Harnack Inequalities for Functional SDEs Driven by Subordinate Brownian Motions*

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Abstract

Using coupling by change of measure and an approximation technique, Wang's Harnack inequalities are established for a class of functional SDEs driven by subordinate Brownian motions. The results cover the corresponding ones in the case without delay.

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1 Introduction

The dimension-free Harnack inequality was firstly introduced by Wang [13] to derive the log-Sobolev inequality on Riemannian manifolds. As a weaker version of the power-Harnack inequality, the log-Harnack inequality was considered in [10] for semi-linear SDEs. These two Harnack-type inequalities have been intensively investigated and applied for various finite- and infinite-dimensional SDEs and SPDEs driven by Brownian noise; we refer to the monograph by F.-Y. Wang [14] for a systematic theory on dimension-free Harnack inequalities and applications. For the functional SDEs and SPDEs, the Harnack inequalities are also investigated in [1, 2], see also [12] for SDEs with non-Lipschitz coefficients and [7, 8] for SDEs with Dini drifts. However, the noise in all the above results is assumed to contain a Brownian motion part. The central aim of this work is to establish Harnack inequalities

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for functional SDEs driven by subordinate Brownian motions, which form a very large class of Lévy processes. It turns out that our results cover the corresponding ones in the case without delay derived by J. Wang and F.-Y. Wang [15] (cf. [4] for an improved estimate).

Fix a constant $r_0 \geq 0$. Denote by $\mathscr{C}$ the family of all right continuous functions $f : [-r_0, 0] \to \mathbb{R}^d$ with left limits. To characterize the state space, equip $\mathscr{C}$ with the norm $\|\cdot\|_2$ given by

$$\|\xi\|_2^2 := \int_{-r_0}^0 |\xi(s)|^2 \,\mathrm{d}s + |\xi(0)|^2, \quad \xi \in \mathscr{C}.$$

For $f : [-r_0, \infty) \to \mathbb{R}^d$, we will denote $f_t \in \mathscr{C}$, $t \geq 0$, the corresponding segment process, by

$$f_t(s) := f(t+s), \quad s \in [-r_0, 0].$$

Let $S = (S(t))_{t\geq 0}$ be a subordinator (without killing), i.e. a nondecreasing Lévy process on $[0, \infty)$ starting at $S(0) = 0$. Due to the independent and stationary increments property, it is uniquely determined by the Laplace transform

$$\mathbb{E}\,\mathrm{e}^{-uS(t)} = \mathrm{e}^{-t\phi(u)}, \quad u > 0,\, t \geq 0,$$

where the characteristic (Laplace) exponent $\phi : (0, \infty) \to (0, \infty)$ is a Bernstein function with $\phi(0+) := \lim_{r\downarrow 0} \phi(r) = 0$, i.e. a C^∞-function such that $(-1)^{n-1}\phi^{(n)} \geq 0$ for all $n \in \mathbb{N}$. Every such ϕ has a unique Lévy–Khintchine representation (cf. [11, Theorem 3.2])

$$\phi(u) = \kappa u + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-ux}\right) \nu(\mathrm{d}x), \quad u > 0, \tag{1.1}$$

where $\kappa \geq 0$ is the drift parameter and ν is a Lévy measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge x)\, \nu(\mathrm{d}x) < \infty.$$

It is clear that $\tilde{\phi}(u) := \phi(u) - \kappa u$ is the Bernstein function of the subordinator $\tilde{S}(t) := S(t) - \kappa t$ having zero drift and Lévy measure ν.

Consider the following functional SDEs on $\mathbb{R}^d$:

$$\mathrm{d}X(t) = b(X(t))\,\mathrm{d}t + B(X_t)\,\mathrm{d}t + \mathrm{d}W(S(t)), \tag{1.2}$$

where $W = (W(t))_{t\geq 0}$ is a d-dimensional standard Brownian motion with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, $S = (S(t))_{t\geq 0}$ is a subordinator with Bernstein function of the form (1.1) and independent of W, $b : \mathbb{R}^d \to \mathbb{R}^d$ is continuous, and $B : \mathscr{C} \to \mathbb{R}^d$ is measurable.

We shall need the following conditions on b and B:

(H) There exist constants $K \in \mathbb{R}$ and $K_1 \geq 0$ such that

$$\langle x - y, b(x) - b(y)\rangle \leq K|x - y|^2, \quad x, y \in \mathbb{R}^d,$$

and

$$|B(\xi) - B(\eta)| \leq K_1\|\xi - \eta\|_2, \quad \xi, \eta \in \mathscr{C}.$$

Remark 1.1. *The condition* **(H)** *ensures the existence, uniqueness and non-explosion of the solution to* (1.2). *Indeed, letting* $L(t) = W(S(t))$, $\hat{b}(t,x) = b(x + L(t))$ *and* $\hat{B}(t,\xi) = B(\xi + L_t)$, *one has*

$$\langle x - y, \hat{b}(t,x) - \hat{b}(t,y)\rangle \le K|x-y|^2, \quad x, y \in \mathbb{R}^d, t \ge 0$$

and

$$|\hat{B}(t,\xi) - \hat{B}(t,\eta)| \le K_1\|\xi - \eta\|_2, \quad \xi, \eta \in \mathscr{C}, t \ge 0.$$

Then the following (functional) ordinary differential equation

$$\mathrm{d}\hat{X}(t) = \hat{b}(t, \hat{X}(t))\,\mathrm{d}t + \hat{B}(t, \hat{X}_t)\,\mathrm{d}t$$

has a unique solution which does not explode in finite time; setting $X(t) := \hat{X}(t) + L(t)$, *we know that* (1.2) *has a unique non-explosive solution.*

The remaining part of this paper is organized as follows. In Section 2, we state our main results. By using the coupling by change of measure and an approximation technique, we establish in Section 3 the Harnack inequalities for functional SDEs driven by non-random time-changed Brownian motions. Section 4 is devoted to the proofs of Theorem 2.1 and Example 2.4 presented in Section 2.

2 Main results

For $\xi \in \mathscr{C}$, let X_t^ξ be the solution to (1.2) with $X_0 = \xi$. Let P_t be the semigroup associated to X_t^ξ, i.e.

