Uniform Cramér moderate deviations and Berry-Esseen bounds for a supercritical branching process in a random environment

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Abstract

Let $\{Z_n, n \geq 0\}$ be a supercritical branching process in an independent and identically distributed random environment. We prove Cramér moderate deviations and Berry-Esseen bounds for $\ln(Z_{n+n_0}/Z_{n_0})$ uniformly in $n_0 \in \mathbb{N}$, which extend the corresponding results by Grama et al. (Stochastic Process. Appl. 2017) established for $n_0 = 0$. The extension is interesting in theory, and is motivated by applications. A new method is developed for the proofs; some conditions of Grama et al. (2017) are relaxed in our present setting. An example of application is given in constructing confidence intervals to estimate the criticality parameter in terms of $\ln(Z_{n+n_0}/Z_{n_0})$ and n.

Keywords: Branching processes; Random environment; Cramér moderate deviations; Berry-Esseen bounds

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1. Introduction

As an important generalization of the Galton-Watson process, the branching process in a random environment (BPRE) was first introduced by Smith and Wilkinson [19] to modelize the growth of a population submitted to an independent and identically distributed (iid) random environment. Basic results for a BPRE can be found in Athreya and Karlin [2, 3] who considered the stationary and ergodic environment case.

A BPRE can be described as follows. Let $\xi = (\xi_0, \xi_1, ...)$ be a sequence of independent and identically distributed (iid) random variables, where ξ_n stands for the random environment at time n. Each realization of ξ_n corresponds to a probability law $p(\xi_n) = \{p_i(\xi_n) : i \in \mathbb{N}\}$ on $\mathbb{N} = \{0, 1, ...\}$ ($p_i(\xi_n) \geq 0$ and $\sum_{i=0}^{\infty} p_i(\xi_n) = 1$). A branching process $\{Z_n, n \geq 0\}$ in the

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random environment ξ can be defined as follows:

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad \text{for} \ \ n \geq 0,$$

where $X_{n,i}$ is the number of offspring of the i-th individual in generation n. Conditioned on the environment ξ the random variables $X_{n,i}\,(n \geq 0, i \geq 1)$ are independent, and each $X_{n,i}$ has the same law $p(\xi_n)$. Denote by $\mathbb{P}_\xi$ the probability when the environment ξ is given, τ the law of the environment ξ, and

$$\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$$

the total law of the process; $\mathbb{P}_\xi$ can be considered as the conditional law of $\mathbb{P}$ given the environment ξ. The conditional probability $\mathbb{P}_\xi$ is called the quenched law, while the total probability $\mathbb{P}$ is called annealed law. In the sequel $\mathbb{E}_\xi$ and $\mathbb{E}$ denote respectively the quenched and annealed expectations. Set for $n \geq 0$,

$$m_n = \sum_{k=0}^{\infty} k\, p_k(\xi_n) \quad \text{and} \quad \Pi_n = \prod_{i=0}^{n-1} m_i,$$

with the convention that $\Pi_0 = 1$. Then $m_n = \mathbb{E}_\xi X_{n,i}$ for each $i \geq 1$ and $\Pi_n = \mathbb{E}_\xi Z_n$. Let

$$X = \log m_0, \quad \mu = \mathbb{E}X.$$

The process $\{Z_n, n \geq 0\}$ is called supercritical, critical or subcritical according to $\mu > 0$, $\mu = 0$ or $\mu < 0$, respectively. We call μ the criticality parameter.

Limit theorems for BPRE have attracted a lot of attentions. See for example Vatutin [21], Afanasyev et al. [1], Vatutin and Zheng [22] and Bansaye and Vatutin [6] on the survival probability and conditional limit theorems for subcritical BPRE. For supercritical BPRE, a number of researches have studied moderate and large deviations; see, for instance, Kozlo [15], Bansaye and Berestycki [4], Böinghoff and Kersting [8], Bansaye and Böinghoff [5], Huang and Liu [14], Nakashima [18], Böinghoff [7], and Grama, Liu and Miqueu [13].

In this paper, we are interested in Cramér moderate deviations and Berry-Esseen bounds for a supercritical BPRE. For simplicity we assume that

$$p_0(\xi_0) = 0 \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad \sigma^2 = \mathbb{E}(X-\mu)^2 \in (0, \infty), \tag{1.1}$$

which imply that the process is supercritical and $Z_n \to \infty$ a.s. Under the additional conditions: $\mathbb{E}\frac{Z_1^p}{m_0} < \infty$ for a constant $p > 1$ and $\mathbb{E}e^{\lambda_0 X} < \infty$ for a constant $\lambda_0 > 0$, Grama et al. [13] have established the Cramér moderate deviation expansion, which implies in particular that for $0 \leq x = o(\sqrt{n})$ as $n \to \infty$,

$$\left| \ln \frac{\mathbb{P}\big(\frac{\ln Z_n - n\mu}{\sigma\sqrt{n}} \geq x\big)}{1 - \Phi(x)} \right| \leq C \frac{1 + x^3}{\sqrt{n}}, \tag{1.2}$$

where throughout the paper the symbol C, probably supplied with some indices, denotes a positive constant whose value may differ from line to line. Inequality (1.2) is interesting due to the fact that it implies a moderate deviation principle (MDP) and the following result about the equivalence to the normal tail:

$$\frac{\mathbb{P}\big(\frac{\ln Z_n - n\mu}{\sigma\sqrt{n}} \geq x\big)}{1-\Phi(x)} = 1 + o(1), \tag{1.3}$$

for $x \in [0, o(n^{1/6}))$, as $n \to \infty$. Assuming $\mathbb{E}\big(\frac{Z_1}{m_0}\big)^p < \infty$ for a constant $p > 1$ and $\mathbb{E}X^{2+\rho} < \infty$ for a constant $\rho \in (0,1)$, Grama et al. [13] have also obtained the following Berry-Esseen bound for $\ln Z_n$:

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\Big(\frac{\ln Z_n - n\mu}{\sigma\sqrt{n}} \leq x\Big) - \Phi(x)\Big| \leq \frac{C}{n^{\rho/2}}. \tag{1.4}$$

The results (1.2), (1.3) and (1.4) are interesting both in theory and in applications. For example, they can be applied to obtain confidence intervals to estimate the criticality parameter μ in terms of the observation Z_n and the present time n, or to estimate the population size Z_n in terms of μ and n. In the real-world applications, it may happen that we know a historical data Z_{n_0} for some $n_0 > 0$, the current population size Z_{n_0+n}, as well as the increment n of generation numbers, but do not know the generation number $n_0 + n$. In such a case (1.2), (1.3) and (1.4) are no longer applicable to obtain confidence intervals to estimate μ in terms of Z_{n_0}, Z_{n_0+n} and n, while $n_0 > 0$. The same problem exists while we want to construct confidence intervals to preview Z_{n_0+n} in terms of Z_{n_0}, μ and n. Motivated by these problems, we will extend (1.2), (1.3) and (1.4), with $\ln Z_n$ replaced by $\ln \frac{Z_{n_0+n}}{Z_{n_0}}$, uniformly in $n_0 \in \mathbb{N}$ (so that in applications n_0 can be taken as a function of n). This is the main objective of the present paper.

The main results are presented in Section 2. Let us introduce them briefly. Denote by $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ the positive and negative parts of x, respectively. In Theorem 2.1, assuming $\mathbb{E}\frac{Z_1}{m_0}\ln^+ Z_1 < \infty$ and $\mathbb{E}e^{\lambda_0 X} < \infty$ for a constant $\lambda_0 > 0$, we prove that uniformly in $n_0 \in \mathbb{N}$, for $0 \leq x = o(\sqrt{n})$, as $n \to \infty$,

$$\left| \ln \frac{\mathbb{P}\big(\frac{\ln \frac{Z_{n_0+n}}{Z_{n_0}} - n\mu}{\sigma\sqrt{n}} \geq x\big)}{1-\Phi(x)} \right| \leq C(1+x^3)\frac{1+\mathbf{1}_{[0,\sqrt{\ln n})}(x)\ln n}{\sqrt{n}}. \tag{1.5}$$

When $n_0 = 0$, inequality (1.5) reduces nearly to (1.2), with $\ln n$ as an additional factor. Notice that here we do not need the additional condition that $\mathbb{E}\frac{Z_1^p}{m_0} < \infty$ for some $p > 1$ assumed in [13] for (1.2) to hold. As a consequence, we obtain a uniform MDP for $\ln \frac{Z_{n_0+n}}{Z_{n_0}}$, see Corollary 2.1. From (1.5), we also obtain the following equivalence to the normal tail: uniformly in $n_0 \in \mathbb{N}$, for $x \in [0, o(n^{1/6}))$, as $n \to \infty$,

$$\frac{\mathbb{P}\big(\frac{\ln \frac{Z_{n_0+n}}{Z_{n_0}} - n\mu}{\sigma\sqrt{n}} \geq x\big)}{1-\Phi(x)} = 1 + o(1). \tag{1.6}$$

