_1\sqrt{\eta(E)} + \theta_2\eta(E) + \theta_3\eta(E)^2 + \frac{p\eta(E)}{1+\eta(E)} \\ &= \frac{\theta_1\rho(\eta)}{2} + \frac{\theta_2\rho(\eta)^2}{4} + \frac{\theta_3\rho(\eta)^4}{8} + \frac{p\rho(\eta)^2}{4+\rho(\eta)^2}.\end{aligned}$$

(a) If (1) holds, then $\theta_3 < 0$, or $\theta_2 < 1$, or $\theta_3 = \theta_2 - 1 = 0$ and $\theta_1 < 0$. In any case, we have

$$\limsup_{\rho(\eta)\to\infty} \mathscr{L}_{\mathbf{1},V}\rho(\eta) = \limsup_{\rho(\eta)\to\infty} \frac{2}{\rho(\eta)}\Big(\nu(E) - \frac{\rho(\eta)^2}{4} + \nabla_1^{ext} V(\eta)\Big) < 0,$$

so that Corollary 2.4 implies $\mathrm{gap}(\mathscr{L}_{\mathbf{1},V}) > 0$.

(b) Under condition (2), we prove the weak Poincaré inequality for the desired $\alpha(r)$. Since one may always take $\alpha(r) \le 1$ in (1.10) due to $\mathscr{G}^V(F^2) \le \|F\|_\infty^2$, it suffices to prove for small $r > 0$, say $r \in (0,1]$.

It is easy to see that

L01 (2.12) $$\mathscr{G}^V(\rho > N) \le c_0 N^{\nu(E)-p}, \quad N > 0$$

holds for some constant $c_0 > 0$. For $\varepsilon \in (0,1]$, we take $h_\varepsilon(s) = \varepsilon\sqrt{s}$. Since $a = 1$, it is easy to check that

$$\sigma_{1,h_\varepsilon} \le c_1\varepsilon^{-2}$$

for some constant $c_1 > 0$ independent of $\varepsilon \in (0,1]$. Moreover, there is a constant c_2 independent of $\varepsilon \in (0,1]$ such that

$$\sup_{\rho\le N} V_{h_\varepsilon} - \inf_{\rho\le N} V_{h_\varepsilon} \le c_2\big[\|\psi_3\|_\infty N^4 + \|\psi_2 - 1\|_\infty N^2 + \|\psi_1\|_\infty N + \varepsilon N + \log(1+N)\big].$$

Combining this with (2.11), we may find constants $c_3, c_4 > 0$ independent of $\varepsilon \in (0,1]$ such that

$$C(h_\varepsilon) \le c_3\big(\|\psi_3\|_\infty \varepsilon^{-12} + \|\psi_2 - 1\|_\infty \varepsilon^{-8} + \|\psi_1\|_\infty \varepsilon^{-6} + \varepsilon^{-5}\big) \le c_4 \varepsilon^{-\theta}.$$

Taking this into account and applying Theorem 2.5 for

$$N = N_r := \Big(\frac{2c_0}{r}\Big)^{\frac{1}{p-\nu(E)}},$$

such that (2.12) implies $\mathscr{G}^V(\rho > N) \le \frac{r}{2}$ as required for $r \in (0,1]$, we conclude that the weak Poincaré inequality holds for

$$\alpha(r) := \inf_{\varepsilon \in (0,1]} C(h_\varepsilon) \mathrm{e}^{h_\varepsilon(N_r)} \le \inf_{\varepsilon \in (0,1]} c_4 \varepsilon^{-\theta} \exp\big[\varepsilon (2c_0 r^{-1})^{\frac{1}{2(p-\nu(E))}}\big], \quad r \in (0,1].$$

Therefore, by taking $\varepsilon = 1 \wedge r^{\frac{1}{2(p-\nu(E))}}$, we prove (1.10) for the desired $\alpha(r)$.

□

3 The Dirichlet form

For any $F \in \mathscr{F}C_0^\infty$, let

LVV (3.1)
$$\begin{aligned}\mathscr{L}_{\mathscr{A},V} F(\eta) :=& \int_E \mathscr{A}_\eta[\nabla^{ext} F(\eta)](x)(\nu - \eta)(\mathrm{d}x) \\ &+ \int_E \nabla^{ext}\big[\mathscr{A}_\eta(\nabla^{ext} F(\eta))(x)\big](x)\, \eta(\mathrm{d}x) + \big\langle \nabla^{ext} V(\eta), \mathscr{A}_\eta[\nabla^{ext} F(\eta)]\big\rangle_{L^2(\eta)}.\end{aligned}$$

It is easy to see from (1.4) that when $F(\eta) = f(\eta(A_1), \cdots, \eta(A_n))$ for some $n \ge 1, f \in C_0^\infty(\mathbb{R}^n)$ and a measurable partition $\{A_i\}_{1 \le i \le n}$ of E, we have

$$\begin{aligned}\mathscr{L}_{\mathscr{A},V} F(\eta) =& \bigg(\sum_{i=1}^n \Big[(\nu - \eta)(\mathscr{A}_\eta 1_{A_i}) + \eta\big(\nabla^{ext}[\mathscr{A}_\eta 1_{A_i}(\cdot)](\cdot)\big) + \nabla^{ext}_{\mathscr{A}_\eta 1_{A_i}} V(\eta) \Big] \partial_i f \\ &+ \sum_{i,j=1}^n \eta(1_{A_i} \mathscr{A}_\eta 1_{A_j})(\partial_i \partial_j f) \bigg)(\eta(A_1), \cdots, \eta(A_n)).\end{aligned}$$

T2.1 **Theorem 3.1.** *Assume* **(H)**. *Then*

*AB1 (3.2)
$$\mathscr{E}_{\mathscr{A},V}(F,G) = -\int_{\mathbb{M}} (G \mathscr{L}_{\mathscr{A},V} F) \mathrm{d}\mathscr{G}^V, \quad F, G \in \mathscr{F}C_0^\infty.$$

Consequently, $(\mathscr{E}_{\mathscr{A},V}, \mathscr{F}C_0^\infty)$ *is closable in* $L^2(\mathbb{M}, \mathscr{G}^V)$ *whose closure* $(\mathscr{E}_{\mathscr{A},V}, \mathscr{D}(\mathscr{E}_{\mathscr{A},V}))$ *is a symmetric Dirichlet form with generator* $(\mathscr{L}_{\mathscr{A},V}, \mathscr{D}(L_{\mathscr{A},V}))$ *being the Friedrichs extension of* $(\mathscr{L}_{\mathscr{A},V}, \mathscr{F}C_0^\infty)$. *If moreover* (1.8) *holds, then* $1 \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V})$ *and* $\mathscr{E}_{\mathscr{A},V}(1,1) = 0$.

To prove this result, we introduce the divergence operator corresponding to ∇^{ext}. To this end, we formulate the Gamma distribution $\mathscr{G}$ by using the Poisson measure $\pi_{\hat\nu}$ with intensity $\hat\nu(\mathrm{d}x, \mathrm{d}s) := s^{-1}\mathrm{e}^{-s}\nu(dx)\mathrm{d}s$ on $\hat E := E \times (0,\infty)$. Recall that $\pi_{\hat\nu}$ is the unique probability measure on the configuration space

$$\mathbf{\Gamma}(\hat E) := \Big\{\gamma = \sum_{i=1}^\infty \delta_{(x_i,s_i)} :\ \gamma(K) < \infty \text{ for compact } K \subset \hat E, (x_i, s_i) \in \hat E\Big\}$$

such that for any disjoint relatively compact subsets $\{\hat A_i\}_{1\le i\le n}$ of $\hat E$, $\{\gamma \mapsto \gamma(\hat A_i)\}_{1\le i\le n}$ are independent random Poisson random variables with parameters $\{\hat\nu(\hat A_i)\}_{1\le i\le n}$. Since $S(\gamma) := \sum_{i=1}^\infty s_i$ for $\gamma = \sum_{i=1}^\infty s_i\delta_{x_i} \in \mathbf{\Gamma}(\hat E)$ satisfies

$$\int_{\mathbf{\Gamma}(\hat E)} S(\gamma)\pi_{\hat\nu}(\mathrm{d}\gamma) = \int_{\hat E} s\hat\nu(\mathrm{d}x, \mathrm{d}s) = \nu(E) < \infty,$$

the measure $\pi_{\hat\nu}$ is concentrated on the S-finite configuration space

$$\mathbf{\Gamma}_f(\hat E) := \Big\{\gamma = \sum_{i=1}^\infty \delta_{(x_i,s_i)} \in \mathbf{\Gamma}(\hat E) : S(\gamma) := \sum_{i=1}^\infty s_i < \infty\Big\}.$$

LLK **Lemma 3.2.** *The map* $\Phi : \mathbf{\Gamma}_f(\hat E) \ni \gamma = \sum_{i=1}^\infty s_i\delta_{x_i} \mapsto \sum_{i=1}^\infty s_i\delta_{x_i} \in \mathbb{M}$ *is measurable with*

*AC (3.3)
$$\mathscr{G} = \pi_{\hat\nu} \circ \Phi^{-1}.$$

Moreover,

MC (3.4)
$$\begin{aligned}&\int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta)\int_E F(\eta, x)\eta(\mathrm{d}x)\\ &= \int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta)\int_{\hat E} \mathrm{e}^{-s}F(\eta + s\delta_x, x)\nu(\mathrm{d}x)\mathrm{d}s, \quad F \in L^1\big(\mathbb{M} \times E, \mathscr{G}(\mathrm{d}\eta)\eta(\mathrm{d}x)\big).\end{aligned}$$

Proof. Formula (3.3) was proved in [8, Theorem 6.2] for $E = \mathbb{R}^d$ and $\nu(\mathrm{d}x) = \theta\mathrm{d}x$ (which is an infinite measure) with $\theta > 0$, by identifying the Laplace transforms of $\mathscr{G}$ and $\pi_{\hat\nu} \circ \Phi^{-1}$. Below we explain that the same argument works to the present setting.

Firstly, the Laplace transform of $\mathscr{G}$ is

LTR (3.5)
$$\int_{\mathbb{M}} \mathrm{e}^{-\eta(h)}\mathscr{G}(\mathrm{d}\eta) = \mathrm{e}^{-\nu(\log(1+h))}, \quad h \in \mathscr{B}^+(E),$$

where $\mathscr{B}^+(E)$ is the class of nonnegative measurable functions on E. This was given by [18, (7)] when ν is atomless. In general, we decompose ν into $\nu = \nu_0 + \sum_{i=1}^\infty c_i\delta_{x_i}$, where ν_0 is an atomless finite measure on E, $x_i \in E$ with $x_i \ne x_j$ for $i \ne j$, and $c_i \ge 0$ with $\sum_{i=1}^\infty c_i < \infty$. Let $E_0 = E \setminus \{x_i : i \ge 1\}$. By the definition of Gamma distribution,

$$\eta(h \cdot 1_{E_0}), \quad \eta(h \cdot 1_{\{x_i\}}), \quad i \ge 1$$

are independent under $\mathscr{G}$, the distribution of $\eta(h\cdot 1_{E_0})$ under $\mathscr{G}$ coincides with that under $\mathscr{G}_0$ (the Gamma distribution with intensity measure ν_0), and the distribution of $\eta(\{x_i\})$ under $\mathscr{G}$ coincides with the one-dimensional Gamma distribution γ_{c_i} with shape parameter c_i. So, applying (3.5) for ν_0 replacing ν due to [18, (7)], and using the Laplace transform for Gamma distributions on $\mathbb{R}_+$, we derive

$$\begin{aligned}&\int_{\mathbb{M}} \mathrm{e}^{-\eta(h)}\mathscr{G}(\mathrm{d}\eta) = \bigg(\int_{\mathbb{M}} \mathrm{e}^{-\eta(h\cdot 1_{E_0})}\mathscr{G}(\mathrm{d}\eta)\bigg)\cdot\prod_{i=1}^{\infty}\int_{\mathbb{M}} \mathrm{e}^{-h(x_i)\eta(\{x_i\})}\mathscr{G}(\mathrm{d}\eta)\\ &= \mathrm{e}^{-\nu_0(\log(1+h))}\cdot\prod_{i=1}^{\infty}\mathrm{e}^{-c_i\log(1+h(x_i))} = \mathrm{e}^{-\nu(\log(1+h))}.\end{aligned}$$

Therefore, (3.5) holds.

On the other hand, the Laplace transform for $\pi_{\hat\nu}$ (see for instance [1]) is

$$\int_{\mathbf{\Gamma}_{pf}(\hat E)} \mathrm{e}^{-\gamma(\hat h)}\pi_{\hat\nu}(\mathrm{d}\gamma) = \exp\big[-\hat\nu(1-\mathrm{e}^{-\hat h})\big],\quad \hat h\in\mathscr{B}^+(\hat E).$$

By letting $\hat h(x,s) = sh(x)$ for $(x,s)\in\hat E$, we arrive at

$$\begin{aligned}&\int_{\mathbb{M}} \mathrm{e}^{-\eta(h)}(\pi_{\hat\nu}\circ\Phi^{-1})(\mathrm{d}\eta) = \int_{\mathbf{\Gamma}_{pf}(\hat E)} \mathrm{e}^{-\gamma(\hat h)}\pi_{\hat\nu})(\mathrm{d}\gamma)\\ &= \exp\big[-\hat\nu\big(1-\mathrm{e}^{-\hat h}\big)\big] = \mathrm{e}^{-\nu(\log(1+h))},\quad h\in\mathscr{B}^+(E).\end{aligned}$$

Combining this with (3.5) we prove (3.3).