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi), \quad f \in \mathscr{B}_b(\mathscr{C}).$$

As usual, we make the conventions: $\frac{1}{0} = \infty$ and $0 \cdot \infty = 0$.

Theorem 2.1. *Assume* **(H)** *and let* $T > r_0$ *and* S *be a subordinator with Bernstein function* ϕ *of the form* (1.1).

i) *For any* $\xi, \eta \in \mathscr{C}$ *and* $f \in \mathscr{B}_b(\mathscr{C})$ *with* $f \ge 1$,

$$\begin{aligned} P_T \log f(\eta) \le \log P_T f(\xi) &+ |\xi(0) - \eta(0)|^2\, \mathbb{E}\left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}S(t)\right)^{-1} \\ &+ \frac{K_1^2}{\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}|\xi(0)-\eta(0)|^2\right). \end{aligned}$$

ii) *For any* $p > 1$, $\xi, \eta \in \mathscr{C}$ *and non-negative* $f \in \mathscr{B}_b(\mathscr{C})$,

$$\begin{aligned} (P_T f)^p(\eta) \le P_T f^p(\xi) &\left(\mathbb{E}\exp\left[\frac{p}{(p-1)^2}\,|\xi(0)-\eta(0)|^2\left(\int_0^{T-r_0}\mathrm{e}^{-2Kt}\,\mathrm{d}S(t)\right)^{-1}\right]\right)^{p-1} \\ &\times \exp\left[\frac{p}{p-1}\frac{K_1^2}{\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}\,|\xi(0)-\eta(0)|^2\right)\right]. \end{aligned}$$

Remark 2.2. *If $B = 0$, then we can choose $r_0 = 0$ and $K_1 = 0$, and thus the assertions in Theorem 2.1 reduce to the ones derived in [15] for the case without delay.*

For a measurable space $(E, \mathscr{F})$, let $\mathscr{P}(E)$ denote the family of all probability measures on $(E, \mathscr{F})$. For $\mu, \nu \in \mathscr{P}(E)$, the entropy $\mathrm{Ent}(\nu|\mu)$ is defined by

$$\mathrm{Ent}(\nu|\mu) := \begin{cases} \int (\log \frac{\mathrm{d}\nu}{\mathrm{d}\mu})\, \mathrm{d}\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu, \\ \infty, & \text{otherwise;} \end{cases}$$

the total variation distance $\|\mu - \nu\|_{\mathrm{var}}$ is defined by

$$\|\mu - \nu\|_{\mathrm{var}} := \sup_{A \in \mathscr{F}} |\mu(A) - \nu(A)|.$$

By Pinsker's inequality (see [3, 9]),

$$\|\mu - \nu\|_{\mathrm{var}}^2 \le \frac{1}{2} \mathrm{Ent}(\nu|\mu), \quad \mu, \nu \in \mathscr{P}(E). \tag{2.1}$$

For $\xi \in \mathscr{C}$, let $P_T(\xi, \cdot)$ be the distribution of X_T^ξ. The following corollary is a direct consequence of Theorem 2.1, see [14, Theorem 1.4.2] for the proof; we also refer to [14, Subsection 1.4.1] for an in-depth explanation of the applications of the Harnack inequalities.

Corollary 2.3. *Let the assumptions in Theorem 2.1 hold. Then the following assertions hold.*

i) *For any $\xi, \eta \in \mathscr{C}$, $P_T(\xi, \cdot)$ is equivalent to $P_T(\eta, \cdot)$ and*

$$\begin{aligned} \mathrm{Ent}\big(P_T(\xi, \cdot)|P_T(\eta, \cdot)\big) \le\ & |\xi(0) - \eta(0)|^2 \mathbb{E} \left(\int_0^{T - r_0} \mathrm{e}^{-2Kt}\, \mathrm{d}S(t) \right)^{-1} \\ & + \frac{K_1^2}{\kappa} \left(r_0 \|\xi - \eta\|_2^2 + (T+1) \frac{\mathrm{e}^{2K(T - r_0)} - 1}{2K} |\xi(0) - \eta(0)|^2 \right), \end{aligned}$$

which together with Pinsker's inequality (2.1) *implies that*

$$\begin{aligned} 2\|P_T(\xi, \cdot) - P_T(\eta, \cdot)\|_{\mathrm{var}}^2 \le\ & |\xi(0) - \eta(0)|^2 \mathbb{E} \left(\int_0^{T - r_0} \mathrm{e}^{-2Kt}\, \mathrm{d}S(t) \right)^{-1} \\ & + \frac{K_1^2}{\kappa} \left(r_0 \|\xi - \eta\|_2^2 + (T+1) \frac{\mathrm{e}^{2K(T - r_0)} - 1}{2K} |\xi(0) - \eta(0)|^2 \right). \end{aligned}$$

ii) *For any $p > 1$ and $\xi, \eta \in \mathscr{C}$,*

$$\begin{aligned} P_T \left\{ \left(\frac{\mathrm{d}P_T(\xi, \cdot)}{\mathrm{d}P_T(\eta, \cdot)} \right)^{1/(p-1)} \right\} (\xi) \le\ & \mathbb{E} \exp \left[\frac{p}{(p-1)^2} |\xi(0) - \eta(0)|^2 \left(\int_0^{T - r_0} \mathrm{e}^{-2Kt}\, \mathrm{d}S(t) \right)^{-1} \right] \\ & \times \exp \left[\frac{p}{(p-1)^2} \frac{K_1^2}{\kappa} \left(r_0 \|\xi - \eta\|_2^2 + (T+1) \frac{\mathrm{e}^{2K(T - r_0)} - 1}{2K} |\xi(0) - \eta(0)|^2 \right) \right]. \end{aligned}$$