When the exponential moment condition $\mathbb{E}e^{\lambda_0 X} < \infty$ is relaxed to the sub-exponential moment condition that $\mathbb{E}\exp\{\lambda_0 X^{\frac{4\gamma}{1-2\gamma}}\} < \infty$ for some $\gamma \in (0, \frac{1}{6}]$, we prove that (1.6) still holds for $x \in [0, o(n^\gamma))$; see Theorem 2.2 for a result of type Linnik [16]. Using (1.6), we can prove, under the exponential moment condition, the following uniform Berry-Esseen bound: uniformly in $n_0 \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\Big(\frac{\ln \frac{Z_{n_0+n}}{Z_{n_0}} - n\mu}{\sigma\sqrt{n}} \leq x \Big) - \Phi(x) \Big| \leq \frac{C \ln n}{\sqrt{n}}. \tag{1.7}$$

Compared to the best rate $\frac{C}{\sqrt{n}}$ of the Berry-Esseen bound for random walks, here the factor $\ln n$ is added. We believe that this factor $\ln n$ can be removed from (1.7), just as in the case $n_0 = 0$ considered in Grama et al. [13]. In fact for $n_0 = 0$, the more general Berry-Esseen bound $\frac{C}{n^{\rho/2}}$ was established in [13] under the moment condition $\mathbb{E}X^{2+\rho} < \infty$ with $\rho \in (0, 1]$. In this paper, we prove that if $\mathbb{E}X^{2+\rho} < \infty$ for some $\rho \in (0, \frac{\sqrt{5}-1}{2})$, then uniformly in $n_0 \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\big(Z_{n_0,n} \leq x \big) - \Phi(x) \Big| \leq \frac{C}{n^{\rho/2}}. \tag{1.8}$$

See Theorem 2.4. Clearly, inequality (1.8) with $n_0 = 0$ reduces to (1.4), which was obtained in [13] under the additional condition that $\mathbb{E}\big(\frac{Z_1}{m_0}\big)^p < \infty$ for some $p > 1$.

In Section 3, some applications of the main results are demonstrated. We construct confidence intervals for estimating the criticality parameter μ in terms of $\frac{Z_{n_0+n}}{Z_{n_0}}$ and n; see Propositions 3.1 and 3.2. The proofs of the main results are given in Sections 4 - 8, by developing a method different to that used in [13] .

2. Main results

It is well-known that the normalized population size

$$W_n = \frac{Z_n}{\Pi_n}, \quad n \geq 0,$$

is a nonnegative martingale both under the quenched law $\mathbb{P}_\xi$ and under the annealed law $\mathbb{P}$, with respect to the natural filtration $\mathcal{F}_0 = \sigma\{\xi\}$, $\mathcal{F}_n = \sigma\{\xi, X_{k,i}, 0 \leq k \leq n-1, i \geq 1\}, n \geq 1$. Then the limit

$$W = \lim_{n\to\infty} W_n$$

exists $\mathbb{P}$-a.s. by Doob's convergence theorem, and satisfies $\mathbb{E}W \leq 1$ by Fatou's lemma. Throughout the paper, assume that

$$\mathbb{E}\frac{Z_1}{m_0} \ln^+ Z_1 < \infty. \tag{2.1}$$

Together with the condition that $p_0(\xi_0) = 0$ a.s., condition (2.1) implies that $\mathbb{P}(W > 0) = \mathbb{P}(Z_n \to \infty) = \lim_{n\to\infty} \mathbb{P}(Z_n > 0) = 1$, and that the martingale W_n converges to W in $\mathbb{L}^1(\mathbb{P})$

(see Athreya and Karlin [3] and also Tanny [20]). Clearly, the following decomposition holds:

$$\ln Z_n = \sum_{i=1}^{n} X_i + \ln W_n, \tag{2.2}$$

where $X_i = \ln m_{i-1} (i \geq 1)$ are iid random variables depending only on the environment ξ. The asymptotic behavior of $\ln Z_n$ is crucially affected by the *associated random walk*

$$S_n = \sum_{i=1}^{n} X_i = \ln \Pi_n, \quad n \geq 0.$$

By our notation and hypothesis (see (1.1)), it follows that $X = X_1$, $\mu = \mathbb{E}X > 0$ and $\sigma^2 = \mathbb{E}(X-\mu)^2 \in (0,\infty)$; the later implies that the random walk $\{S_n, n \geq 0\}$ is non-degenerate.

We will need the following Cramér condition on the associated random walk.

A1. The random variable $X = \ln m_0$ has an exponential moment, i.e. there exists a constant $\lambda_0 > 0$ such that

$$\mathbb{E}e^{\lambda_0 X} = \mathbb{E}m_0^{\lambda_0} < \infty.$$

Our first result concerns the uniform Cramér moderate deviations for

$$Z_{n_0,n} := \frac{\ln \frac{Z_{n_0+n}}{Z_{n_0}} - n\mu}{\sigma\sqrt{n}}, \quad n_0 \in \mathbb{N}. \tag{2.3}$$

Theorem 2.1. *Assume condition* ***A1****. Then the following results hold uniformly in $n_0 \in \mathbb{N}$: for $n \geq 2$ and $0 \leq x < \sqrt{\ln n}$,*

$$\left| \ln \frac{\mathbb{P}\big(Z_{n_0,n} \geq x\big)}{1-\Phi(x)} \right| \leq C(1+x^3)\frac{\ln n}{\sqrt{n}}; \tag{2.4}$$

for $n \geq 2$ and $\sqrt{\ln n} \leq x = o(\sqrt{n})$ as $n \to \infty$,

$$\left| \ln \frac{\mathbb{P}\big(Z_{n_0,n} \geq x\big)}{1-\Phi(x)} \right| \leq C\frac{x^3}{\sqrt{n}}. \tag{2.5}$$

The results remain valid when $\frac{\mathbb{P}(Z_{n_0,n}\geq x)}{1-\Phi(x)}$ is replaced by $\frac{\mathbb{P}(-Z_{n_0,n}\geq x)}{\Phi(-x)}$.

The uniformity in n_0 is interesting in applications. Due to the uniformity, in (2.4) and (2.5) we can take n_0 as a function of n. Inequality (2.5) coincides with the corresponding result for the random walk (cf. [9] or inequality (1) of [10]), while in inequality (2.4) there is the additional factor $\ln n$ for BPRE. When $n_0 = 0$, the inequalities (2.4) and (2.5) but without the factor $\ln n$ have been proved by Grama et al. [13] under the additional condition that $\mathbb{E}\frac{Z_1^p}{m_0} < \infty$ for some $p > 1$.

Theorem 2.1 implies the following uniform MDP for $Z_{n_0,n}$.

Corollary 2.1. *Assume condition **A1**. Let a_n be any sequence of real numbers satisfying $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$ as $n \to \infty$. Then, for each Borel set B,*

$$\begin{aligned} -\inf_{x \in B^o} \frac{x^2}{2} &\leq \liminf_{n\to\infty} \frac{1}{a_n^2} \ln \inf_{n_0 \in \mathbb{N}} \mathbb{P}\bigg(\frac{Z_{n_0,n}}{a_n} \in B \bigg) \\ &\leq \limsup_{n\to\infty} \frac{1}{a_n^2} \ln \sup_{n_0 \in \mathbb{N}} \mathbb{P}\bigg(\frac{Z_{n_0,n}}{a_n} \in B \bigg) \leq -\inf_{x\in\overline{B}} \frac{x^2}{2}, \end{aligned} \tag{2.6}$$

where B^o and $\overline{B}$ denote the interior and the closure of B, respectively.

The MDP for $Z_{0,n}$ has been established by Huang and Liu [14] (see Theorem 1.6 therein) when the random variable $X = \ln m_0$ satisfies $A_1 \leq m_0$ and $m_0(1+\delta) \leq A^{1+\delta}$ for constants δ, A_1 and A_2 satisfying $\delta > 0$ and $1 < A_1 < A$, and by Wang and Liu [23] under the same condition **A1** but in a more general setting.

From Theorem 2.1, using the inequality $|e^y - 1| \leq e^C |y|$ valid for $|y| \leq C$, we obtain the following result about the uniform equivalence to the normal tail.

Corollary 2.2. *Assume condition **A1**. Then, uniformly for $n_0 \in \mathbb{N}$, as $n \to \infty$,*

$$\frac{\mathbb{P}\big(Z_{n_0,n} \geq x\big)}{1 - \Phi(x)} = 1 + o(1) \tag{2.7}$$

for $x \in [0,\, o(n^{1/6}))$. The result remains valid when $\frac{\mathbb{P}(Z_{n_0,n} \geq x)}{1-\Phi(x)}$ is replaced by $\frac{\mathbb{P}(-Z_{n_0,n}\geq x)}{\Phi(-x)}$.