Finally, (3.4) follows from (3.3) and the Mecke formula [10, Satz 3.1] for Poisson measures. □

To establish the integration by parts formula for $\nabla^{ext}_\phi F$, we introduce the divergence operator div^{ext} as follows.

Let $\phi:\mathbb{M}\times E\to\mathbb{R}$ be measurable. If for any $x\in E$, $\phi(\cdot,x)\in\mathscr{D}(\nabla^{ext})$ such that

$$(\mathscr{G}\times\nu)(|\phi|) + \int_{\mathbb{M}}\eta\big(|\phi(\eta,\cdot)| + |\nabla^{ext}\phi(\eta,\cdot)(\cdot)|\big)\mathscr{G}(\mathrm{d}\eta) < \infty,$$

where $\eta(\cdot)$ stands for the integral with respect to η as in (1.1), then we write $\phi\in\mathscr{D}(\mathrm{div}^{ext})$ and denote

DIV (3.6)
$$\mathrm{div}^{ext}(\phi)(\eta) = (\eta-\nu)\big(\phi(\eta,\cdot)\big) - \eta\big(\nabla^{ext}\phi(\eta,\cdot)(\cdot)\big).$$

When $\phi(\eta,x)=\phi(x)$ does not depend on η, the following integration by parts formula follows from [9, Theorem 14]. We include below a complete proof for the η-dependent ϕ.

L2.2 **Lemma 3.3.** *Let $\phi\in\mathscr{D}(\mathrm{div}^{ext})$. Then*

BBT* (3.7)
$$\int_{\mathbb{M}}(\nabla^{ext}_\phi F)\,\mathrm{d}\mathscr{G} = \int_{\mathbb{M}}[F\mathrm{div}^{ext}(\phi)]\,\mathrm{d}\mathscr{G},\quad F\in\mathscr{F}C_0^\infty.$$

Proof. By (3.4) and the Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{M}} (\nabla_\phi^{ext} F)\,\mathrm{d}\mathscr{G} = \int_{\mathbb{M}\times E} \Big(\lim_{\varepsilon\downarrow 0} \frac{F(\eta+\varepsilon\delta_x)-F(\eta)}{\varepsilon}\Big)\phi(\eta,x)\,\eta(\mathrm{d}x)\mathscr{G}(\mathrm{d}\eta)$$
$$= \int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta) \lim_{\varepsilon\downarrow 0}\int_{\hat{E}} \frac{1}{\varepsilon}\mathrm{e}^{-s}\big[F(\eta+(s+\varepsilon)\delta_x)-F(\eta+s\delta_x)\big]\phi(\eta+s\delta_x,x)\,\nu(\mathrm{d}x)\mathrm{d}s$$
$$= \int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta) \int_{\hat{E}} \mathrm{e}^{-s}\big[\partial_s F(\eta+s\delta_x,x)\big]\phi(\eta+s\delta_x,x)\,\nu(\mathrm{d}x)\mathrm{d}s$$
$$= \int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta) \int_{\hat{E}} \Big(\partial_s\big[\mathrm{e}^{-s}F(\eta+s\delta_x)\phi(\eta+s\delta_x,x)\big] - F(\eta+s\delta_x)\partial_s\big[\mathrm{e}^{-s}\phi(\eta+s\delta_x,x)\big]\Big)\,\nu(\mathrm{d}x)\mathrm{d}s.$$

Noting that $F\in\mathscr{F}C_0^\infty$ implies $F(\eta+s\delta_x)=0$ for large s, we have

$$\int_0^\infty \partial_s\big[\mathrm{e}^{-s}F(\eta+s\delta_x)\phi(\eta+s\delta_x,x)\big]\,\mathrm{d}s = -F(\eta)\phi(\eta,x).$$

Hence, by using (3.4) again,

$$\int_{\mathbb{M}} (\nabla_\phi^{ext} F)\,\mathrm{d}\mathscr{G} + \int_{\mathbb{M}} F(\eta)\nu(\phi(\eta,\cdot))\,\mathscr{G}(\mathrm{d}\eta)$$
$$= -\int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta)\int_{\hat{E}} F(\eta+s\delta_x)\mathrm{e}^{-s}\big[\partial_s\phi(\eta+s\delta_x,x)-\phi(\eta+s\delta_x,x)\big]\,\nu(\mathrm{d}x)\mathrm{d}s$$
$$= \int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta)\int_{\hat{E}} \big[\phi(\eta+s\delta_x,x)-\nabla^{ext}\phi(\cdot,x)(\eta+s\delta_x)(x)\big]\mathrm{e}^{-s}F(\eta+s\delta_x)\nu(\mathrm{d}x)\mathrm{d}s$$
$$= \int_{\mathbb{M}} F(\eta)\mathscr{G}(\mathrm{d}\eta)\int_E [\phi(\eta,x)-\nabla^{ext}\phi(\eta,x)(x)]\eta(\mathrm{d}x).$$

Therefore, (3.7) holds. □

Proof of Theorem 3.1. We first prove (3.2), which implies the closability of $(\mathscr{E}_{\mathscr{A},V},\mathscr{F}C_0^\infty)$ and that the closure is a symmetric Dirichlet form in $L^2(\mathscr{G}^V)$, see [4]. By the definition of $\mathscr{E}_{\mathscr{A},V}$ and Lemma 3.3, for any $F,G\in\mathscr{F}C_0^\infty$ we have

$$\mathscr{E}_{\mathscr{A},V}(F,G) = \int_{\mathbb{M}} \Gamma_{\mathscr{A}}(F,G)\mathrm{d}\mathscr{G}^V = \int_{\mathbb{M}} \big(\nabla^{ext}_{\mathrm{e}^{V(\eta)}\mathscr{A}_\eta\nabla^{ext}F(\eta)}G\big)(\eta)\,\mathscr{G}(\mathrm{d}\eta)$$
$$= \int_{\mathbb{M}} G(\eta)\mathrm{div}^{ext}\big(\mathrm{e}^{V(\eta)}\mathscr{A}_\eta[\nabla^{ext}F(\eta)](\cdot)\big)\,\mathscr{G}(\mathrm{d}\eta).$$

Therefore, by (3.6), (3.2) holds for

$$\mathscr{L}_{\mathscr{A},V}F(\eta) := -\mathrm{e}^{-V(\eta)}\mathrm{div}^{ext}\big(\mathrm{e}^{V(\eta)}\mathscr{A}_\eta[\nabla^{ext}F(\eta)](\cdot)\big)$$
$$= \int_E \Big([\nabla^{ext}V(\eta)(x)]\mathscr{A}_\eta[\nabla^{ext}F(\eta)](x) + \nabla^{ext}\big(\mathscr{A}_\eta[\nabla^{ext}F(\eta)](x)\big)(x)\Big)\,\eta(\mathrm{d}x)$$
$$+ \int_E \mathscr{A}_\eta[\nabla^{ext}F(\eta)](x)\,(\nu-\eta)(\mathrm{d}x).$$

Next, assume that (1.8) holds. It remains to find a sequence $\{F_n\}_{n\ge1}\subset\mathscr{D}(\mathscr{E}_{\mathscr{A},V})$ such that

$$\lim_{n\to\infty}\big[\mathscr{G}^V(|F_n-1|^2)+\mathscr{E}_{\mathscr{A},V}(F_n,F_n)\big]=0.$$

To this end, we consider $\rho_n:=\sqrt{n^{-1}+\rho^2},\ n\ge1$. By (2.3), we have $\rho_n\in\mathscr{D}(\nabla^{ext})$ with

$$\Gamma_{\mathbf{1}}(\rho_n,\rho_n)=\frac{\rho^2}{\rho_n^2}\le1.$$

Let $h\in C_0^\infty([0,\infty))$ such that $h(r)=1$ for $r\le1$ and $h(r)=0$ for $r\ge2$. We have

$$F_n:=h\big(n^{-1}\log[1+\rho_n]\big)\subset\mathscr{F}C_0^\infty,\quad n\ge1.$$

It is easy to see that $\mathscr{G}^V(|F_n-1|^2)\to0$ as $n\to\infty$ and due to (1.8),

$$\limsup_{n\to\infty}\mathscr{E}_{\mathscr{A},V}(F_n,F_n)\le\limsup_{n\to\infty}\int_{\mathbb{M}}\frac{\|\mathscr{A}_\eta\|_{L^2(\eta)}\|h'\|_\infty^2}{n^2(1+\rho)^2}\mathscr{G}^V(\mathrm{d}\eta)=0.$$

□

4 Proofs of the main results

In this section, we prove Theorems 2.1, 2.2, 2.5 and Corollary 2.3.

4.1 Proof of Theorem 2.1 and a local Poincaré inequality

Proof of Theorem 2.1. The invalidity of the super Poincaré inequality will be included in the proof of Theorem 2.2(3) for a more general case. So, we only prove (1) and (3).

(a) We first prove gap$(\mathscr{L}_{\mathbf{1},0})=1$, i.e. $\lambda=1$ is the optimal constant for the Poincaré inequality

PC2 (4.1)
$$\mathscr{G}(F^2)\le\frac{1}{\lambda}\mathscr{E}_{\mathbf{1},0}(F,F)+\mathscr{G}(F)^2,\quad F\in\mathscr{F}C_0^\infty$$

to hold. Let $F(\eta)=f(\eta(A_1),\cdots,\eta(A_n))$ for some $f\in C_0^\infty(\mathbb{R}^n)$ and disjoint $A_1,\cdots,A_n$. This Poincaré inequality reduces to

$$\mu^n(f^2)-\mu^n(f)^2\le\mu^n\Big(\sum_{i=1}^n x_i|\partial_i f(x_1,\cdots,x_n)|^2\Big),$$

where according to (1.2),

*PC (4.2)
$$\mu^n(\mathrm{d}x):=\prod_{i=1}^n\mu_i(\mathrm{d}x_i),\quad \mu_i(\mathrm{d}s)=\gamma_{\nu(A_i)}(\mathrm{d}s):=1_{[0,\infty)}(s)\frac{s^{\nu(A_i)-1}\mathrm{e}^{-s}}{\Gamma(\nu(A_i))}\mathrm{d}s,\ 1\le i\le n.$$

By the additive property of the Poincaré inequality, it suffices to prove that for every $1 \le i \le n$, $\lambda = 1$ is the largest constant satisfying

$$\mu_i(f^2) - \mu_i(f)^2 \le \frac{1}{\lambda} \int_0^\infty r f'(r)^2 \mu_i(\mathrm{d}r), \quad f \in C_0^\infty([0,\infty)).$$

This follows from the fact that the generator of the Dirichlet form

$$\mathscr{E}_i(f,g) := \int_0^\infty r f'(r) g'(r) \mu_i(\mathrm{d}r), \quad f, g \in W^{1,2}([0,\infty), \mu_i)$$

is

$$\mathscr{L}_i f(r) := r f''(r) + (\nu(A_i) - r) f'(r), \quad r \in [0,\infty),$$

which has spectral gap 1 with the first eigenfunction $u_i(r) = r - \nu(A_i)$.

(b) Let $\operatorname{supp} \nu = \{x_1, \cdots, x_n\}$, we have $\delta = \min\{\nu(\{x_i\}) : 1 \le i \le n\} > 0$. It suffices to find a universal constant $c_0 > 0$ such that (2.1) holds for

$$F(\eta) := f(\eta(\{x_1\}), \cdots, \eta(\{x_n\})), \quad f \in C_0^\infty(\mathbb{R}^n).$$

Letting μ^n and μ_i be as in (4.2) for $A_i = \{x_i\}$, (2.1) for this F becomes

$$\mu^n(f^2 \log f^2) \le \frac{c_0}{1 \wedge \delta} \sum_{i=1}^n \int_{[0,\infty)^n} s_i (\partial_i f)^2 (s_1, \cdots, s_n) \mu^n(\mathrm{d}s_1, \cdots, \mathrm{d}s_n) + \mu^n(f^2) \log \mu^n(f^2).$$

By the additive property of the log-Sobolev inequality, this follows from the following Lemma 4.1. □

LNN **Lemma 4.1.** *For any $a, b > 0$, let $\mu_a(\mathrm{d}s) := 1_{[0,\infty)}(s) \frac{s^{a-1} \mathrm{e}^{-s}}{\Gamma(a)} \mathrm{d}s$ and $\mu_{a,b}(\mathrm{d}s) := 1_{[0,b]}(s) \frac{\mu_a(\mathrm{d}s)}{\mu_a([0,b])}$. Then there exists a constant $c_0 > 0$ such that for any $a, b > 0$,*

LSI' (4.3)
$$\mu_{a,b}(f^2 \log f^2) \le \frac{c_0}{a \wedge 1} \int_0^b s f'(s)^2 \mu_{a,b}(\mathrm{d}s), \quad f \in C^1([0,b]), \mu_{a,b}(f^2) = 1.$$

Proof. (a) Let $a \ge 2$. We will use the Bakry–Émery criterion on Riemannian manifolds with convex boundary which in particular includes $[0,b]$ for $b > 0$. More precisely, let $\mathscr{L}_a f(s) = s f''(s) + (a - s) f'(s)$ and $\Gamma_1(f,g)(s) = s f'(s) g'(s)$. By [25, Theorem 1.1(4)] with $\sigma = 0$ and $t \to \infty$, if

$$\Gamma_2(f,f) := \frac{1}{2} \mathscr{L}_a \Gamma_1(f,f) - \Gamma_1(\mathscr{L}_a f, f) \ge K \Gamma_1(f,f)$$

holds for some constant $K > 0$ and all $f \in C^2([0,b])$, then

$$\mu_{a,b}(f^2 \log f^2) \le \frac{2}{K} \int_0^b s f'(s)^2 \mu_{a,b}(\mathrm{d}s), \quad f \in C^1([0,b]), \mu_{a,b}(f^2) = 1.$$

So, the desired inequality (4.3) with $c_0 = 4$ follows since

CUV (4.4)
$$\begin{aligned}\Gamma_2(f,f)(s) &= s^2 f''(s)^2 + \frac{a+s}{2} f'(s)^2 + 2sf'(s)f''(s)\\ &\geq \frac{a+s-2}{2s}\Gamma_1(f,f)(s), \quad s \geq 0,\end{aligned} \tag{4.4}$$

so that $\Gamma_2(f,f) \geq \frac{1}{2}\Gamma_1(f,f)$ when $a \geq 2$.