Example 2.4. *Assume that* **(H)** *holds with $K = 0$. Let $T > r_0$, and S be a subordinator with Bernstein function $\phi(u) \geq \kappa u + cu^\alpha$ ($\kappa \geq 0$, $c > 0$, $0 < \alpha < 1$).*
i) *There exists $C = C(\alpha, c) > 0$ such that for any $\xi, \eta \in \mathscr{C}$ and $f \in \mathscr{B}_b(\mathscr{C})$ with $f \geq 1$,*

$$P_T \log f(\eta) \leq \log P_T f(\xi) + \frac{C|\xi(0) - \eta(0)|^2}{[\kappa(T - r_0)] \vee (T - r_0)^{1/\alpha}} + \frac{K_1^2}{\kappa}\left(r_0\|\xi - \eta\|_2^2 + (T+1)(T - r_0)|\xi(0) - \eta(0)|^2\right).$$

ii) *If in addition $1/2 < \alpha < 1$, then there exists $C = C(\alpha, c) > 0$ such that for any $p > 1$, $\xi, \eta \in \mathscr{C}$ and non-negative $f \in \mathscr{B}_b(\mathscr{C})$,*

$$\begin{aligned}(P_T f)^p(\eta) \leq P_T f^p(\xi) \cdot \exp\left[\frac{p}{p-1}\frac{K_1^2}{\kappa}\left(r_0\|\xi - \eta\|_2^2 + (T+1)(T - r_0)|\xi(0) - \eta(0)|^2\right)\right] \\ \times \exp\left[C\left(\frac{p|\xi(0) - \eta(0)|^2}{(p-1)(T - r_0)^{1/\alpha}} + \frac{[p|\xi(0) - \eta(0)|^2]^{\alpha/(2\alpha - 1)}}{[(p-1)(T - r_0)]^{1/(2\alpha - 1)}}\right) \wedge \frac{p|\xi(0) - \eta(0)|^2}{(p-1)\kappa(T - r_0)}\right].\end{aligned}$$

3 Harnack inequalities under deterministic time-change

Let $\ell : [0, \infty) \to [0, \infty)$ be a sample path of S (with Bernstein function ϕ of the form (1.1)), which is a non-decreasing and càdlàg function with $\ell(0) = 0$. By **(H)** and the same explanation as in Remark 1.1, for any $\xi \in \mathscr{C}$, the following functional SDE has a unique non-explosive solution with $X_0^\ell = \xi$:

$$\mathrm{d}X^\ell(t) = b(X^\ell(t))\,\mathrm{d}t + B(X_t^\ell)\,\mathrm{d}t + \mathrm{d}W(\ell(t)). \tag{3.1}$$

We denote the solution by $X_t^{\ell,\xi}$. Let

$$P_t^\ell f(\xi) = \mathbb{E} f(X_t^{\ell,\xi}), \quad t \geq 0, f \in \mathscr{B}_b(\mathscr{C}), \xi \in \mathscr{C}.$$

Proposition 3.1. *Assume* **(H)** *and let $T > r_0$.*
i) *For any $\xi, \eta \in \mathscr{C}$ and $f \in \mathscr{B}_b(\mathscr{C})$ with $f \geq 1$,*

$$\begin{aligned}P_T^\ell \log f(\eta) \leq \log P_T^\ell f(\xi) + |\xi(0) - \eta(0)|^2 \left(\int_0^{T - r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell(t)\right)^{-1} \\ + \frac{K_1^2}{\kappa}\left(r_0\|\xi - \eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T - r_0)} - 1}{2K}|\xi(0) - \eta(0)|^2\right).\end{aligned}$$

ii) *For any $p > 1$, $\xi, \eta \in \mathscr{C}$ and non-negative $f \in \mathscr{B}_b(\mathscr{C})$,*

$$\begin{aligned}\left(P_T^\ell f(\eta)\right)^p \leq P_T^\ell f^p(\xi) \exp\left[\frac{p}{p-1}|\xi(0) - \eta(0)|^2 \left(\int_0^{T - r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell(t)\right)^{-1}\right] \\ \times \exp\left[\frac{p}{p-1}\frac{K_1^2}{\kappa}\left(r_0\|\xi - \eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T - r_0)} - 1}{2K}|\xi(0) - \eta(0)|^2\right)\right].\end{aligned}$$

Following the line of [4, 6, 15, 16, 17], for $\varepsilon \in (0,1)$, consider the following regularization of ℓ:

$$\ell^\varepsilon(t) := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \ell(s)\,\mathrm{d}s + \varepsilon t = \int_0^1 \ell(\varepsilon s + t)\,\mathrm{d}s + \varepsilon t, \quad t \geq 0.$$

It is clear that, for each $\varepsilon \in (0,1)$, the function ℓ^ε is absolutely continuous, strictly increasing and satisfies for any $t \geq 0$

$$\ell^\varepsilon(t) \downarrow \ell(t) \quad \text{as } \varepsilon \downarrow 0. \tag{3.2}$$

For $\xi \in \mathscr{C}$, let $X_t^{\ell^\varepsilon,\xi}$ be the solution to the following functional SDE with initial value ξ:

$$\mathrm{d}X^{\ell^\varepsilon,\xi}(t) = b(X^{\ell^\varepsilon,\xi}(t))\,\mathrm{d}t + B(X_t^{\ell^\varepsilon,\xi})\,\mathrm{d}t + \mathrm{d}W(\ell^\varepsilon(t) - \ell^\varepsilon(0)).$$

The associated semigroup is denoted by $P_t^{\ell^\varepsilon}$. Note that this SDE is indeed driven by Brownian motions and thus the method of coupling and Girsanov's transformation can be used to establish the dimension-free Harnack inequalities for $P_t^{\ell^\varepsilon}$.

Lemma 3.2. *Fix $\varepsilon \in (0,1)$, assume* **(H)** *and let $T > r_0$.*
i) *For any $\xi, \eta \in \mathscr{C}$ and $f \in \mathscr{B}_b(\mathscr{C})$ with $f \geq 1$,*

$$\begin{aligned} P_T^{\ell^\varepsilon} \log f(\eta) \leq \log P_T^{\ell^\varepsilon} f(\xi) &+ |\xi(0) - \eta(0)|^2 \left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell^\varepsilon(t)\right)^{-1} \\ &+ \frac{K_1^2}{\kappa}\left(r_0\|\xi - \eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)} - 1}{2K}|\xi(0) - \eta(0)|^2\right). \end{aligned}$$

ii) *For any $p > 1$, $\xi, \eta \in \mathscr{C}$ and non-negative $f \in \mathscr{B}_b(\mathscr{C})$,*

$$\begin{aligned} \left(P_T^{\ell^\varepsilon} f(\eta)\right)^p \leq P_T^{\ell^\varepsilon} f^p(\xi) &\exp\left[\frac{p}{p-1}|\,\xi(0) - \eta(0)|^2 \left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell^\varepsilon(t)\right)^{-1}\right] \\ &\times \exp\left[\frac{p}{p-1}\frac{K_1^2}{\kappa}\left(r_0\|\xi - \eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)} - 1}{2K}|\xi(0) - \eta(0)|^2\right)\right]. \end{aligned}$$