Inequality (2.7) states that the relative error for normal approximation tends to zero uniformly for $x \in [0, o(n^{1/6}))$. Notice that the normal range $x \in [0, o(n^{1/6}))$ coincides with the random walk case, under Cramér's condition **A1**. In the following Cramér moderate deviation result of type Linnik [16], we give a normal range when the exponential moment condition **A1** is relaxed to

A2. The random variable $X = \ln m_0$ has a sub-exponential moment, i.e. there exist two constants $\lambda_0 > 0$ and $\gamma \in (0, \frac{1}{6}]$ such that

$$\mathbb{E} \exp\{\lambda_0 X^{\frac{4\gamma}{1-2\gamma}}\} < \infty.$$

Theorem 2.2. *Assume condition **A2**. Then (2.7) holds uniformly in $n_0 \in \mathbb{N}$, for $x \in [0,\, o(n^{\gamma}))$, as $n \to \infty$. The result remains valid when $\frac{\mathbb{P}(Z_{n_0,n}\geq x)}{1-\Phi(x)}$ is replaced by $\frac{\mathbb{P}(-Z_{n_0,n}\geq x)}{\Phi(-x)}$.*

Notice that when $\gamma = \frac{1}{6}$, Theorem 2.2 reduces to Corollary 2.2.

We now consider the uniform Berry-Esseen bound for $Z_{n_0,n}$ and $-Z_{n_0,n}$. The following result under the exponential moment condition **A1** can be obtained as a corollary to Theorem 2.1.

Theorem 2.3. *Assume condition **A1**. Then the following holds uniformly in $n_0 \in \mathbb{N}$: for $n \geq 2$,*

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\big(Z_{n_0,n} \leq x\big) - \Phi(x) \Big| \leq C \frac{\ln n}{\sqrt{n}} \tag{2.8}$$

and

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\big(-Z_{n_0,n} \leq x\big) - \Phi(x) \Big| \leq C \frac{\ln n}{\sqrt{n}}. \tag{2.9}$$

In (2.8) and (2.9) there is the additional factor $\ln n$ for BPRE compared to the Berry-Esseen bound for random walks, for which the best rate is $\frac{C}{\sqrt{n}}$. We conjecture that the factor $\ln n$ in (2.8) and (2.9) can be removed, just as in the case where $n_0 = 0$ considered in Grama et al. [13]. Actually Grama et al. [13] gave the more general Berry-Esseen bound $\frac{C}{n^{\rho/2}}$ for $Z_{0,n}$ under a moment condition of order $2+\rho$ on X, with $\rho \in (0,1]$. We shall prove the same bound for $Z_{n_0,n}$ when $\rho \in (0, \frac{\sqrt{5}-1}{2})$, namely, when the following moment condition holds:

A3. There exists a constant $\rho \in (0, \frac{\sqrt{5}-1}{2})$ such that

$$\mathbb{E}X^{2+\rho} < \infty.$$

Theorem 2.4. *Assume condition **A3**. Then uniformly in $n_0 \in \mathbb{N}$,*

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\big(Z_{n_0,n} \leq x\big) - \Phi(x) \Big| \leq \frac{C}{n^{\rho/2}} \tag{2.10}$$

and

$$\sup_{x\in\mathbb{R}} \Big| \mathbb{P}\big(-Z_{n_0,n} \leq x\big) - \Phi(x) \Big| \leq \frac{C}{n^{\rho/2}}. \tag{2.11}$$

For $n_0 = 0$, the inequalities (2.10) and (2.11) have been established by Grama et al. [13, Theorem 1.1] assuming $\mathbb{E}X^{2+\rho} < \infty$ for some $\rho \in (0,1]$ and $\mathbb{E}(\frac{Z_1}{m_0})^p < \infty$ for some $p > 1$.

3. Applications to construction of confidence intervals

Cramér moderate deviations can be applied to constructing confidence intervals for the criticality parameter μ. Assume that σ is known. The following two propositions give two confidence intervals for μ.

Proposition 3.1. *Assume condition **A1**. Let $\kappa_n \in (0,1)$. Assume that*

$$\big| \ln \kappa_n \big| = o\big(n^{1/3}\big). \tag{3.1}$$

Let

$$\Delta_n = \frac{\sigma}{\sqrt{n}} \Phi^{-1}(1 - \kappa_n/2).$$

Then $[A_n, B_n]$, *with*

$$A_n = \frac{1}{n} \ln\Big(\frac{Z_{n_0+n}}{Z_{n_0}}\Big) - \Delta_n \quad \textit{and} \quad B_n = \frac{1}{n} \ln\Big(\frac{Z_{n_0+n}}{Z_{n_0}}\Big) + \Delta_n,$$

is a $1-\kappa_n$ *confidence interval for* μ*, for* n *large enough.*

Proof. By Corollary 2.2, for $0 \le x = o(n^{1/6})$,

$$\frac{\mathbb{P}(Z_{n_0,n} \ge x)}{1-\Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}(Z_{n_0,n} \le -x)}{\Phi(-x)} = 1 + o(1). \tag{3.2}$$

Clearly, the upper $(\kappa_n/2)$th quantile of a standard normal distribution

$$\Phi^{-1}(1-\kappa_n/2) = -\Phi^{-1}(\kappa_n/2) = O(\sqrt{|\ln \kappa_n|}),$$

which, by (3.1), is of order $o\big(n^{1/6}\big)$. Then applying the last equality to (3.2), we have

$$\mathbb{P}\big(Z_{n_0,n} \ge \Phi^{-1}(1-\kappa_n/2)\big) \sim \kappa_n/2 \quad \text{and} \quad \mathbb{P}\big(Z_{n_0,n} \le -\Phi^{-1}(1-\kappa_n/2)\big) \sim \kappa_n/2 \tag{3.3}$$

as $n \to \infty$. Clearly, $Z_{n_0,n} \le \Phi^{-1}(1-\kappa_n/2)$ means that $\mu \ge A_n$, while $Z_{n_0,n} \ge -\Phi^{-1}(1-\kappa_n/2)$ means $\mu \le B_n$. This completes the proof of Proposition 3.1. □

When the risk probability κ_n goes to 0, we have the following result.

Proposition 3.2. *Assume condition* ***A1****. Let* $\kappa_n \in (0,1)$ *such that* $k_n \to 0$*. Assume that*

$$\big|\ln \kappa_n\big| = o\big(n\big). \tag{3.4}$$

Let

$$\Delta_n = \frac{\sigma}{\sqrt{n}} \sqrt{2|\ln(\kappa_n/2)|}.$$

Then $[A_n, B_n]$, *with*

$$A_n = \frac{1}{n} \ln\Big(\frac{Z_{n_0+n}}{Z_{n_0}}\Big) - \Delta_n \quad \textit{and} \quad B_n = \frac{1}{n} \ln\Big(\frac{Z_{n_0+n}}{Z_{n_0}}\Big) + \Delta_n,$$

is a $1-\kappa_n$ *confidence interval for* μ*, for* n *large enough.*

Proof. By Theorem 2.1, we have

$$\frac{\mathbb{P}(Z_{n_0,n} \ge x)}{1-\Phi(x)} = \exp\Big\{\theta_1 C \frac{(\ln n)^3 + x^3}{n^{1/2}}\Big\} \text{ and } \frac{\mathbb{P}(-Z_{n_0,n} \ge x)}{\Phi(-x)} = \exp\Big\{\theta_2 C \frac{(\ln n)^3 + x^3}{n^{1/2}}\Big\} \tag{3.5}$$

uniformly for $0 \le x = o(n^{1/2})$, where $\theta_1, \theta_2 \in [-1, 1]$. Notice that

$$1-\Phi(x_n) = \Phi(-x_n) \sim \frac{1}{x_n\sqrt{2\pi}} e^{-x_n^2/2} = \exp\Big\{-\frac{x_n^2}{2}\Big(1 + \frac{2}{x_n^2}\ln(x_n\sqrt{2\pi})\Big)\Big\}, \ x_n \to \infty.$$

When $k_n \to 0$, the upper $(\kappa_n/2)$th quantile of the distribution

$$1-\Big(1-\Phi\left(x\right)\Big)\exp\left\{\theta_1 C\frac{(\ln n)^3+x^{2+\rho}}{n^{\rho/2}}\right\}$$

has the same order as $\sqrt{2|\ln(\kappa_n/2)|}$, which by (3.4) is of order $o\big(n^{1/2}\big)$ as $n\to\infty$. Then applying (3.5) to $Z_{n_0,n}$ and $-Z_{n_0,n}$, by an argument similar to the proof of Proposition 3.1, we obtain the desired result. □

4. Proof of Theorem 2.1

We should prove Theorem 2.1 for the case of $\frac{\mathbb{P}(Z_{n_0,n}\geq x)}{1-\Phi(x)}$, $x\geq 0$. Thanks to the existence a harmonic moment (see Lemma 4.2), the case of $\frac{\mathbb{P}(-Z_{n_0,n}\geq x)}{\Phi(-x)}$ can be proved in the similar way. To this end, we start with the proofs of Lemmas 4.1 and 4.3, and conclude with the proof of Theorem 2.1. In the sequel, we denote

$$\eta_{n,i}=\frac{X_i-\mu}{\sigma\sqrt{n}},\qquad i=1,...,n_0+n.$$

Then it is easy to see that $\sum_{i=1}^{n}\mathbb{E}\eta_{n,n_0+i}^2=1$. Denote

$$W_{n_0,n}=\frac{W_{n_0+n}}{W_{n_0}}\quad\text{and}\quad W_{n_0,\infty}=\frac{W}{W_{n_0}}.$$

Then $(W_{n_0,n})_{n\geq 0}$ is also a nonnegative martingales both under the quenched law $\mathbb{P}_\xi$ and under the annealed law $\mathbb{P}$ with respect to the natural filtration.