(b) Let $a \in (0, \frac{1}{2}]$. By (4.4) we have $\Gamma_2(f,f)(s) \geq \frac{a\wedge 2}{4}\Gamma_1(f,f)(s)$ for $s \geq 2$. So, by the Bakry−Émery criterion,

LSI2 (4.5)
$$\mu_{a,b_1}(1_{[2,b_1]} f^2 \log f^2) \leq \frac{8}{a\wedge 2}\mu_{a,b_1}(1_{[2,b_1]}\Gamma_1(f,f)) + \mu_{a,b_1}(1_{[2,b_1]}f^2)\log \mu_{a,b_1}(1_{[2,b_1]}f^2) \tag{4.5}$$

holds for any $b_1 > 2$ and all $f \in C^1([0,b_1])$.

On the other hand, for any $b_2 > 0$ and $f \in C^1([0,b_2])$ with $\mu_{a,b_2}(f) = 0$, there exists $r_0 \in [0,b_2]$ such that $f(r_0) = 0$. So, for any $r \in [0,b_2]$ we have

$$\begin{aligned}|f(r)| &= \left|\int_{r_0}^r f'(s)\mathrm{d}s\right| \leq \left(\int_0^{b_2} sf'(s)^2\mu_{a,b_2}(\mathrm{d}s)\right)^{\frac{1}{2}}\left(\int_0^{b_2} s^{-a}\mathrm{e}^s\Gamma(a)\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leq \Big(\frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1-a}\mu_{a,b_2}(\Gamma_1(f,f))\Big)^{\frac{1}{2}}, \quad r \in [0,b_2].\end{aligned}$$

Therefore, for $\mu_{a,b_2}(f^2) = 1$ with $\mu_{a,b_2}(f) = 0$ we have

$$\begin{aligned}\mu_{a,b_2}(f^2\log f^2) &\leq \mu_{a,b_2}(f^2)\log\Big[\frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1-a}\mu_{a,b_2}(\Gamma_1(f,f))\Big]\\ &\leq \frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1-a}\mu_{a,b_2}(\Gamma_1(f,f)) - 1.\end{aligned}$$

This implies

AAA (4.6)
$$\begin{aligned}&\mu_{a,b_2}(f^2\log f^2) - \mu_{a,b_2}(f^2)\log\mu_{a,b_2}(f^2)\\ &\leq \frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1-a}\mu_{a,b_2}(\Gamma_1(f,f)) - \mu_{a,b_2}(f^2), \quad f \in C^1([0,b_2]),\ \mu_{a,b_2}(f) = 0.\end{aligned} \tag{4.6}$$

In general, for a non-zero function $f \in C^1([0,b_2])$, let $\tilde f = f - \mu_{a,b_2}(f)$. We have (see [2])

CCC (4.7)
$$\begin{aligned}&\mu_{a,b_2}(f^2\log f^2) - \mu_{a,b_2}(f^2)\log\mu_{a,b_2}(f^2)\\ &\leq \mu_{a,b_2}(\tilde f^2\log\tilde f^2) - \mu_{a,b_2}(\tilde f^2)\log\mu_{a,b_2}(\tilde f^2) + 2\mu_{a,b_2}(\tilde f^2).\end{aligned} \tag{4.7}$$

Combining this with (4.6) and using the Poincaré inequality (4.15) below, we arrive at

BBB (4.8)
$$\begin{aligned}&\mu_{a,b_2}(f^2\log f^2) - \mu_{a,b_2}(f^2)\log\mu_{a,b_2}(f^2)\\ &\leq \Big(\frac{\Gamma(a)b_2^{1-a}\mathrm{e}^{b_2}}{1-a} + 1\Big)\mu_{a,b_2}(\Gamma_1(f,f)), \quad b_2 > 0, f \in C_b^1([0,b_2]).\end{aligned} \tag{4.8}$$

In conclusion, when $b \le 4$, the desired inequality (4.3) for $a \in (0, \frac{1}{2}]$ follows from (4.8). Finally, for $b \ge 4$ we deduce from (4.5) and (4.8) that for any $f \in C^1([0,b])$ with $\mu_{a,b}(f^2) = 1$,

$$\begin{aligned}
\mu_{a,b}(f^2 \log f^2) &= \frac{\int_0^2 s^{a-1} \mathrm{e}^{-s} \mathrm{d}s}{\int_0^b s^{a-1} \mathrm{e}^{-s} \mathrm{d}s} \mu_{a,2}(f^2 \log f^2) + \mu_{a,b}(1_{[2,b]} f^2 \log f^2) \\
&\le \Big(\frac{\Gamma(a) 2^{1-a} \mathrm{e}^2}{1-a} + 1 \Big) \mu_{a,b}(1_{[0,2]} \Gamma_1(f,f)) + \mu_{a,b}(1_{[0,2]} f^2) \log \frac{\Gamma(a)}{\int_0^2 s^{a-1} \mathrm{e}^{-s} \mathrm{d}s} \\
&\quad + \frac{8}{a \wedge 2} \mu_{a,b}(1_{[2,b]} \Gamma_1(f,f)) + \mu_{a,b}(1_{[2,b]} f^2) \log \frac{\Gamma(a)}{\int_2^b s^{a-1} \mathrm{e}^{-s} \mathrm{d}s} \\
&\le \frac{c_1}{a} \int_0^b s f'(s)^2 \, \mu_{a,b}(\mathrm{d}s) + \frac{c_1}{a} \mu_{a,b}(f^2),
\end{aligned}$$

where $c_1 > 0$ is a universal constant independent of $a \in (0, \frac{1}{2}]$ and $b \ge 4$. Combining this with (4.7) and the Poincaré inequality (4.15) below, we prove the inequality (4.3) for some universal constant $c_0 > 0$ and all $a \in (0, \frac{1}{2}]$ and $b \ge 4$.

(c) Let $a \in (\frac{1}{2}, 2)$. In this case, we have $a' := \frac{a}{4} \in (0, \frac{1}{2}]$, so that by (b) there exists a constant $c_0 > 0$ such that

GGG
$$\mu_{a',b}(f^2 \log f^2) \le \frac{c_0}{a} \int_0^b s f'(s)^2 \mu_{a',b}(\mathrm{d}s), \quad a \in \Big(\frac{1}{2}, 2\Big), f \in C^1([0,b]), \mu_{a',b}(f^2) = 1. \tag{4.9}$$

Let $\bar{\mu}_{a',\infty}(\mathrm{d}s_1, \mathrm{d}s_2, \mathrm{d}s_3, \mathrm{d}s_4) = \prod_{i=1}^4 \mu_{a',\infty}(\mathrm{d}s_i)$, where $\mu_{a',\infty} := \lim_{b \to \infty} \mu_{a',b}$ is the Gamma distribution with parameter a'. By the property of Gamma distributions we have

$$\int_{[0,\infty)^n} f(s_1 + s_2 + s_3 + s_4) \bar{\mu}_{a',\infty}(\mathrm{d}s_1, \mathrm{d}s_2, \mathrm{d}s_3, \mathrm{d}s_4) = \int_{[0,\infty)} f(s) \mu_{a,\infty}(\mathrm{d}s), \quad f \in \mathscr{B}_b([0,\infty)).$$

Using (4.9) with $b \to \infty$ and the additivity property of the log-Sobolev inequality, we obtain

$$\begin{aligned}
&\bar{\mu}_{a',\infty}(F^2 \log F^2) - \bar{\mu}_{a',\infty}(F^2) \log \bar{\mu}_{a',\infty}(F^2) \\
&\le \frac{c_0}{a} \int_0^b \sum_{i=1}^4 s_i \partial_i F(s_1, \cdots, s_4)^2 \bar{\mu}_{a',b}(\mathrm{d}s_1, \cdots, \mathrm{d}s_4), \quad F \in C_b^1([0,\infty)^4).
\end{aligned}$$

By an approximation argument we may apply this inequality to

$$F(s_1, \cdots, s_4) := f(b \wedge (s_1 + \cdots + s_4))$$

for $f \in C^1([0,b])$, so that (4.3) is derived. □

To prove Theorem 2.2, we consider the local Poincaré inequality for $\mathscr{E}_{\mathbf{1},0}$ on the set $\mathbf{B}_N$, by decomposing η into the radial part $\eta(E)$ and the simplicial part $\bar{\eta} := \frac{\eta}{\eta(E)}$. It is well known that under $\mathscr{G}$ these two parts are independent with

OPQ
$$\mathscr{G}(\eta(E) < r, \bar{\eta} \in \mathbf{A}) = \mathbf{Dir}(\mathbf{A}) \gamma_{\nu(E)}([0,r)), \quad r > 0, \mathbf{A} \in \mathscr{B}(\mathbb{M}_1), \tag{4.10}$$

where $\gamma_{\nu(E)}(\mathrm{d}s) := 1_{[0,\infty)}(s)\frac{s^{\nu(E)-1}\mathrm{e}^{-s}}{\Gamma(\nu(E))}\,\mathrm{d}s$, and **Dir** is the Dirichlet measure with intensity measure ν, see for instance [17] for details. According to [16] (see also [17, Proposition 3.3]), we have the Poincaré inequality

PDI
$$\mathbf{Dir}(F^2) \le \mathbf{Dir}(\Gamma^D(F,F)) + \mathbf{Dir}(F)^2, \quad F \in \mathscr{F}C_0^\infty, \tag{4.11}$$

where for $F(\eta) = f(\eta(A_1), \cdots, \eta(A_n))$ and $\eta \in \mathbb{M}_1$,

GGD
$$\Gamma^D(F,F)(\eta) := \sum_{i,j=1}^n \big[\delta_{ij}\eta(A_i) - \eta(A_i)\eta(A_j)\big] \cdot [(\partial_i f)(\partial_j f)](\eta(A_1), \cdots, \eta(A_n)). \tag{4.12}$$

CN **Lemma 4.2.** *For any $N > 0$,*

DIR2
$$\mathscr{G}(1_{\mathbf{B}_N} F^2) \le \Big(\frac{N^2}{4\nu(E)} \vee 1\Big)\mathscr{G}(1_{\mathbf{B}_N}\Gamma_{\mathbf{1}}(F,F)), \quad F \in \mathscr{F}C_0^\infty, \mathscr{G}(1_{\mathbf{B}_N}F) = 0. \tag{4.13}$$

Proof. Since $\mathbf{B}_N = \{\eta(E) \le N^2/4\}$, (4.10) implies

TT0
$$\int_{\mathbb{M}} \big[1_{\mathbf{B}_N}F\big](\eta)\mathscr{G}(\mathrm{d}\eta) = \int_{\mathbb{M}_1 \times [0,N^2/4]} F(s\bar\eta)\mathbf{Dir}(\mathrm{d}\bar\eta)\gamma_{\nu(E)}(\mathrm{d}s), \quad F \in L^1(1_{\mathbf{B}_N}\mathscr{G}). \tag{4.14}$$

We observe that (2.1) implies

TT1
$$\gamma_{\nu(E)}(1_{[0,r]}f^2) \le \int_0^r s f'(s)^2 \gamma_{\nu(E)}(\mathrm{d}s), \quad r > 0, f \in C^1([0,r]), \gamma_{\nu(E)}(1_{[0,r]}f) = 0. \tag{4.15}$$

Indeed, applying the Poincaré inequality

$$\mathscr{G}(F^2) \le \mathscr{E}_{\mathbf{1},0}(F,F) + \mathscr{G}(F)^2$$

to $F(\eta) := f(\eta(E) \wedge r)$, and noting that for $\tilde f(s) := f(s \wedge r)$ we have

$$\begin{aligned} &\mathscr{G}(F^i) = \gamma_{\nu(E)}(\tilde f^i) = \gamma_{\nu(E)}(1_{[0,r]}f^i) + \gamma_{\nu(E)}((r,\infty))f(r), \ i = 1,2, \\ &\mathscr{E}_{\mathbf{1},0}(F,F) = \int_0^\infty s\tilde f'(s)^2 \mathrm{d}s = \int_0^N s f'(s)^2 \mathrm{d}s, \end{aligned}$$

it follows that

$$\begin{aligned} &\gamma_{\nu(E)}(1_{[0,r]}f^2) = \gamma(\tilde f^2) - \gamma_{\nu(E)}((r,\infty))f(r)^2 \\ &\le \int_0^N s f'(s)^2 \mathrm{d}s + \gamma_{\nu(E)}((r,\infty))^2 f(r)^2 - \gamma_{\nu(E)}((r,\infty))f(r)^2 \le \int_0^N s f'(s)^2 \mathrm{d}s. \end{aligned}$$

