Riesz transform via heat kernel and harmonic functions on non-compact manifolds

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Abstract. Let M be a complete non-compact manifold satisfying the volume doubling condition, with doubling index N and reverse doubling index n, $n \le N$, both for large balls. Assume a Gaussian upper bound for the heat kernel, and an L^2-Poincaré inequality outside a compact set.
If $2 < n$, then we show that for $p \in (2, n)$, (R_p): L^p-boundedness of the Riesz transform, (G_p): L^p-boundedness of the gradient of the heat semigroup, and (RH_p): reverse L^p-Hölder inequality for the gradient of harmonic functions, are equivalent to each other. Our characterization implies that for $p \in (2, n)$, (R_p) has an open ended property and is stable under gluing operations. This substantially extends the well known equivalence of (R_p) and (G_p) from [4] to more general settings, and is optimal in the sense that (R_p) does not hold for any $p \ge n > 2$ on manifolds having at least two Euclidean ends of dimension n.
For $p \in (\max\{N, 2\}, \infty)$, the fact that (R_p), (G_p) and (RH_p) are equivalent essentially follows from [22]; moreover, if M is non-parabolic, then any of these conditions implies that M has only one end.
For the proof, we develop a new criteria for boundedness of the Riesz transform, which was nontrivially adapted from [4], and make an essential application of results from [22]. Our result allows extensions to non-smooth settings.

Contents

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1 Introduction

1.1 Background and motivations

Let M be a complete, connected and non-compact Riemannian manifold. Denote by d the geodesic distance, by μ the Riemannian measure, and by $\mathcal{L}$ the non-negative Laplace-Beltrami operator on M. Let $\{e^{-t\mathcal{L}}\}_{t>0}$ be the heat semigroup. The inverse of the square root of $\mathcal{L}$ is given by

$$\mathcal{L}^{-1/2} = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-s\mathcal{L}}\frac{ds}{\sqrt{s}}.$$

Denote by ∇ the Riemannian gradient.

The study of the Riesz transform $\nabla\mathcal{L}^{-1/2}$ is one of the central topics of analysis on manifolds. Strichartz in 1983 [48], and then Bakry in 1987 [5], provided sufficient conditions on non-compact manifolds such that the Riesz transform is bounded for all $1 < p < \infty$ (see Chen [14] for the case $p = 1$). Since then, many sufficient, or even in some cases necessary and sufficient, conditions for the boundedness of the Riesz transform have been provided; see for instance [1, 3, 4, 11, 13, 16, 20, 21, 40, 43]. Let us review some related results. Since the boundedness of Riesz transform on compact manifolds is not an issue (cf. [48]), we will only consider non-compact cases.

For each $p \in (1,\infty)$, we say that (R_p) holds, if the Riesz transform $|\nabla\mathcal{L}^{-1/2}|$ is bounded on $L^p(M)$. Notice that (R_2) holds automatically which can be seen by integration by parts. In the metric measure space (M, d, μ), denote by $B(x, r)$ the open ball with centre $x \in M$ and radius $r > 0$ and by $V(x, r)$ its volume $\mu(B(x, r))$. One says that M satisfies the volume doubling property (in short is doubling) if there exists a constant $C_D > 1$ such that

$$(D) \qquad V(x, 2r) \le C_D V(x, r),$$

for all $r > 0$ and $x \in M$. The heat semigroup has a smooth positive and symmetric kernel $p_t(x, y)$, meaning that

$$e^{-t\mathcal{L}}f(x) = \int_M p_t(x, y)f(y)\,d\mu(y)$$

for suitable functions f. One says that the heat kernel satisfies a Gaussian upper bound if there exist $C, c > 0$ such that for all $t > 0$ and $x, y \in M$,

$$(UE) \qquad p_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\left\{-\frac{d^2(x,y)}{ct}\right\}.$$

Coulhon and Duong [20] showed that the doubling condition together with a Gaussian upper bound of heat kernel is sufficient for (R_p) for all $p \in (1,2)$. Recently, Chen et al. [15] showed, a bit surprisingly, that a sub-Gaussian upper bound of the heat kernel could replace the Gaussian upper bound in the above result; see [39] for further developments.

The case $p > 2$ is more difficult. Notice that if (R_p) holds, then it follows from the analytic property of the heat semigroup that

$$(G_p) \qquad |||\nabla e^{-t\mathcal{L}}|||_{p\to p} \le |||\nabla \mathcal{L}^{-1/2}\mathcal{L}^{1/2} e^{-t\mathcal{L}}|||_{p\to p} \le \frac{C}{\sqrt{t}}$$

for all $t > 0$; see [47] or [4]. Above and in what follows, we use the notation $\|\cdot\|_p$ to denote the L^p norm over M, and the notation $\|\cdot\|_{p\to p}$ for the operator norm from L^p to L^p, for any $p \in [1,\infty]$. A natural and longstanding question is as following.

Question 1.1. *Let $p \in (2,\infty)$. Does (G_p) imply (R_p)?*

Auscher, Coulhon, Duong and Hofmann in 2004 [4] established a remarkable result, which shows that, under (D) and a scale-invariant L^2-Poincaré inequality, (G_{p_0}) implies (R_p) for all $p \in (2, p_0)$, where $p_0 \in (2,\infty]$. The scale invariant L^2-Poincaré inequality means that there exists $C > 0$ such that for every ball B and each $f \in C^1(\bar{B})$, it holds

$$(P_2) \qquad \fint_B |f - f_B|^2\, d\mu \le C r_B^2 \fint_B |\nabla f|^2\, d\mu,$$

where f_B denotes the average of the integral of f on B. Notice that (D) together with (P_2) is equivalent to a two-sided Gaussian bound for the heat kernel; see [28, 45]. By recent results from [7, 22], one finally sees that $(G_p) \Longleftrightarrow (R_p)$ for each $p \in (2,\infty)$, under (D) and (P_2).

However, Question 1.1 in generality is still open; see [4, Subsection 1.4] and also [7]. The requirement of (P_2) is not necessary by looking at a manifold obtained by gluing two Euclidean ends through a compact manifold smoothly; see [13, 20, 29]. Here and below, an *end* means, an unbounded component of a complete non-compact manifold M outside a compact subset M_0.

In [13], Carron, Coulhon and Hassell showed that the Riesz transform is L^p-bounded for $2 < p < n$, $n \ge 3$, if M is an n-dimensional manifold with a finite number of Euclidean ends; the result has been further generalized to manifolds with conic ends by Guillarmou and Hassell [32], and by Carron [12] to manifolds with quadratic Ricci curvature decay, i.e., for a fixed $x_M \in M$ and $C_M \ge 0$, it holds

$$(QD) \qquad Ric_M(x) \ge -\frac{C_M}{[d(x,x_M)+1]^2}.$$

Moreover, in [13], it has been showed that if M has at least two ends, then the Riesz transform is not L^p-bounded for any $p \geq n$. Indeed, by using L^p-cohomology, the following non-trivial result was proved in [13].

Theorem 1.2 ([13]). *Suppose that M has Ricci curvature bounded from below, and for some $N > 2$ $V(x,r) \lesssim r^N$, for all $x \in M$ and $r \geq 1$. If there exists $C > 0$ such that for any $f \in C_c^\infty(M)$ it holds*

$$(S_{\frac{2N}{N-2},2}), \qquad \|f\|_{\frac{2N}{N-2}} \leq C\||\nabla f|\|_2.$$

and M has at least two ends, then the Riesz transform is not bounded on $L^p(M)$ for any $p \geq N$.

Notice that the Sobolev inequality $(S_{\frac{2N}{N-2},2})$ together with $V(x,r) \lesssim r^N$ implies (UE) (cf. [28, 29]), conversely (UE) only implies a local Sobolev inequality (cf. [9, 28, 29]). In particular, under (UE), $(S_{\frac{2N}{N-2},2})$ may not hold; see [49].

The above result has been further refined by Carron [12, Theorem C]. Notice that, in particular, in the above theorem and Carron's theorem, the ends are not necessarily Euclidean or conic. In view of this, in [13], several questions, regarding relaxing the requirement that ends are Euclidean, had been proposed; see following Question 1.13, Question 6.2 and Question 6.3.

In this paper, we provide a solution to Question 1.1 by relaxing the requirement of (P_2), but only for p in the intervals $(2,n)$ and $(\max\{2,N\},\infty)$; see Theorem 1.5 and Theorem 1.11 below. As an application, we obtain stability under gluing operation and open ended property for the Riesz transform on manifolds with general ends. Notice that the case $p \in (1,2)$ was well understood by [20, 15], as we recalled above. We will only consider the case $p > 2$ in this work.

Throughout the paper, we assume that M is a non-compact, connected and complete manifold, that satisfies the doubling condition (D). We shall simply recognize M as the union of a compact set M_0 and one or more but finitely many ends $\{E_i\}_i$. We fix a point $x_M \in M_0$ and assume without loss of generality that $\mathrm{diam}(M_0) = 1$.

The doubling condition (D) together with connectedness implies that there exist $0 < \upsilon \leq \Upsilon < \infty$ such that for any $x \in M$ and all $0 < r < R < \infty$ it holds

$$(1.1) \qquad \left(\frac{R}{r}\right)^{\upsilon} \lesssim \frac{V(x,R)}{V(x,r)} \lesssim \left(\frac{R}{r}\right)^{\Upsilon};$$

see for instance [35, p. 213, Remark 8.1.15]. This further implies that there exists $0 < N < \infty$ such that

$$(D_N) \qquad \frac{V(x,R)}{V(x,r)} \lesssim \left(\frac{R}{r}\right)^{N}, \ \forall\, x \in M \ \&\ \forall\, 1 < r < R < \infty,$$

and there exists $0 < n \leq N$ such that

$$(RD_n) \qquad \left(\frac{R}{r}\right)^{n} \lesssim \frac{V(x_M,R)}{V(x_M,r)}, \ \forall\, 1 < r < R < \infty,$$

where $x_M \in M_0$ is a fixed point. In what follows, we call n the lower dimension, and N the upper dimension, of M. Moreover, we simply use (D_N) to indicate that μ is a doubling measure with N being the upper dimension.

Remark 1.3. (i) It holds obviously $\upsilon \le n \le N \le \Upsilon$. The examples of cocompact covering Riemannian manifolds with polynomial growth deck transformation group and Lie groups of polynomial growth show it may happen that $\upsilon < n$ and $N < \Upsilon$; see [1, 26, 34, 49] for instance.

(ii) Notice that we only need (RD_n) for a fixed point $x_M \in M_0$ and $R > r > 1$. Take weighted lines $(\mathbb{R}, (1+|x|)^\alpha\,dx)$, $\alpha > 0$, for example. A small calculation shows that $(D_{\alpha+1})$ and $(RD_{\alpha+1})$ hold, but (1.1) holds with $\upsilon = 1$ and $\Upsilon = \alpha + 1$; see [34] and also [12]. Moreover, by using the doubling property and the fact M_0 is compact, one sees that (RD_n) holds if and only it holds for each $o \in M_0$ and all $1 < r < R < \infty$ that $(R/r)^n \lesssim V(o,R)/V(o,r)$.

(iii) In many cases, such as manifolds with conic ends, or with ends like cocompact covering Riemannian manifolds with polynomial growth deck transformation group or Lie groups of polynomial growth, one has $n = N$.

By Theorem 1.2 and [12, Theorem C], we already see that the (homogenous) dimension plays a key role in the Riesz transform. It is then naturally to split the case $p > 2$ into two categories: p less than the dimension and p bigger than the dimension. We will provide necessary and sufficient conditions for boundedness of the Riesz transform in both cases.

We first consider $p > 2$ that is smaller than the lower dimension n, which means that $n > 2$ and the ends are non-parabolic; see Subsection 1.3 for the definition and [12, 41] for more materials. Recall that if a manifold has two Euclidean or conic ends of dimension two, then the Riesz transform is not L^p-bounded for any $p > 2$ by [12, 20].

For $p > 2$ that is bigger than the upper dimension N, we will consider manifolds with general ends (including small ones). Notice that in this case boundedness of the Riesz transform will imply that the manifold can have only *one* end, if M is non-parabolic; see Theorem 1.11 below.

1.2 Necessary and sufficient conditions for small p

In this part, we provide a necessary and sufficient condition for L^p-boundedness of the Riesz transform for small p, i.e., p less than the lower dimension. Our approach depends heavily on recent developments on the relation of regularities of harmonic functions and heat kernels from [22, 36, 37], and is a nontrivial adaption of the criteria for the boundedness of the Riesz transform established in [4] (see also [2]) to our settings.

Definition 1.4 (Poincaré inequality). *We say that a Poincaré inequality holds on ends ((P_2^E), for short) of M, if there exists $C > 0$ such that for any ball B with $2B \cap M_0 = \emptyset$, and each $f \in C^1(\bar{B})$,*

$$(P_2^E) \qquad \fint_B |f - f_B|^2\,d\mu \le C r_B^2 \fint_B |\nabla f|^2\,d\mu.$$

Our first main result provides a solution to Question 1.1 for $p \in (2, n)$. Notice that, under the doubling condition, our assumptions (UE) and (P_2^E) below are much weaker than (P_2). For example, (UE) and (P_2^E) hold on a manifold obtained by gluing two copies of Eulcidean space $\mathbb{R}^n$ together, $n \ge 2$, while (P_2) does not hold; see [11, 12, 13, 20, 32] for instance.

Theorem 1.5. *Assume that* (D_N) *and* (RD_n) *hold on* M *with* $2 < n \le N < \infty$. *Suppose that* (UE) *and* (P_2^E) *hold. Let* $p \in (2, n)$. *Then the following statements are equivalent.*

(i) (R_p) *holds;*

(ii) (G_p) *holds;*

(iii) (RH_p) *holds, where* (RH_p) *means that there exists* $C > 0$ *such that for any ball* B *with radius* r_B *and any harmonic function* u *on* $3B$*, it holds*

$$(RH_p) \qquad \left(\fint_B |\nabla u|^p \, d\mu\right)^{1/p} \le \frac{C}{r_B} \fint_{2B} |u| \, d\mu.$$

Remark 1.6. (i) Under assumptions of the theorem, (R_p) holds for all $p \in (1, 2]$ from Coulhon-Duong [20], and does not hold for any $p \ge N$, if the manifold has at least two ends, by Carron [12, Theorem C].

(ii) The condition (RH_p) is different from the true reverse Hölder inequalities used in [3, 46]. Our formulation is natural since in case of manifolds with two Euclidean/conic ends, the true reverse Hölder inequalities fail for any $p > 2$, but (RH_p) holds for $p \in (2, n)$; see [22, Section 7].

(iii) The equivalence $(G_p) \Longleftrightarrow (RH_p)$ was proved in [22] under a local Poincaré inequality $(P_{2,\,\mathrm{loc}})$ instead of (P_2^E). Notice that (P_2^E) implies $(P_{2,\,\mathrm{loc}})$; see Lemma 2.2 below.