Proof. Due to the existence of the delay part B, we will construct couplings as follows. Let Y_t solve the equation

$$\begin{aligned} \mathrm{d}Y(t) &= b(Y(t))\,\mathrm{d}t + B(X_t^{\ell^\varepsilon,\xi})\,\mathrm{d}t \\ &\quad + \lambda(t)\mathbb{1}_{[0,\tau)}(t)\frac{X^{\ell^\varepsilon,\xi}(t) - Y(t)}{|X^{\ell^\varepsilon,\xi}(t) - Y(t)|}|\xi(0) - \eta(0)|\,\mathrm{d}\ell^\varepsilon(t) + \mathrm{d}W(\ell^\varepsilon(t) - \ell^\varepsilon(0)) \end{aligned} \tag{3.3}$$

with $Y_0 = \eta$, where

$$\lambda(t) := \frac{\mathrm{e}^{-Kt}}{\int_0^{T-r_0} \mathrm{e}^{-2Ks}\,\mathrm{d}\ell^\varepsilon(s)}, \quad t \geq 0,$$

and

$$\tau := T \wedge \inf\{t \geq 0\,;\, X^{\ell^\varepsilon,\xi}(t) = Y(t)\}$$

is the coupling time. It is clear that $(X^{\ell^\varepsilon,\xi}(t), Y(t))$ is well defined for $t < \tau$. By **(H)**, we have

$$\mathrm{d}|X^{\ell^\varepsilon,\xi}(t) - Y(t)| \le K|X^{\ell^\varepsilon,\xi}(t) - Y(t)|\,\mathrm{d}t - \lambda(t)|\xi(0) - \eta(0)|\,\mathrm{d}\ell^\varepsilon(t), \quad t \in [0,\tau).$$

Thus, for $t \in [0,\tau)$,

$$\begin{aligned}
|X^{\ell^\varepsilon,\xi}(t) - Y(t)| &\le \mathrm{e}^{Kt}|\xi(0) - \eta(0)| \left(1 - \int_0^t \mathrm{e}^{-Ks}\lambda(s)\,\mathrm{d}\ell^\varepsilon(s)\right) \\
&\le \frac{\mathrm{e}^{Kt}\int_t^{T-r_0} \mathrm{e}^{-2Ks}\,\mathrm{d}\ell^\varepsilon(s)}{\int_0^{T-r_0} \mathrm{e}^{-2Ks}\,\mathrm{d}\ell^\varepsilon(s)}\,|\xi(0) - \eta(0)| \\
&=: \Gamma(t)|\xi(0) - \eta(0)|.
\end{aligned} \tag{3.4}$$

If $\tau(\omega) > T - r_0$ for some $\omega \in \Omega$, we can take $t = T - r_0$ in the above inequality to get

$$0 < |X^{\ell^\varepsilon,\xi}(t)(\omega) - Y(t)(\omega)| \le 0,$$

which is absurd. Therefore, $\tau \le T - r_0$. Letting $Y(t) = X^{\ell^\varepsilon,\xi}(t)$ for $t \in [\tau, T]$, $Y(t)$ solves (3.3) for $t \in [\tau, T]$. In particular, $X_T^{\ell^\varepsilon,\xi} = Y_T$. Moreover, by (3.4) and $\tau \le T - r_0$, we have

$$|X^{\ell^\varepsilon,\xi}(t) - Y(t)|^2 \le |\xi(0) - \eta(0)|^2\Gamma(t)^2\mathbb{1}_{[0,T-r_0]}(t), \quad t \in [0,T]. \tag{3.5}$$

Denote by $\gamma^\varepsilon : [\ell^\varepsilon(0), \infty) \to [0,\infty)$ the inverse function of ℓ^ε. Then $\ell^\varepsilon(\gamma^\varepsilon(t)) = t$ for $t \ge \ell^\varepsilon(0)$, $\gamma^\varepsilon(\ell^\varepsilon(t)) = t$ for $t \ge 0$, and $t \mapsto \gamma^\varepsilon(t)$ is absolutely continuous and strictly increasing. Let

$$\widetilde{W}(t) := \int_0^t \Psi(u)\,\mathrm{d}u + W(t) \quad \text{and} \quad M_t := -\int_0^t \langle \Psi(u), \mathrm{d}W(u)\rangle, \quad t \ge 0,$$

where $\Psi(u) := \Phi \circ \gamma^\varepsilon(u + \ell^\varepsilon(0))$ and

$$\Phi(u) := [B(X_u^{\ell^\varepsilon,\xi}) - B(Y_u)]\frac{1}{(\ell^\varepsilon)'(u)} + \lambda(u)\mathbb{1}_{[0,\tau)}(u)\frac{X^{\ell^\varepsilon,\xi}(u)) - Y(u)}{|X^{\ell^\varepsilon,\xi}(u) - Y(u)|}\,|\xi(0) - \eta(0)|.$$

By **(H)**, the compensator of the martingale M_t satisfies, for $t \ge 0$,

$$\begin{aligned}
\langle M\rangle_t &= \int_0^t |\Psi(u)|^2\,\mathrm{d}u \le \int_0^T |\Phi(s)|^2\,\mathrm{d}\ell^\varepsilon(s) \\
&\le 2K_1^2 \int_0^T \|X_t^{\ell^\varepsilon,\xi} - Y_t\|_2^2 \frac{1}{(\ell^\varepsilon)'(t)}\,\mathrm{d}t + 2|\xi(0) - \eta(0)|^2 \int_0^{T-r_0} |\lambda(t)|^2\,\mathrm{d}\ell^\varepsilon(t).
\end{aligned} \tag{3.6}$$