By the additivity property of the Poincaré inequality, (4.11), (4.14) and (4.15), we obtain that for any $F \in \mathscr{F}C_0^\infty$ with $\mathscr{G}(1_{\mathbf{B}_N}F) = 0$,

$$\mathscr{G}(1_{\mathbf{B}_N}F^2) \le \int_{\mathbb{M}_1 \times [0,N^2/4]} \Big[\frac{1}{\nu(E)}\Gamma^D(F(s\cdot), F(s\cdot))(\bar\eta) + s\Big|\frac{\partial}{\partial s}F(s\bar\eta)\Big|^2\Big]\mathbf{Dir}(\mathrm{d}\bar\eta)\gamma_{\nu(E)}(\mathrm{d}s)$$

$$= \int_{\mathbf{B}_N} \Big[\frac{1}{\nu(E)}\Gamma^D(F(\eta(E)\cdot), F(\eta(E)\cdot))(\bar\eta) + \eta(E)\Big|\frac{\partial}{\partial \eta(E)}F(\eta(E)\bar\eta)\Big|^2\Big]\mathscr{G}(\mathrm{d}\eta).$$

So, it remains to prove

TPP (4.16)
$$\begin{aligned} I(\eta) &:= \frac{1}{\nu(E)}\Gamma^D(F(\eta(E)\cdot), F(\eta(E)\cdot))(\bar\eta) + \eta(E)\Big|\frac{\partial}{\partial \eta(E)}F(\eta(E)\bar\eta)\Big|^2 \\ &\le \Big(\frac{N^2}{4\nu(E)} \vee 1\Big)\Gamma_{\mathbf{1}}(F,F)(\eta), \quad \eta(E) \le \frac{N^2}{4}. \end{aligned}$$

For $F \in \mathscr{F}C_0^\infty$ with $F(\eta) = f\big(\eta(A_1), \cdots, f(A_n)\big) = f\big(\eta(E)\bar\eta(A_1), \cdots, \eta(E)\bar\eta(A_n)\big)$, by (4.12) we have

$$\begin{aligned} &\Gamma^D\big(F(\eta(E)\cdot), F(\eta(E)\cdot)\big)(\bar\eta) \\ &= \sum_{i,j=1}^n \big[\delta_{ij}\bar\eta(A_i) - \bar\eta(A_i)\bar\eta(A_j)\big]\eta(E)^2[(\partial_i f)(\partial_j f)](\eta(A_1), \cdots, \eta(A_n)) \\ &= \sum_{i,j=1}^n \big[\delta_{ij}\eta(A_i)\eta(E) - \eta(A_i)\eta(A_j))[(\partial_i f)(\partial_j f)](\eta(A_1), \cdots, \eta(A_n)). \end{aligned}$$

Moreover,

$$\begin{aligned} \eta(E)\Big|\frac{\partial}{\partial \eta(E)}F(\eta(E)\bar\eta)\Big|^2 &= \eta(E)\Big|\sum_{i=1}^n \bar\eta(A_i)(\partial_i f)(\eta(A_1), \cdots, \eta(A_n))\Big|^2 \\ &= \frac{1}{\eta(E)}\sum_{i,j=1}^n \eta(A_i)\eta(A_j)[(\partial_i f)(\partial_j f)](\eta(A_1), \cdots, \eta(A_n)). \end{aligned}$$

So, when $\eta(E) \le \frac{N^2}{4}$ (i.e. $\rho(E) \le N$),

$$\begin{aligned} I(\eta) &\le \frac{\eta(E)\vee\nu(E)}{\nu(E)}\Big(\frac{1}{\eta(E)}\Gamma^D(F(\eta(E)\cdot), F(\eta(E)\cdot))(\bar\eta) + \eta(E)\Big|\frac{\partial}{\partial \eta(E)}F(\eta(E)\bar\eta)\Big|^2\Big) \\ &= \Big(1 \vee \frac{N^2}{4\nu(E)}\Big)\Gamma_{\mathbf{1}}(F,F)(\eta). \end{aligned}$$

This implies (4.16), and hence finishes the proof. □

4.2 Proofs of Theorem 2.2 and Corollary 2.3

Proof of Theorem 2.2. We will make a standard split argument by using the local Poincaré inequality (4.13) and the principal eigenvalue of $\mathscr{L}_{\mathscr{A},V}$ outside $\mathbf{B}_N$. To estimate the principal eigenvalue, we recall Hardy's criterion for the first mixed eigenvalue. Consider the following differential operator on $[0,\infty)$:

$$\mathscr{L}f(r) = \underline{a}(r)f''(r) + \gamma(r)f'(r), \quad r \ge 0.$$

For any $k > 0$ and $n \geq 1$, let $\lambda_{k,n}$ be the first mixed eigenvalue of $\mathscr{L}$ on $[k, k+n]$ with Dirichlet boundary condition at k and Neumann boundary condition at $k+n$. Define

$$\sigma_{k,n} = \sup_{t\in(k,n+k)} \int_t^{n+k} \mathrm{e}^{\int_k^r \frac{\gamma(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r \int_k^t \frac{1}{\underline{a}(r)} \mathrm{e}^{-\int_k^r \frac{\gamma(s)}{\underline{a}(s)}\mathrm{d}s}\mathrm{d}r.$$

By Hardy's criterion, see for instance [24, Theorem 1.4.2], we have

*A1
$$\frac{1}{\sigma_{k,n}} \geq \lambda_{k,k+n} \geq \frac{1}{4\sigma_{k,n}}, \quad n \geq 1, k > 0. \tag{4.17}$$

Below we prove assertions (1)-(3) respectively.

(1) By (4.13) and a standard perturbation argument, we have

*A5
$$\mathscr{G}^V(1_{\{\rho\leq N\}}F^2) \leq \mathscr{G}^V(1_{\{\rho\leq N\}}F)^2 + \Phi(N)\mathscr{G}^V(1_{\{\rho\leq N\}}\Gamma_{\mathscr{A}}(F,F)), \quad F \in \mathscr{F}C_0^2. \tag{4.18}$$

If $\sigma_k < \infty$ for some $k > 0$, it suffices to prove the Poincaré inequality

*A6
$$\mathscr{G}^V(F^2) \leq C\mathscr{G}^V(\Gamma_{\mathscr{A}}(F,F)), \quad F \in \mathscr{F}C_0^2, \mathscr{G}^V(F) = 0 \tag{4.19}$$

for

$$C = 2\Phi\big(\psi^{-1}\Big(\psi(k) + 8\lambda_k^{-1} + 1\big) + \lambda_k\Big) + 8\lambda_k^{-1},$$

where according to (4.17),

PPK0
$$\lambda_k := \lim_{n\to\infty} \lambda_{k,n} \geq \frac{1}{4\sigma_k}. \tag{4.20}$$

Let $F \in \mathscr{F}C_0^2$ such that $\operatorname{supp} F \subset \mathbf{B}_{N_1}$ for some constant $N_1 > k$. For any $N \geq k$, let

$$F_N = F[(\psi(\rho) - \psi(N))^+ \wedge 1].$$

Then $F_N = 0$ for $\rho \leq N$ and $F_N = F$ for $\psi(\rho) \geq \psi(N) + 1$. For $n > N_1$, let $u_n \geq 0$ be the first mixed eigenfunction of $\mathscr{L}$ on $[k, k+n]$ with Dirichlet boundary condition at k and Neumann boundary condition at $k+n$, such that

$$u_n(k) = u_n'(k+n) = 0, \quad u_n'(r) > 0 \text{ for } r \in (k, k+n), \quad \mathscr{L}u_n = -\lambda_{k,n}u_n \leq 0.$$

Combining this with the definition of $\mathscr{L}$ we obtain

$$\mathscr{L}_{\mathscr{A},V}(u_n \circ \rho) \geq (\mathscr{L}u_n) \circ \rho, \quad \rho \in [k, k+n].$$

So,

*CD1
$$\begin{aligned}\lambda_{k,n}\mathscr{G}^V(F_N^2) &= -\int_{\{k<\rho<n+k\}} \frac{F_N^2}{u_n^2 \circ \rho}(-\mathscr{L}u_n) \circ \rho \, \mathrm{d}\mathscr{G}^V \\ &\leq -\int_{\{k<\rho<n+k\}} \frac{F_N^2}{u_n \circ \rho}\big[-\mathscr{L}_{\mathscr{A},V}(u_n \circ \rho)\big]\mathrm{d}\mathscr{G}^V.\end{aligned} \tag{4.21}$$

To apply the integration by parts formula, we approximate u_n as follows. Since $u_n(k) = u_n'(k+n) = 0$, we may construct a sequence $\{u_{n,m}\}_{m\geq 1} \subset C^\infty([0,\infty))$ such that

$$\begin{aligned}
&u_{n,m}(r) = u_n(r) \text{ for } r \in [k+m^{-1}, k+n-m^{-1}],\\
&u_{n,m}(r) = 0 \text{ for } r \leq k, \quad u_{n,m}'(r) = 0 \text{ for } \ r \geq k+n,\\
&\sup_{m\geq 1}\sup_{r\geq k}\big(|u_{n,m}'(r)| + |u_{n,m}''(r)|\big) < \infty.
\end{aligned}$$

Since $F_N = 0$ for $\rho \leq N$, (4.21) implies that for any $k < N$,

*CD2 (4.22)

$$\begin{aligned}
\lambda_{k,n}\mathscr{G}^V(F_N^2) &= -\lim_{m\to\infty}\int_{\{k<\rho<n+k\}} \frac{F_N^2}{u_{n,m}^2\circ\rho}(-\mathscr{L}u_{n,m})\circ\rho\,\mathrm{d}\mathscr{G}^V\\
&= \lim_{m\to\infty}\int_{\mathbb{M}}\Big\langle \mathscr{A}_\eta\nabla^{ext}\frac{F_N^2}{u_{n,m}\circ\rho}(\eta), \nabla^{ext}(u_{n,m}\circ\rho)(\eta)\Big\rangle_{L^2(\eta)}\mathrm{d}\mathscr{G}^V.
\end{aligned} \tag{4.22}$$

On the other hand, since $\mathscr{A}_\eta$ is positive definite due to **(H)**, for any $u \in C^2([0,\infty))$ with $u(r) > 0$ for $r \geq N$, we have

$$\begin{aligned}
&\Big\langle \mathscr{A}_\eta\nabla^{ext}\frac{F_N^2}{u\circ\rho}(\eta), \nabla^{ext}(u\circ\rho)(\eta)\Big\rangle_{L^2(\eta)}\\
&= \big\langle \mathscr{A}_\eta\nabla^{ext}F_N(\eta), \nabla^{ext}F_N(\eta)\big\rangle_{L^2(\eta)}\\
&\quad - \Big\langle \mathscr{A}_\eta\Big[\nabla^{ext}F_N - \frac{F_N}{u\circ\rho}\nabla^{ext}(u\circ\rho)\Big](\eta), \nabla^{ext}F_N(\eta) - \frac{F_N}{u\circ\rho}\nabla^{ext}(u\circ\rho)(\eta)\Big\rangle_{L^2(\eta)}\\
&\leq \big\langle \mathscr{A}_\eta\nabla^{ext}F_N(\eta), \nabla^{ext}F_N(\eta)\big\rangle_{L^2(\eta)}.
\end{aligned}$$

Combining this with (4.22) and the definition of F_N, we obtain

$$\begin{aligned}
\lambda_{k,n}\mathscr{G}^V(F_N^2) &\leq \int_{\mathbb{M}}\langle \mathscr{A}_\eta\nabla^{ext}F_N, \nabla^{ext}F_N\rangle_{L^2(\eta)}\mathrm{d}\mathscr{G}^V\\
&\leq 2\mathscr{E}_{\mathscr{A},V}(F,F) + 2\int_{\{\psi(N)<\psi(\rho)<\psi(N)+1\}} F^2\Gamma_{\mathscr{A}}(\psi(\rho),\psi(\rho))\mathrm{d}\mathscr{G}^V.
\end{aligned}$$

Multiplying by $\lambda_{k,n}^{-1}$ and letting $n \to \infty$ leads to

IMP (4.23)

$$\int_{\mathbb{M}} F_N^2\mathrm{d}\mathscr{G}^V \leq \frac{2}{\lambda_k}\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{2}{\lambda_k}\int_{\{\psi(N)<\psi(\rho)<\psi(N)+1\}} F^2\Gamma_{\mathscr{A}}(\psi(\rho),\psi(\rho))\mathrm{d}\mathscr{G}^V. \tag{4.23}$$

By the definition of ψ and $\bar{a}$, and noting that $\Gamma_{\mathbf{1}}(\rho,\rho) = 1$, we have

GMM (4.24)

$$\Gamma_{\mathscr{A}}(\psi(\rho),\psi(\rho))(\eta) = \frac{\langle \mathscr{A}_\eta\nabla^{ext}\rho(\eta), \nabla^{ext}\rho(\eta)\rangle_{L^2(\eta)}}{\bar{a}(\rho(\eta))} \leq \Gamma_{\mathbf{1}}(\rho,\rho)(\eta) = 1. \tag{4.24}$$

So, (4.23) implies

IMP' (4.25)

$$\int_{\mathbb{M}} F_N^2\mathrm{d}\mathscr{G}^V \leq \frac{2}{\lambda_k}\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{2}{\lambda_k}\int_{\{\psi(N)<\psi(\rho)<\psi(N)+1\}} F^2\mathrm{d}\mathscr{G}^V. \tag{4.25}$$