The following lemma gives the upper bound of Theorem 2.1.

Lemma 4.1. *Assume condition **A1**. Then the following holds uniformly in $n_0\in\mathbb{N}$: for $n\geq 2$ and $0\leq x<\sqrt{\ln n}$,*

$$\ln\frac{\mathbb{P}\big(Z_{n_0,n}\geq x\big)}{1-\Phi(x)}\leq C(1+x^3)\frac{\ln n}{\sqrt{n}};\tag{4.1}$$

and for $n\geq 2$ and $\sqrt{\ln n}\leq x=o(\sqrt{n})$,

$$\ln\frac{\mathbb{P}\big(Z_{n_0,n}\geq x\big)}{1-\Phi(x)}\leq C\frac{x^3}{\sqrt{n}}.\tag{4.2}$$

Proof. We first give a proof for (4.2). Clearly, by (2.2), it holds for $x\in\mathbb{R}$,

$$\begin{aligned}\mathbb{P}\Big(Z_{n_0,n}\geq x\Big) &= \mathbb{P}\bigg(\sum_{i=1}^{n}\eta_{n,n_0+i}+\frac{\ln W_{n_0,n}}{\sigma\sqrt{n}}\geq x\bigg)\leq\mathbb{P}\bigg(\sum_{i=1}^{n}\eta_{n,n_0+i}+\frac{(\ln W_{n_0,n})^+}{\sigma\sqrt{n}}\geq x\bigg)\\ &\leq I_1+I_2,\end{aligned}\tag{4.3}$$

where

$$I_1 = \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} \geq (x - \frac{x^2}{\sigma\sqrt{n}}) \Big) \quad \text{and} \quad I_2 = \mathbb{P}\Big(\frac{(\ln W_{n_0,n})^+}{\sigma\sqrt{n}} \geq \frac{x^2}{\sigma\sqrt{n}} \Big).$$

Next, we give some estimations for I_1 and I_2. Notice that $\sum_{i=1}^n \eta_{n,n_0+i}$ is a sum of iid random variables. By upper bound of Cramér moderate deviations for sums of iid random variables (cf. inequality (1.1) of [12]), we obtain for $\sqrt{\ln n} \leq x = o(\sqrt{n})$,

$$I_1 \leq \Big(1 - \Phi(x - \frac{x^2}{\sigma\sqrt{n}})\Big) \exp\Big\{ \frac{C}{\sqrt{n}} (x - \frac{x^2}{\sigma\sqrt{n}})^3 \Big\}.$$

Using the following inequalities

$$\frac{1}{\sqrt{2\pi}(1+x)} e^{-x^2/2} \leq 1 - \Phi\left(x\right) \leq \frac{1}{\sqrt{\pi}(1+x)} e^{-x^2/2}, \quad x \geq 0, \tag{4.4}$$

we deduce that for $x \geq \ln 2$ and $\varepsilon_n \geq 0$,

$$\begin{aligned}
\frac{1 - \Phi\left(x(1-\varepsilon_n)\right)}{1 - \Phi\left(x\right)} &= 1 + \frac{\int_{x(1-\varepsilon_n)}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}{1 - \Phi\left(x\right)} \\
&\leq 1 + \frac{\frac{1}{\sqrt{2\pi}} e^{-x^2(1-\varepsilon_n)^2/2} x\varepsilon_n}{\frac{1}{\sqrt{2\pi}(1+x)} e^{-x^2/2}} \\
&\leq 1 + Cx^2\varepsilon_n e^{Cx^2\varepsilon_n} \\
&\leq \exp\Big\{ Cx^2\varepsilon_n \Big\}.
\end{aligned} \tag{4.5}$$

Hence, for $\sqrt{\ln n} \leq x = o(\sqrt{n})$,

$$I_1 \leq \Big(1 - \Phi(x)\Big) \exp\Big\{ C\frac{x^3}{\sqrt{n}} \Big\}. \tag{4.6}$$

By Markov's inequality and (4.4), it is easy to see that for $\sqrt{\ln n} \leq x = o(\sqrt{n})$,

$$\begin{aligned}
I_2 &= \mathbb{P}\Big(W_{n_0,n} \geq \exp\big\{x^2\big\} \Big) \\
&\leq \exp\big\{ -x^2 \big\} \mathbb{E} W_{n_0,n} = \exp\big\{ -x^2 \big\} \\
&\leq C\frac{1+x}{\sqrt{n}} \Big(1 - \Phi(x)\Big).
\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) together, we obtain for $\sqrt{\ln n} \leq x = o(\sqrt{n})$,

$$\begin{aligned}
\mathbb{P}\Big(Z_{n_0,n} \geq x \Big) &\leq \Big(1 - \Phi(x)\Big) \exp\Big\{ C_1\frac{x^3}{\sqrt{n}} \Big\} + C_2\frac{(1+x)}{\sqrt{n}} \Big(1 - \Phi(x)\Big) \\
&\leq \Big(1 - \Phi(x)\Big) \exp\Big\{ C_3\frac{x^3}{\sqrt{n}} \Big\},
\end{aligned}$$

which gives the desired inequality for $\sqrt{\ln n} \leq x = o(n^{1/2})$.

Next, we give a proof for (4.1). By an argument similar to that of (4.3), we have for $x \in \mathbb{R}$,

$$\mathbb{P}\big(Z_{n_0,n} \geq x\big) \ \ \leq \ \ I_3 + I_4, \tag{4.8}$$

where

$$I_3 = \mathbb{P}\bigg(\sum_{i=1}^{n} \eta_{n,n_0+i} \geq (x - \frac{x^2 \ln n}{\sigma\sqrt{n}})\bigg) \quad \text{ and } \quad I_4 = \mathbb{P}\bigg(\frac{(\ln W_{n_0,n})^+}{\sigma\sqrt{n}} \geq \frac{x^2 \ln n}{\sigma\sqrt{n}}\bigg).$$

With arguments similar to that of (4.6) and (4.7), we get for $1 \leq x < \sqrt{\ln n}$,

$$\begin{aligned} I_3 \ &\leq \ \Big(1 - \Phi(x - \frac{x^2 \ln n}{\sigma\sqrt{n}})\Big) \exp\Big\{\frac{C_1}{\sqrt{n}}(x - \frac{x^2 \ln n}{\sigma\sqrt{n}})^3\Big\} \\ &\leq \ \Big(1 - \Phi(x)\Big) \exp\Big\{C_2 x^3 \frac{\ln n}{\sqrt{n}}\Big\} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} I_4 \ &= \ \mathbb{P}\Big(W_{n_0,n} \geq \exp\Big\{x^2 \ln n\Big\}\Big) \\ &\leq \ \exp\Big\{-x^2 \ln n\Big\}\mathbb{E}W_{n_0,n} = \exp\Big\{-x^2 \ln n\Big\} \\ &\leq \ C\frac{1+x}{\sqrt{n}}\Big(1 - \Phi(x)\Big). \end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10) together, we obtain the desired inequality for $1 \leq x < \sqrt{\ln n}$.

Again by an argument similar to that of (4.3), we have for $x \in \mathbb{R}$,

$$\mathbb{P}\big(Z_{n_0,n} \geq x\big) \ \ \leq \ \ I_5 + I_6, \tag{4.11}$$

where

$$I_5 = \mathbb{P}\bigg(\sum_{i=1}^{n} \eta_{n,n_0+i} \geq (x - \frac{\ln n}{\sigma\sqrt{n}})\bigg) \quad \text{ and } \quad I_6 = \mathbb{P}\bigg(\frac{(\ln W_{n_0,n})^+}{\sigma\sqrt{n}} \geq \frac{\ln n}{\sigma\sqrt{n}}\bigg).$$

By the Berry-Esseen bound for a sum of iid random variables, we get for $0 \leq x < 1$,

$$\begin{aligned} I_5 \ &\leq \ \Big(1 - \Phi(x - \frac{\ln n}{\sigma\sqrt{n}})\Big)\Big(1 + \frac{C_1}{\sqrt{n}}\Big) \\ &\leq \ \Big(1 - \Phi(x)\Big)\Big(1 + C_2\frac{\ln n}{\sqrt{n}}\Big)\Big(1 + \frac{C_1}{\sqrt{n}}\Big) \\ &\leq \ \Big(1 - \Phi(x)\Big)\Big(1 + C_3\frac{\ln n}{\sqrt{n}}\Big) \end{aligned} \tag{4.12}$$

and, with an arguments similar to that of (4.7),

$$I_6 = \mathbb{P}\Big(W_{n_0,n} \geq \exp\Big\{\ln n\Big\}\Big) \leq \exp\Big\{-\ln n\Big\}\mathbb{E}W_{n_0,n} = \frac{1}{n}. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) together, we obtain for $0 \leq x < 1$,

$$\begin{aligned}\mathbb{P}\Big(Z_{n_0,n} \geq x\Big) &\leq \Big(1-\Phi(x)\Big)\Big(1+C_3\frac{\ln n}{\sigma\sqrt{n}}\Big)+\frac{1}{n} \\ &\leq \Big(1-\Phi(x)\Big)\Big(1+C_4\frac{\ln n}{\sqrt{n}}\Big) \\ &\leq \Big(1-\Phi(x)\Big)\exp\Big\{C_4\frac{\ln n}{\sqrt{n}}\Big\},\end{aligned}$$

which gives the desired inequality for $0 \leq x < 1$. This completes the proof of Lemma 4.1. □

To prove the lower bound of Theorem 2.1, we shall make use of the following lemma (see Theorem 3.1 of Grama et al. [13]). The lemma shows that condition **A1** implies the existence of harmonic moments of order $a > 0$.