Recalling that ℓ is a sample path of the subordinator S with drift parameter $\kappa \ge 0$, one has

$$(\ell^\varepsilon)'(t) = \frac{\ell(t+\varepsilon) - \ell(t)}{\varepsilon} + \varepsilon > \kappa,$$

and therefore

$$\int_0^T \|X_t^{\ell^\varepsilon,\xi} - Y_t\|_2^2 \frac{1}{(\ell^\varepsilon)'(t)}\,\mathrm{d}t \le \frac{1}{\kappa}\int_0^T \|X_t^{\ell^\varepsilon,\xi} - Y_t\|_2^2\,\mathrm{d}t. \tag{3.7}$$

Next, we focus on the estimate of $\int_0^T \|X_t^{\ell^\varepsilon,\xi} - Y_t\|_2^2\,\mathrm{d}t$. Firstly, it is clear that for any $t \in [0,T]$,

$$\begin{aligned}\int_{-r_0}^0 |X_t^{\ell^\varepsilon,\xi}(s) - Y_t(s)|^2\,\mathrm{d}s &= \int_{-r_0}^0 |X^{\ell^\varepsilon,\xi}(t+s) - Y(t+s)|^2\,\mathrm{d}s\\ &= \int_{t-r_0}^t |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s.\end{aligned}$$

This implies that for $t \in [0, r_0]$,

$$\begin{aligned}\int_{-r_0}^0 |X_t^{\ell^\varepsilon,\xi}(s) - Y_t(s)|^2\,\mathrm{d}s &= \left(\int_{t-r_0}^0 + \int_0^t\right) |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s\\ &\le \left(\int_{-r_0}^0 + \int_0^{r_0}\right) |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s\\ &= \int_{-r_0}^0 |\xi(s) - \eta(s)|^2\,\mathrm{d}s + \int_0^{r_0} |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s,\end{aligned} \tag{3.8}$$

and by (3.5), for $t \in [r_0, T]$,

$$\int_{-r_0}^0 |X_t^{\ell^\varepsilon,\xi}(s) - Y_t(s)|^2\,\mathrm{d}s \le \int_0^{T-r_0} |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s. \tag{3.9}$$

Combining (3.5), (3.8) and (3.9), we obtain

$$\begin{aligned}&\int_0^T \|X_t^{\ell^\varepsilon,\xi} - Y_t\|_2^2\,\mathrm{d}t\\ &= \int_0^{r_0}\left(\int_{-r_0}^0 |X_t^{\ell^\varepsilon,\xi}(s) - Y_t(s)|^2\,\mathrm{d}s\right)\mathrm{d}t + \int_{r_0}^T\left(\int_{-r_0}^0 |X_t^{\ell^\varepsilon,\xi}(s) - Y_t(s)|^2\,\mathrm{d}s\right)\mathrm{d}t\\ &\quad + \int_0^T |X^{\ell^\varepsilon,\xi}(t) - Y(t)|^2\,\mathrm{d}t\\ &\le r_0\left(\int_{-r_0}^0 |\xi(s) - \eta(s)|^2\,\mathrm{d}s + \int_0^{r_0} |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s\right)\\ &\quad + (T-r_0)\left(\int_0^{T-r_0} |X^{\ell^\varepsilon,\xi}(s) - Y(s)|^2\,\mathrm{d}s\right) + \int_0^{T-r_0} |X^{\ell^\varepsilon,\xi}(t) - Y(t)|^2\,\mathrm{d}t\\ &\le r_0\|\xi - \eta\|_2^2 + (T+1)|\xi(0) - \eta(0)|^2 \int_0^{T-r_0} \Gamma(s)^2\,\mathrm{d}s\\ &\le r_0\|\xi - \eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)} - 1}{2K}\,|\xi(0) - \eta(0)|^2,\end{aligned} \tag{3.10}$$

where in the last inequality we have used $\Gamma(s) \le \mathrm{e}^{Ks}$ for $s \in [0, T-r_0]$. By the definition of $\lambda(t)$, it is easy to see that

$$2|\xi(0)-\eta(0)|^2 \int_0^{T-r_0} |\lambda(t)|^2 \,\mathrm{d}\ell^\varepsilon(t) \le 2|\xi(0)-\eta(0)|^2 \left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell^\varepsilon(t)\right)^{-1}.$$

This, together with (3.6), (3.7) and (3.10), yields that for any $t \ge 0$

$$\begin{aligned}\langle M\rangle_t \le{}& \frac{2K_1^2}{\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}|\xi(0)-\eta(0)|^2\right)\\ &+ 2|\xi(0)-\eta(0)|^2\left(\int_0^{T-r_0}\mathrm{e}^{-2Kt}\,\mathrm{d}\ell^\varepsilon(t)\right)^{-1}.\end{aligned}\tag{3.11}$$

By Novikov's criterion, we have $\mathbb{E}R = 1$, where

$$R := \exp\left[M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{1}{2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right].$$

According to Girsanov's theorem, $(\widetilde{W}(t))_{0\le t\le \ell^\varepsilon(T)-\ell^\varepsilon(0)}$ is a d-dimensional Brownian motion under the new probability measure $R\mathbb{P}$. Rewrite (3.3) as

$$\mathrm{d}Y(t) = b(Y(t))\,\mathrm{d}t + B(Y_t)\,\mathrm{d}t + \mathrm{d}\widetilde{W}(\ell^\varepsilon(t)-\ell^\varepsilon(0)).$$

Thus, the distribution of $(Y_t)_{0\le t\le T}$ under $R\mathbb{P}$ coincides with that of $(X_t^{\ell^\varepsilon,\eta})_{0\le t\le T}$ under $\mathbb{P}$; in particular, it holds that for any $f \in \mathscr{B}_b(\mathscr{C})$,

$$\mathbb{E}f(X_T^{\ell^\varepsilon,\eta}) = \mathbb{E}_{R\mathbb{P}}f(Y_T) = \mathbb{E}\left[Rf(Y_T)\right] = \mathbb{E}\big[Rf(X_T^{\ell^\varepsilon,\xi})\big].\tag{3.12}$$