Letting $\lfloor s\rfloor = \sup\{k\in\mathbb{Z}:\ k\le s\}$ be the integer part of a real number s, we have

$$\int_{\mathbb{M}} F^2 \mathrm{d}\mathscr{G}^V \ge \sum_{i=1}^{1+\lfloor 8\lambda_k^{-1}\rfloor} \int_{\{\psi(k)+i-1<\psi(\rho)<\psi(k)+i]} F^2\mathrm{d}\mathscr{G}^V.$$

Then there exists $N\in\big[k,\psi^{-1}\big(\psi(k)+8\lambda_k^{-1}\big)\big]$ such that

$$\int_{\{\psi(N)<\psi(\rho)<\psi(N)+1\}} F^2\mathrm{d}\mathscr{G}^V \le \frac{\lambda_k}{8}\int_{\mathbb{M}} F^2\mathrm{d}\mathscr{G}^V,$$

so that (4.25) yields

PPK (4.26) $$\int_{\mathbb{M}} F_N^2\mathrm{d}\mathscr{G}^V \le \frac{2}{\lambda_k}\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{1}{4}\int_{\mathbb{M}} F^2\mathrm{d}\mathscr{G}^V.$$

Combining this with (4.18) and noting that $\mathscr{G}^V(F)=0$, we may find $N\in[k,\psi^{-1}(\psi(k)+8\lambda_k^{-1})]$ such that

$$\begin{aligned}
&\int_{\mathbb{M}} F^2\mathrm{d}\mathscr{G}^V \le \int_{\psi(\rho)\le\psi(N)+1} F^2\mathrm{d}\mathscr{G}^V + \int_{\mathbb{M}} F_N^2\mathrm{d}\mathscr{G}^V\\
&\le \Phi(\psi^{-1}(\psi(N)+1))\mathscr{E}_{\mathscr{A},V}(F,F) + \mathscr{G}^V(1_{\{\psi(\rho)\ge\psi(N)+1\}}F)^2 + \int_{\mathbb{M}} F_N^2\mathrm{d}\mathscr{G}^V\\
&\le \Phi(\psi^{-1}(\psi(N)+1))\mathscr{E}_{\mathscr{A},V}(F,F) + 2\int_{\mathbb{M}} F_N^2\mathrm{d}\mathscr{G}^V\\
&\le \Big(\Phi(\psi^{-1}(\psi(N)+1)) + \frac{4}{\lambda_k}\Big)\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{1}{2}\int_{\mathbb{M}} F^2\mathrm{d}\mathscr{G}^V.
\end{aligned}$$

Since $\Phi(N)$ is increasing in $N\in\big[k,\psi^{-1}\big(\psi(k)+8\lambda_k^{-1}\big)\big]$, this implies (4.19) with

$$C = 2\Phi(\psi^{-1}(\psi(N)+1)) + \frac{8}{\lambda_k} \le 2\Phi\Big(\psi^{-1}\Big(\psi(k)+8\lambda_k^{-1}+1\Big)+\lambda_k\Big) + 8\lambda_k^{-1}.$$

Then the proof is finished by (4.20).

(2) Assume that $\operatorname{supp}\nu$ is an infinite set. To disprove the super Poincaré inequality, it suffices to construct a sequence $\{F_n\}\subset\mathscr{D}(\mathscr{E}_{\mathscr{A},V})$ such that $\mathscr{G}^V(F_n^2)>0$ and

AAO (4.27) $$C := \sup_{n\ge1}\frac{\mathscr{E}_{\mathscr{A},V}(F_n,F_n)}{\mathscr{G}^V(F_n^2)} < \infty,\quad \lim_{n\to\infty}\frac{\mathscr{G}^V(|F_n|)^2}{\mathscr{G}^V(F_n^2)} = 0.$$

Indeed, if (1.11) holds for some $\beta:(0,\infty)\to(0,\infty)$, then

$$1\le r\frac{\mathscr{E}_{\mathscr{A},V}(F,F_n)}{\mathscr{G}^V(F_n^2)} + \beta(r)\frac{\mathscr{G}^V(|F_n|)^2}{\mathscr{G}^V(F_n^2)},\quad n\ge1, r>0.$$

Combining this with (4.27) and letting $n\to\infty$, we obtain $1\le rC$ for all $r>0$ which is impossible.

We now show that (4.27) holds for $F_n(\eta) := (1-\eta(E))^+ \frac{\eta(A_n)}{\eta(E)}$, where $\{A_n\}_{n\ge 1}$ are measurable subsets of E such that $\frac{1}{2}\nu(E) > p_n := \nu(A_n) \downarrow 0$ as $n \uparrow \infty$, which exist since supp ν is an infinite set.

Obviously, $\{F_n\}_{n\ge1} \subset \mathscr{D}(\mathscr{E}_{\mathscr{A},V})$. Since $\|\mathscr{A}_\eta\|_{L^2(\eta)} + \mathrm{e}^{V(\eta)} + \mathrm{e}^{-V(\eta)}$ is bounded on the set $\{\eta(E) \le 1\}$, we may find constants $K_i, C_i > 0, i = 1,2,3$ such that (4.10) implies for all $n \ge 1$ that

$$\begin{aligned}
&\mathscr{G}^V(F_n^2) \ge K_1 \mathscr{G}(F_n^2) = K_1 \int_0^1 \frac{(1-s)^2 s^{\nu(E)-1}\mathrm{e}^{-s}}{\Gamma(\nu(E))}\mathrm{d}s \int_0^1 \frac{t^{p_n+1}(1-t)^{\nu(E)-p_n-1}}{\Gamma(p_n)\Gamma(\nu(E)-p_n)}\mathrm{d}t \ge C_1 p_n,\\
&\mathscr{G}^V(|F_n|)^2 \le K_2 \mathscr{G}(|F_n|)^2 = K_2\bigg(\int_0^1 \frac{(1-s)s^{\nu(E)-1}\mathrm{e}^{-s}}{\Gamma(\nu(E))}\mathrm{d}s \int_0^1 \frac{t^{p_n}(1-t)^{\nu(E)-p_n-1}}{\Gamma(p_n)\Gamma(\nu(E)-p_n)}\mathrm{d}t\bigg)^2 \le C_2 p_n^2,\\
&\mathscr{E}_{\mathscr{A},V}(F_n,F_n) \le K_3 \mathscr{G}(\|\nabla^{ext} F_n\|^2_{L^2(\eta)}) = K_3 \int_{\{\eta(E)\le 1\}} \mathscr{G}(\mathrm{d}\eta) \int_E |(1-\eta(E)) - \eta(A_n)|^2 \mathrm{d}\eta\\
&\quad \le 2K_3 \int_{\{\eta(E)\le 1\}} \big[(1-\eta(E))^2\eta(A_n) + \eta(A_n)^2\eta(E)\big]\mathscr{G}(\mathrm{d}\eta)\\
&\quad \le 4K_3 \int_{\{\eta(E)\le 1\}} \frac{\eta(A_n)}{\eta(E)}\mathscr{G}(\mathrm{d}\eta) \le C_3 p_n.
\end{aligned}$$

Since $p_n \downarrow 0$ as $n \uparrow \infty$, we prove (4.27).

(3) The local Poincaré inequality (4.13) implies that for any $F \in \mathscr{F}C_0^\infty$ with $\mathscr{G}^V(F) = 0$,

$$\begin{aligned}
\mathscr{G}^V(F^2) &= \mathscr{G}^V(1_{\mathbf{B}_N}F^2) + \mathscr{G}^V(F^2 \cdot 1_{\mathbf{B}_N^c})\\
&\le 2\Phi(N)\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{1}{\mathscr{G}^V(\mathbf{B}_N)}\mathscr{G}^V(1_{\mathbf{B}_N^c}F)^2 + \mathscr{G}^V(\rho > N)\|F\|_\infty^2\\
&\le 2\Phi(N)\mathscr{E}_{\mathscr{A},V}(F,F) + \Big(\frac{\mathscr{G}^V(\rho>N)^2}{1-\mathscr{G}^V(\rho>N)} + \mathscr{G}^V(\rho > N)\Big)\|F\|_\infty^2, \quad N > 0.
\end{aligned}$$

So, for any $r > 0$, taking $N > 0$ such that $\frac{\mathscr{G}^V(\rho>N)^2}{1-\mathscr{G}^V(\rho>N)} + \mathscr{G}^V(\rho > N) \le r$, i.e. $\mathscr{G}^V(\rho > N) \le \frac{r}{1+r}$, we prove (1.10).

□

Proof of Corollary 2.3. Let $r_0 \in (0,\infty)$ such that $\mathscr{A}_\eta = a(\rho(\eta))\mathbf{1}$ and $V(\eta) = v(\rho(\eta))$ for large $\rho(\eta) \ge r_0$. By (2.2) we have

$$\begin{aligned}
&(\nu-\eta)(\mathscr{A}_\eta 1) = a(\rho(\eta))\big(\nu(E)-\eta(E)\big) = a(\rho(\eta))\Big(\nu(E) - \frac{\rho(\eta)^2}{4}\Big),\\
&\eta\big(\nabla^{ext}[\mathscr{A}_\eta 1(\cdot)](\cdot)\big) = a'(\rho(\eta))\eta\Big(\frac{1}{\sqrt{\eta(E)}}\Big) = \frac{\rho(\eta)}{2}a'(\rho(\eta)),\\
&\nabla^{ext}_{\mathscr{A}_\eta 1}V(\eta) := \eta\big([\mathscr{A}_\eta 1]\nabla^{ext}V(\eta)\big) = (av')(\rho(\eta))\frac{2\eta(E)}{\rho(\eta)} = \frac{\rho(\eta)}{2}(av')(\rho(\eta)),\\
&\frac{4}{\rho(\eta)^3}\eta(\mathscr{A}_\eta 1) = \frac{4a(\rho(\eta))\eta(E)}{\rho(\eta)^3} = \frac{a(\rho(\eta))}{\rho(\eta)}, \quad \rho(\eta) \ge r_0.
\end{aligned}$$

Then (2.4) implies

PKKN (4.28)
$$\mathscr{L}_{\mathscr{A}_\eta,V}\rho(\eta)=\xi(\rho(\eta)) \tag{4.28}$$

for the given function ξ. So, when $\sigma_k<\infty$ for some $k>0$, Theorem 2.2(1) implies $\text{gap}(\mathscr{L}_{\mathscr{A},V})>0$.

On the other hand, let $\sigma_k=\infty$ for all $k>0$. We have

** (4.29)
$$\lambda_k:=\lim_{n\to\infty}\lambda_{k,n}=0,\quad k>0, \tag{4.29}$$

where $\lambda_{k,n}$ is given in the proof of Theorem 2.2. Let $u_{k,n}$ be the corresponding first mixed eigenfunction of $\mathscr{L}$ on $[k,k+n]$ with $u_{k,n}(r)>0$ in $(k,k+n]$, and let

$$\Theta_v(\mathrm{d}s)=\frac{\mathrm{e}^{v(s)-s}s^{\nu(E)-1}}{\Gamma(\nu(E))}\,\mathrm{d}s,$$

such that $\mathscr{L}$ is symmetric in $L^2([k,k+n],\Theta_v)$ under the mixed boundary conditions. Then

$$\int_k^{k+n}u_{k,n}(r)^2\Theta_v(\mathrm{d}r)=\frac{1}{\lambda_{k,n}}\int_k^{k+n}r\underline{a}(r)|u_{k,n}'(r)|^2\Theta_v(\mathrm{d}r).$$

Letting $F_{k,n}(\eta)=u_{k,n}((\eta(E)\vee k)\wedge(k+n))$, for large enough $k>0$ such that $\mathscr{A}_\eta=a(\rho(\eta))\mathbf{1}$ and $V(\eta)=v(\rho(\eta))$ for $\eta(E)\ge k$, the above formula implies

$$\mathscr{G}^V(F_{k,n}^2)-\mathscr{G}^V(F_{k,n})^2\ge\mathscr{G}^V(F_{k,n}^2\cdot 1_{\{k\le\rho\le k+n\}})=\frac{1}{\lambda_{k,n}}\mathscr{E}_{\mathscr{A},V}(F_{n,k},F_{n,k}),\quad n\ge1.$$

Obviously, due to (4.29) this implies $\text{gap}(\mathscr{L}_{\mathscr{A},V})=0$.