Lemma 4.2. *Assume condition **A1**. There exists a constant $a_0 > 0$ such that for $a \in (0, a_0)$,*

$$\mathbb{E}W^{-a} < \infty. \tag{4.14}$$

The following lemma gives the lower bound of Theorem 2.1.

Lemma 4.3. *Assume condition **A1**. Then the following holds uniformly in $n_0 \in \mathbb{N}$: for $n \geq 2$ and $0 \leq x < \sqrt{\ln n}$,*

$$\ln\frac{\mathbb{P}\big(Z_{n_0,n} \geq x\big)}{1-\Phi(x)} \geq -C(1+x^3)\frac{\ln n}{\sqrt{n}}; \tag{4.15}$$

and for $n \geq 2$ and $\sqrt{\ln n} \leq x = o(\sqrt{n})$, $n \to \infty$,

$$\ln\frac{\mathbb{P}\big(Z_{n_0,n} \geq x\big)}{1-\Phi(x)} \geq -C\frac{x^3}{\sqrt{n}}. \tag{4.16}$$

Proof. We first give a proof for (4.16). Clearly, it holds for all $x \in \mathbb{R}$,

$$\mathbb{P}\Big(Z_{n_0,n} \geq x\Big) = \mathbb{P}\Big(\sum_{i=1}^{n}\eta_{n,n_0+i} + \frac{\ln W_{n_0,n}}{\sigma\sqrt{n}} \geq x\Big) \geq \mathbb{P}\Big(\sum_{i=1}^{n}\eta_{n,n_0+i} - \frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}} \geq x\Big).$$

Notice that

$$\Big\{\sum_{i=1}^{n}\eta_{n,n_0+i}\geq x+\frac{4x^2}{a\sigma\sqrt{n}}\Big\}=\Big\{\sum_{i=1}^{n}\eta_{n,n_0+i}-\frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}}+\frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}}\geq x+\frac{4x^2}{a\sigma\sqrt{n}}\Big\}$$
$$\subset\Big\{\sum_{i=1}^{n}\eta_{n,n_0+i}-\frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}}\geq x\Big\}\cup\Big\{\frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}}\geq\frac{4x^2}{a\sigma\sqrt{n}}\Big\},$$

where a is a constant satisfying $a\in(0,\min\{a_0,1\})$ with a_0 given by Lemma 4.2. Thus, we have

$$\begin{aligned}\mathbb{P}\Big(Z_{n_0,n}\geq x\Big) &\geq \mathbb{P}\Big(\sum_{i=1}^{n}\eta_{n,n_0+i}\geq x+\frac{4x^2}{a\sigma\sqrt{n}}\Big)-\mathbb{P}\Big(\frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}}\geq\frac{4x^2}{a\sigma\sqrt{n}}\Big)\\ &=: P_1-P_2. \end{aligned}\tag{4.17}$$

Next, we give estimations for terms P_1 and P_2. By lower bound of Cramér moderate deviations for sums of iid random variables (cf. inequality (1.1) of [10]), we obtain for $\sqrt{\ln n}\leq x=o(\sqrt{n})$,

$$P_1\geq\Big(1-\Phi(x+\frac{4x^2}{a\sigma\sqrt{n}})\Big)\exp\Big\{-\frac{C}{\sqrt{n}}(x+\frac{4x^2}{a\sigma\sqrt{n}})^3\Big\}.$$

By an argument similar to that of (4.5), we deduce that for $x\geq\ln 2$ and $0\leq\varepsilon_n\leq 1$,

$$\frac{1-\Phi\left(x(1+\varepsilon_n)\right)}{1-\Phi\left(x\right)}\geq\exp\Big\{-Cx^2\varepsilon_n\Big\}.\tag{4.18}$$

Hence, for $\sqrt{\ln n}\leq x=o(\sqrt{n})$,

$$P_1\geq\Big(1-\Phi(x)\Big)\exp\Big\{-C\frac{x^3}{\sqrt{n}}\Big\}.\tag{4.19}$$

By Markov's inequality, it is easy to see that for $\sqrt{\ln n}\leq x=o(\sqrt{n})$,

$$\begin{aligned}P_2 &= \mathbb{P}\Big(\ln W_{n_0+n}-\ln W_{n_0}\leq -4x^2/a\Big)\\ &\leq \mathbb{P}\Big(\ln W_{n_0+n}\leq -2x^2/a\Big)+\mathbb{P}\Big(-\ln W_{n_0}\leq -2x^2/a\Big)\\ &\leq \exp\Big\{-2x^2\Big\}\mathbb{E}W_{n_0+n}^{-a}+\exp\Big\{-2x^2/a\Big\}\mathbb{E}W_{n_0}\\ &= \exp\Big\{-2x^2\Big\}\mathbb{E}W_{n_0+n}^{-a}+\exp\Big\{-2x^2/a\Big\}.\end{aligned}\tag{4.20}$$

By (2.1), it is known that $W_n\to W$ in $\mathbb{L}^1$. Then we have $W_n=\mathbb{E}[W|\mathcal{F}_n]$ a.s. By Jensen's inequality, we get

$$W_{n_0+n}^{-a}=(\mathbb{E}[W|\mathcal{F}_{n_0+n}])^{-a}\leq\mathbb{E}[W^{-a}|\mathcal{F}_{n_0+n}].$$

Taking expectations with respect to $\mathbb{P}$ on both sides of the last inequality, we deduce that

$$\mathbb{E}W_{n_0+n}^{-a} \leq \mathbb{E}W^{-a}. \tag{4.21}$$

By Lemma 4.2, we have for $\sqrt{\ln n} \leq x = o(\sqrt{n})$,

$$\begin{aligned} P_2 &\leq \exp\Big\{-2x^2\Big\}\mathbb{E}W^{-a} + \exp\Big\{-2x^2\Big\} \\ &\leq C_1 \exp\Big\{-2x^2\Big\} \\ &\leq C_2 \frac{x}{\sqrt{n}}\Big(1-\Phi(x)\Big)\exp\Big\{-x^2\Big\}. \end{aligned} \tag{4.22}$$

Combining (4.17), (4.19) and (4.22) together, we obtain for $\sqrt{\ln n} \leq x = o(\sqrt{n})$,

$$\begin{aligned} \mathbb{P}\big(Z_{n_0,n} \geq x\big) &\geq \Big(1-\Phi(x)\Big)\exp\Big\{-C_1\frac{x^3}{\sqrt{n}}\Big\} - C_2\frac{x}{\sqrt{n}}\Big(1-\Phi(x)\Big)\exp\Big\{-x^2\Big\} \\ &\geq \Big(1-\Phi(x)\Big)\exp\Big\{-C_3\frac{x^3}{\sqrt{n}}\Big\}, \end{aligned}$$

which gives the desired inequality for $\sqrt{\ln n} \leq x = o(\sqrt{n})$.

For $0 \leq x < \sqrt{\ln n}$, the assertion of Lemma 4.3 follows by a similar argument, but in (4.17) with $\frac{4x^2}{a\sigma\sqrt{n}}$ replaced by $\frac{4x^2\ln n}{a\sigma\sqrt{n}}$ when $1 \leq x < \sqrt{\ln n}$ and $\frac{4x^2}{a\sigma\sqrt{n}}$ replaced by $\frac{4\ln n}{a\sigma\sqrt{n}}$ when $0 \leq x < 1$, and accordingly in the subsequent statements. Then we get the desired inequality for $0 \leq x < \sqrt{\ln n}$. This completes the proof of Lemma 4.3. □

5. Proof of Corollary 2.1

We only give a proof for the case of $Z_{n_0,n}$. We first show that for each Borel set B,

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \ln \sup_{n_0\in\mathbb{N}} \mathbb{P}\bigg(\frac{Z_{n_0,n}}{a_n} \in B\bigg) \leq -\inf_{x\in\overline{B}} \frac{x^2}{2}. \tag{5.1}$$

When $B = \emptyset$, the last inequality is obvious, with the convention $-\inf_{x\in\emptyset}\frac{x^2}{2} = \infty$. Thus, we may assume that $B \neq \emptyset$. Given a Borel set $B \subset \mathbb{R}$, let $x_0 = \inf_{x\in B}|x|$. Clearly, we have $x_0 \geq \inf_{x\in\overline{B}}|x|$. Then, by Theorem 2.1,

$$\begin{aligned} \sup_{n_0\in\mathbb{N}} \mathbb{P}\Big(Z_{n_0,n} \in a_n B\Big) &\leq \sup_{n_0\in\mathbb{N}} \mathbb{P}\Big(\big|Z_{n_0,n}\big| \geq a_n x_0\Big) \\ &\leq 2\Big(1-\Phi\left(a_n x_0\right)\Big)\exp\bigg\{C\left(1+(a_n x_0)^3\right)\frac{1+\mathbf{1}_{[0,\sqrt{\ln n})}(a_n x_0)\ln n}{\sqrt{n}}\bigg\}. \end{aligned}$$

Using (4.4), after some calculations, we get

$$\limsup_{n\to\infty} \frac{1}{a_n^2} \ln \sup_{n_0\in\mathbb{N}} \mathbb{P}\bigg(\frac{Z_{n_0,n}}{a_n} \in B\bigg) \leq -\frac{x_0^2}{2} \leq -\inf_{x\in\overline{B}} \frac{x^2}{2},$$

which gives (5.1).