By (3.12), the Young inequality (cf. [14, p. 24]), and the observation that

$$\begin{aligned}\log R &= -\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} \langle \Psi(u), \mathrm{d}W(u)\rangle - \frac{1}{2}\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} |\Psi(u)|^2\,\mathrm{d}u\\ &= -\int_0^{\ell^\varepsilon(T)-\ell^\varepsilon(0)} \langle \Psi(u), \mathrm{d}\widetilde{W}(u)\rangle + \frac{1}{2}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)},\end{aligned}$$

we get that, for any $f \in \mathscr{B}_b(\mathscr{C})$ with $f \ge 1$,

$$\begin{aligned}P_T^{\ell^\varepsilon}\log f(\eta) &= \mathbb{E}\log f(X_T^{\ell^\varepsilon,\eta})\\ &= \mathbb{E}\big[R\log f(X_T^{\ell^\varepsilon,\xi})\big]\\ &\le \log\mathbb{E}f(X_T^{\ell^\varepsilon,\xi}) + \mathbb{E}[R\log R]\\ &= \log P_T^{\ell^\varepsilon}f(\xi) + \mathbb{E}_{R\mathbb{P}}\log R\\ &= \log P_T^{\ell^\varepsilon}f(\xi) + \frac{1}{2}\,\mathbb{E}_{R\mathbb{P}}\langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}.\end{aligned}$$

Combining this with (3.11), we obtain the desired log-Harnack inequality.

Next, we prove the second assertion of the theorem. For any non-negative $f \in \mathscr{B}_b(\mathscr{C})$, we find with (3.12) and the Hölder inequality

$$
\begin{aligned}
(P_T^{\ell^\varepsilon} f)^p(\eta) &= \big(\mathbb{E} f(X_T^{\ell^\varepsilon,\eta})\big)^p \\
&= \big(\mathbb{E}\big[R f(X_T^{\ell^\varepsilon,\xi})\big]\big)^p \\
&\le P_T^{\ell^\varepsilon} f^p(\xi) \cdot \big(\mathbb{E}\big[R^{p/(p-1)}\big]\big)^{p-1}.
\end{aligned}
\tag{3.13}
$$

Since by (3.11)

$$
\begin{aligned}
R^{p/(p-1)} &= \exp\left[\frac{p}{p-1} M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p}{2(p-1)} \langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right] \\
&= \exp\left[\frac{p}{2(p-1)^2} \langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right] \\
&\quad \times \exp\left[\frac{p}{p-1} M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p^2}{2(p-1)^2} \langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right] \\
&\le \exp\left[\frac{p}{(p-1)^2} |\xi(0)-\eta(0)|^2 \left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\, \mathrm{d}\ell^\varepsilon(t)\right)^{-1}\right] \\
&\quad \times \exp\left[\frac{pK_1^2}{(p-1)^2\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}|\xi(0)-\eta(0)|^2\right)\right] \\
&\quad \times \exp\left[\frac{p}{p-1} M_{\ell^\varepsilon(T)-\ell^\varepsilon(0)} - \frac{p^2}{2(p-1)^2} \langle M\rangle_{\ell^\varepsilon(T)-\ell^\varepsilon(0)}\right],
\end{aligned}
$$

and noting the fact that $\exp\left[\frac{p}{p-1} M_{\ell^\varepsilon(t)-\ell^\varepsilon(0)} - \frac{p^2}{2(p-1)^2} \langle M\rangle_{\ell^\varepsilon(t)-\ell^\varepsilon(0)}\right]$, $0 \le t \le T$, is a martingale with mean 1 – this is due to Novikov's criterion – we know that

$$
\begin{aligned}
\mathbb{E}\left[R^{p/(p-1)}\right] \le \exp&\left[\frac{p}{(p-1)^2} |\,\xi(0)-\eta(0)|^2 \left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\, \mathrm{d}\ell^\varepsilon(t)\right)^{-1}\right] \\
\times \exp&\left[\frac{pK_1^2}{(p-1)^2\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}\,|\xi(0)-\eta(0)|^2\right)\right].
\end{aligned}
$$

Inserting this estimate into (3.13), we get the power-Harnack inequality. □

To prove Proposition 3.1 by using Lemma 3.2, we first prove the following lemma.

Lemma 3.3. *Let $\varepsilon \in (0,1)$ and $T > 0$. If $g^{(\varepsilon)} : [-r_0,\infty) \to [0,\infty)$ satisfies $g^{(\varepsilon)}(s) = 0$ for $s \in [-r_0, 0]$, $\int_0^T \|g_t^{(\varepsilon)}\|_2^2\, \mathrm{d}t < \infty$, and*

$$
|g^{(\varepsilon)}(t)|^2 \le C \int_0^t \|g_r^{(\varepsilon)}\|_2^2\, \mathrm{d}r + h^{(\varepsilon)}(t), \quad t \in [0,T],
$$

where $C>0$ is a constant and $h^{(\varepsilon)}:[0,T]\to[0,\infty)$ is measurable such that

$$\sup_{\varepsilon\in(0,1),t\in[0,T]} h^{(\varepsilon)}(t)<\infty$$

and $\lim_{\varepsilon\downarrow 0} h^{(\varepsilon)}(t)=0$ for any $t\in[0,T]$. Then we have

$$\lim_{\varepsilon\downarrow 0}\|g_t^{(\varepsilon)}\|_2=0,\quad t\in[0,T].$$

Proof. Since $g^{(\varepsilon)}(s)=0$ for $s\in[-r_0,0]$, it holds that

$$\begin{aligned}\int_{-r_0}^0 |g^{(\varepsilon)}(t+s)|^2\,\mathrm{d}s &= \left(\int_{-r_0+t}^0+\int_0^t\right)|g^{(\varepsilon)}(s)|^2\,\mathrm{d}s\\ &\le \int_0^t |g^{(\varepsilon)}(s)|^2\,\mathrm{d}s\\ &\le C\int_0^t\left(\int_0^s \|g_r^{(\varepsilon)}\|_2^2\,\mathrm{d}r\right)\mathrm{d}s+\int_0^t h^{(\varepsilon)}(s)\,\mathrm{d}s\\ &\le Ct\int_0^t \|g_r^{(\varepsilon)}\|_2^2\,\mathrm{d}r+\int_0^t h^{(\varepsilon)}(s)\,\mathrm{d}s.\end{aligned}$$