In view of Theorem 1.2 and [12, Theorem C], the above result is rather optimal if the manifold has at least two ends. It is worth to note that our method are completely different from those from [11, 12, 13, 32], in particular, our assumptions (D), (UE) and (P_2^E) all are stable under quasi-isometries. As a consequence, our results work with the Laplace-Beltrami operator replaced by any *uniformly elliptic operator* of divergence form, and more generally, work on Dirichlet metric measure spaces; see Section 5.

The condition (P_2^E) is satisfied on an end, if the Ricci curvature has quadratic decay (QD) (see Buser [10] or Theorem 2.6), or the end is quasi-isometric to one of the following manifold removing a compact set: a co-compact covering manifold with polynomial growth deck transformation group, Lie group of polynomial growth as well as conic manifold; see [1, 17, 23, 26, 38, 49, 50] for instance.

The condition (UE) is a global condition and seems to be more restrictive. However, recent results of Grigor'yan and Saloff-Coste [29, 30] shed some light on this point. In particular, by [30] one sees that if each end E_i is isometric to $\widetilde{M}_i \setminus K_i$, where $\widetilde{M}_i$ is a complete manifold satisfying (UE) and K_i is a compact set, then the manifold M satisfies (UE); see the final section.

We next provide some further necessary and sufficient conditions for the boundedness of the Riesz transform.

Definition 1.7. *Let* $p \in (2, \infty]$. *We say that the reverse* L^p*-Hölder inequality for gradients of harmonic functions holds on ends of* M *(for short,* (RH_p^E)*), if there exists* $C > 0$ *such that for each ball* B *with* $3B \cap M_0 = \emptyset$*, and each harmonic function* u *on* $3B$*, it holds*

$$(RH_p^E) \qquad \left(\fint_B |\nabla u|^p \, d\mu\right)^{1/p} \le \frac{C}{r_B} \fint_{2B} |u| \, d\mu.$$

The observation below is that, (RH_p) is stable under gluing operation, if $p < n$; see Lemma 3.9 and Lemma 3.10 below.

Theorem 1.8. *Assume that (D_N) and (RD_n) hold on M with $2 < n \le N < \infty$. Suppose that (UE) and (P_2^E) hold. Let $p \in (2,n)$. Then (R_p) holds on M, if and only if, (RH_p^E) holds.*

The advantage is that (RH_p^E) is a condition much easier to verify than (G_p). An immediate consequence of the above result is that compact metric perturbation does not affect (R_p), if $p < n$. We also note that, the above result implies the stability of (R_p) $(p < n)$ under gluing operations, see Theorem 1.14 and Corollary 1.15 below.

An open-ended property of the Riesz transform follows from the above result.

Corollary 1.9. *Assume that (D_N) and (RD_n) hold on M with $2 < n \le N < \infty$. Suppose that (UE) and (P_2^E) hold. Let $p \in (2,n)$. If (R_p) holds, then there exists $\epsilon > 0$ such that $p + \epsilon < n$ and $(R_{p+\epsilon})$ holds.*

Further, (RH_∞^E) and (P_2^E) hold if the Ricci curvature has quadratic decay.

Corollary 1.10. *Assume that (D_N) and (RD_n) hold on M with $2 < n \le N < \infty$. If (UE) holds, and there exists $C_M > 0$ such that for each $x \in M$,*

$$(QD) \qquad Ric_M(x) \ge -\frac{C_M}{[d(x,x_M)+1]^2},$$

then (R_p) holds for $p \in (1,n)$.

Carron [12, Thoerem A] established that if a manifold satisfies a volume comparison condition (VC), the (RCE) condition (relatively connected to an end), (QD) and (RD_n), then (R_p) holds for $p \in (1,n)$. See also Devyver [24, Theorem 5] for a related result. Note that Carron's assumptions imply (D) and (UE); see [12, Section 2]. However, after a careful reading of [12, Section 3 and Section 4] we find that Carron's proof indeed works under our assumptions in the above Corollary. The approach [12] used Li-Yau's Harnack inequality (cf. [42]) to deduce point-wise behaviors of the Riesz kernel, which depends on the smooth structure. Our approach (after applying Theorem 1.5 and Theorem 1.8) needs to verify the regularity of harmonic functions on the ends, and can be applied to deal with general uniformly elliptic operators on such manifolds (see Section 5).

1.3 Necessary and sufficient conditions for large p

Theorem 1.5 seems to be rather optimal if the manifold has at least two ends, however, it is less satisfied if the manifold has only one end, where in general the Riesz transform may be bounded on $L^p(M)$ for some $p > N$; see [12, 32, 38] for instance. We next provide a necessary and sufficient condition for $p > N$ under the same requirements as Theorem 1.5 except that we do not need the reverse doubling condition, which however holds automatically. The result in this part essentially follows from [22]. We shall denote $\max\{A, B\}$ by $A \vee B$.

Let us recall some notation regarding parabolic and hyperbolic manifolds; see [12] for instance. Let $p \in (1, \infty)$. For a bounded open set $O \subset M$, define its p-capacity by

$$\mathrm{Cap}_p(O) := \inf \left\{ \int_M |\nabla \psi|^p \, d\mu, \ \psi \in C_c^\infty(M), \psi \geq 1 \text{ on } O \right\}.$$

We say that M is p-hyperbolic if the p-capacity of some (equivalently, any) bounded open subsets is positive. A non-p-hyperbolic manifold is called p-parabolic. A 2-hyperbolic manifold is called non-parabolic.

Theorem 1.11. *Assume that (D_N) holds on M with $0 < N < \infty$, and that (UE) and (P_2^E) hold. Let $p \in (N \vee 2, \infty)$. Then the following statements are equivalent.*

(i) (R_p) holds;

(ii) (RH_p) holds;

(iii) (G_p) holds.

Moreover, if M is non-parabolic, then any of the three conditions implies that M can have only one end.

Note that we did not assume (P_2) above, however (P_2) follows as a consequence of the proof; see Remark 4.1.

For the proof we will show that the validity (P_2^E) guarantees scale-invariant Poincaré inequalities (P_p) for any $p > N \vee 2$ (see Theorem 2.3). The validity of these Poincaré inequalities allows us to use [22, Theorem 1.9], and then [12, Theorem C] to conclude the theorem.

We have the following unboundedness of the Riesz transform as an application of the above result.

Corollary 1.12. *Assume that (D_N) holds on M with $0 < N < \infty$, and that (UE) and (P_2^E) hold. If there exists a non-constant harmonic function u on M with the growth*

$$u(x) = O(d(x, o)^\alpha) \text{ as } d(x, o) \to \infty$$

for some $\alpha \in [0, 1)$ and a fixed $o \in M$, then (R_p) does not hold for any $p > N \vee 2$ satisfying $p(1 - \alpha) \geq N$.

Using the Poincaré inequality (P_p) for any $p > N \vee 2$ established in Theorem 2.3 together with Theorem 1.11 allows us to conclude the claim via arguing by contradiction. We refer the reader to [12, 41] for more on existence and non-existence of non-constant harmonic functions of sublinear growth.

1.4 Applications and comments

As applications of our main results, in this part, we address the questions of stability of boundedness of the Riesz transform under gluing operations and make some final comments on our result.

The following question was asked in [13].

Question 1.13 (Part of Open Problem 8.2 [13]). *Under which conditions is boundedness of the Riesz transform on L^p stable under the gluing operation on manifolds?*

We refer the reader to [30, Section 3] and also [29] for a detailed description of the gluing operation. Here we only need to know that the gluing operation is smooth, and only changes structure and metric in a compact set. As shown by Theorem 1.2, [12, Theorem C] and Theorem 1.11, the L^p-boundedness of the Riesz transform is not stable under the gluing operations if p is not less than the dimension N and bigger than two. Previously, Carron [11] and Devyver [25] had addressed this question under the requirement of lower Ricci curvature bound and Sobolev inequalities; see also [12] for a description of Devyver's result.

Our Theorem 1.8 provides a solution to the above question in a different manner than [11, 25].

Theorem 1.14. *Let $k \geq 2$. Suppose that for each $1 \leq i \leq k$, M_i is a complete non-compact manifold where (D), (UE) and (P_2^E) hold. Assume that the gluing manifold $M := M_1\#\cdots\#M_k$ satisfies (D_N) and (RD_n) for some $2 < n \leq N < \infty$. Then if for some $p \in (2, n)$, (R_p) holds on each M_i, (R_p) holds on M.*

It is worth to note that, our assumptions, (UE) and (P_2^E), are stable under gluing operations. Indeed, under gluing operations, it is straight to see that (P_2^E) is stable, on the other hand, the stability of (UE) follows from [29, 30] (see Theorem 6.1).

Since (P_2) implies (UE) and (P_2^E), we obtain the following corollary.

Corollary 1.15. *Let $k \geq 2$. Suppose that for each $1 \leq i \leq k$, M_i is a complete non-compact manifold where (D) and (P_2) hold. Assume that the gluing manifold $M := M_1\#\cdots\#M_k$ satisfies (D_N) and (RD_n) for some $2 < n \leq N < \infty$.*

(i) There exists $\epsilon > 0$ such that $2 + \epsilon < n$ and $(R_{2+\epsilon})$ holds.

(ii) If for some $p \in (2, n)$, (R_p) holds on each M_i, then (R_p) holds on M.

In [13], some open questions regarding manifolds with conic ends or ends isometric to simply connected nilpotent Lie groups at infinity were also proposed. These two questions were solved by Guillarmou and Hassell [32] and Carron [11], respectively; see also Carron [12]. Our results also provide a new proof to the two questions; see Section 6.

Finally, let us make some comments. Notice that our main results, Theorem 1.5, Theorem 1.8 and Theorem 1.11, together with [12, Theorem C] and [20], give a more or less satisfactory solution for the Riesz transform on manifolds with ends, for the two cases: (i) $1 < p < \max\{2, n\}$, (ii) $p > N \vee 2$. Recall that the case $p = 2$ is trivially true.

Note that for manifolds with ends like Euclidean ends, conic ends, or ends at infinity isometric to Lie groups of polynomial growth or cocompact covering Riemannian manifolds with polynomial growth deck transformation group, it holds that $n = N$. It turns out that on these settings, (R_p) is stable under gluing operation for $p < n = N$ by Theorem 1.14, and not stable for $p \geq N$ by [12, Theorem C]. It is then somehow not restrictive to assume (P_2) for $p \geq N$ or may necessary to have (P_2), under which (R_p) for $p > 2$ is well understood by [4], see also [22] and Theorem 1.11.

However, for manifolds where one only has $n < N$, the case $p \in (2, \infty) \cap [n, N]$ is still unclear, and certainly deserves further study.

Question 1.16. *Let M be a complete non-compact manifold, which satisfies (D_N) and (RD_n) for some $0 < n \le N < \infty$ and $N > 2$. Suppose that (UE) and (P_2^E) hold. Then is (R_p) equivalent to (G_p) or (RH_p) for $p \in (2,\infty) \cap [n,N]$?*

For each $p \in (2,\infty)$, it was known from [22] that $(G_p) \Longleftrightarrow (RH_p)$, and it holds automatically that $(R_p) \Longrightarrow (G_p)$ (cf. [4]). So the only question left is, does (G_p) or (RH_p) imply (R_p) for $p \in (2,\infty) \cap [n,N]$?

1.5 Structure of the paper

The paper is organized as follows. In Section 2, we provide various versions of Poincaré inequalities for later use. In Section 3, we study the Riesz transform for p less than the lower dimension, while in Section 4, we study the case p bigger than the upper dimension. In Section 5, we provide some extensions of the main results to non-smooth settings. In the final section, we shall discuss the validity of (UE), and provide examples that our results can be applied to, in particular, we give the proof of Theorem 1.14 and Corollary 1.15.

Throughout the work, we denote by C, c positive constants which are independent of the main parameters, but which may vary from line to line. For a ball B, unless otherwisely specified, we denote its radius and center by r_B and x_B, respectively.

2 Poincaré inequality

In this section, we shall provide various versions of Poincaré inequalities for later use.

Definition 2.1 (Hardy-Littlewood maximal function)**.** *For any locally integrable function f on M, its Hardy-Littlewood maximal function is defined as*

$$\mathcal{M}f(x) := \sup_{B:\, x\in B} \fint_B |f|\, d\mu,$$

where B is any ball that contains x. For $p > 1$, we define the p-Hardy-Littlewood maximal function as

$$\mathcal{M}_p f(x) := \sup_{B:\, x\in B} \left(\fint_B |f|^p \, d\mu \right)^{1/p}.$$

We say that M supports a local L^2-Poincaré inequality (for short, $(P_{2,\mathrm{loc}})$), if for all $r_0 > 0$ there exists $C_P(r_0) > 0$ such that, for every ball B with $r_B < r_0$ and each $f \in C^1(\bar{B})$,

$$(P_{2,\mathrm{loc}}) \qquad \fint_B |f - f_B|^2 \, d\mu \le C_P(r_0) r_B^2 \fint_B |\nabla f|^2 \, d\mu.$$

Lemma 2.2. *Assume that (P_2^E) holds on M, then $(P_{2,\mathrm{loc}})$ holds on M.*

Proof. For any $r_0 > 0$, the Ricci curvature on the set $\{x \in M : \operatorname{dist}(x, M_0) < 3r_0\}$ is bounded below by a constant $K(r_0)$ depending on r_0. Therefore, by Buser [10] (see also [33]), there exists $C_P(r_0)$ such that for every ball $B = B(x, r)$ with $r < r_0$ and $\operatorname{dist}(x, M_0) < 2r_0$, and each $f \in C^1(\bar{B})$, it holds

$$(P_{2,\mathrm{loc}})\qquad \fint_B |f - f_B|^2\, d\mu \le C_P(r_0) r^2 \fint_B |\nabla f|^2\, d\mu.$$

On the other hand, by (P_2^E), one sees that there exists C such that for any ball $B(x, r)$ with center $x \notin \{y \in M : \operatorname{dist}(y, M_0) < 2r_0\}$ and $r < r_0$, it holds for each $f \in C^1(\bar{B})$ that

$$\fint_B |f - f_B|^2\, d\mu \le C r^2 \fint_B |\nabla f|^2\, d\mu,$$

as desired. □

For a real number $\gamma > 0$ we denote by $[\log_2 \gamma]$ the biggest integer not bigger than $\log_2 \gamma$.