□

4.3 Proof of Theorem 2.5

Let $h\in\mathscr{H}$, i.e. $h\in C^1([0,\infty))$ with $h(r),h'(r)>0$ for $r>0$ such that (2.8) and (2.9) hold. By (2.10) and noting that $V_h=V-h(\rho)+c(h)$ where $c(h)$ is a constant such that $\mathscr{G}^{V_h}$ is a probability measure, for any $F\in\mathscr{F}C_0^\infty$ we have

$$\begin{aligned}&\mathscr{G}^V(F^2\cdot 1_{\mathbf{B}_N})-\frac{\mathscr{G}^V(F\cdot 1_{\mathbf{B}_N})^2}{\mathscr{G}^V(\mathbf{B}_N)}=\inf_{c\in\mathbb{R},|c|\le\|F\|_\infty}\mathscr{G}^V(|F-c|^2\cdot 1_{\mathbf{B}_N})\\
&\le\mathrm{e}^{h(N)-c(h)}\inf_{c\in\mathbb{R},|c|\le\|F\|_\infty}\mathscr{G}^{V_h}(|F-c|^2\cdot 1_{\mathbf{B}_N})\le\mathrm{e}^{h(N)-c(h)}\inf_{c\in\mathbb{R},|c|\le\|F\|_\infty}\mathscr{G}^{V_h}(|F-c|^2)\\
&=\mathrm{e}^{h(N)-c(h)}\big[\mathscr{G}^{V_h}(F^2)-\mathscr{G}^{V_h}(F)^2\big]\le C(h)\mathrm{e}^{h(N)-c(h)}\mathscr{G}^{V_h}(\Gamma_{\mathscr{A}}(F,F))\\
&\le C(h)\mathrm{e}^{h(N)}\mathscr{G}^V(\Gamma_{\mathscr{A}}(F,F))=C(h)\mathrm{e}^{h(N)}\mathscr{E}_{\mathscr{A},V}(F,F).\end{aligned}$$

This implies

$$\mathscr{G}^V(F^2\cdot 1_{\mathbf{B}_N})\le C(h)\mathrm{e}^{h(N)}\mathscr{E}_{\mathscr{A},V}(F,F)+\frac{\mathscr{G}^V(F\cdot 1_{\mathbf{B}_N})^2}{\mathscr{G}^V(\mathbf{B}_N)},\quad F\in\mathscr{D}(\mathscr{E}_{\mathscr{A},V}).$$

Then for any $F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V})$ with $\mathscr{G}^V(F) = 0$, we have $\mathscr{G}^V(F \cdot 1_{\mathbf{B}_N})^2 = \mathscr{G}^V(F \cdot 1_{\{\rho > N\}})^2$ and

$$\begin{aligned}
\mathscr{G}^V(F^2) &\leq \mathscr{G}^V(F^2 \cdot 1_{\mathbf{B}_N}) + \mathscr{G}^V(F^2 \cdot 1_{\{\rho > N\}}) \\
&\leq C(h)\mathrm{e}^{h(N)}\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{\mathscr{G}^V(F \cdot 1_{\mathbf{B}_N})^2}{\mathscr{G}^V(\mathbf{B}_N)} + \mathscr{G}^V(F^2 \cdot 1_{\{\rho > N\}}) \\
&\leq C(h)\mathrm{e}^{h(N)}\mathscr{E}_{\mathscr{A},V}(F,F) + \Big(\frac{\mathscr{G}^V(\rho > N)^2}{\mathscr{G}^V(\mathbf{B}_N)} + \mathscr{G}^V(\rho > N)\Big)\|F\|_\infty^2 \\
&\leq C(h)\mathrm{e}^{h(N)}\mathscr{E}_{\mathscr{A},V}(F,F) + \frac{\mathscr{G}^V(\rho > N)}{\mathscr{G}^V(\mathbf{B}_N)}\|F\|_\infty^2.
\end{aligned}$$

So, for any $r > 0$ and $N > 0$ such that $\frac{\mathscr{G}^V(\rho > N)}{\mathscr{G}^V(\mathbf{B}_N)} \leq r$, equivalently $\mathscr{G}^V(\rho > N) \leq \frac{r}{1+r}$, we have

$$\mathscr{G}^V(F^2) \leq C(h)\mathrm{e}^{h(N)}\mathscr{E}_{\mathscr{A},V}(F,F) + r\|F\|_\infty^2, \quad F \in \mathscr{D}(\mathscr{E}_{\mathscr{A},V}), \mathscr{G}^V(F) = 0, h \in \mathscr{H}.$$

Therefore, the weak Poincaré inequality (1.10) holds for

$$\alpha(r) := \inf\Big\{C(h)\mathrm{e}^{h(N)} : \ h \in \mathscr{H}, \mathscr{G}^V(\rho > N) \leq \frac{r}{1+r}\Big\}, \quad r > 0.$$

5 Extensions to the space of finite signed measures

Consider the space of finite signed measures

$$\mathbb{M}_{\mathbf{s}} := \{\eta - \eta' : \ \eta, \eta' \in \mathbb{M}\}$$

equipped with the topology induced by the map

$$\eta \mapsto (\eta^+, \eta^-) \in \mathbb{M} \times \mathbb{M},$$

where η^+ and η^- are the positive and negative parts of η in the Hahn decomposition respectively, and $\mathbb{M} \times \mathbb{M}$ is equipped with the weak topology. So, under this topology $\mathbb{M}_{\mathbf{s}}$ is a Polish space. Note that this topology maybe different from the weak topology, i.e. $\eta_n \to \eta$ if $\eta_n(f) := \int_E f \mathrm{d}\eta_n \to \eta(f)$ holds for any $f \in C_b(E)$, since the latter on $\mathbb{M}_{\mathbf{s}}$ might be not metrizable, see [19].

To extend the Dirichlet form $(\mathscr{E}_{\mathscr{A},V}, \mathscr{D}(\mathscr{E}_{\mathscr{A},V}))$ from $L^2(\mathscr{G}^V)$ to $L^2(\mathscr{G}_{\mathbf{s}}^V)$ for a probability measure $\mathscr{G}_{\mathbf{s}}^V$ with a potential V on $\mathbb{M}_{\mathbf{s}}$, we introduce below the measure $\mathscr{G}_{\mathbf{s}}^V$, the extrinsic derivative and the operator $\mathscr{A}$ respectively.

In [18], an analogue to the Lebesgue measure was introduced on $\mathbb{M}_{\mathbf{s}}$ by using the convolution of two weighted Gamma distributions. In the same spirit, we extend the measure $\mathscr{G}$ to $\mathscr{G}_{\mathbf{s}}$ on $\mathbb{M}_{\mathbf{s}}$ as follows:

MC' (5.1)
$$\int_{\mathbb{M}_{\mathbf{s}}} f(\eta)\mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta) = \int_{\mathbb{M}\times\mathbb{M}} f(\eta^+ - \eta^-)\mathscr{G}(\mathrm{d}\eta^+)\mathscr{G}(\mathrm{d}\eta^-), \quad f \in \mathscr{B}_b(\mathbb{M}_{\mathbf{s}}).$$

Let $\tau(\eta) = \{x \in E : \eta(\{x\}) \neq 0\}$. To ensure that $\tau(\eta^+)$ and $\tau(\eta^-)$ are disjoint such that $\eta = \eta^+ - \eta^-$ is the Hahn decomposition of η, we will assume that ν is atomless. In this case, $\tau(\eta^+) \cap \tau(\eta^-) = \emptyset$ for $\mathscr{G} \times \mathscr{G}$-a.e. (η^+, η^-).

Next, we define the extrinsic derivative operator $(\nabla^{ext}, \mathscr{D}(\nabla^{ext}))$ as in Definition 1.1 for $\mathbb{M}_{\mathbf{s}}$ replacing $\mathbb{M}$:

*PW

$$\nabla^{ext} F(\eta)(x) = \lim_{0 \neq s \to 0} \frac{F(\eta + s\delta_x) - F(\eta)}{s}, \quad \eta \in \mathbb{M}_{\mathbf{s}}. \tag{5.2}$$

Let $\mathscr{F}_{\mathbf{s}} C_0^\infty$ be the class of cylindrical functions of type

*PY0-1

$$F(\eta) := f(\eta^+(A_1), \cdots, \eta^+(A_n), \eta^-(A_1), \cdots, \eta^-(A_n)), \quad n \geq 1, f \in C_0^\infty(\mathbb{R}^{2n}), \tag{5.3}$$

where $\{A_i\}_{1 \leq i \leq n}$ is a measurable partition of E, and $\eta = \eta^+ - \eta^-$ is the Hahn decomposition. Let

*PY0

$$A_\eta := \{x \in E : \ \eta(\{x\}) < 0\}, \quad \eta \in \mathbb{M}_{\mathbf{s}}. \tag{5.4}$$

It is easy to see that such a function F is extrinsically differentiable with

*PY

$$\nabla^{ext} F(\eta) = \sum_{i=1}^{2n} (1 - 2 \cdot 1_{\{i > n\}}) (\partial_i f)(\eta^+(A_1), \cdots, \eta^+(A_n), \eta^-(A_1), \cdots, \eta^-(A_n)) 1_{A_\eta^{i,n}}, \tag{5.5}$$

where

$$A_\eta^{i,n} := \begin{cases} A_i \cap A_\eta^c, & \text{if } i \leq n, \\ A_i \cap A_\eta, & \text{if } i > n. \end{cases}$$

Since for any $\eta \in \mathbb{M}_{\mathbf{s}}$, $A_{\eta + \varepsilon \delta_x} = A_\eta$ holds for small $\varepsilon > 0$ and all $x \in E$, $\nabla^{ext} F(\eta)(x)$ is again extrinsically differentiable in η with

*PY'

$$\begin{aligned} \nabla^{ext}[\nabla^{ext} F(\eta)(x)](y) = \sum_{i,j=1}^{2n} \Big[&(1 - 2 \cdot 1_{\{i > n\}})(1 - 2 \cdot 1_{\{j > n\}}) \\ &\times (\partial_i \partial_j f)(\eta^+(A_1), \cdots, \eta^+(A_n), \eta^-(A_1), \cdots, \eta^-(A_n)) 1_{A_\eta^{i,n}}(x) 1_{A_\eta^{j,n}}(y) \Big]. \end{aligned} \tag{5.6}$$

Finally, For any $\eta \in \mathbb{M}_{\mathbf{s}}$, let $\mathscr{A}_\eta$ be a positive definite bounded linear operator on $L^2(|\eta|)$, where $|\eta| := \eta^+ + \eta^-$ is the total variation of η. Consider the pre-Dirichlet form

PDs

$$\mathscr{E}^{\mathbf{s}}_{\mathscr{A}, V}(F, G) := \int_{\mathbb{M}_{\mathbf{s}}} \langle \mathscr{A}_\eta \nabla^{ext} F(\eta), \nabla^{ext} G(\eta) \rangle_{L^2(|\eta|)} \, \mathrm{d}\mathscr{G}^V_{\mathbf{s}}, \quad F, G \in \mathscr{F} C_0^\infty. \tag{5.7}$$

To ensure the closability of this bilinear form, we assume

(H') ν is atomless, $V \in \mathscr{D}(\nabla^{ext})$ such that $\mathscr{G}^V_{\mathbf{s}}$ is a probability measure. Moreover, for any $A \in \mathscr{B}(E)$ and $x \in E$, $\mathscr{A}_\eta 1_{A \cap A_\eta^c}(x)$ and $\mathscr{A}_\eta 1_{A \cap A_\eta}(x)$ are extrinsically differentiable in η with

$$\begin{aligned} \int_{\mathbb{M}_{\mathbf{s}}} \Big[&|\eta| \big(|\nabla^{ext}[\mathscr{A}_\eta 1_{A \cap A_\eta^c}]| + |\nabla^{ext}[\mathscr{A}_\eta 1_{A \cap A_\eta}]| \big) \\ &+ |\eta| \big((|\mathscr{A}_\eta 1_{A \cap A_\eta^c}| + |\mathscr{A}_\eta 1_{A \cap A_\eta}|) |\nabla^{ext} V(\eta)| \big) \Big] \mathscr{G}^V_{\mathbf{s}}(\mathrm{d}\eta) < \infty. \end{aligned}$$

Obviously, this assumption is satisfied if $\mathscr{A}_\eta = F(\eta)\mathbf{1}$ for some positive bounded extrinsically differentiable function F such that $\mathscr{G}_{\mathbf{s}}^V$ is a probability measure with

$$\int_{\mathbb{M}_{\mathbf{s}}} |\eta|(|\nabla^{ext} F(\eta)| + |\nabla^{ext} V(\eta)|)\mathscr{G}_{\mathbf{s}}^V(\mathrm{d}\eta) < \infty.$$

5.1 Integration by parts formula

T4.1 **Theorem 5.1.** *Assume* **(H')**. *Then*

DRC (5.8)
$$\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(F,G) = -\int_{\mathbb{M}_{\mathbf{s}}} (G\mathscr{L}^{\mathbf{s}}_{\mathscr{A},V}F)\,\mathrm{d}\mathscr{G}_{\mathbf{s}}^V, \quad F,G \in \mathscr{F}_{\mathrm{s}}C_0^\infty \tag{5.8}$$

holds for

$$\begin{aligned}\mathscr{L}^{\mathbf{s}}_{\mathscr{A},V}F(\eta) := &\int_E \Big(\big[\nabla^{ext} V(\eta)(x)\big]\mathscr{A}_\eta[\nabla^{ext} F(\eta)](x) + \nabla^{ext}\big(\mathscr{A}_\eta[\nabla^{ext} F(\eta)](x)\big)(x)\Big)\,|\eta|(\mathrm{d}x)\\ &- \int_E \mathscr{A}_\eta[\nabla^{ext} F(\eta)](x)\eta(\mathrm{d}x).\end{aligned}$$

Consequently, $(\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}, \mathscr{F}_{\mathrm{s}}C_0^\infty)$ *is closable in* $L^2(\mathscr{G}_{\mathbf{s}}^V)$ *and its closure* $(\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}, \mathscr{D}(\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}))$ *is a symmetric Dirichlet form with* $1 \in \mathscr{D}(\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V})$ *and* $\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(1,1) = 0$.

To prove this result, we introduce the divergence operator associated with ∇^{ext}.