Thus, we find that for any $t\in[0,T]$

$$\begin{aligned}\|g_t^{(\varepsilon)}\|_2^2 &= \int_{-r_0}^0 |g^{(\varepsilon)}(t+s)|^2\,\mathrm{d}s+|g^{(\varepsilon)}(t)|^2\\ &\le C(t+1)\int_0^t \|g_r^{(\varepsilon)}\|_2^2\,\mathrm{d}r+H^{(\varepsilon)}(t)\\ &\le C(T+1)\int_0^t \|g_r^{(\varepsilon)}\|_2^2\,\mathrm{d}r+H^{(\varepsilon)}(t),\end{aligned}$$

where

$$H^{(\varepsilon)}(t):=h^{(\varepsilon)}(t)+\int_0^t h^{(\varepsilon)}(s)\,\mathrm{d}s.$$

Now we can apply Gronwall's inequality to get that, for all $t\in[0,T]$,

$$\|g_t^{(\varepsilon)}\|_2^2\le H^{(\varepsilon)}(t)+C(T+1)\int_0^t H^{(\varepsilon)}(s)\,\mathrm{e}^{C(T+1)(t-s)}\,\mathrm{d}s.$$

By our assumptions, we know that $\lim_{\varepsilon\downarrow 0} H^{(\varepsilon)}(t)=0$ for all $t\in[0,T]$. Letting $\varepsilon\downarrow 0$ on both sides of the above inequality and using the dominated convergence theorem, we complete the proof. □

Proof of Proposition 3.1. Fix $T>r_0$. By a standard approximation argument, we may and do assume that $f\in C_b(\mathscr{C})$.

Step 1: First, we assume that b is globally Lipschitzian: there exists a constant $C > 0$ such that

$$|b(x) - b(y)| \le C|x - y|, \quad x, y \in \mathbb{R}^d.$$

By the Lipschitz continuity of b and B, and noting that $|X^{\ell^\varepsilon,\xi}(r) - X^{\ell,\xi}(r)| \le \|X_r^{\ell^\varepsilon,\xi} - X_r^{\ell,\xi}\|_2$, we have for $t \ge 0$

$$\begin{aligned} |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)| &\le C\int_0^t |X^{\ell^\varepsilon,\xi}(r) - X^{\ell,\xi}(r)|\,\mathrm{d}r + K_1\int_0^t \|X_r^{\ell^\varepsilon,\xi} - X_r^{\ell,\xi}\|_2\,\mathrm{d}r \\ &\quad + |W(\ell^\varepsilon(t) - \ell^\varepsilon(0)) - W(\ell(t))| \\ &\le (C + K_1)\int_0^t \|X_r^{\ell^\varepsilon,\xi} - X_r^{\ell,\xi}\|_2\,\mathrm{d}r + |W(\ell^\varepsilon(t) - \ell^\varepsilon(0)) - W(\ell(t))|. \end{aligned}$$

By the elementary inequality

$$(u + v)^2 \le 2u^2 + 2v^2, \quad u, v \ge 0,$$

and the Hölder inequality, we get that for $t \in [0, T]$,

$$\begin{aligned} |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|^2 &\le 2(C + K_1)^2 t\int_0^t \|X_r^{\ell^\varepsilon,\xi} - X_r^{\ell,\xi}\|_2^2\,\mathrm{d}r + 2|W(\ell^\varepsilon(t) - \ell^\varepsilon(0)) - W(\ell(t))|^2 \\ &\le 2(C + K_1)^2 T\int_0^t \|X_r^{\ell^\varepsilon,\xi} - X_r^{\ell,\xi}\|_2^2\,\mathrm{d}r + 2|W(\ell^\varepsilon(t) - \ell^\varepsilon(0)) - W(\ell(t))|^2. \end{aligned}$$

Applying Lemma 3.3 with $g^{(\varepsilon)}(t) = |X^{\ell^\varepsilon,\xi}(t) - X^{\ell,\xi}(t)|$ and $h^{(\varepsilon)}(t) = 2|W(\ell^\varepsilon(t) - \ell^\varepsilon(0)) - W(\ell(t))|^2$, we conclude that $X_T^{\ell^\varepsilon,\xi} \to X_T^{\ell,\xi}$ in $\mathscr{C}$ as $\varepsilon \downarrow 0$, and so

$$\lim_{\varepsilon\downarrow 0} P_T^{\ell^\varepsilon} f = P_T^{\ell} f, \quad f \in C_b(\mathscr{C}).$$

Since ℓ is of bounded variation, it is easy to get from (3.2) that

$$\lim_{\varepsilon\downarrow 0}\int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell^\varepsilon(t) = \int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}\ell(t).$$

Letting $\varepsilon \downarrow 0$ in Lemma 3.2, we obtain the desired inequalities.

Step 2: For the general case, we shall make use of the approximation argument proposed in [15, part (c) of proof of Theorem 2.1]. Let

$$\tilde{b}(x) := b(x) - Kx, \quad x \in \mathbb{R}^d.$$

Then $\tilde{b}$ satisfies the dissipative condition:

$$\langle \tilde{b}(x) - \tilde{b}(y), x - y\rangle \le 0, \quad x, y \in \mathbb{R}^d,$$

and it is easy to see that the mapping $\mathrm{id} - \varepsilon\tilde{b} : \mathbb{R}^d \to \mathbb{R}^d$ is injective for any $\varepsilon > 0$. For $\varepsilon > 0$, let $\tilde{b}^{(\varepsilon)}$ be the Yoshida approximation of $\tilde{b}$, i.e.

$$\tilde{b}^{(\varepsilon)}(x) := \frac{1}{\varepsilon}\left[\left(\mathrm{id} - \varepsilon\tilde{b}\right)^{-1}(x) - x\right], \quad x \in \mathbb{R}^d.$$

Then $\tilde{b}^{(\varepsilon)}$ is dissipative and globally Lipschitzian, $|\tilde{b}^{(\varepsilon)}| \le |\tilde{b}|$ and $\lim_{\varepsilon\downarrow 0} \tilde{b}^{(\varepsilon)} = \tilde{b}$. Let $b^{(\varepsilon)}(x) := \tilde{b}^{(\varepsilon)}(x) + Kx$. Then $b^{(\varepsilon)}$ is also Lipschitzian and

$$\langle b^{(\varepsilon)}(x) - b^{(\varepsilon)}(y), x - y\rangle \le K|x-y|^2.$$