Theorem 2.3. *Assume that (D_N) holds on M with $0 < N < \infty$. If (P_2^E) holds on M, then for any $p > N \vee 2$ there is a Poincaré inequality (P_p), i.e., there exists $C > 0$ such that for any ball B and any $f \in C^1(\bar{B})$ it holds*

$$(P_p)\qquad \fint_B |f - f_B|\, d\mu \le C r_B \left(\fint_B |\nabla f|^p\, d\mu \right)^{1/p}.$$

Proof. Since (M, d) is a geodesic space, by Hajłasz-Koskela [33, Section 9], it suffices to prove the following weaker version, i.e., for $f \in C^1(\overline{8B})$,

$$(\widetilde{P_p})\qquad \fint_B |f - f_B|\, d\mu \le C r_B \left(\fint_{8B} |\nabla f|^p\, d\mu \right)^{1/p}.$$

By Lemma 2.2, a local Poincaré inequality $(P_{2,\mathrm{loc}})$ holds. If $r_B \le 100$, then the required estimate $(\widetilde{P_p})$ follows from $(P_{2,\mathrm{loc}})$.

Assume now $r_B > 100$. If $2B \cap M_0 = \emptyset$, then (P_p) and hence $(\widetilde{P_p})$ follows from (P_2^E).

Suppose $2B \cap M_0 \neq \emptyset$. Let $f \in C^1(\overline{8B})$ and write

$$\fint_B |f - f_B|\, d\mu \le \fint_B \fint_B |f(x) - f(y)|\, d\mu(x)\, d\mu(y).$$

Claim: For each $q \in (N \vee 2, \infty)$, there is a constant $C > 0$ such that for all $x, y \in B$ it holds

$$|f(x) - f(y)| \le C r_B \left[\mathcal{M}_q (|\nabla f| \chi_{8B})(x) + \mathcal{M}_q (|\nabla f| \chi_{8B})(y) \right].$$

If the claim holds, then by taking $q \in (N \vee 2, p)$, we conclude that

$$\fint_B |f - f_B|\, d\mu \le C r_B \fint_B \fint_B \left[\mathcal{M}_q (|\nabla f| \chi_{8B})(x) + \mathcal{M}_q (|\nabla f| \chi_{8B})(y) \right] d\mu(x)\, d\mu(y)$$

$$\leq Cr_B \left(\fint_B \left[\mathcal{M}_q \left(|\nabla f| \chi_{8B} \right)(x) \right]^p \, d\mu(x) \right)^{1/p}$$
$$\leq Cr_B \left(\fint_{8B} |\nabla f|^p \, d\mu \right)^{1/p},$$

where the last inequality follows from the fact that $\mathcal{M}_q$ is L^p-bounded for $p > q$. The above estimate completes the proof of $(\widetilde{P_p})$ and therefore the theorem.

Let us prove the claim. Take $x_{M_0} \in M_0 \cap 2B$ and set $B_{x_{M_0}} = B(x_{M_0}, 1)$. Note that $B(x_{M_0}, 1) \subset 3B$ since $r_B > 100$. Recall that we assume $\operatorname{diam}(M_0) = 1$. For all $x, y \in B$, we write

$$|f(x) - f(y)| \leq |f(x) - f_{B_{x_{M_0}}}| + |f(y) - f_{B_{x_{M_0}}}|. \tag{2.1}$$

Step 1. Suppose first that $d(x, x_{M_0}) \leq 100$. We choose a sequence of balls $\{B_j\}_{j=0}^{\infty}$ such that $B_j = B(x, 2^{-j} * 102)$ for each $j \geq 0$. As $x \in B$ and $r_B > 100$, we have $B_j \subset 3B \subset 8B$. We write

$$\left| f(x) - f_{B_{x_{M_0}}} \right| \leq \left| f(x) - f_{B_0} \right| + |f_{B_0} - f_{B_{x_{M_0}}}|. \tag{2.2}$$

For the first term, note that $B_{j+1} \subset B_j$ for each $j \geq 0$. By using $(P_{2,\mathrm{loc}})$, $q > N \vee 2$ and the Hölder inequality, we conclude that

$$\left| f(x) - f_{B_0} \right| = \lim_{j\to\infty} \left| f_{B_j} - f_{B_0} \right| \leq \sum_{j=0}^{\infty} \left| f_{B_j} - f_{B_{j+1}} \right| \leq \sum_{j=0}^{\infty} \fint_{B_{j+1}} \left| f - f_{B_j} \right| d\mu$$
$$\leq C \sum_{j=0}^{\infty} \fint_{B_j} \left| f - f_{B_j} \right| d\mu \leq \sum_{j=0}^{\infty} C 2^{-j} * 102 \left(\fint_{B_j} |\nabla f|^q \, d\mu \right)^{1/q}$$
$$\leq \sum_{j=0}^{\infty} C 2^{-j} \mathcal{M}_q \left(|\nabla f| \chi_{8B} \right)(x)$$
$$\leq Cr_B \mathcal{M}_q \left(|\nabla f| \chi_{8B} \right)(x), \tag{2.3}$$

where in the last step we used the fact $r_B > 100$.

For the remaining term in (2.2), note that $B_{x_{M_0}} = B(x_{M_0}, 1) \subset B(x, 102) = B_0 \subset 3B \subset 8B$ since $d(x, x_{M_0}) \leq 100$. From this and using (D_N), $(P_{2,\mathrm{loc}})$, $q > N \vee 2$ and the Hölder inequality, we conclude that

$$\left| f_{B_0} - f_{B_{x_{M_0}}} \right| \leq \fint_{B_{x_{M_0}}} \left| f - f_{B_0} \right| d\mu \leq C \fint_{B_0} \left| f - f_{B_0} \right| d\mu$$
$$\leq C \left(\fint_{B_0} |\nabla f|^q \, d\mu \right)^{1/q} \leq C \mathcal{M}_q \left(|\nabla f| \chi_{8B} \right)(x)$$
$$\leq Cr_B \mathcal{M}_q \left(|\nabla f| \chi_{8B} \right)(x), \tag{2.4}$$

since $r_B > 100$. The estimates (2.3) and (2.4) yield that for $x \in B$ with $d(x, x_{M_0}) \leq 100$,

$$|f(x) - f_{B_{x_{M_0}}}| \leq Cr_B \mathcal{M}_q \left(|\nabla f| \chi_{8B} \right)(x). \tag{2.5}$$

Step 2. Suppose $d(x, x_{M_0}) > 100$ and let $k_0 \in \mathbb{N}$ be such that

$$9\left(\frac{9}{8}\right)^{k_0} < d(x, x_{M_0}) + 8 \le 9\left(\frac{9}{8}\right)^{k_0+1}. \tag{2.6}$$

Note that (2.6) together with $d(x, x_{M_0}) > 100$ implies

$$8\left(\frac{9}{8}\right)^{k_0} < d(x, x_{M_0}) < 9\left(\frac{9}{8}\right)^{k_0+1}. \tag{2.7}$$

Take a geodesic γ connecting x to x_{M_0}. On the geodesic, we choose a sequence of points $\{x_j\}_{j=0}^{k_0}$

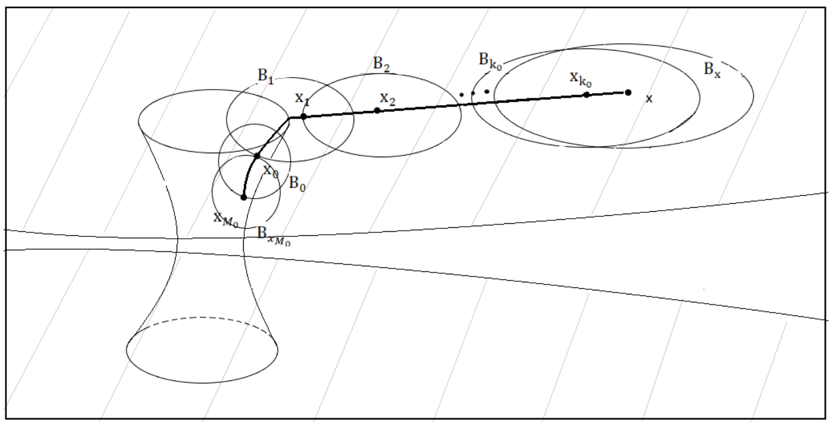

Figure 1: The chosen points and balls along the geodesic.

such that $d(x_0, x_{M_0}) = 1$, $d(x_j, x_{j+1}) = (9/8)^{j+1}$ for $0 \le j \le k_0 - 1$. As the set $\{x_j\}_{j=1}^{k_0}$ belongs to the geodesic γ, we have

$$d(x, x_{k_0}) = d(x, x_{M_0}) - \sum_{j=0}^{k_0}\left(\frac{9}{8}\right)^j = d(x, x_{M_0}) - 8\left[\left(\frac{9}{8}\right)^{k_0+1} - 1\right],$$

which together with (2.6) yields that

$$0 < d(x, x_{k_0}) \le \left(\frac{9}{8}\right)^{k_0+1}. \tag{2.8}$$

Let $B_j = B(x_j, (9/8)^j)$ for $0 \le j \le k_0$ and $B_x = B(x, (9/8)^{k_0+1})$. We write

$$\begin{aligned}\left|f(x) - f_{B_{x_{M_0}}}\right| &\le \left|f_{B_{x_{M_0}}} - f_{B_0}\right| + \sum_{j=0}^{k_0-1}\left|f_{B_j} - f_{B_{j+1}}\right| + \left|f_{B_x} - f_{B_{k_0}}\right| + \left|f(x) - f_{B_x}\right| \\ &=: I + II + III + IV.\end{aligned}$$

For the term I, by $(P_{2,\mathrm{loc}})$, (D_N) and the Hölder inequality, we obtain

$$\begin{aligned}
I &\le \left|f_{B_{x_{M_0}}} - f_{B(x_{M_0},2)}\right| + \left|f_{B_0} - f_{B(x_{M_0},2)}\right| \\
&\le \fint_{B_{x_{M_0}}} |f - f_{B(x_{M_0},2)}|\, d\mu + \fint_{B_0} |f - f_{B(x_{M_0},2)}|\, d\mu \\
&\le \frac{C}{V(x_{M_0},2)} \int_{B(x_{M_0},2)} |f - f_{B(x_{M_0},2)}|\, d\mu \\
&\le \frac{C}{V(x_{M_0},2)^{1/2}} \left(\int_{B(x_{M_0},2)} |\nabla f|^2\, d\mu\right)^{1/2} \le \frac{C}{V(x_{M_0},2)^{1/q}} \left(\int_{B(x_{M_0},2)} |\nabla f|^q\, d\mu\right)^{1/q} \\
&\le \frac{Cd(x,x_{M_0})^{N/q}}{V(x_{M_0},2d(x,x_{M_0}))^{1/q}} \left(\int_{B(x,d(x,x_{M_0})+2)} |\nabla f|^q\, d\mu\right)^{1/q} \\
&\le \frac{Cd(x,x_{M_0})^{N/q}}{V(x,2d(x,x_{M_0}))^{1/q}} \left(\int_{B(x,2d(x,x_{M_0}))} |\nabla f|^q\, d\mu\right)^{1/q} \\
&\le Cd(x,x_{M_0})^{N/q} \mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x) \le Cr_B \mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x).
\end{aligned}$$

Above, in the last second inequality we used that $d(x,x_{M_0}) < 3r_B$, $B(x,2d(x,x_{M_0})) \subset 8B$ since $x \in B$ and $x_{M_0} \in 2B$, and in the last inequality we used that $100 < d(x,x_{M_0}) < 3r_B$.

Let us estimate the second term II. For each $0 \le j < k_0$, we have $B_j, B_{j+1} \subset B(x_{j+1}, \frac{17}{8}(\frac{9}{8})^j)$ and $B(x_{j+1}, \frac{17}{8}(\frac{9}{8})^j) \subset B(x, 2d(x,x_{M_0}))$ by (2.7). By the doubling property, we have

$$\begin{aligned}
\left|f_{B_j} - f_{B_{j+1}}\right| &\le \left|f_{B_j} - f_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)}\right| + \left|f_{B_{j+1}} - f_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)}\right| \\
&\le \fint_{B_j} |f - f_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)}|\, d\mu + \fint_{B_{j+1}} |f - f_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)}|\, d\mu \\
&\le \frac{C}{V(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)} \int_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)} |f - f_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)}|\, d\mu.
\end{aligned}$$

For the j's such that

$$\left(\frac{9}{8}\right)^j + \left(\frac{9}{8}\right)^{j+1} = \frac{17}{8}\left(\frac{9}{8}\right)^j \le 100,$$

by using $(P_{2,\mathrm{loc}})$, (D_N), (2.7) and the Hölder inequality, we conclude that

$$\begin{aligned}
\left|f_{B_j} - f_{B_{j+1}}\right| &\le \frac{C\frac{17}{8}(\frac{9}{8})^j}{V(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)^{1/q}} \left(\int_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)} |\nabla f|^q\, d\mu\right)^{1/q} \\
&\le \frac{C(\frac{9}{8})^{j+(k_0-j)N/q}}{V(x_{j+1},(\frac{9}{8})^{k_0})^{1/q}} \left(\int_{B(x,2d(x,x_{M_0}))} |\nabla f|^q\, d\mu\right)^{1/q} \\
&\le \frac{C(\frac{9}{8})^{j(1-N/q)} d(x,x_{M_0})^{N/q}}{V(x,2d(x,x_{M_0}))^{1/q}} \left(\int_{B(x,2d(x,x_{M_0}))} |\nabla f|^q\, d\mu\right)^{1/q}, \qquad (2.9)
\end{aligned}$$

where in the last inequality we used that $V(x_{j+1},(\frac{9}{8})^{k_0}) \sim V(x_{j+1},2d(x,x_{M_0})) \sim V(x,2d(x,x_{M_0}))$ which follows from (2.7).