Next, we show that

$$\liminf_{n\to\infty} \frac{1}{a_n^2} \ln \inf_{n_0\in\mathbb{N}} \mathbb{P}\Big(\frac{Z_{n_0,n}}{a_n} \in B\Big) \geq - \inf_{x\in B^o} \frac{x^2}{2}. \tag{5.2}$$

When $B^o = \emptyset$, the last inequality is obvious, with the convention $\inf_{x\in\emptyset} \frac{x^2}{2} = \infty$. Therefore, we may assume that $B^o \neq \emptyset$. For any given small $\varepsilon_1 > 0$, there exists an $x_0 \in B^o$, such that

$$0 < \frac{x_0^2}{2} \leq \inf_{x\in B^o} \frac{x^2}{2} + \varepsilon_1.$$

Since B^o is an open set, for $x_0 \in B^o$ and all small enough $\varepsilon_2 \in (0, |x_0|]$, it holds $(x_0-\varepsilon_2, x_0+\varepsilon_2] \subset B^o$. Therefore, $|x_0| \geq \inf_{x\in B^o} |x|$. Without loss of generality, we may assume that $x_0 > 0$. Obviously, we have

$$\begin{aligned}
\inf_{n_0\in\mathbb{N}} \mathbb{P}\big(Z_{n_0,n} \in a_n B\big) &\geq \inf_{n_0\in\mathbb{N}} \mathbb{P}\big(Z_{n_0,n} \in (a_n(x_0-\varepsilon_2), a_n(x_0+\varepsilon_2)]\big) \\
&= \inf_{n_0\in\mathbb{N}} \Big(\mathbb{P}\big(Z_{n_0,n} \geq a_n(x_0-\varepsilon_2)\big) - \mathbb{P}\big(Z_{n_0,n} \geq a_n(x_0+\varepsilon_2)\big)\Big).
\end{aligned}$$

Again by Theorem 2.1, it is easy to see that

$$\lim_{n\to\infty} \frac{\sup_{n_0\in\mathbb{N}} \mathbb{P}\big(Z_{n_0,n} \geq a_n(x_0+\varepsilon_2)\big)}{\inf_{n_0\in\mathbb{N}} \mathbb{P}\big(Z_{n_0,n} \geq a_n(x_0-\varepsilon_2)\big)} = 0.$$

Therefore, by Theorem 2.1, it holds for all n large enough,

$$\begin{aligned}
\inf_{n_0\in\mathbb{N}} \mathbb{P}\Big(\frac{Z_{n_0,n}}{a_n} \in B\Big) &\geq \inf_{n_0\in\mathbb{N}} \frac{1}{2}\mathbb{P}\Big(Z_{n_0,n} \geq a_n(x_0-\varepsilon_2)\Big) \\
&\geq \frac{1}{2}\Big(1 - \Phi\left(a_n(x_0-\varepsilon_2)\right)\Big) \\
&\quad \times \exp\bigg\{ - C\left(1 + (a_n(x_0-\varepsilon_2))^3\right) \frac{1 + \mathbf{1}_{[0,\sqrt{\ln n})}(a_n(x_0-\varepsilon_2)) \ln n}{\sqrt{n}} \bigg\}.
\end{aligned}$$

Using (4.4), after some calculations, we get

$$\liminf_{n\to\infty} \frac{1}{a_n^2} \ln \inf_{n_0\in\mathbb{N}} \mathbb{P}\Big(\frac{Z_{n_0,n}}{a_n} \in B\Big) \geq -\frac{1}{2}(x_0-\varepsilon_2)^2.$$

Letting $\varepsilon_2 \to 0$, we deduce that

$$\liminf_{n\to\infty} \frac{1}{a_n^2} \ln \inf_{n_0\in\mathbb{N}} \mathbb{P}\Big(\frac{Z_{n_0,n}}{a_n} \in B\Big) \geq -\frac{x_0^2}{2} \geq - \inf_{x\in B^o} \frac{x^2}{2} - \varepsilon_1.$$

Since ε_1 can be arbitrarily small, we get (5.2). Combining (5.1) and (5.2) together, we complete the proof of Corollary 2.1. □

6. Proof of Theorem 2.2

To prove Theorem 2.2, we shall make use of the following lemma.

Lemma 6.1. *Assume condition* ***A2****. There exists a constant $b_0 > 0$ such that for $b \in (0, b_0)$,*

$$\mathbb{E}\exp\{b\,|\ln W|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{W\le 1\}} < \infty. \tag{6.1}$$

Proof. Denote

$$\phi(t) = \mathbb{E}e^{-tW},$$

for $t \ge 0$. From inequality (2.7) of Grama et al. [13], there exists a positive constant K such that for all $A > 1$, $n \ge 1$ and $t \ge KA^n$,

$$\phi(t) \le \alpha^n + \mathbb{P}(\Pi_n \ge A^n), \tag{6.2}$$

where $\alpha \in (0, 1)$. Choose A such that $\ln A > \mu$. By condition **A2** and Theorem 2.1 of [11], there exists a constant $C > 0$ such that for all $n \ge 1$,

$$\mathbb{P}(\Pi_n \ge A^n) = \mathbb{P}\big(S_n - n\mu \ge n(\ln A - \mu)\big) \le \exp\big\{ - Cn^{\frac{4\gamma}{1-2\gamma}}\big\}.$$

From (6.2), we get for all $n \ge 1$ and $t \ge KA^n$,

$$\phi(t) \le \exp\{-Cn^{\frac{4\gamma}{1-2\gamma}}\}. \tag{6.3}$$

Now for any $t \ge KA$, there exists an integer n_0 depending on t such that

$$KA^{n_0+1} > t \ge KA^{n_0},$$

so that

$$n_0 > \frac{\ln(t/K)}{\ln A} - 1.$$

Then, for any $t \ge KA^2$,

$$\begin{aligned}\phi(t) &\le \exp\{-Cn_0^{\frac{4\gamma}{1-2\gamma}}\} \le \exp\Big\{ - C(\frac{\ln(t/K)}{\ln A} - 1)^{\frac{4\gamma}{1-2\gamma}}\Big\} \\ &\le \exp\Big\{ - C_1(\ln t)^{\frac{4\gamma}{1-2\gamma}}\Big\},\end{aligned} \tag{6.4}$$

where the last line follows by the fact that $(\frac{\ln(t/K)}{\ln A} - 1)/\ln t \to 1/\ln A$ as $t \to \infty$. By the facts that $\mathbb{P}(W \le t^{-1}) \le e\phi(t), t > 0$, and

$$\mathbb{E}\exp\{b\,|\ln W|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{W\le 1\}} = \frac{4b\gamma}{1-2\gamma}\int_1^\infty \frac{1}{t}\mathbb{P}(W \le t^{-1})(\ln t)^{\frac{6\gamma-1}{1-2\gamma}}\exp\{b\,(\ln t)^{\frac{4\gamma}{1-2\gamma}}\}dt,$$

it follows that $\mathbb{E}\exp\{b\,|\ln W|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{W\le 1\}} < \infty$ for $b \in [0, C_1)$. □

Now we are in position to prove Theorem 2.2. We only give a proof of Theorem 2.2 for the case of $\frac{\mathbb{P}(Z_{n_0,n}\geq x)}{1-\Phi(x)}$. For the case of $\frac{\mathbb{P}(-Z_{n_0,n}\geq x)}{\Phi(-x)}$, Theorem 2.2 can be proved in a similar way. We first consider the case of $\sqrt{\ln n} \leq x = o(n^\gamma)$. Clearly, it holds for $\sqrt{\ln n} \leq x = o(n^\gamma)$,

$$\begin{aligned}
\mathbb{P}\Big(Z_{n_0,n} \geq x\Big) &= \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} + \frac{\ln W_{n_0,n}}{\sigma\sqrt{n}} \geq x\Big) \\
&\geq \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} \geq x + \frac{2x^2}{\sigma n^{3\gamma}}\Big) - \mathbb{P}\Big(\frac{\ln W_{n_0,n}}{\sigma\sqrt{n}} \geq \frac{2x^2}{\sigma n^{3\gamma}}\Big) \\
&=: T_1 - T_2. \qquad (6.5)
\end{aligned}$$