Let $X_t^{\ell,(\varepsilon),\xi}$ solve the SDE (3.1) with b replaced by $b^{(\varepsilon)}$ and $X_0^{\ell,(\varepsilon),\xi} = \xi \in \mathscr{C}$. Denote by $P_t^{\ell,(\varepsilon)}$ the associated semigroup. Due to the first part of the proof, the statements of Proposition 3.1 hold with P_t^ℓ replaced by $P_t^{\ell,(\varepsilon)}$. If

$$\lim_{\varepsilon\downarrow 0} P_T^{\ell,(\varepsilon)} f = P_T^\ell f, \quad f \in C_b(\mathscr{C}), \tag{3.14}$$

then we complete the proof by applying Proposition 3.1 with P_t^ℓ replaced by $P_t^{\ell,(\varepsilon)}$ and letting $\varepsilon \downarrow 0$. Indeed, noting that

$$\begin{aligned}
&\mathrm{d}|X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t)|^2\\
&\quad= 2\langle X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t), b^{(\varepsilon)}(X^{\ell,(\varepsilon),\xi}(t)) - b^{(\varepsilon)}(X^{\ell,\xi}(t))\rangle\,\mathrm{d}t\\
&\quad\quad + 2\langle X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t), b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))\rangle\,\mathrm{d}t\\
&\quad\quad + 2\langle X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t), B(X_t^{\ell,(\varepsilon),\xi}) - B(X_t^{\ell,\xi})\rangle\,\mathrm{d}t\\
&\quad\le (2K+1)|X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t)|^2\,\mathrm{d}t + |b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))|^2\,\mathrm{d}t\\
&\quad\quad + 2K_1\|X_t^{\ell,(\varepsilon),\xi} - X_t^{\ell,\xi}\|_2^2\,\mathrm{d}t\\
&\quad\le (2|K| + 2K_1 + 1)\|X_t^{\ell,(\varepsilon),\xi} - X_t^{\ell,\xi}\|_2^2\,\mathrm{d}t + |b^{(\varepsilon)}(X^{\ell,\xi}(t)) - b(X^{\ell,\xi}(t))|^2\,\mathrm{d}t,
\end{aligned}$$

one has for $t \in [0,T]$

$$\begin{aligned}
&|X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t)|^2\\
&\quad\le (2|K| + 2K_1 + 1)\int_0^t \|X_r^{\ell,(\varepsilon),\xi} - X_r^{\ell,\xi}\|_2^2\,\mathrm{d}r + \int_0^t |b^{(\varepsilon)}(X^{\ell,\xi}(r)) - b(X^{\ell,\xi}(r))|^2\,\mathrm{d}r\\
&\quad= (2|K| + 2K_1 + 1)\int_0^t \|X_r^{\ell,(\varepsilon),\xi} - X_r^{\ell,\xi}\|_2^2\,\mathrm{d}r + \int_0^t |\tilde{b}^{(\varepsilon)}(X^{\ell,\xi}(r)) - \tilde{b}(X^{\ell,\xi}(r))|^2\,\mathrm{d}r.
\end{aligned}$$

Applying Lemma 3.3 with $g^{(\varepsilon)}(t) = |X^{\ell,(\varepsilon),\xi}(t) - X^{\ell,\xi}(t)|$ and $h^{(\varepsilon)}(t) = \int_0^t |\tilde{b}^{(\varepsilon)}(X^{\ell,\xi}(r)) - \tilde{b}(X^{\ell,\xi}(r))|^2\,\mathrm{d}r$, we find that $X_T^{\ell,(\varepsilon),\xi} \to X_T^{\ell,\xi}$ in $\mathscr{C}$ as $\varepsilon \downarrow 0$, and thus (3.14) follows. □

4 Proofs of Theorem 2.1 and Example 2.4

Proof of Theorem 2.1. Since the processes W and S are independent, we have

$$P_T f(\cdot) = \mathbb{E}\left[P_T^\ell f(\cdot)\,|_{\ell=S}\right], \quad f \in \mathscr{B}_b(\mathscr{C}). \tag{4.1}$$

By the first assertion of Proposition 3.1, for all $f \in \mathscr{B}_b(\mathscr{C})$ with $f \ge 1$,

$$P_T \log f(\eta) = \mathbb{E}\left[P_T^\ell \log f(\eta)\,|_{\ell=S}\right]$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\log P_T^\ell f(\xi)\,|_{\ell=S}\right] + |\xi(0)-\eta(0)|^2\,\mathbb{E}\left(\int_0^{T-r_0} \mathrm{e}^{-2Kt}\,\mathrm{d}S(t)\right)^{-1}\\
&\quad + \frac{K_1^2}{\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}|\xi(0)-\eta(0)|^2\right),
\end{aligned}
$$

which, together with the Jensen inequality and (4.1), implies the log-Harnack inequality. Analogously, by the second assertion of Proposition 3.1, for all non-negative $f \in \mathscr{B}_b(\mathscr{C})$,

$$
\begin{aligned}
P_T f(\eta) &= \mathbb{E}\left[P_T^\ell f(\eta)\,|_{\ell=S}\right]\\
&\leq \mathbb{E}\left[\left(P_T^\ell f^p(\xi)\right)^{1/p}\exp\left[\frac{1}{p-1}|\xi(0)-\eta(0)|^2\left(\int_0^{T-r_0}\mathrm{e}^{-2Kt}\,\mathrm{d}\ell(t)\right)^{-1}\right]\Bigg|_{\ell=S}\right]\\
&\quad\times\exp\left[\frac{1}{p-1}\frac{K_1^2}{\kappa}\left(r_0\|\xi-\eta\|_2^2 + (T+1)\frac{\mathrm{e}^{2K(T-r_0)}-1}{2K}|\xi(0)-\eta(0)|^2\right)\right].
\end{aligned}
$$

It remains to use the Hölder inequality and (4.1) to derive the power-Harnack inequality. □

Proof of Example 2.4. By the assumption, one has

$$
S(t) \geq \kappa t + \tilde{S}(t) \geq (\kappa t) \vee \tilde{S}(t), \quad t \geq 0,
$$

where $\tilde{S}$ is an α-stable subordinator with Bernstein function $\tilde{\phi}(u) = cu^\alpha$. Combining this with Theorem 2.1 and the moment estimates for subordinators in [5, Theorem 3.8 (a) and (b)], we get the desired estimates. □

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