For the j's such that

$$\left(\frac{9}{8}\right)^j + \left(\frac{9}{8}\right)^{j+1} = \frac{17}{8}\left(\frac{9}{8}\right)^j > 100,$$

notice that

$$d(x_{j+1},x_{M_0}) = \sum_{j=0}^{j+1}\left(\frac{9}{8}\right)^k = \frac{\left(\frac{9}{8}\right)^{j+2}-1}{\frac{9}{8}-1} = 8\left(\frac{9}{8}\right)^{j+2} - 8,$$

which together with $\mathrm{diam}(M_0) = 1$ implies

$$\mathrm{dist}\,(x_{j+1},M_0) - 2\left[\left(\frac{9}{8}\right)^j + \left(\frac{9}{8}\right)^{j+1}\right] \geq d(x_{j+1},x_{M_0}) - 1 - \frac{17}{4}\left(\frac{9}{8}\right)^j \geq \frac{47}{8}\left(\frac{9}{8}\right)^j - 9 > \frac{47}{8}\frac{800}{17} - 9 > 100.$$

Therefore, $2B(x_{j+1},\frac{17}{8}\left(\frac{9}{8}\right)^j) \cap M_0 = \emptyset$. Applying (P_2^E), (D_N), (2.7) and the Hölder inequality, we conclude that for such j's

$$\begin{aligned}\left|f_{B_j} - f_{B_{j+1}}\right| &\leq \frac{C\frac{17}{8}(\frac{9}{8})^j}{V(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)^{1/q}}\left(\int_{B(x_{j+1},\frac{17}{8}(\frac{9}{8})^j)}|\nabla f|^q\,d\mu\right)^{1/q}\\ &\leq \frac{C(\frac{9}{8})^{j+(k_0-j)N/q}}{V(x_{j+1},(\frac{9}{8})^{k_0})^{1/q}}\left(\int_{B(x,2d(x,x_{M_0}))}|\nabla f|^q\,d\mu\right)^{1/q}\\ &\leq \frac{C(\frac{9}{8})^{j(1-N/q)}d(x,x_{M_0})^{N/q}}{V(x,2d(x,x_{M_0}))^{1/q}}\left(\int_{B(x,2d(x,x_{M_0}))}|\nabla f|^q\,d\mu\right)^{1/q}.\end{aligned}\tag{2.10}$$

Combining (2.9) and (2.10), we further deduce from (2.7) and the fact $q > N$ that

$$\begin{aligned}II &\leq \sum_{j=0}^{k_0-1}\frac{C(\frac{9}{8})^{j(1-N/q)}d(x,x_{M_0})^{N/q}}{V(x,2d(x,x_{M_0}))^{1/q}}\left(\int_{B(x,2d(x,x_{M_0}))}|\nabla f|^q\,d\mu\right)^{1/q}\\ &\leq \sum_{j=0}^{k_0-1}C(\frac{9}{8})^{j(1-N/q)}d(x,x_{M_0})^{N/q}\mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x)\\ &\leq C(\frac{9}{8})^{k_0(1-N/q)}d(x,x_{M_0})^{N/q}\mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x)\\ &\leq Cr_B\mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x).\end{aligned}$$

For the term III, by the choice of the points $\{x_j\}_{j=0}^{k_0}$, we see that

$$d(x,x_{M_0}) - 2\left[\left(\frac{9}{8}\right)^{k_0+1} + \left(\frac{9}{8}\right)^{k_0}\right] \geq \sum_{j=0}^{k_0}\left(\frac{9}{8}\right)^j - \frac{17}{4}\left(\frac{9}{8}\right)^{k_0} \geq \frac{\left(\frac{9}{8}\right)^{k_0+1}-1}{\frac{9}{8}-1} - \frac{17}{4}\left(\frac{9}{8}\right)^{k_0} \geq \frac{19}{4}\left(\frac{9}{8}\right)^{k_0} - 8.$$

By (2.6) one has $108 < d(x, x_{M_0}) + 8 \le 9\,(9/8)^{k_0+1}$, which implies

$$\frac{19}{4}\left(\frac{9}{8}\right)^{k_0} - 8 = \frac{19}{4}\frac{8}{9}\left(\frac{9}{8}\right)^{k_0+1} - 8 > 12\frac{38}{9} - 8 > 40,$$

and hence,

$$\operatorname{dist}(x, M_0) - 2\left[\left(\frac{9}{8}\right)^{k_0+1} + \left(\frac{9}{8}\right)^{k_0}\right] \ge d(x, x_{M_0}) - 1 - 2\left[\left(\frac{9}{8}\right)^{k_0+1} + \left(\frac{9}{8}\right)^{k_0}\right] \ge 39, \tag{2.11}$$

from which it follows that $2B(x, \frac{17}{8}(\frac{9}{8})^{k_0}) \cap M_0 = \emptyset$. By (2.7), $B(x, \frac{17}{8}(\frac{9}{8})^{k_0}) \subset B(x, d(x, x_{M_0}))$, where $d(x, x_{M_0}) < 3r_B$. Thus, by applying (P_2^E), (D_N) and the Hölder inequality, we conclude that

$$\begin{aligned} III &\le \left|f_{B_x} - f_{B(x,\frac{17}{8}(\frac{9}{8})^{k_0})}\right| + \left|f_{B_{k_0}} - f_{B(x,\frac{17}{8}(\frac{9}{8})^{k_0})}\right| \\ &\le \frac{C}{V(x, \frac{17}{8}(\frac{9}{8})^{k_0})}\int_{B(x,\frac{17}{8}(\frac{9}{8})^{k_0})} |f - f_{B(x,\frac{17}{8}(\frac{9}{8})^{k_0})}|\,d\mu \\ &\le \frac{C(\frac{9}{8})^{k_0}}{V(x, \frac{17}{8}(\frac{9}{8})^{k_0})^{1/q}}\left(\int_{B(x,\frac{17}{8}(\frac{9}{8})^{k_0})} |\nabla f|^q\,d\mu\right)^{1/q} \\ &\le C r_B \mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x). \end{aligned}$$

For the term IV, (2.11) implies $2B_x \cap M_0 = \emptyset$, where $B_x = B(x, (9/8)^{k_0+1})$. Therefore, by applying (P_2^E), (D_N) and the approach similar to (2.3), we find

$$IV = |f(x) - f_{B_x}| \le C r_B \mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x).$$

For $x \in B$ with $d(x, x_{M_0}) > 100$, from the estimates for I, II, III, IV, it follows that

$$|f(x) - f_{B_{x_{M_0}}}| \le C r_B \mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(x). \tag{2.12}$$

Apparently, the same proofs in Step 1 and Step 2 work for $y \in B$ and yield that

$$|f(y) - f_{B_{x_{M_0}}}| \le C r_B \mathcal{M}_q\left(|\nabla f|\chi_{8B}\right)(y). \tag{2.13}$$

A combination of (2.1), (2.5), (2.12) and (2.13) completes the proof of the Claim. □

Remark 2.4. The approach used in proving (2.5) is called "telescopic approach" in the literature, see [35, p. 211, proof of Theorem 8.1.7, and p. 243, Section 8.5].

Proposition 2.5. *Assume that (D_N) holds on M with $0 < N < \infty$ and that $(P_{2,\mathrm{loc}})$ holds. Then there exist $N_\mu > 0$ and $C > 0$ such that for any ball $B = B(x_0, r)$, $r > 1$, and any $f \in C^1(\overline{2B})$, it holds*

$$\fint_B |f - f_B|^2\,d\mu \le C r^{2N_\mu+2+N} \fint_{2B} |\nabla f|^2\,d\mu. \tag{P_G}$$

Proof. Based on the validity of $(P_{2,\mathrm{loc}})$, we only need to show (P_G) for balls $B = B(x_0, r)$ when r is sufficiently large. Let us assume $r > 100$. Set $B_0 = B(x_0, 1)$. By (D_N), we can find a sequence of balls via an ϵ-net with $\epsilon = 1/2$, $\{B_i\}_{1\le i\le C(r)}$, where $C(r)$ is an integer not bigger than Cr^{N_μ}, $N_\mu > 0$, such that each ball B_i is of radius one and the center of B_i is located in B, $\frac{1}{2}B_i \cap \frac{1}{2}B_j = \emptyset$ for any $i \neq j$, $0 \le i, j \le C(r)$; see [35, p. 102] for instance. By the choice of B_i we have

$$B \subset \cup_{i=0}^{C(r)} B_i \subset \cup_{i=0}^{C(r)} 3B_i \subset B(x_0, r+3) \subset 2B. \tag{2.14}$$

Write

$$\left(\fint_B |f - f_B|^2\, d\mu\right)^{1/2} \le 2\left(\fint_B |f - f_{B_0}|^2\, d\mu\right)^{1/2} \le \frac{2}{\mu(B)^{1/2}} \sum_{i=0}^{C(r)} \left(\int_{B_i} |f - f_{B_0}|^2\, d\mu\right)^{1/2}.$$

If $i = 0$, then $(P_{2,\mathrm{loc}})$ implies

$$\int_{B_0} |f - f_{B_0}|^2\, d\mu \le C \int_{B_0} |\nabla f|^2\, d\mu.$$

For other i's, let x_i be the center of the ball B_i, and there exists a geodesic $\gamma(x_0, x_i)$ that links x_i to x_0 with length equaling $d(x_0, x_i)$. As $x_i \in B$, one has $d(x_0, x_i) < r$ and $\gamma(x_0, x_i) \subset B$. Along $\gamma(x_0, x_i)$, we may find a sequence of balls $\{B_{i,j}\}_{1\le j\le C(i)}$ with $C(i)$ be an integer not bigger than $2d(x_0, x_i)$, such that each ball $B_{i,j}$ is of radius one and has center on $\gamma(x_0, x_i)$, and $B_{i,1} \cap B_0 \neq \emptyset$, $B_{i,C(i)} \cap B_i \neq \emptyset$ and $B_{i,j} \cap B_{i,j-1} \neq \emptyset$ if $2 \le j \le C(i)$. As the balls $B_{i,j}$ are of radius one and have center on $\gamma(x_0, x_i)$, where $\gamma(x_0, x_i) \subset B$, we have

$$\cup_{1\le j\le C(i)} B_{i,j} \subset \cup_{1\le j\le C(i)} 3B_{i,j} \subset B(x_0, r+3) \subset 2B. \tag{2.15}$$

A chain argument implies

$$\left(\fint_{B_i} |f - f_{B_0}|^2\, d\mu\right)^{1/2} \le \left(\fint_{B_i} |f - f_{B_i}|^2\, d\mu\right)^{1/2} + |f_{B_i} - f_{B_{i,C(i)}}| + |f_{B_0} - f_{B_{i,1}}| + \sum_{j=2}^{C(i)} |f_{B_{i,j}} - f_{B_{i,j-1}}|.$$

Notice that, as $B_0 \cap B_{i,1} \neq \emptyset$, $B_{i,1} \subset 3B_0$, and therefore,

$$|f_{B_0} - f_{B_{i,1}}| \le |f_{3B_0} - f_{B_0}| + |f_{3B_0} - f_{B_{i,1}}| \le C \fint_{3B_0} |f - f_{3B_0}|\, d\mu \le C\left(\fint_{3B_0} |\nabla f|^2\, d\mu\right)^{1/2}.$$

Similarly, we conclude via (D_N), (2.14) and (2.15) that for each $1 \le i \le C(r)$,

$$\begin{aligned}&\left(\fint_{B_i} |f - f_{B_0}|^2\, d\mu\right)^{1/2}\\ &\le C\left(\fint_{3B_0} |\nabla f|^2\, d\mu\right)^{1/2} + C\left(\fint_{3B_{i,C(i)}} |\nabla f|^2\, d\mu\right)^{1/2} + C \sum_{j=1}^{C(i)-1} \left(\fint_{3B_{i,j}} |\nabla f|^2\, d\mu\right)^{1/2}\end{aligned}$$

$$\leq C\frac{r^{N/2}}{\mu(2B)^{1/2}}\left[\left(\int_{3B_0}|\nabla f|^2\,d\mu\right)^{1/2}+\left(\int_{3B_{i,C(i)}}|\nabla f|^2\,d\mu\right)^{1/2}+\sum_{j=1}^{C(i)-1}\left(\int_{3B_{i,j}}|\nabla f|^2\,d\mu\right)^{1/2}\right]$$
$$\leq C\frac{C(i)r^{N/2}}{\mu(2B)^{1/2}}\left(\int_{2B}|\nabla f|^2\,d\mu\right)^{1/2}\leq C\frac{r^{N/2+1}}{\mu(2B)^{1/2}}\left(\int_{2B}|\nabla f|^2\,d\mu\right)^{1/2}.$$

Summarizing these estimates, we conclude that

$$\begin{aligned}\left(\fint_B|f-f_B|^2\,d\mu\right)^{1/2} &\leq \frac{2}{\mu(B)^{1/2}}\sum_{i=0}^{C(r)}\left(\int_{B_i}|f-f_{B_0}|^2\,d\mu\right)^{1/2}\\ &\leq \frac{C}{\mu(B)^{1/2}}\left[\left(\int_{3B_0}|\nabla f|^2\,d\mu\right)^{1/2}+\sum_{i=1}^{C(r)}\frac{r^{N/2+1}\mu(B_i)^{1/2}}{\mu(2B)^{1/2}}\left(\int_{2B}|\nabla f|^2\,d\mu\right)^{1/2}\right]\\ &\leq \frac{CC(r)r^{N/2+1}}{\mu(B)^{1/2}}\left(\int_{2B}|\nabla f|^2\,d\mu\right)^{1/2}\\ &\leq Cr^{N_\mu+N/2+1}\left(\fint_{2B}|\nabla f|^2\,d\mu\right)^{1/2},\end{aligned}$$

where in the third inequality we used $\mu(B_i)/\mu(2B)\leq 1$. This gives the desired estimate. □

Theorem 2.6. *If there exists $C_M>0$ such that for each $x\in M$, it holds*

$$Ric_M(x)\geq -\frac{C_M}{[d(x,x_M)+1]^2},$$

then (P_2^E) holds on M.

Proof. By Buser's inequality (cf. [10, 33]), there exists a constant $C>0$ depending only on the dimension such that, for any $f\in C^1(\bar{B})$,

$$\int_B|f-f_B|\,d\mu\leq Ce^{\sqrt{K}r_B}r_B\int_B|\nabla f|\,d\mu;$$

where $K\geq 0$ and the Ricci curvature on B is not less than $-K$.