Definition 5.1. A measurable function ϕ on $\mathbb{M}_{\mathbf{s}} \times E$ is said in the domain $\mathscr{D}(\mathrm{div}^{ext}_{\mathbf{s}})$, if for any $x \in E$ we have $\phi(\cdot,x) \in \mathscr{D}(\nabla^{ext})$ and

BBT2 (5.9)
$$\int_{\mathbb{M}_{\mathbf{s}}} \bigg(\int_E \big(|\nabla^{ext}\phi(\eta,x)(x)| + |\phi(\eta,x)|\big)|\eta|(\mathrm{d}x)\bigg)\,\mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta) < \infty. \tag{5.9}$$

In this case, the divergence operator is given by

DIVs (5.10)
$$\mathrm{div}^{ext}_{\mathbf{s}}(\phi)(\eta) := \int_E \phi(\eta,x)\eta(\mathrm{d}x) - \int_E \nabla^{ext}\phi(\eta,x)(x)\,|\eta|(\mathrm{d}x), \quad \eta \in \mathbb{M}_{\mathbf{s}}. \tag{5.10}$$

We have the following integration by parts formula for the directional derivative

$$\nabla^{ext}_\phi F(\eta) := \int_E \big[\phi(\eta,x)\nabla^{ext}F(\eta)(x)\big]|\eta|(\mathrm{d}x), \quad \phi \in \mathscr{D}(\mathrm{div}^{ext}_{\mathbf{s}}), F \in \mathscr{D}(\nabla^{ext}).$$

L4.2 **Lemma 5.2.** *Let* $\phi \in \mathscr{D}(\mathrm{div}^{ext}_{\mathbf{s}})$. *Then*

$$\int_{\mathbb{M}_{\mathbf{s}}} (\nabla^{ext}_\phi F)\,\mathrm{d}\mathscr{G}_{\mathbf{s}} = \int_{\mathbb{M}_{\mathbf{s}}} [F\mathrm{div}^{ext}_{\mathbf{s}}(\phi)]\,\mathrm{d}\mathscr{G}_{\mathbf{s}}, \quad F \in \mathscr{F}_{\mathbf{s}}C_0^\infty.$$

Proof. By a simple approximation argument, we may and do assume that ϕ is bounded so that $(\mathscr{G}_{\mathbf{s}}\times\nu)(|\phi|)<\infty$. For $F\in\mathscr{F}_{\rm s}C_0^\infty$, (5.2) implies

ABC (5.11) $$\nabla^{ext}F(\eta)(x)=\nabla^{ext}F(\cdot-\eta^-)(\eta^+)(x)=-\nabla^{ext}F(\eta^+-\cdot)(\eta^-)(x),\quad F\in\mathscr{D}(\nabla^{ext}).$$

Next, for any $\eta'\in\mathbb{M}$, let

$$\phi_{+,\eta'}(\eta,x):=\phi(\eta'-\eta,x),\quad \phi_{-,\eta'}(\eta,x):=\phi(\eta-\eta',x),\quad (\eta,x)\in\mathbb{M}\times E,$$

By (3.6) and (5.11) we obtain

$$\begin{aligned}
&\mathrm{div}^{ext}(\phi_{-,\eta^-})(\eta^+)-\mathrm{div}^{ext}(\phi_{+,\eta^+})(\eta^-)\\
&=\int_E\big[\phi(\eta^+-\eta^-,x)-\nabla^{ext}\phi(\cdot-\eta^-,x)(\eta^+)(x)\big]\eta^+(\mathrm{d}x)-\nu(\phi(\eta,\cdot))\\
&\qquad-\int_E\big[\phi(\eta^+-\eta^-,x)+\nabla^{ext}\phi(\eta^+-\cdot,x)(\eta^-)(x)\big]\eta^-(\mathrm{d}x)-\nu(\phi(\eta,\cdot))\\
&=\int_E\phi(\eta^+-\eta^-,x)(\eta^+-\eta^-)(\mathrm{d}x)-\int_E\big[\nabla^{ext}(\cdot,x)(\eta^+-\eta^-)(x)\big](\eta^++\eta^-)(\mathrm{d}x)\\
&=\mathrm{div}^{ext}_{\mathbf{s}}(\phi)(\eta),\quad \eta=\eta^+-\eta^-\ \text{with}\ \tau(\eta^+)\cap\tau(\eta^-)=\emptyset.
\end{aligned}$$

Combining this with Lemma 3.3, (5.1) and (5.11), we obtain

$$\begin{aligned}
&\int_{\mathbb{M}_{\mathbf{s}}}(\nabla^{ext}_\phi F)\,\mathrm{d}\mathscr{G}_{\mathbf{s}}=\int_{\mathbb{M}\times\mathbb{M}}\mathscr{G}(\mathrm{d}\eta^+)\mathscr{G}(\mathrm{d}\eta^-)\int_E\big[\phi(\eta^+-\eta^-,x)\nabla^{ext}F(\eta^+-\eta^-)(x)\big](\eta^++\eta^-)(\mathrm{d}x)\\
&=\int_{\mathbb{M}}\mathscr{G}(\eta^-)\int_{\mathbb{M}\times E}\big[\phi(\eta^+-\eta^-,x)\nabla^{ext}F(\cdot-\eta^-)(\eta^+)(x)\big]\eta^+(\mathrm{d}x)\\
&\qquad-\int_{\mathbb{M}}\mathscr{G}(\eta^+)\int_{\mathbb{M}\times E}\big[\phi(\eta^+-\eta^-,x)\nabla^{ext}F(\eta^+-\cdot)(\eta^-)(x)\big]\eta^-(\mathrm{d}x)\\
&=\int_{\mathbb{M}\times\mathbb{M}}F(\eta^+-\eta^-)\big[\mathrm{div}^{ext}(\phi_{-,\eta^-})(\eta^+)-\mathrm{div}^{ext}(\phi_{+,\eta^+})(\eta^-)\big]\mathscr{G}(\mathrm{d}\eta^+)\mathscr{G}(\eta^-)\\
&=\int_{\mathbb{M}_{\mathbf{s}}}F(\eta)\mathrm{div}^{ext}_{\mathbf{s}}(\phi)(\eta)\,\mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta).
\end{aligned}$$

□

Proof of Theorem 5.1. Let $F\in\mathscr{F}_{\mathbf{s}}C_0^\infty$ be given in (5.3), and let

$$\begin{aligned}
\phi(\eta,x)&:=\mathrm{e}^{V(\eta)}\mathscr{A}_\eta[\nabla^{ext}F(\eta)](x)\\
&=\mathrm{e}^{V(\eta)}\sum_{i=1}^{2n}(1-2\cdot 1_{\{i>n\}})(\partial_i f)(\eta(A_1,\cdots,\eta(A_n))\mathscr{A}_\eta 1_{A_i^\eta}(x),\quad (\eta,x)\in\mathbb{M}_{\mathbf{s}}\times E.
\end{aligned}$$

Then **(H')** and (5.5) imply $\phi\in\mathscr{D}(\mathrm{div}^{ext}_{\mathbf{s}})$. By the definition of $\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}$ and Lemma 5.2, for any $G\in\mathscr{F}_{\mathbf{s}}C_0^\infty$ we have

$$\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(F,G)=\int_{\mathbb{M}_{\mathbf{s}}}\big\langle\mathscr{A}_\eta\nabla^{ext}F(\eta),\nabla^{ext}G(\eta)\big\rangle_{L^2(|\eta|)}\mathscr{G}^V_{\mathbf{s}}(\mathrm{d}\eta)$$

$$= \int_{\mathbb{M}_{\mathbf{s}}} \big\langle \phi(\eta,\cdot), \nabla^{ext} G(\eta)\big\rangle_{L^2(|\eta|)} \mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta) = \int_{\mathbb{M}_{\mathbf{s}}} G(\eta) \mathrm{div}^{ext}_{\mathbf{s}}(\phi)\, \mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta).$$

This together with (5.10) implies (5.8) for

$$\begin{aligned}\mathscr{L}^{\mathbf{s}}_{\mathscr{A},V} F(\eta) &:= -\mathrm{e}^{-V(\eta)} \mathrm{div}^{ext}_{\mathbf{s}}(\phi) = -\mathrm{e}^{-V(\eta)} \mathrm{div}^{ext}_{\mathbf{s}}\big(\mathrm{e}^{V(\eta)} \mathscr{A}_\eta[\nabla^{ext} F(\eta)](\cdot)\big)\\ &= \int_E \Big(\big[\nabla^{ext} V(\eta)(x)\big] \mathscr{A}_\eta[\nabla^{ext} F(\eta)](x) + \nabla^{ext}\big(\mathscr{A}_\eta[\nabla^{ext} F(\eta)](x)\big)(x)\Big)\, |\eta|(\mathrm{d}x)\\ &\quad - \int_E \mathscr{A}_\eta[\nabla^{ext} F(\eta)](x)\eta(\mathrm{d}x).\end{aligned}$$

Next, to prove that $1 \in \mathscr{D}(\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V})$ with $\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(1,1) = 0$, we take $\{f_n\}_{n\ge 1} \subset C_0^\infty(\mathbb{R})$ such that $f_n(s) = 1$ for $|s| \le n$, $0 \le f_n \le 1$ and $\|f_n'\|_\infty \le 1$. Let $F_n(\eta) := f_n(\eta(E)), n \ge 1$. Then $F_n \in \mathscr{F}C_0^\infty$. By **(H')** we have $\mathscr{G}^V_{\mathbf{s}}(|F_n - 1|^2) \to 0$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(F_n, F_n) = \limsup_{n\to\infty} \int_{\{|\eta(E)|>n\}} \|\mathscr{A}_\eta 1\|_{L^1(|\eta|)}\, \mathscr{G}^V_{\mathbf{s}}(\mathrm{d}\eta) = 0.$$

Therefore, $1 \in \mathscr{D}(\mathscr{E}^2_{\mathscr{A},V})$ and $\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(1,1) = 0$. □

5.2 Functional inequalities for $\mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}$

For any $N > 0$, let $\tilde{\mathbf{B}}^{\mathbf{s}}_N = \{\eta \in \mathbb{M}_{\mathbf{s}} : \eta^+(E) \vee \eta^-(E) \le N\}$.

T5.2 **Theorem 5.3.** *Let $\mathscr{A} = \mathbf{1}$ and $V = 0$.*

(1) $\mathrm{gap}(\mathscr{L}^{\mathbf{s}}_{\mathscr{A},V}) = 1$, *i.e. the following Poincaré inequality*

P00 (5.12) $$\mathscr{G}_{\mathbf{s}}(F^2) \le \mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}(F,F) + \mathscr{G}_{\mathbf{s}}(F)^2, \quad F \in \mathscr{D}(\mathscr{E}^{\mathbf{s}}_{\mathbf{1},0})$$

holds, and the constant 1 in front of $\mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}(F,F)$ is optimal.

(2) *If* $\mathrm{supp}\,\nu$ *is infinite, then $\mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}$ does not satisfy the super Poincaré inequality. On the other hand, there exists a constant $c_0 > 0$ such that when* $\mathrm{supp}\,\nu$ *is a finite set, the log-Sobolev inequality*

LSI0* (5.13) $$\mathscr{G}_{\mathbf{s}}(F^2 \log F^2) \le \frac{c_0}{1\wedge\delta} \mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}(F,F), \quad F \in \mathscr{D}(\mathscr{E}_{\mathbf{1},0}), \mathscr{G}(F^2) = 1$$

holds, where $\delta := \min\{\nu(\{x\}) :\ x \in \mathrm{supp}\,\nu\}$.

(3) *For any $N > 0$ and $F \in \mathscr{F}C_0^\infty$ with $\mathscr{G}_{\mathbf{s}}(1_{\tilde{\mathbf{B}}^{\mathbf{s}}_N} F) = 0$,*

LSP* (5.14) $$\mathscr{G}_{\mathbf{s}}(1_{\tilde{\mathbf{B}}^{\mathbf{s}}_N} F^2) \le \Big(2 \vee \frac{N^2}{2\nu(E)}\Big) \mathscr{G}_{\mathbf{s}}\big(1_{\tilde{B}^{\mathbf{s}}_N} \|\nabla^{ext} F\|^2_{L^2(|\eta|)}\big).$$

Proof. By taking $F(\eta)$ depending only on η^+, it is easy to see that a Poincaré inequality for $\mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}$ implies the same inequality for $\mathscr{E}_{\mathbf{1},0}$. So, the optimality of (5.12), and the invalidity of the super Poincaré inequality when $\operatorname{supp}\nu$ is infinite, follow from Theorem 2.1. It remains to prove the inequalities (5.12), (5.13) and (5.14). According to the additivity property of the Poincaré and log-Sobolev inequalities, these inequalities follow from the corresponding ones of $\mathscr{E}_{\mathbf{1},0}$. For simplicity, below we only prove the first inequality.