Next, we give estimations for terms T_1 and T_2. By lower bound of Linnik type Cramér moderate deviations for sums of iid random variables (cf. Linnik [16]), we deduce that

$$T_1 \geq \Big(1 - \Phi(x + \frac{2x^2}{\sigma n^{3\gamma}})\Big)\Big(1 - g_n(x)\Big),$$

where $g_n(x) \geq 0$ and $g_n(x) \to 0$ uniformly for $0 \leq x = o(n^\gamma)$ as $n \to \infty$. Hence, by (4.18), we get for $\sqrt{\ln n} \leq x = o(n^\gamma)$,

$$\begin{aligned}
T_1 &\geq \Big(1-\Phi(x)\Big)\exp\Big\{-C\frac{x^3}{n^{3\gamma}}\Big\}\Big(1-g_n(x)\Big) \\
&\geq \Big(1-\Phi(x)\Big)\Big(1-g_n(x)-C\frac{x^3}{n^{3\gamma}}\Big). \qquad (6.6)
\end{aligned}$$

By Markov's inequality, it is easy to see that for $\sqrt{\ln n} \leq x = o(n^\gamma)$,

$$\begin{aligned}
T_2 &= \mathbb{P}\Big(\ln W_{n_0+n} - \ln W_{n_0} \geq 2x^2 n^{\frac{1}{2}-3\gamma}\Big) \\
&\leq \mathbb{P}\Big(\ln W_{n_0+n} \geq x^2 n^{\frac{1}{2}-3\gamma}\Big) + \mathbb{P}\Big(-\ln W_{n_0} \geq x^2 n^{\frac{1}{2}-3\gamma}\Big) \\
&\leq \exp\Big\{-x^2 n^{\frac{1}{2}-3\gamma}\Big\}\mathbb{E}W_{n_0+n} \\
&\quad + \exp\Big\{-\frac{b_0}{2}(x^2 n^{\frac{1}{2}-3\gamma})^{\frac{4\gamma}{1-2\gamma}}\Big\}\mathbb{E}\exp\{\frac{b_0}{2}\,|\ln W_{n_0}|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{W_{n_0}\leq 1\}},
\end{aligned}$$

where b_0 is given by Lemma 6.1. Recall that $W_n = \mathbb{E}[W|\mathcal{F}_n]$ a.s. Since the function

$$f(x) = \exp\{\frac{b_0}{2}\,|\ln x|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{x\leq 1\}}$$

is convex in the interval $(0,1]$, by Jensen's inequality, we get

$$f(W_n) = f(\mathbb{E}[W|\mathcal{F}_n]) \leq \mathbb{E}[f(W)|\mathcal{F}_n].$$

Taking expectations with respect to $\mathbb{P}$ on both sides of the last inequality, we deduce that

$$\mathbb{E}[\exp\{\frac{b_0}{2}\,|\ln W_n|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{W_n\leq 1\}}] \leq \mathbb{E}[\exp\{\frac{b_0}{2}\,|\ln W|^{\frac{4\gamma}{1-2\gamma}}\}\mathbf{1}_{\{W\leq 1\}}].$$

By the fact $\mathbb{E}W_{n_0+n}=1$ and Lemma 6.1, we have for $\sqrt{\ln n}\leq x=o(n^\gamma)$,

$$\begin{aligned} T_2 &\leq C\exp\Big\{-\frac{\min\{2,b_0\}}{2}(x^2 n^{\frac{1}{2}-3\gamma})^{\frac{4\gamma}{1-2\gamma}}\Big\} \\ &\leq \frac{C_1}{\sqrt{n}}\Big(1-\Phi(x)\Big). \end{aligned} \tag{6.7}$$

Combining (6.5), (6.6) and (6.7) together, we obtain for $\sqrt{\ln n}\leq x=o(n^\gamma)$,

$$\begin{aligned} \mathbb{P}\big(Z_{n_0,n}\geq x\big) &\geq \big(1-\Phi(x)\big)\Big(1-g_n(x)-C\frac{x^3}{n^{3\gamma}}\Big)-\frac{C_1}{\sqrt{n}}\big(1-\Phi(x)\big) \\ &= \big(1-\Phi(x)\big)\Big(1-g_n(x)-C\frac{x^3}{n^{3\gamma}}-\frac{C_1}{\sqrt{n}}\Big). \end{aligned} \tag{6.8}$$

Similarly, we can prove that for $\sqrt{\ln n}\leq x=o(n^\gamma)$,

$$\mathbb{P}\big(Z_{n_0,n}\geq x\big) \leq \big(1-\Phi(x)\big)\Big(1+g_n(x)+C\frac{x^3}{n^{3\gamma}}+\frac{C_1}{\sqrt{n}}\Big). \tag{6.9}$$

Combining (6.8) and (6.9) together, we have

$$\mathbb{P}\big(Z_{n_0,n}\geq x\big) = \big(1-\Phi(x)\big)\big(1+o(1)\big)$$

uniformly for $\sqrt{\ln n}\leq x=o(n^\gamma)$. This completes the proof of Theorem 2.2 for $\sqrt{\ln n}\leq x=o(n^\gamma)$. For $0\leq x\leq\sqrt{\ln n}$, Theorem 2.2 can be proved in a similar way, but in (6.5) with $\frac{2x^2}{\sigma n^{3\gamma}}$ replaced by $\frac{2\ln n}{\sigma n^{3\gamma}}$, and accordingly in the subsequent statements. □

7. Proof of Theorem 2.3

We only give a proof of (2.8). Inequality (2.9) can be proved in a similar way. Clearly, it holds

$$\begin{aligned} &\sup_{x\in\mathbb{R}}\big|\mathbb{P}\big(Z_{n_0,n}\leq x\big)-\Phi(x)\big| \\ &\quad\leq \sup_{x>n^{1/8}}\big|\mathbb{P}\big(Z_{n_0,n}\leq x\big)-\Phi(x)\big| + \sup_{0\leq x\leq n^{1/8}}\big|\mathbb{P}\big(Z_{n_0,n}\leq x\big)-\Phi(x)\big| \\ &\quad+ \sup_{-n^{1/8}\leq x\leq 0}\big|\mathbb{P}\big(Z_{n_0,n}\leq x\big)-\Phi(x)\big| + \sup_{x<-n^{1/8}}\big|\mathbb{P}\big(Z_{n_0,n}\leq x\big)-\Phi(x)\big| \\ &\quad=: H_1+H_2+H_3+H_4. \end{aligned} \tag{7.1}$$

By Theorem 2.1 and (4.4), it is easy to see that

$$\begin{aligned} H_1 &= \sup_{x>n^{1/8}} \Big|\mathbb{P}\big(Z_{n_0,n} > x\big) - \big(1-\Phi\left(x\right)\big)\Big| \\ &\leq \sup_{x>n^{1/8}} \mathbb{P}\big(Z_{n_0,n} > x\big) + \sup_{x>n^{1/8}} \big(1-\Phi\left(x\right)\big) \\ &\leq \mathbb{P}\big(Z_{n_0,n} > n^{1/8}\big) + \big(1-\Phi\left(n^{1/8}\right)\big) \\ &\leq \big(1-\Phi\left(n^{1/8}\right)\big)e^{C} + \exp\{-\frac{1}{2}n^{1/4}\} \\ &\leq C_1 \frac{\ln n}{\sqrt{n}} \end{aligned}$$

and

$$\begin{aligned} H_4 &\leq \sup_{x<-n^{1/8}} \mathbb{P}\big(Z_{n_0,n} \leq x\big) + \sup_{x<-n^{1/8}} \Phi\left(x\right) \\ &\leq \mathbb{P}\big(Z_{n_0,n} \leq -n^{1/8}\big) + \Phi\left(-n^{1/8}\right) \\ &\leq \Phi\left(-n^{1/8}\right)e^{C} + \exp\{-\frac{1}{2}n^{1/4}\} \\ &\leq C_2 \frac{\ln n}{\sqrt{n}}. \end{aligned}$$

By Theorem 2.1 and the inequality $|e^x - 1| \leq |x|e^{|x|}$, we have

$$\begin{aligned} H_2 &= \sup_{0\leq x\leq n^{1/8}} \Big|\mathbb{P}\big(Z_{n_0,n} > x\big) - \big(1-\Phi\left(x\right)\big)\Big| \\ &\leq \sup_{0\leq x\leq n^{1/8}} \big(1-\Phi(x)\big)\Big|e^{C(1+x^3)(\ln n)/\sqrt{n}} - 1\Big| \\ &\leq C_3 \frac{\ln n}{\sqrt{n}} \end{aligned}$$

and

$$\begin{aligned} H_3 &= \sup_{-n^{1/8}\leq x\leq 0} \Big|\mathbb{P}\big(Z_{n_0,n} \leq x\big) - \Phi\left(x\right)\Big| \\ &\leq \sup_{-n^{1/8}\leq x\leq 0} \Phi(x)\Big|e^{C(1+|x|^3)(\ln n)/\sqrt{n}} - 1\Big| \\ &\leq C_4 \frac{\ln n}{\sqrt{n}}. \end{aligned}$$

Applying the bounds of H_1, H_2, H_3 and H_4 to (7.1), we obtain inequality (2.8). This completes the proof of Theorem 2.3. □

8. Proof of Theorem 2.4

We should prove Theorem 2.4 for the case of $Z_{n_0,n}$. The cases of $-Z_{n_0,n}$ can be proved in the similar way. To prove the lower bound of Theorem 2.4, we shall make use of the following lemma, which is an improvement on Lemma 2.3 of Grama et al. [13], in which $p \in (0, 1+\rho/2)$ instead of $p \in (0, 1+\rho)$.