For any $B\subset M$ with $2B\cap M_0=\emptyset$, we then have

$$Ric_M(x)\geq -\frac{C_M}{[r_B+1]^2},\ \forall\ x\in B.$$

This together with Buser's inequality implies

$$\int_B|f-f_B|\,d\mu\leq Ce^{\sqrt{\frac{C_M}{r_B^2}}r_B}r_B\int_B|\nabla f|\,d\mu\leq Cr_B\int_B|\nabla f|\,d\mu,$$

which together with [33, Theorem 5.1] further implies

$$\fint_B|f-f_B|^2\,d\mu\leq Cr_B^2\fint_B|\nabla f|^2\,d\mu,$$

as desired. □

3 Riesz transform for p below the lower dimension

3.1 Riesz transform via heat kernel regularity

In this section, we study the behavior of the Riesz transform on $L^p(M)$, where $p \in (2, n)$. In what follows, let $A_r := I - (I - e^{-r^2\mathcal{L}})^m$, where $m \in \mathbb{N}$ is chosen such that $m > N/4$; see [4, p. 932]. Let $T := \nabla\mathcal{L}^{-1/2}$. The sharp maximal function $\mathcal{M}^{\#}_{T,A} f$ for every locally integrable function f is given as

$$\mathcal{M}^{\#}_{T,A} f(x) := \sup_{B:\, x\in B} \left(\fint_B |T(1 - A_{r_B})f|^2 \, d\mu \right)^{1/2}.$$

Lemma 3.1. *Assume that (D_N) holds on M with $0 < N < \infty$. There exists $C > 0$ such that for any $f \in L^2(M)$, any ball B and $x \in B$, it holds*

$$\left(\fint_B |T(I - A_{r_B})f|^2 \, d\mu \right)^{1/2} \le \mathcal{M}^{\#}_{T,A} f(x) \le C\mathcal{M}_2(|f|)(x). \tag{3.1}$$

Proof. See [4, Lemma 3.1]. □

Lemma 3.2. *Assume that (D_N) holds on M with $0 < N < \infty$, and that (UE) and (G_{p_0}) for some $p_0 \in (2, \infty)$ hold. Then for every $p \in (2, p_0)$, there exist $C, \tau > 0$ such that for every ball B with radius r_B and every $f \in L^2(M)$ supported in $U_i = 2^{i+1}B \setminus 2^iB$, $i \ge 2$, or $U_1 = 4B$, one has*

$$\left(\fint_B |\nabla A_{r_B} f|^p \, d\mu \right)^{1/p} \le \frac{Ce^{-\tau 4^i}}{r_B} \left(\frac{1}{\mu(2^iB)} \int_{U_i} |f|^2 \, d\mu \right)^{1/2}. \tag{3.2}$$

Proof. The lemma was proved in [4, Lemma 3.2]. Notice that, although in the statement of [4, Lemma 3.2], (P_2) was assumed, its proof indeed only needs (D_N), (UE) and (G_{p_0}). □

Recall that $x_M \in M_0$ is fixed, and we assume that $\mathrm{diam}(M_0) = 1$. The reverse doubling condition only requires that for all $1 < r < R < \infty$ it holds

$$\left(\frac{R}{r}\right)^n \lesssim \frac{V(x_M, R)}{V(x_M, r)}. \tag{RD_n}$$

Lemma 3.3. *Assume that (D_N) and (RD_n) hold on M with $1 < n \le N < \infty$, and that (UE) holds. Let $C_0 > 10$ be fixed. Then for any $p \in (1, n)$, there exists $C > 0$, depending only on C_0, n, N, such that for any ball B, with $r_B > 1$ and $C_0B \cap M_0 \ne \emptyset$, and any $f \in L^p(M)$, it holds*

$$\left(\fint_B |\mathcal{L}^{-1/2} f|^p \, d\mu \right)^{1/p} \le \frac{Cr_B}{\mu(B)^{1/p}} \|f\|_p.$$

Proof. For each $x \in B$, write

$$|\mathcal{L}^{-1/2} f(x)| \le \frac{\sqrt{\pi}}{2} \int_0^{(2C_0r_B)^2} \left| e^{-s\mathcal{L}} f(x) \right| \frac{ds}{\sqrt{s}} + \frac{\sqrt{\pi}}{2} \int_{(2C_0r_B)^2}^{\infty} \left| e^{-s\mathcal{L}} f(x) \right| \frac{ds}{\sqrt{s}} =: I_1 + I_2.$$

For the term I_1, one has via the Minkowski inequality that

$$\left(\int_B |I_1|^p\,d\mu\right)^{1/p} \le C\int_0^{(2C_0r_B)^2} \|e^{-s\mathcal{L}}f\|_p \frac{ds}{\sqrt{s}} \le C\int_0^{(2C_0r_B)^2} \|f\|_p \frac{ds}{\sqrt{s}} \le Cr_B\|f\|_p.$$

For the term I_2, notice that by (D_N), the assumptions $C_0B\cap M_0\neq\emptyset$ and $\operatorname{diam}(M_0)=1$, it holds

$$V(x_B,\sqrt{t})\sim V(x_M,\sqrt{t})\sim V(x,\sqrt{t})$$

for any $x\in B$ and $t\ge(2C_0r_B)^2$. From this together with (UE), (RD_n) and the Hölder inequality, we deduce that for each $x\in B$

$$\begin{aligned}
|I_2| &\le \int_{(2C_0r_B)^2}^\infty \int_M \frac{C}{V(x,\sqrt{s})}\exp\left\{-\frac{d^2(x,y)}{cs}\right\}|f(y)|\,d\mu(y)\frac{ds}{\sqrt{s}}\\
&\le C\int_{(2C_0r_B)^2}^\infty \|f\|_p\left(\int_M \frac{1}{V(x,\sqrt{s})^{p/(p-1)}}\exp\left\{-\frac{pd^2(x,y)}{c(p-1)s}\right\}d\mu(y)\right)^{(p-1)/p}\frac{ds}{\sqrt{s}}\\
&\le C\|f\|_p\int_{(2C_0r_B)^2}^\infty \frac{C}{V(x_M,\sqrt{s})^{1/p}}\frac{ds}{\sqrt{s}}\\
&\le C\|f\|_p\int_{(2C_0r_B)^2}^\infty \frac{Cr_B^{n/p}}{V(x_M,2C_0r_B)^{1/p}s^{n/(2p)}}\frac{ds}{\sqrt{s}} \le \frac{Cr_B}{\mu(B)^{1/p}}\|f\|_p,
\end{aligned}$$

where in the last inequality we used the fact that $p<n$. Combining the estimates of I_1 and I_2, we conclude that

$$\left(\fint_B |\mathcal{L}^{-1/2}f|^p\,d\mu\right)^{1/p} \le \left(\fint_B |I_1+I_2|^p\,d\mu\right)^{1/p} \le \frac{Cr_B}{\mu(B)^{1/p}}\|f\|_p,$$

as desired. □

Using the the previous mapping property of the Riesz potentials together with Lemma 3.2, we deduce the following estimates.

Proposition 3.4. *Assume that (D_N) and (RD_n) hold on M with $2<n\le N<\infty$, and that (UE) and (P_2^E) hold. Suppose that (G_{p_0}) for some $p_0\in(2,n)$ holds. Let $p\in(2,p_0)$ and $q\in(2,n)$. Let $10<C_0,\alpha,\beta<\infty$. Then for each $B=B(x_B,r_B)\subset M$ and each $f\in C_c^\infty(M)$, the followings hold.*

(i) If $r_B<\alpha$, then there exists $C_1=C_1(n,N,p,p_0,\alpha)$ such that it holds

$$\left(\fint_B |TA_{r_B}f|^p\,d\mu\right)^{1/p} \le C_1\inf_{y\in B}\mathcal{M}_2(|Tf|)(y). \tag{3.3}$$

(ii) If $r_B\ge\beta$ and $d(x_B,x_M)<C_0r_B$, then there exists $C_2=C_2(n,N,p,q,p_0,C_0,\beta)$ such that it holds

$$\left(\fint_B |TA_{r_B}f|^p\,d\mu\right)^{1/p} \le \frac{C_2}{\mu(B)^{1/q}}\|f\|_q. \tag{3.4}$$

(iii) If $r_B \geq \beta$ and $d(x_B, x_M) \geq C_0 r_B$, then there exist C_3, C_4, depending on $n, N, p, q, p_0, C_0, \beta$, such that it holds

$$\left(\fint_B |TA_{r_B} f|^p \, d\mu\right)^{1/p} \leq \frac{C_3 \|f\|_q}{V(x_M, d(x_B, x_M) + 1)^{1/q}} + C_4 \inf_{y \in B} \mathcal{M}_2(|Tf|)(y). \tag{3.5}$$

Proof. Let $g := \mathcal{L}^{-1/2} f$ for $f \in C_c^\infty(M)$. Let $U_i = 2^{i+1}B \setminus 2^i B$, $i \geq 2$, and $U_1 = 4B$.

(i) The case $r_B < \alpha$ follows from a proof similar to [4, p. 935], we provide a proof for completeness. By Lemma 3.2 one has

$$\begin{aligned}\left(\fint_B |TA_{r_B} f|^p \, d\mu\right)^{1/p} &\leq \sum_{i \geq 1} \left(\fint_B |\nabla A_{r_B}[(g - g_{4B})\chi_{U_i}]|^p \, d\mu\right)^{1/p} \\ &\leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i}}{r_B} \left(\frac{1}{\mu(2^i B)} \int_{U_i} |g - g_{4B}|^2 \, d\mu\right)^{1/2}.\end{aligned}$$

By using $(P_{2,\mathrm{loc}})$ from Lemma 2.2 and (P_G) from Proposition 2.5, one finds

$$\begin{aligned}\left(\fint_B |TA_{r_B} f|^p \, d\mu\right)^{1/p} &\leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i}}{r_B} \left(\frac{1}{\mu(2^i B)} \int_{U_i} |g - g_{4B}|^2 \, d\mu\right)^{1/2} \\ &\leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i}}{r_B} \left[\left(\fint_{2^{i+1}B} |g - g_{2^{i+1}B}|^2 \, d\mu\right)^{1/2} + \sum_{j=2}^{i} |g_{2^j B} - g_{2^{j+1}B}|\right] \\ &\leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i}}{r_B} \sum_{j=2}^{i+1} (2^{j+1} r_B)[(2^{j+1} r_B) \vee 1]^{N_\mu + \frac{N}{2}} \left(\fint_{2^{j+1}B} |\nabla g|^2 \, d\mu\right)^{1/2} \\ &\overset{r_B < \alpha}{\leq} \sum_{i \geq 1} Ce^{-\tau 4^i} \sum_{j=1}^{i+1} 2^{j(N_\mu + 1 + \frac{N}{2})} \inf_{y \in B} \mathcal{M}_2(|\nabla g|)(y) \\ &\leq C_1 \inf_{y \in B} \mathcal{M}_2(|Tf|)(y).\end{aligned}$$

(ii) Suppose now $d(x_B, x_M) < C_0 r_B$ and $r_B \geq \beta$. By Lemma 3.2, Lemma 3.3 together with the Hölder inequality, one has

$$\begin{aligned}\left(\fint_B |TA_{r_B} f|^p \, d\mu\right)^{1/p} &\leq \sum_{i \geq 1} \left(\fint_B |\nabla A_{r_B}(g\chi_{U_i})|^p \, d\mu\right)^{1/p} \leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i}}{r_B} \left(\frac{1}{\mu(2^i B)} \int_{U_i} |g|^2 \, d\mu\right)^{1/2} \\ &\leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i}}{r_B} \left(\frac{1}{\mu(2^i B)} \int_{U_i} |g|^q \, d\mu\right)^{1/q} \\ &\leq \sum_{i \geq 1} \frac{Ce^{-\tau 4^i} 2^i r_B}{r_B \mu(2^i B)^{1/q}} \|f\|_q \leq \frac{C}{\mu(B)^{1/q}} \|f\|_q.\end{aligned}$$

(iii) If $d(x_B, x_M) \geq C_0 r_B$ and $r_B \geq \beta$, then the ball B is included in one end. Let $k \in \mathbb{N}$ such that $2^{k+1} r_B \leq d(x_B, x_M) < 2^{k+2} r_B$ (recall that $C_0 > 10$).

By Lemma 3.2 again one has

$$\begin{aligned}\left(\fint_B |TA_{r_B}f|^p\,d\mu\right)^{1/p} &\le \sum_{i\ge1}\left(\fint_B |\nabla A_{r_B}[(g-g_{4B})\chi_{U_i}]|^p\,d\mu\right)^{1/p}\\ &\le \sum_{i=1}^k \frac{Ce^{-\tau 4^i}}{r_B}\left(\frac{1}{\mu(2^iB)}\int_{U_i}|g-g_{4B}|^2\,d\mu\right)^{1/2}+\sum_{i>k}\cdots\\ &=: I_1+I_2.\end{aligned}$$

Since $d(x_B,x_M)<2^{k+2}r_B$, for each $i>k$, $2^{i+1}B\cap M_0\neq\emptyset$. By Lemma 3.3, $q<n$, (D_N) and (RD_n), we obtain

$$\begin{aligned}I_2 &\le \sum_{i>k}\frac{Ce^{-\tau 4^i}}{r_B}\left(\frac{1}{\mu(2^iB)}\int_{U_i}|g-g_{4B}|^2\,d\mu\right)^{1/2}\\ &\le \sum_{i>k}\frac{Ce^{-\tau 4^i}}{r_B}\left[|g|_{4B}+\left(\frac{1}{\mu(2^iB)}\int_{U_i}|g|^2\,d\mu\right)^{1/2}\right]\\ &\le \sum_{i>k}\frac{Ce^{-\tau 4^i}}{r_B}\left[\left(\frac{1}{\mu(B)}\int_{2^{k+1}B}|g|^q\,d\mu\right)^{1/q}+\left(\frac{1}{\mu(2^iB)}\int_{U_i}|g|^q\,d\mu\right)^{1/q}\right]\\ &\le \sum_{i>k}Ce^{-\tau 4^i}\left[\frac{2^k}{\mu(B)^{1/q}}\|f\|_q+\frac{2^i}{\mu(2^iB)^{1/q}}\|f\|_q\right]\\ &\le \sum_{i>k}C\|f\|_q e^{-\tau 4^i}\left[\frac{2^{k+kN/q}}{\mu(2^kB)^{1/q}}+\frac{2^i}{V(x_M,2^ir_B)^{1/q}}\right]\\ &\le \sum_{i>k}C\|f\|_q e^{-\tau 4^i}\left[\frac{2^{k+kN/q}}{V(x_M,2^kr_B)^{1/q}}+\frac{2^{k+(i-k)(1-n/q)}}{V(x_M,2^kr_B)^{1/q}}\right]\\ &\le \frac{Ce^{-c\tau 4^k}2^{k+kN/q}}{V(x_M,d(x_B,x_M)+1)^{1/q}}\|f\|_q\\ &\le \frac{C_3}{V(x_M,d(x_B,x_M)+1)^{1/q}}\|f\|_q.\end{aligned}$$

Using (P_2^E), we can estimate the term I_1 as

$$\begin{aligned}I_1 &= \sum_{i=1}^k\frac{Ce^{-\tau 4^i}}{r_B}\left(\frac{1}{\mu(2^iB)}\int_{U_i}|g-g_{4B}|^2\,d\mu\right)^{1/2}\\ &\le \sum_{i=1}^k\frac{Ce^{-\tau 4^i}}{r_B}\left[\left(\fint_{2^{i+1}B}|g-g_{2^{i+1}B}|^2\,d\mu\right)^{1/2}+\sum_{j=2}^i|g_{2^jB}-g_{2^{j+1}B}|\right]\\ &\le \sum_{i=1}^k\frac{Ce^{-\tau 4^i}}{r_B}\sum_{j=2}^{i+1}(2^jr_B)\left(\fint_{2^jB}|\nabla g|^2\,d\mu\right)^{1/2}\end{aligned}$$

$$\leq C_4 \inf_{y\in B} \mathcal{M}_2(|Tf|)(y).$$

Using the estimates of I_1 and I_2, one can finally conclude that

$$\left(\fint_B |TA_{r_B} f|^p \, d\mu\right)^{1/p} \leq \frac{C_3}{V(x_M, d(x_B, x_M)+1)^{1/q}} \|f\|_q + C_4 \inf_{x\in B} \mathcal{M}_2(|Tf|)(x),$$

as desired. □

Using Proposition 3.4 and adapting the argument from [4], we are able to provide a modified good-λ inequality. The key ingredient is that for large balls, Proposition 3.4 allows us to deduce a small error term in the good-λ inequality; see Proposition 3.6 below.