Let $\mathscr{F} \in \mathscr{F}_{\mathbf{s}} C_0^\infty$. By Theorem 2.1, (5.1), (5.7) for $\mathscr{A} = \mathbf{1}$ and $V = 0$, and using (5.11), we obtain

$$\begin{aligned}
\mathscr{G}_{\mathbf{s}}(F^2) &= \int_{\mathbb{M}} \mathscr{G}(\mathrm{d}\eta^-) \int_{\mathbb{M}} F(\eta^+ - \eta^-)^2 \mathscr{G}(\mathrm{d}\eta^+) \\
&\le \int_{\mathbb{M}\times\mathbb{M}} \|\nabla^{ext} F(\cdot - \eta^-)(\eta^+)\|^2_{L^2(\eta^+)} \mathscr{G}(\mathrm{d}\eta^+)\mathscr{G}(\mathrm{d}\eta^-) + \int_{\mathbb{M}} \bigg(\int_{\mathbb{M}} F(\eta^+ - \eta^-)\mathscr{G}(\mathrm{d}\eta^+)\bigg)^2 \mathscr{G}(\mathrm{d}\eta^-) \\
&\le \int_{\mathbb{M}_{\mathbf{s}}} \|\nabla^{ext} F(\eta)\|^2_{L^2(\eta^+)} \mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta) + \bigg(\int_{\mathbb{M}\times\mathbb{M}} F(\eta^+ - \eta^-)\mathscr{G}(\mathrm{d}\eta^+)\mathscr{G}(\mathrm{d}\eta^-)\bigg)^2 \\
&\quad + \int_{\mathbb{M}} \bigg\|\nabla^{ext}\bigg[\int_{\mathbb{M}} F(\eta^+ - \cdot)\mathscr{G}(\mathrm{d}\eta^+)\bigg](\eta^-)\bigg\|^2_{L^2(\eta^-)} \mathscr{G}(\mathrm{d}\eta^-).
\end{aligned}$$

By the Jensen inequality, we have

$$\bigg\|\nabla^{ext}\bigg[\int_{\mathbb{M}} F(\eta^+ - \cdot)\mathscr{G}(\mathrm{d}\eta^+)\bigg](\eta^-)\bigg\|^2_{L^2(\eta^-)} \le \int_{\mathbb{M}} \|\nabla^{ext} F(\eta^+ - \cdot)(\eta^-)\|^2_{L^2(\eta^-)} \mathscr{G}(\mathrm{d}\eta^+).$$

Therefore,

$$\mathscr{G}_{\mathbf{s}}(F^2) \le \mathscr{G}_{\mathbf{s}}(F)^2 + \int_{\mathbb{M}_{\mathbf{s}}} \|\nabla^{ext} F(\eta)\|^2_{L^2(\eta^+ + \eta^-)} \mathscr{G}_{\mathbf{s}}(\mathrm{d}\eta) = \mathscr{G}_{\mathbf{s}}(F)^2 + \mathscr{E}^{\mathbf{s}}_{\mathbf{1},0}(F,F).$$

□

5.3 Functional inequalities for $\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}$

According to the proof of Theorem 2.2 and the local Poincaré inequality (5.14), it seems that we should take

$$\tilde{\rho}_{\mathbf{s}}(\eta) := 2\sqrt{\eta^+(E) \vee \eta^-(E)}, \quad \eta \in \mathbb{M}_{\mathbf{s}}$$

to replace the function ρ on M. But by (5.5) we have

$$\nabla^{ext}\tilde{\rho}_{\mathbf{s}}(\eta)(x) = \frac{2}{\tilde{\rho}_{\mathbf{s}}(\eta)}\Big(1_{\{\eta(E)\ge 0\}} 1_{A^c_\eta}(x) - 1_{\{\eta(E)<0\}} 1_{A_\eta}(x)\Big),$$

which is however not extrinsically differentiable in η, so that $\mathscr{L}_{\mathscr{A},V}\tilde{\rho}_{\mathbf{s}}$ is not well defined as required. To avoid this problem, below we will use both $\tilde{\rho}_{\mathbf{s}}$ and

BBG (5.15) $$\rho_{\mathbf{s}}(\eta) := 2\sqrt{|\eta|(E)}, \quad \eta \in \mathbb{M}_{\mathbf{s}},$$

which satisfies $\|\nabla^{ext}\rho_{\mathbf{s}}(\eta)\|_{L^2(|\eta|)} = 1$ according to the following lemma.

L5.1 **Lemma 5.4.** *Let ρ be defined in* (5.15) *and let* $\mathbf{s}(\eta,\cdot) := 1 - 2\cdot 1_{A_\eta}$ *for* A_η *in* (5.4)*. Then*

$$\nabla^{ext}\rho_{\mathbf{s}}(\eta) = \frac{2\mathbf{s}(\eta,\cdot)}{\rho_{\mathbf{s}}(\eta)}, \quad \nabla^{ext}\mathbf{s}(\eta,x)(y) = 0, \quad \eta\in\mathbb{M}_{\mathbf{s}}, x,y\in E.$$

Consequently, if $\mathscr{A}_\eta\mathbf{s}(\eta,\cdot)$ *is extrinsically differentiable in* $\eta\in\mathbb{M}_{\mathbf{s}}$ *with*

LOT (5.16)
$$\sup_{|\eta|(E)\le r}|\eta|\big(|\mathscr{A}_\eta\mathbf{s}(\eta,\cdot)| + |\nabla^{ext}[\mathscr{A}_\eta\mathbf{s}(\eta,\cdot)](\cdot)|\big) < \infty, \quad r\in(0,\infty),$$

then

LRR (5.17)
$$\mathscr{L}^{\mathbf{s}}_{\mathscr{A},V}\rho_{\mathbf{s}}(\eta) = \frac{2}{\rho_{\mathbf{s}}(\eta)}\Big[|\eta|\Big([\nabla^{ext}V(\eta)]\mathscr{A}_\eta\mathbf{s}(\eta,\cdot)\big) + \nabla^{ext}\big[\mathscr{A}_\eta\mathbf{s}(\eta,\cdot)\big](\cdot)\Big) - \eta\big(\mathscr{A}_\eta\mathbf{s}(\eta,\cdot)\big)\Big].$$

This lemma can be proved by simple calculations using (5.2) and the definition of $\mathscr{L}_{\mathscr{A},V}$ in Theorem 5.1, so we omit the details.

By Lemma 5.4, we have

$$\Gamma^{\mathbf{s}}_{\mathbf{1}}(\rho_{\mathbf{s}},\rho_{\mathbf{s}}) := |\eta|(|\nabla^{ext}\rho_{\mathbf{s}}|^2) = 1, \quad \nabla^{ext}[\nabla^{ext}\rho_{\mathbf{s}}(\eta)(x)](x) = -\frac{4}{\rho_{\mathbf{s}}(\eta)^3}.$$

These coincide with the corresponding properties of ρ on $\mathbb{M}$.

Similarly to (2.5) and (2.6), let

*FY (5.18)
$$\begin{aligned}
&\xi_{\mathbf{s}}(r) = \inf_{\rho_{\mathbf{s}}(\eta)=r}\mathscr{L}^{\mathbf{s}}_{\mathscr{A},V}\rho_{\mathbf{s}}(\eta), \quad \underline{a}_{\mathbf{s}}(r) = \inf_{\rho_{\mathbf{s}}(\eta)=r}\inf_{\|\phi\|_{L^2(|\eta|)}=1}\langle\mathscr{A}_\eta\phi,\phi\rangle_{L^2(|\eta|)},\\
&\bar{a}_{\mathbf{s}}(r) = \sup_{\rho_{\mathbf{s}}(\eta)=r}\sup_{\|\phi\|_{L^2(|\eta|)}=1}\langle\mathscr{A}_\eta\phi,\phi\rangle_{L^2(|\eta|)}, \quad r>0,\\
&\sigma_{k,\mathbf{s}} = \sup_{t\ge k}\int_t^\infty \mathrm{e}^{\int_k^r\frac{\xi_{\mathbf{s}}(s)}{\underline{a}_{\mathbf{s}}(s)}\mathrm{d}s}\mathrm{d}r\int_k^t\frac{1}{\underline{a}_{\mathbf{s}}(r)}\mathrm{e}^{-\int_k^r\frac{\xi_{\mathbf{s}}(s)}{\underline{a}_{\mathbf{s}}(s)}\mathrm{d}s}\mathrm{d}r, \quad k>0.
\end{aligned}$$

Assume that

00* (5.19)
$$\psi(t) := \int_0^t[\bar{a}_{\mathbf{s}}(r)]^{-\frac{1}{2}}\,\mathrm{d}r\uparrow\infty \text{ as } t\uparrow\infty.$$

As in the proof of Theorem 2.2, we may use $\sigma_{k,\mathbf{s}}$ to estimate $\mathscr{G}^V_{\mathbf{s}}(F_N^2)$ for

$$F_N := \big[(\psi(\rho_{\mathbf{s}})-\psi(N))^+\wedge 1\big]\cdot F, \quad N>0, F\in\mathscr{F}_{\mathbf{s}}C_0^\infty.$$

More precisely, as in (4.20) and (4.26) we conclude that for any $k>0$ there exists $N\in[k,\psi^{-1}(\psi(k)+32\sigma_{k,\mathbf{s}})]$ such that

PPK2 (5.20)
$$\begin{aligned}
\int_{\mathbb{M}_{\mathbf{s}}}F_N^2\mathrm{d}\mathscr{G}^V_{\mathbf{s}} &\le \frac{2}{\lambda_k}\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(F,F) + \frac{1}{4}\int_{\mathbb{M}}F^2\mathrm{d}\mathscr{G}^V_{\mathbf{s}}\\
&\le 8\sigma_{k,\mathbf{s}}\,\mathscr{E}^{\mathbf{s}}_{\mathscr{A},V}(F,F) + \frac{1}{4}\int_{\mathbb{M}}F^2\mathrm{d}\mathscr{G}^V_{\mathbf{s}}.
\end{aligned}$$

On the other hand, we estimate $\mathscr{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\rho_{\mathbf{s}} \le N\}})$ by using the local Poincaré inequality (5.14). Since the bounded set in (5.14) is $\tilde{\mathbf{B}}_N^{\mathbf{s}} := \{\tilde{\rho}_{\mathbf{s}} \le N\}$ rather than $\mathbf{B}_N^{\mathbf{s}} := \{\rho_{\mathbf{s}} \le N\}$, we change the definition of $\Phi(N)$ into

$$\Phi_{\mathbf{s}}(N) := \Big(2 \vee \frac{N^2}{2\nu(E)}\Big) \exp\Big[\sup_{\tilde{\rho}_{\mathbf{s}} \le N} V - \inf_{\tilde{\rho}_{\mathbf{s}} \le N} V\Big] \sup_{\tilde{\rho}_{\mathbf{s}}(\eta) \le N} \sup_{\|\phi\|_{L^2(|\eta|)}=1} \frac{1}{\langle \mathscr{A}_\eta \phi, \phi \rangle_{L^2(|\eta|)}}, \quad N > 0.$$

Noting that $1_{\{\rho_{\mathbf{s}} \le N\}} \le 1_{\{\tilde{\rho}_{\mathbf{s}} \le N\}}$, we may apply Theorem 5.3 to bound $\mathscr{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\rho_{\mathbf{s}} \le N\}})$. For instance, corresponding to (4.18) we have

$$\mathscr{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\rho_{\mathbf{s}} \le N\}}) \le \mathscr{G}_{\mathbf{s}}^V(F^2 \cdot 1_{\{\tilde{\rho}_{\mathbf{s}} \le N\}}) \le \mathscr{G}_{\mathbf{s}}^V(1_{\{\tilde{\rho}_{\mathbf{s}} \le N\}} F)^2 + \Phi_{\mathbf{s}}(N) \mathscr{E}_{\mathscr{A},V}^{\mathbf{s}}(F, F).$$

Combining this with (5.20) we may extend assertions of Theorem 2.2 to the present setting as follows, where when $\operatorname{supp}\nu$ is infinite the super Poincaré can be disproved as in the proof of Theorem 2.2(2) by taking $F_n(\eta) = (1 - \eta^+(E))^+ \frac{\eta^+(A_n)}{\eta^+(E)}$ for $0 < \nu(A_n) \downarrow 0$. Moreover, one may also extend Corollaries 2.3-2.4 and Theorem 2.5. We omit the details to save space.

TSP* **Theorem 5.5.** *In addition to* **(H')**, *assume that* $\mathscr{A}_\eta \mathbf{s}(\eta, \cdot)$ *is extrinsically differentiable in* η *such that* (5.16) *holds. Moreover, assume that* $\underline{a}_{\mathbf{s}}$ *and* $\bar{a}_{\mathbf{s}}$ *in* (5.18) *are such that* $\underline{a}_{\mathbf{s}}^{-1}(r)$ *is locally bounded in* $r \ge 0$ *and* (5.19) *holds.*

(1) *If* $\lim_{k\to\infty} \sigma_{k,\mathbf{s}} < \infty$, *then*

$$\operatorname{gap}(\mathscr{L}_{\mathscr{A},V}^{\mathbf{s}}) \ge \sup\Big\{\frac{1}{2\Phi_{\mathbf{s}}\big(\psi^{-1}(\psi(k) + 32\sigma_{k,\mathbf{s}} + 1)\big) + 32\sigma_{k,\mathbf{s}}} : \ k > 0\Big\} > 0.$$

(2) *If* $\operatorname{supp}\nu$ *contains infinitely many points, then* $\mathscr{E}_{\mathscr{A},V}^{\mathbf{s}}$ *does not satisfy the super Poincaré inequality.*

(3) *The weak Poincaré inequality* (1.10) *holds for* $(\mathscr{E}_{\mathscr{A},V}^{\mathbf{s}}, \mathscr{G}_{\mathbf{s}}^V)$ *replacing* $(\mathscr{E}_{\mathscr{A},V}, \mathscr{G}^V)$ *and*

$$\alpha(r) := \inf\Big\{2\Phi_{\mathbf{s}}(N) : \ \mathscr{G}_{\mathbf{s}}^V(\rho > N) \le \frac{r}{1+r}\Big\}, \quad r > 0.$$

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