Lemma 8.1. *Assume condition **A3**. Then for* $p \in (0, 1+\rho)$,

$$\mathbb{E}|\ln W|^p < \infty \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \mathbb{E}|\ln W_n|^p < \infty. \tag{8.1}$$

Proof. By Jensen's inequality, it is enough to prove Lemma 8.1 for $p \in [1, 2+\rho)$. Recall that $\mu = \mathbb{E}X$ and $S_n = \ln \Pi_n = \sum_{i=1}^n X_i$. Then S_n is a sum of iid random variables with $(2+\rho)$-moments. Choose A such that $\ln A > \mu$. Clearly, we have $A > 1$. By Nagaev's inequality (see Corollary 1.8 of Nagaev [17] or Corollary 2.5 of [11]), there exists a constant $C' > 0$ such that for $n \geq 1$,

$$\mathbb{P}(\Pi_n \geq A^n) = \mathbb{P}\big(S_n - n\mu \geq n(\ln A - \mu)\big) \leq \frac{C'}{n^{1+\rho}}.$$

From (6.2), we get for all $n \geq 1$ and $t \geq KA^n$,

$$\phi(t) \leq \frac{C}{n^{1+\rho}}. \tag{8.2}$$

Now for any $t \geq KA$, set n_0 be the integer such that $KA^{n_0+1} > t \geq KA^{n_0}$, so that

$$n_0 > \frac{\ln(t/K)}{\ln A} - 1.$$

Thus, by (8.2), for any $t \geq KA^2$,

$$\phi(t) \leq \frac{C}{n_0^{1+\rho}} \leq C\Big(\frac{\ln(t/K)}{\ln A} - 1\Big)^{-1-\rho} \leq C_0(\ln t)^{-1-\rho}. \tag{8.3}$$

By the facts that $\mathbb{P}(W \leq t^{-1}) \leq e\phi(t), t > 0$, and

$$\mathbb{E}|\ln W|^p \mathbf{1}_{\{W\leq 1\}} = p\int_1^\infty \frac{1}{t}(\ln t)^{p-1}\mathbb{P}(W \leq t^{-1})dt,$$

it follows that $\mathbb{E}|\ln W|^p \mathbf{1}_{\{W\leq 1\}} < \infty$ for $p \in [1, 1+\rho)$. Using the inequality $|\ln x|^p \leq Cx, x > 1$, we deduce that $\mathbb{E}|\ln W|^p \mathbf{1}_{\{W>1\}} \leq C\mathbb{E}W \leq C\mathbb{E}W_n = C$. Thus, we have

$$\mathbb{E}|\ln W|^p = \mathbb{E}|\ln W|^p \mathbf{1}_{\{W\leq 1\}} + \mathbb{E}|\ln W|^p \mathbf{1}_{\{W>1\}} < \infty.$$

Notice that $x \mapsto |\ln x|^p \mathbf{1}_{\{0<x\leq 1\}}$ is a non-negative and convex function for $p \in [1, 1+\rho)$. By Lemma 2.1 of Huang and Liu [14], we have $\sup_n \mathbb{E}|\ln W_n|^p \mathbf{1}_{\{W_n \leq 1\}} = \mathbb{E}|\ln W|^p \mathbf{1}_{\{W\leq 1\}} < \infty$. It is also easy to see that for $p \in [1, 1+\rho)$,

$$\begin{aligned}\sup_n \mathbb{E}|\ln W_n|^p &\leq \sup_n \mathbb{E}|\ln W_n|^p \mathbf{1}_{\{W_n\leq 1\}} + \sup_n \mathbb{E}|\ln W_n|^p \mathbf{1}_{\{W_n>1\}} \\ &\leq \mathbb{E}|\ln W|^p \mathbf{1}_{\{W\leq 1\}} + C \sup_n \mathbb{E} W_n = \mathbb{E}|\ln W|^p \mathbf{1}_{\{W\leq 1\}} + C < \infty.\end{aligned}$$

This completes the proof of Lemma 8.1. □

Now we are in position to prove Theorem 2.4. We first prove that for $x \in \mathbb{R}$,

$$\mathbb{P}\Big(Z_{n_0,n} \leq x\Big) - \Phi(x) \leq \frac{C}{n^{\rho/2}}. \tag{8.4}$$

It is easy to see that

$$\mathbb{P}\Big(Z_{n_0,n} \leq x\Big) \leq \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} - \frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}} \leq x\Big) \leq R_1 + R_2, \tag{8.5}$$

where

$$R_1 = \mathbb{P}\Big(\sum_{i=1}^{n} \eta_{n,n_0+i} \leq x + \frac{2}{\sigma n^{\rho/2}}\Big) \quad \text{and} \quad R_2 = \mathbb{P}\Big(\frac{(\ln W_{n_0,n})^-}{\sigma\sqrt{n}} \geq \frac{2}{\sigma n^{\rho/2}}\Big).$$

Next, we give estimations for R_1 and R_2. By the Berry-Esseen bound for a sum of iid random variables, we obtain

$$\begin{aligned}R_1 &\leq \Phi(x + \frac{2}{\sigma n^{\rho/2}}) + \frac{C_1}{n^{\rho/2}} \\ &\leq \Phi(x) + \frac{C_2}{n^{\rho/2}}. \end{aligned} \tag{8.6}$$

Notice that when $\rho \in (0, (\sqrt{5}-1)/2)$, we have $p := \frac{\rho}{1-\rho} < 1+\rho$. By Markov's inequality and Lemma 8.1, it is easy to see that

$$\begin{aligned}R_2 &\leq \mathbb{P}\Big(|\ln(W_{n_0+n}/W_{n_0})| \geq 2n^{(1-\rho)/2}\Big) \\ &\leq \mathbb{P}\Big(|\ln W_{n_0+n}| + |\ln W_{n_0}| \geq 2n^{(1-\rho)/2}\Big) \\ &\leq \mathbb{P}\Big(|\ln W_{n_0+n}| \geq n^{(1-\rho)/2}\Big) + \mathbb{P}\Big(|\ln W_{n_0}| \geq n^{(1-\rho)/2}\Big) \\ &\leq n^{-p(1-\rho)/2}\mathbb{E}|\ln W_{n_0+n}|^p + n^{-p(1-\rho)/2}\mathbb{E}|\ln W_{n_0}|^p \leq 2n^{-\rho/2} \sup_n \mathbb{E}|\ln W_n|^p \\ &\leq \frac{C}{n^{\rho/2}}. \end{aligned} \tag{8.7}$$

Applying the upper bounds of R_1 and R_2 to (8.5), we obtain (8.4).

Next, we prove that for $x \in \mathbb{R}$,

$$\mathbb{P}\big(Z_{n_0,n} \le x\big) - \Phi(x) \ge -\frac{C}{n^{\rho/2}}. \tag{8.8}$$

Clearly, it holds

$$\mathbb{P}\bigg(Z_{n_0,n} \le x\bigg) \ge \mathbb{P}\bigg(\sum_{i=1}^{n} \eta_{n,n_0+i} + \frac{(\ln W_{n_0,n})^+}{\sigma\sqrt{n}} \le x\bigg) \ge R_3 - R_4, \tag{8.9}$$

where

$$R_3 = \mathbb{P}\bigg(\sum_{i=1}^{n} \eta_{n,n_0+i} \le x - \frac{1}{\sigma n^{\rho/2}}\bigg) \quad \text{and} \quad R_4 = \mathbb{P}\bigg(\frac{(\ln W_{n_0,n})^+}{\sigma\sqrt{n}} \ge \frac{1}{\sigma n^{\rho/2}}\bigg).$$

Again by the Berry-Esseen bound for a sum of iid random variables, we obtain

$$\begin{aligned} R_3 &\ge \Phi(x - \frac{1}{\sigma n^{\rho/2}}) - \frac{C_1}{n^{\rho/2}} \\ &\ge \Phi(x) - \frac{C_2}{n^{\rho/2}}. \end{aligned} \tag{8.10}$$

Again by Markov's inequality, we get

$$\begin{aligned} R_4 &\le \mathbb{P}\Big(W_{n_0,n} \ge \exp\{n^{(1-\rho)/2}\}\Big) \\ &\le \exp\{-n^{(1-\rho)/2}\}\mathbb{E}W_{n_0,n} = \exp\{-n^{(1-\rho)/2}\} \\ &\le \frac{C}{n^{\rho/2}}. \end{aligned} \tag{8.11}$$

Applying the upper bounds of R_3 and R_4 to (8.9), we obtain (8.8).

Combining (8.4) and (8.8) together, we get

$$\Big|\mathbb{P}\big(Z_{n_0,n} \le x\big) - \Phi(x)\Big| \le \frac{C}{n^{\rho/2}}, \tag{8.12}$$

which gives the desired inequality. □

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