Proposition 3.5. *Assume that* (D_N) *and* (RD_n) *hold on* M *with* $2 < n \leq N < \infty$, *and that* (UE) *and* (P_2^E) *hold. Assume that* (G_{p_0}) *holds for some* $p_0 \in (2, n)$. *Let* $\alpha > 10$ *and* $2 < q < p_0$. *There exist* $K_0, C > 0$ *only depending on* n, N, α, q, p_0, *such that for each* $f \in C_c^\infty(M)$, *every* $\lambda > 0$, $K > K_0$ *and* $\gamma > 0$, *and every ball* $B_0 = B(x_B, r_B)$, $r_B < \alpha$, *if there exists* $x_0 \in B_0$ *such that* $\mathcal{M}_2(|Tf|)(x_0) \leq \lambda$, *then it holds*

$$\mu\left(\left\{x \in B_0 : \mathcal{M}_2(|Tf|)(x) > K\lambda,\ \mathcal{M}^{\#}_{T,A} f(x) \leq \gamma\lambda\right\}\right) \leq C(\gamma^2 + K^{-q})\mu(B_0). \tag{3.6}$$

Proof. By using (i) of Proposition 3.4, the conclusion follows from [4, Lemma 2.2]. □

Recall again that we fix $x_M \in M_0$ and assume $\mathrm{diam}(M_0) = 1$.

Proposition 3.6. *Assume that* (D_N) *and* (RD_n) *hold on* M *with* $2 < n \leq N < \infty$, *and that* (UE) *and* (P_2^E) *hold. Suppose that* (G_{p_0}) *holds for some* $p_0 \in (2, n)$. *Let* $\beta > 10$ *and* $2 < p < q < p_0$. *There exist* $K_0 > 1$ *and* $C, C_E > 0$ *only depending on* n, N, p, q, p_0, β, *such that for every* $f \in C_c^\infty(M)$, *every* $\lambda > 0$, $K > K_0$ *and* $\gamma > 0$, *and every ball* $B_0 = B(x_B, r_B)$, $r_B > \beta$, *if there exists* $x_0 \in B_0$ *such that* $\mathcal{M}_2(|Tf|)(x_0) \leq \lambda$, *then it holds*

$$\begin{aligned} &\mu\left(\left\{x \in B_0 : \mathcal{M}_2(|Tf|)(x) > K\lambda,\ \mathcal{M}^{\#}_{T,A} f(x) \leq \gamma\lambda,\ \frac{\|f\|_p}{V(x_M, d(x, x_M)+1)^{1/p}} \leq C_E\lambda\right\}\right) \\ &\leq C\left(\gamma^2 + K^{-q}\right)\mu(B_0). \end{aligned} \tag{3.7}$$

Proof. Let $J, K > 1$ and $\gamma, C_E > 0$ to be fixed later. For $\lambda > 0$ let

$$E := \left\{x \in B_0 : \mathcal{M}_2(|Tf|)(x) > K\lambda,\ \mathcal{M}^{\#}_{T,A} f(x) \leq \gamma\lambda,\ \frac{\|f\|_p}{V(x_M, d(x, x_M)+1)^{1/p}} \leq C_E\lambda\right\}$$

and

$$\Omega := \left\{x \in B_0 : \mathcal{M}_2(|TA_{3r_B} f|\chi_{3B_0})(x) > J\lambda,\ \frac{\|f\|_p}{V(x_M, d(x, x_M)+1)^{1/p}} \leq C_E\lambda\right\}.$$

Claim 1. There exists $C > 0$ such that

$$\mu(\Omega) \leq CJ^{-q}\mu(B_0). \tag{3.8}$$

Let us prove the claim. First assume $d(x_B, x_M) < 100r_B$. Notice that, if $\|f\|_p \ge \mu(B_0)^{1/p}\lambda$, then it follows from the doubling property (D_N) that for each $x \in B_0$

$$\begin{aligned} V(x_M, d(x, x_M) + 1) &\le V(x_M, d(x, x_M) + r_B) \le V(x_B, 2d(x_B, x_M) + 2r_B) \\ &\le C\left(\frac{2d(x_B, x_M) + 2r_B}{r_B}\right)^N V(x_B, r_B) \le C\mu(B_0), \end{aligned}$$

and hence, there exists $c_1 > 0$ such that for each $x \in B_0$ it holds

$$\lambda \le \|f\|_p \mu(B_0)^{-1/p} \le c_1 \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}}.$$

By choosing $C_E < 1/c_1$ we see that

$$\mu(\Omega) = 0. \tag{3.9}$$

Suppose now $\|f\|_p < \mu(B_0)^{1/p}\lambda$. Recall that the 2-Hardy-Littlewood maximal operator $\mathcal{M}_2$ is bounded on $L^q(M)$ for all $q > 2$. By using (ii) of Proposition 3.4 and (D_N) one has

$$\left(\fint_{3B_0} |TA_{3r_B} f|^q \, d\mu\right)^{1/q} \le \frac{C_2\|f\|_p}{\mu(3B_0)^{1/p}} < C\lambda,$$

which together with the (q, q) boundedness of $\mathcal{M}_2$ implies that

$$\mu(\Omega) \le \frac{1}{(J\lambda)^q} \int_{B_0} \mathcal{M}_2(|TA_{3r_B} f|\chi_{3B_0})^q \, d\mu \le \frac{C}{(J\lambda)^q} \int_{3B_0} |TA_{3r_B} f|^q \, d\mu \le \frac{C}{J^q}\mu(B_0). \tag{3.10}$$

If $d(x_B, x_M) \ge 100r_B$, then by using (iii) of Proposition 3.4, one has via the fact $\mathcal{M}_2(Tf)(x_0) \le \lambda$ that

$$\begin{aligned} \left(\fint_{3B_0} |TA_{3r_B} f|^q \, d\mu\right)^{1/q} &\le \frac{C_3\|f\|_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}} + C_4 \inf_{y \in B_0} \mathcal{M}_2(|Tf|)(y) \\ &\le C_4\lambda + \frac{C_3\|f\|_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}}. \end{aligned} \tag{3.11}$$

If $\|f\|_p \ge V(x_M, d(x_B, x_M) + 1)^{1/p}\lambda$, then by the fact that $d(x_B, x_M) \ge 100r_B$, we see that for each $x \in B_0$ it holds

$$\lambda \le \frac{\|f\|_p}{V(x_M, d(x_B, x_M) + 1)^{1/p}} \le c_2 \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}}.$$

By choosing $C_E < 1/c_2$ we find

$$\mu(\Omega) = 0. \tag{3.12}$$

Suppose now $\|f\|_p < V(x_M, d(x_B, x_M) + 1)^{1/p}\lambda$. Then by the (q, q) boundedness of $\mathcal{M}_2$ and (3.11) we obtain

$$\mu(\Omega) \le \frac{1}{(J\lambda)^q} \int_{B_0} \mathcal{M}_2(|TA_{3r_B} f|\chi_{3B_0})^q \, d\mu \le \frac{C}{(J\lambda)^q} \int_{3B_0} |TA_{3r_B} f|^q \, d\mu$$

$$\leq \frac{C}{(J\lambda)^q}\left(\lambda + \frac{\|f\|_p}{V(x_M, d(x_B, x_M)+1)^{1/p}}\right)^q \mu(3B_0)$$

$$\leq \frac{C}{J^q}\mu(B_0). \tag{3.13}$$

By choosing $0 < C_E < \min\{1/c_1, 1/c_2\}$, the above estimates (3.9)–(3.13) confirm Claim 1.

Let us estimate the measure of the set $E \setminus \Omega$. Notice that there exists $c_0 > 0$ only depending on the measure such that if $c_0K^2 > 1$ then

$$\mathcal{M}_2(|Tf|\chi_{3B_0})(x) > K\lambda,$$

if $x \in E$. Indeed, since $\mathcal{M}_2(|Tf|)(x) > K\lambda$ for $x \in E$ and $\mathcal{M}_2(|Tf|)(x_0) \leq \lambda$, there exists a ball $B = B(z, r)$ such that $x \in B$, $x_0 \notin B$ and

$$\int_B |Tf|^2\, d\mu > K^2\lambda^2\mu(B),$$

and hence

$$\int_{B(x,2r)} |Tf|^2\, d\mu \geq \int_B |Tf|^2\, d\mu > K^2\lambda^2\mu(B) \geq c_0K^2\lambda^2 V(x, 2r) > \lambda^2 V(x, 2r).$$

This implies that $r < r_B$. To see this, let us assume $r \geq r_B$. Then since $x \in E \subset B_0 = B(x_B, r_B)$, one has $x_0 \in B_0 \subset B(x, 2r)$. This together with the above inequality implies that $\mathcal{M}_2(|Tf|)(x_0) > \lambda$, which contradicts the assumption $\mathcal{M}_2(|Tf|)(x_0) \leq \lambda$. Therefore it holds $r < r_B$, $B \subset 3B_0$, and hence $\mathcal{M}_2(|Tf|\chi_{3B_0})(x) > K\lambda$.

Therefore, there exists $K_0 > 0$ large enough, such that for any $K > K_0$ and $x \in E$ it holds

$$\mathcal{M}_2(|Tf|\chi_{3B_0})(x) > K\lambda.$$

By letting $K = J + 1 > K_0$ we find

$$\begin{aligned}
&\mu(E \setminus \Omega)\\
&\leq \mu\left(\left\{x \in B_0 :\ \mathcal{M}_2(|TA_{3r_B}f|\chi_{3B_0})(x) \leq J\lambda,\ \mathcal{M}_2(|Tf|\chi_{3B_0})(x) > K\lambda,\ \mathcal{M}^{\#}_{T,A}f(x) \leq \gamma\lambda\right\}\right)\\
&\leq \mu\left(\left\{x \in B_0 :\ \mathcal{M}_2(|T(I - A_{3r_B})f|\chi_{3B_0})(x) > (K-J)\lambda,\ \mathcal{M}^{\#}_{T,A}f(x) \leq \gamma\lambda\right\}\right)\\
&\leq \frac{C}{(K-J)^2\lambda^2}\int_{3B_0} |T(I - A_{3r_B})f|^2\, d\mu \leq \frac{C\mu(B_0)}{\lambda^2} \inf_{x \in B_0:\, \mathcal{M}^{\#}_{T,A}f(x) \leq \gamma\lambda} \left(\mathcal{M}^{\#}_{T,A}f(x)\right)^2\\
&\leq C\gamma^2\mu(B_0).
\end{aligned}$$

This together with the estimate (3.8) with $J = K - 1$ for Ω gives the desired result. □

We next show that (G_{p_0}) implies (R_p) for $p < p_0$.

Theorem 3.7. *Assume that (D_N) and (RD_n) hold on M with $2 < n \leq N < \infty$, and that (UE) and (P_2^E) hold. Suppose that (G_{p_0}) for some $p_0 \in (2, n)$ hold. Then (R_p) holds for all $p \in (2, p_0)$.*

Proof. Let $p \in (2, p_0)$ and fix $q \in (p, p_0)$. Let $f \in C_c^\infty(M)$. Then we have $|\nabla \mathcal{L}^{-1/2} f| \in L^2(M)$. For each $\lambda > 0$, let

$$E_\lambda := \left\{ x \in M : \mathcal{M}_2 |\nabla \mathcal{L}^{-1/2} f|(x) > \lambda \right\}.$$

Then $\mu(E_\lambda) < \infty$ for each $\lambda > 0$.

By [19, Chapter III, Theorem 1.3], we can find a sequence of balls $\{B_i\}_i$ with finite overlap property, such that $E_\lambda = \cup_i B_i$. Moreover, there exists $C_W > 1$ such that there exists $\tilde{x}_i \in C_W B_i$, $\mathcal{M}_2(|\nabla \mathcal{L}^{-1/2}|)(\tilde{x}_i) \le \lambda$ for each i.

Fix a sufficient large $K_0 > 0$ such that Propositions 3.5 and 3.6 hold for any $K > K_0$. Let $\gamma > 0$ to be fixed later. Set

$$F_{\gamma\lambda} := \left\{ x \in M : \mathcal{M}^{\#}_{T,A} f(x) > \gamma\lambda \right\}$$

and

$$G_\lambda := \left\{ x \in M : \frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}} > C_E \lambda \right\},$$

where C_E is the constant from Proposition 3.6. By noticing that $E_{K\lambda} \subset E_\lambda$, we find

$$\begin{aligned} \mu(E_{K\lambda} \setminus (F_{\gamma\lambda} \cup G_\lambda)) \le & \sum_{B_i :\, r_{B_i} < 100} \mu((B_i \cap E_{K\lambda}) \setminus F_{\gamma\lambda}) + \sum_{B_i :\, r_{B_i} \ge 100} \mu((B_i \cap E_{K\lambda}) \setminus (F_{\gamma\lambda} \cup G_\lambda)) \\ =: & \; I_3 + I_4. \end{aligned}$$

For each B_i with $r_{B_i} < 100$, by applying Proposition 3.5 to the ball $C_W B_i$, one concludes that

$$\mu((B_i \cap E_{K\lambda}) \setminus F_{\gamma\lambda}) \le \mu((C_W B_i \cap E_{K\lambda}) \setminus F_{\gamma\lambda}) \le C\left(\gamma^2 + K^{-q}\right)\mu(C_W B_i) \le C\left(\gamma^2 + K^{-q}\right)\mu(B_i),$$

and hence by the bounded overlap property of $\{B_i\}$, we obtain

$$I_3 \le \sum_{B_i :\, r_{B_i} < 100} C\left(\gamma^2 + K^{-q}\right)\mu(B_i) \le C\left(\gamma^2 + K^{-q}\right)\mu(E_\lambda).$$

Meanwhile, Proposition 3.6 gives for each B_i with $r_{B_i} \ge 100$ that

$$\mu((B_i \cap E_{K\lambda}) \setminus (F_{\gamma\lambda} \cup G_\lambda)) \le \mu((C_W B_i \cap E_{K\lambda}) \setminus (F_{\gamma\lambda} \cup G_\lambda))) \le C\left(\gamma^2 + K^{-q}\right)\mu(B_i),$$

and hence by applying the bounded overlap property of $\{B_i\}$ once more, we obtain

$$I_4 \le \sum_{B_i :\, r_{B_i} \ge 100} C\left(\gamma^2 + K^{-q}\right)\mu(B_i) \le C\left(\gamma^2 + K^{-q}\right)\mu(E_\lambda).$$

By the estimates of I_3 and I_4, we conclude that

$$\mu(E_{K\lambda}) \le C\left(\gamma^2 + K^{-q}\right)\mu(E_\lambda) + \mu(F_{\gamma\lambda}) + \mu(G_\lambda). \tag{3.14}$$

It follows from Lemma 3.1 that

$$\mu(F_{\gamma\lambda}) \le \frac{1}{(\gamma\lambda)^p} \int_M (\mathcal{M}^{\#}_{T,A} f)^p \, d\mu \le \frac{C}{(\gamma\lambda)^p} \int_M (\mathcal{M}_2 f)^p \, d\mu \le \frac{C\|f\|_p^p}{(\gamma\lambda)^p}.$$

Let us estimate $\mu(G_\lambda)$. If $V(x_M, 1)(C_E\lambda)^p \ge \|f\|_p^p$, then since for any $x \in M$ it holds

$$\frac{\|f\|_p}{V(x_M, d(x, x_M) + 1)^{1/p}} \le \frac{\|f\|_p}{V(x_M, 1)^{1/p}} \le C_E\lambda,$$

we conclude that $G_\lambda = \emptyset$.

If $V(x_M, 1)(C_E\lambda)^p < \|f\|_p^p$, then we have

$$G_\lambda = \left\{x \in M : \ V(x_M, d(x_M, x) + 1) < \left(\frac{\|f\|_p}{C_E\lambda}\right)^p\right\},$$

and hence

$$\mu(G_\lambda) \le C\left(\frac{\|f\|_p}{C_E\lambda}\right)^p.$$

Inserting the estimates of $\mu(F_{\gamma\lambda})$ and $\mu(G_\lambda)$ into the estimate (3.14), we see that

$$\mu(E_{K\lambda}) \le C\left(\gamma^2 + K^{-q}\right)\mu(E_\lambda) + \frac{C\|f\|_p^p}{(\gamma\lambda)^p} + \left(\frac{\|f\|_p}{C_E\lambda}\right)^p.$$

This implies that for each $\lambda > 0$

$$(K\lambda)^p\mu(E_{K\lambda}) \le C\left(\gamma^2 + K^{-q}\right)(K\lambda)^p\mu(E_\lambda) + CK^p\gamma^{-p}\|f\|_p^p + CK^p\|f\|_p^p.$$

By taking K large enough first and then γ small enough, we see that

$$\|\mathcal{M}_2(|\nabla\mathcal{L}^{-1/2}f|)\|_{L^{p,\infty}}^p \le \frac{1}{2}\|\mathcal{M}_2(|\nabla\mathcal{L}^{-1/2}f|)\|_{L^{p,\infty}}^p + C(K, \gamma, p)\|f\|_p^p,$$

which implies the Riesz transform is bounded from $L^p(M)$ to $L^{p,\infty}(M)$ for $p \in (2, p_0)$.

Since the Riesz transform is naturally L^2-bounded, we conclude that the Riesz transform is L^p-bounded for any $p \in (2, p_0)$ via the Marcinkiewicz interpolation theorem. □

3.2 Harmonic functions and Riesz transform

We need the following lemmas to conclude Theorem 1.5 and Theorem 1.8. Recall that M is a complete, non-compact manifold with one or more but finitely many ends.

Let $p \in (2, \infty]$. We say that the local reverse L^p-Hölder inequality for gradients of harmonic functions holds on M, if for all $r_0 > 0$ there exists $C_H(r_0) > 0$ such that, for all balls B with $r_B < r_0$, and each u satisfying $\mathcal{L}u = 0$ in $3B$, it holds

$$\left(\fint_B |\nabla u|^p \, d\mu\right)^{1/p} \le \frac{C_H(r_0)}{r_B}\fint_{2B} |u| \, d\mu. \tag{$RH_{p,\mathrm{loc}}$}$$

Notice that, if the constant $C_H(r_0)$ can be taken independent of r_0, then ($RH_{p,\mathrm{loc}}$) becomes (RH_p).

We shall need the mean value property for harmonic functions (see [22, Proposition 2.1] for instance).

Lemma 3.8. *Assume that* (D_N) *holds on* M *with* $0 < N < \infty$*, and that* (UE) *holds. Then for any* $\beta \in (0,1)$ *there exists* $C > 0$ *depending on* β *such that if* $\mathcal{L}u = 0$ *in* $B(x_0, r)$*, then*

$$\|u\|_{L^\infty(B(x_0,\beta r))} \le C \fint_{B(x_0,r)} |u|\, d\mu.$$

Proof. The case $\beta = 1/2$ is a well-known fact as a consequence of Sobolev inequality; see [22, Proposition 2.1] for instance. The general case for $\beta \in (0,1)$ follows from a simple covering argument. □

Lemma 3.9. *Assume that* (D_N) *and* (RD_n) *hold on* M *with* $2 < n \le N < \infty$*, and that* (UE) *and* (P_2^E) *hold. Let* $p \in (2, n)$*. Then* (RH_p) *holds if and only if* $(RH_{p,\mathrm{loc}})$ *and* (RH_p^E) *hold.*

Proof. It is obvious that (RH_p) implies $(RH_{p,\mathrm{loc}})$ and (RH_p^E). Let us prove the converse side.

If $3B \cap M_0 = \emptyset$, then (RH_p) holds by (RH_p^E), and if $r_B \le 100$, then (RH_p) holds by $(RH_{p,\mathrm{loc}})$.

Assume now $3B \cap M_0 \neq \emptyset$ and $r_B > 100$. For any $x \in B$, we set

$$r_x := \max\{10, \min\{d(x, x_M)/10,\ r_B/10\}\}.$$

If $d(x, x_M) \le 100$, then $r_x = 10$, and by applying $(RH_{p,\mathrm{loc}})$ to the ball $B_x := B(x, r_x)$, we see that

$$\begin{aligned}\left(\fint_{B_x} |\nabla u|^p\, d\mu\right)^{1/p} &\le \frac{C}{10} \fint_{2B_x} |u|\, d\mu \le \frac{C}{d(x, x_M) + 1} \fint_{2B_x} |u|\, d\mu \\ &\le \frac{C}{d(x, x_M) + 1} \|u\|_{L^\infty(2B_x)} \le \frac{C}{d(x, x_M) + 1} \fint_{2B} |u|\, d\mu,\end{aligned}$$

where the last inequality follows from the fact $2B_x \subset \frac{6}{5}B \subset 2B$ and Lemma 3.8.

If $d(x, x_M) > 100$, then $r_x = \min\{d(x, x_M)/10,\ r_B/10\}$. Notice that since $\operatorname{diam} M_0 = 1$, $3B(x, r_x) \cap M_0 = \emptyset$. By applying (RH_p^E) to $B_x := B(x, r_x)$, we conclude via Lemma 3.8 once more that

$$\left(\fint_{B_x} |\nabla u|^p\, d\mu\right)^{1/p} \le \frac{C}{r_x} \fint_{2B_x} |u|\, d\mu \le \frac{C}{r_x} \|u\|_{L^\infty(2B_x)} \le \frac{C}{r_x} \|u\|_{L^\infty(\frac{11}{10}B)} \le \frac{C}{r_x} \fint_{2B} |u|\, d\mu.$$

Noticing that $d(x, x_M) \le d(x, x_B) + d(x_B, x_M) < 4r_B + 1 < 5r_B$, and combining the above two estimates, we conclude that for each $x \in B$, it holds

$$\fint_{B_x} |\nabla u|^p\, d\mu \le \frac{C}{[1 + d(x, x_M)]^p} \left(\fint_{2B} |u|\, d\mu\right)^p. \tag{3.15}$$

Let $k_0 = [\log_2(5r_B)]$. Since $p < n$, by using (RD_n) and (D_N), we have

$$\int_B \frac{1}{[1 + d(x, x_M)]^p}\, d\mu(x) \le \int_{B(x_M, 5r_B)} \frac{1}{[1 + d(x, x_M)]^p}\, d\mu(x)$$

$$\le \sum_{k=0}^{-k_0} \int_{2^k B(x_M,5r_B)\setminus 2^{k-1}B(x_M,5r_B)} \frac{1}{[1+d(x,x_M)]^p}\,d\mu(x) + \int_{2^{-k_0}B(x_M,5r_B)} \frac{1}{[1+d(x,x_M)]^p}\,d\mu(x)$$
$$\le \sum_{k=0}^{-k_0} \frac{C}{[1+2^k r_B]^p} V(x_M, 2^k 5r_B) + V(x_M,1)$$
$$\le \sum_{k=0}^{-k_0} \frac{C}{[1+2^k r_B]^p} 2^{kn} V(x_M, 5r_B) + C2^{-k_0 n} V(x_M, 5r_B)$$
$$\le \sum_{k=0}^{-k_0} C2^{k(n-p)} r_B^{-p} V(x_M, 5r_B) + C r_B^{-n} V(x_M, 5r_B)$$
$$\le C\mu(B) r_B^{-p}. \tag{3.16}$$

For each $x \in B$, and every $y \in B_x = B(x, r_x)$, one has

$$d(x, x_M) - r_x \le d(y, x_M) \le d(x, x_M) + r_x.$$

As a consequence of $d(x, x_M) > 100$ and $r_B > 100$, it holds $9r_x/10 \le r_y \le 11r_x/10$, where $r_y = \max\{10, \min\{d(y, x_M)/10,\ r_B/10\}\}$. This together with the doubling condition leads to

$$\begin{aligned}\int_B \fint_{B_x} |\nabla u(y)|^p\,d\mu(y)\,d\mu(x) &= \int_B \int_B |\nabla u(y)|^p \frac{\chi_{B(x,r_x)}(y)}{V(x,r_x)}\,d\mu(y)\,d\mu(x)\\ &\ge C \int_B \int_B |\nabla u(y)|^p \frac{\chi_{B(x,r_x)}(y)}{V(y,r_y)}\,d\mu(y)\,d\mu(x)\\ &\ge C \int_B \int_B |\nabla u(y)|^p \frac{\chi_{B(y,10r_y/11)}(x)}{V(y,r_y)}\,d\mu(x)\,d\mu(y)\\ &\ge C \int_B \nabla u(y)|^p\,d\mu(y).\end{aligned}$$

This together with (3.15) and (3.16) gives that

$$\begin{aligned}\int_B |\nabla u(y)|^p\,d\mu(y) \le \int_B \fint_{B_x} |\nabla u(y)|^p\,d\mu(y)\,d\mu(x) &\le \int_B \frac{C}{[1+d(x,x_M)]^p}\left(\fint_{2B}|u|\,d\mu\right)^p d\mu(x)\\ &\le C\mu(B) r_B^{-p}\left(\fint_{2B}|u|\,d\mu\right)^p,\end{aligned}$$

which is nothing but (RH_p). □

Lemma 3.10. *Assume that (D_N) holds on M with $0 < N < \infty$, and that (P_2^E) and (UE) hold. Let $p \in (2, \infty]$. If (RH_p^E) holds on M, then $(RH_{p,\,\mathrm{loc}})$ holds on M.*

Proof. The proof is similar to that of Lemma 2.2, by using Yau's gradient estimates for harmonic functions.

Recall that Yau's gradient estimate states that if u is a harmonic function on $2B$, then it holds

$$\sup_{x\in B}\frac{|\nabla u(x)|}{u(x)}\le C(N)\left(\frac{1}{r_B}+\sqrt{K}\right),$$

where $K\ge 0$, if every point in $2B$ has Ricci curvature not less than $-K$; see [17, 50].

For any $r_0>0$, the Ricci curvature on the set $\{x\in M:\ \operatorname{dist}(x,M_0)<6r_0\}$ is bounded below by a constant $-K(r_0)$ depending on r_0, $K(r_0)\ge 0$. Suppose that u is a harmonic function on $3B$, where $B=B(x,r)$ with $r<r_0$.

If $\operatorname{dist}(x,M_0)\le 3r_0$, then for an arbitrary $\varepsilon>0$, applying the pointwise Yau's gradient estimate to $u+\|u\|_{L^\infty(\frac32 B)}+\varepsilon$, one has for each $y\in B$

$$|\nabla u(y)|\le C\left(u+\|u\|_{L^\infty(\frac32 B)}+\varepsilon\right)\left(\frac1r+\sqrt{K(r_0)}\right)\le\frac{C(r_0)}{r}\left(\|u\|_{L^\infty(\frac32 B)}+\varepsilon\right).$$

By Lemma 3.8 and letting $\varepsilon\to 0$, we see that

$$|\nabla u(x)|\le\frac{C(r_0)}{r}\fint_{2B}|u|\,d\mu,$$

and hence,

$$\left(\fint_B|\nabla u|^p\,d\mu\right)^{1/p}\le\frac{C(r_0)}{r}\fint_{2B}|u|\,d\mu.$$

If $\operatorname{dist}(x,M_0)>3r_0$, then $B(x,3r)\cap M_0=\emptyset$ for any $r<r_0$. By using (RH_p^E), one sees that

$$\left(\fint_B|\nabla u|^p\,d\mu\right)^{1/p}\le\frac{C}{r}\fint_{2B}|u|\,d\mu,$$

as desired. □

The above lemma leads to the following open-ended character of condition (RH_p) for $p<n$.

Lemma 3.11. *Assume that* (D_N) *and* (RD_n) *hold on* M *with* $2<n\le N<\infty$, *and that* (UE) *and* (P_2^E) *hold. Let* $p\in(2,n)$. *Then if* (RH_p) *holds, there exists* $\epsilon>0$ *such that* $p+\epsilon<n$ *and* $(RH_{p+\epsilon})$ *holds.*

Proof. For each ball $B=B(x_B,r_B)$ with $3B\cap M_0=\emptyset$, we can find via the ϵ-net argument (cf. [35, p.102]) a sequence of balls $\{B_i\}_{1\le i\le k_0}$, where $1\le k_0\le 8^{N_\mu}$ and $N_\mu>0$ depending only on the measure μ (see the proof of Proposition 2.5), such that each B_i has radius of $r_B/4$, $\frac12 B_i\cap\frac12 B_j=\emptyset$ whenever $i\ne j$, and $\cup_{1\le i\le k_0}\frac12 B_i\subset B\subset\cup_{1\le i\le k_0}B_i$.

As B_i has radius of $r_B/4$, $\frac12 B_i\subset B$ and $3B\cap M_0=\emptyset$, we find that $4B_i\subset 2B$ and $4B_i\cap M_0=\emptyset$.

Let v be a harmonic function on $3B$. Using (P_2^E), one can conclude from (RH_p) that for each $1\le i\le k_0$, it holds

$$\left(\fint_{B_i}|\nabla v|^p\,d\mu\right)^{1/p}\le\frac{C}{r_B}\left(\fint_{2B_i}|v-v_{2B_i}|^2\,d\mu\right)^{1/2}\le C\left(\fint_{2B_i}|\nabla v|^2\,d\mu\right)^{1/2}.$$

Moreover, this estimate holds for any sub-ball $\tilde{B}_i$ with $2\tilde{B}_i \subset 2B_i$, since $4\tilde{B}_i \cap M_0 = \emptyset$. Applying Gehring's lemma (cf. [31]) and self-improvements of the reverse Hölder inequality (cf. [6, Appendix]), we see that it holds for some $\epsilon > 0$,

$$\left(\fint_{B_i} |\nabla v|^{p+\epsilon}\, d\mu\right)^{1/(p+\epsilon)} \le C\left(\fint_{2B_i} |\nabla v|^2\, d\mu\right)^{1/2} \le \frac{C}{r_B}\fint_{4B_i} |v|\, d\mu \le \frac{C}{r_B}\fint_{2B} |v|\, d\mu,$$

where the second inequality follows from (RH_p), and hence,

$$\left(\fint_{B} |\nabla v|^{p+\epsilon}\, d\mu\right)^{1/(p+\epsilon)} \le \sum_{i=1}^{k_0} \frac{\mu(B_i)^{1/(p+\epsilon)}}{\mu(B)^{1/(p+\epsilon)}}\left(\fint_{B_i} |\nabla v|^{p+\epsilon}\, d\mu\right)^{1/(p+\epsilon)} \le \frac{C}{r_B}\fint_{2B} |v|\, d\mu,$$

i.e., $(RH^E_{p+\epsilon})$ holds. This together with Lemma 3.9 and Lemma 3.10 completes the proof. □

We can now finish the proof of Theorem 1.5.

Proof of Theorem 1.5. Since in our setting, (D) and (UE) hold, and $(P_{2,\mathrm{loc}})$ follows from (P^E_2) by Lemma 2.2, we see that $(RH_p) \Leftrightarrow (G_p)$ holds for any $p \in (2,\infty)$ by [22, Theorem 1.5]. The implication $(R_p) \Rightarrow (G_p)$ holds automatically by the analyticity on L^p of the heat semigroup; see [4] for instance.

Finally, if (RH_p) holds, then by Lemma 3.11 there exists $\epsilon > 0$ such that $p+\epsilon < n$ and $(RH_{p+\epsilon})$ holds. This implies $(G_{p+\epsilon})$, which by Theorem 3.7 gives (R_q) for all $q \in (2, p+\epsilon)$, in particular, (R_p). The proof is complete. □

Theorem 1.8 follows from Lemma 3.9 and Theorem 1.5.

Proof of Theorem 1.8. By Theorem 1.5, (R_p) is equivalent to (RH_p). By Lemma 3.9 and Lemma 3.10 one sees that (RH_p) is equivalent to (RH^E_p). □

Proof of Corollary 1.9. This corollary follows from Theorem 1.8 and Lemma 3.11. □

Lemma 3.12. *Assume that (D_N) holds on M with $0 < N < \infty$, and that (QD) and (UE) hold. Then (RH^E_∞) holds.*

Proof. Suppose that u is a harmonic function on $3B$, with $3B \cap M_0 = \emptyset$. For an arbitrary $\varepsilon > 0$, applying the pointwise Yau's gradient estimate (see the proof of Lemma 3.10) to $u + \|u\|_{L^\infty(\frac{3}{2}B)} + \varepsilon$, one has for each $x \in B$

$$|\nabla u(x)| \le C\left(u + \|u\|_{L^\infty(\frac{3}{2}B)} + \varepsilon\right)\left(\frac{1}{r_B} + \frac{C_M}{r_B+1}\right) \le \frac{C}{r_B}\left(\|u\|_{L^\infty(\frac{3}{2}B)} + \varepsilon\right) \le \frac{C}{r_B}\fint_{2B}(|u|+\epsilon)\, d\mu.$$

Above in the last step we used Lemma 3.8. Letting $\varepsilon \to 0$, we see that (RH^E_∞) holds. □

Proof of Corollary 1.10. By [20] it is known that under (D) and (UE), (R_p) holds for all $p \in (1,2]$.

Notice that under (QD), (P^E_2) holds by Theorem 2.6, and (RH^E_∞) holds by Lemma 3.12. Since (RH^E_∞) implies (RH^E_p) for all $p \in (2,\infty)$, we see that (R_p) holds for all $p \in (2,n)$ by Theorem 1.8. □

4 Riesz transform for p above the upper dimension

In this section, we provide the proofs for Theorem 1.11 and Corollary 1.12. The ideas employed come from recent developments of the elliptic theory for heat kernels from [6, 22].

Proof of Theorem 1.11. By Theorem 2.3, we see that (P_p) holds for any $p > N \vee 2$. Since our measure is doubling, and (UE) holds, we can apply [22, Theorem 1.9] to show that the conditions (R_p), (RH_p) and (G_p) are equivalent.

Notice that, by [20], (D) and (UE) implies that (R_q) holds for all $q \in (1,2)$. Since $p > N$, M is p-parabolic. By [12, Theorem C] and the assumption that M is non-parabolic, (R_p) together with $(R_{\frac{p}{p-1}})$ implies that M can have only one end. □

Remark 4.1. Notice that (P_p) together with (G_p) implies (P_2) by [6]. One may also use (P_2) to show that there exists only one end if $n > 2$.

Proof of Corollary 1.12. Suppose that there exists a non-constant harmonic function u on M with the growth

$$u(x) = O(d(x,o)^{\alpha}) \text{ as } d(x,o) \to \infty$$

for some $\alpha \in [0,1)$ and a fixed $o \in M$.

Assume first $\alpha = 0$. If (R_p) holds for some $p > N \vee 2$, then by Theorem 1.11 we have (RH_p), which implies that for all balls B with $r_B > 1$ and harmonic function v in $3B$ it holds

$$\left(\int_B |\nabla v|^p \, d\mu\right)^{1/p} \le \frac{C\mu(B)^{1/p}}{r_B} \fint_{2B} |v| \, d\mu \le \frac{CV(x_B,1)^{1/p} r_B^{N/p}}{r_B} \fint_{2B} |v| \, d\mu. \tag{4.1}$$

Applying this estimate to u and letting the radius of B tend to infinity, we see that $\||\nabla u|\|_p = 0$, which cannot be true. Therefore the Riesz transform is not bounded on $L^p(M)$ for any $p > N \vee 2$.

Assume now $\alpha \in (0,1)$. Suppose first that $\frac{N}{1-\alpha} \le 2$. Notice that it implies $N < 2$. Assume (R_p) holds for some $p > 2$. Then the estimate (4.1) holds for u, which further implies that

$$\left(\int_B |\nabla u|^p \, d\mu\right)^{1/p} \le \frac{C\mu(B)^{1/p}}{r_B} \fint_{2B} |u| \, d\mu \le CV(x_B,1)^{1/p} r_B^{N/p+\alpha-1} \to 0,$$

as $r_B \to \infty$, since $p > 2 \ge \frac{N}{1-\alpha}$. This implies $\||\nabla u|\|_p = 0$ which contradicts with u being non-constant. Therefore, (R_p) does not hold for any $p > 2$.

For the cases $\frac{N}{1-\alpha} > 2$, we only need to show that the Riesz transform is not bounded on $L^p(M)$ for $p = \frac{N}{1-\alpha}$. Suppose this is not the case. By the validity of (P_p) from Theorem 2.3 and (R_p), we apply [22, Corollary 1.10] to find that there exists $\epsilon > 0$ such that for each v satisfying $\mathcal{L}v = 0$ in $3B$, it holds

$$\left(\fint_B |\nabla v|^{p+\epsilon} \, d\mu\right)^{1/(p+\epsilon)} \le \frac{C}{r_B} \fint_{2B} |v| \, d\mu.$$

This gives

$$\left(\int_B |\nabla v|^{p+\epsilon} \, d\mu\right)^{1/(p+\epsilon)} \le \frac{C\mu(B)^{1/(p+\epsilon)}}{r_B} \fint_{2B} |v| \, d\mu.$$

Applying this estimate to u and using (D_N), we conclude that

$$\left(\int_B |\nabla u|^{p+\epsilon}\, d\mu\right)^{1/(p+\epsilon)} \le CV(x_B,1)^{1/(p+\epsilon)} r_B^{N/(p+\epsilon)-1+\alpha} \sim r_B^{\frac{N(1-\alpha)}{N+\epsilon(1-\alpha)}-1+\alpha} \to 0,$$

as $r_B \to \infty$. This contradicts with u being non-constant. Therefore, the Riesz transform cannot be bounded on $L^p(M)$ for $p = \frac{N}{1-\alpha}$. The proof is completed. □

Carron [12, Theorem D & Proposition E] had provided some sufficient conditions for both boundedness and unboundedness of the Riesz transform for $p > N$, under the requirement of quadratic Ricci curvature decay (QD). As an application of our criteria above, we can relax the requirement of Ricci curvature bound from [12] to (P_2^E) and (RH_∞^E) (see Lemma 2.6 and Lemma 3.12), and show that his condition (HE_α) is also necessary, if $n = N$.

Theorem 4.2. *Assume that (D_N) and (RD_n) hold on M with $1 < n = N < \infty$. Suppose that (UE), (P_2^E) and (RH_∞^E) hold. Let $p \in (N \vee 2, \infty)$. Then the following statements are equivalent.*

(i) (R_p) holds;

(ii) (RH_p) holds;

(iii) (HE_α) holds for some $\alpha \in (1 - \frac{N}{p}, 1]$, i.e., there exists $C > 0$ such that for any ball $B \subset M$ and any harmonic function u on $3B$, it holds for any $x, y \in B$ that

$$(HE_\alpha) \qquad |u(x) - u(y)| \le C\left(\frac{d(x,y)}{r_B}\right)^{\alpha} \fint_{2B} |u|\, d\mu.$$

Proof. The equivalence (R_p) and (RH_p) is a special case of Theorem 1.11. Let us show that $(RH_p) \Leftrightarrow (HE_\alpha)$.

Step 1. $(HE_\alpha) \Rightarrow (RH_p)$**.** The case $\alpha = 1$ is easy, since (HE_α) implies that for any $x \in B$,

$$|\nabla u(x)| \le \limsup_{y:\, d(x,y)\to 0} \frac{|u(x)-u(y)|}{d(x,y)} \le \frac{C}{r_B} \fint_{2B} |u|\, d\mu,$$

which is (RH_∞).

Suppose now $\alpha \in (1 - N/p, 1)$. Recall that $p > N \vee 2$. If $3B \cap M_0 = \emptyset$, then (RH_∞) holds on B by (RH_∞^E). If $r_B \le 100$, then by applying $(RH_{\infty,\mathrm{loc}})$ from Lemma 3.10, one has that for u satisfying $\mathcal{L}u = 0$ on $3B$, it holds

$$\|\nabla u\|_{L^\infty(B)} \le \frac{C}{r_B} \fint_{2B} |u|\, d\mu.$$

Let us consider the remaining case: $3B \cap M_0 \neq \emptyset$ and $r_B > 100$. For any $x \in B$, if $d(x, x_M) \le 10$, then by applying $(RH_{\infty,\mathrm{loc}})$ to the ball $B_x := B(x, d(x,x_M)+1)$ and $u - u_{2B_x}$ we see that

$$\||\nabla u|\|_{L^\infty(B_x)} \le \frac{C}{d(x,x_M)+1} \fint_{2B_x} \fint_{2B_x} |u(y) - u(z)|\, d\mu(y)\, d\mu(z).$$

Applying (HE_α) we find

$$\||\nabla u|\|_{L^\infty(B_x)} \le \frac{C[1+d(x,x_M)]^\alpha}{[1+d(x,x_M)]r_B^\alpha} \fint_{2B} |u|\,d\mu.$$

For any $x \in B$, if $d(x,x_M) > 10$, then by applying (RH_∞^E) to $B_x := B(x,r_x)$, where $r_x = \min\{d(x,x_M)/10,\ r_B/10\}$, and using (HE_α), we conclude

$$\||\nabla u|\|_{L^\infty(B_x)} \le \frac{C}{r_x}\fint_{2B_x}\fint_{2B_x} |u(y)-u(z)|\,d\mu(y)\,d\mu(z) \le \frac{Cr_x^{\alpha-1}}{r_B^\alpha}\fint_{2B}|u|\,d\mu.$$

Noticing that $d(x,x_M) < 5r_B$, and combining the above two estimates, we conclude that for each $x \in B$, it holds

$$|\nabla u(x)| \le \frac{C}{[1+d(x,x_M)]^{1-\alpha} r_B^\alpha}\fint_{2B}|u|\,d\mu.$$

Notice that $p(1-\alpha) < N$. One has via (3.16) that

$$\begin{aligned}\left(\int_B |\nabla u|^p\,d\mu\right)^{1/p} &\le C\fint_{2B}|u|\,d\mu\left(\int_B \frac{1}{[1+d(x,x_M)]^{(1-\alpha)p} r_B^{p\alpha}}\,d\mu(x)\right)^{1/p}\\ &\le \frac{C}{r_B^\alpha}\fint_{2B}|u|\,d\mu\left(\int_{B(x_M,5r_B)}\frac{1}{[1+d(x,x_M)]^{(1-\alpha)p}}\,d\mu(x)\right)^{1/p}\\ &\le C\mu(B)^{1/p} r_B^{-1}\fint_{2B}|u|\,d\mu.\end{aligned}$$

That is nothing but (RH_p).

Step 2. $(RH_p) \Rightarrow (HE_\alpha)$**.**

As (RH_∞^E) and hence $(RH_{\infty,\,\mathrm{loc}})$ hold, for a ball B satisfying $3B\cap M_0 = \emptyset$ or $r_B \le 100$, one sees that for each v satisfying $\mathcal{L}v = 0$ in $3B$, it holds that

$$|v(x)-v(y)| \le C\frac{d(x,y)}{r_B}\fint_{2B}|v|\,d\mu.$$

Thus we only need to verify (HE_α) for balls B with large radius and $3B\cap M_0 \neq \emptyset$. By Theorem 2.3, (P_p) holds since $p > N \vee 2$. By using (RH_p) and (P_p), we apply [22, Corollary 1.10] to see that $(RH_{p+\epsilon})$ holds for some $\epsilon > 0$. Therefore, for each v satisfying $\mathcal{L}v = 0$ in $3B$, it holds

$$\left(\fint_B |\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)} \le \frac{C}{r_B}\fint_{2B}|v|\,d\mu.$$

Let $x, y \in B$. If $d(x,y) \ge r_B/100$, then by Lemma 3.8 one has that

$$|v(x)-v(y)| \le C\left(\frac{d(x,y)}{r_B}\right)^\alpha (|v(x)|+|v(y)|) \le C\left(\frac{d(x,y)}{r_B}\right)^\alpha \fint_{2B}|v|\,d\mu.$$

Suppose that $d(x,y) < r_B/100$. Using $(P_{p+\epsilon})$, (D_N), $(RH_{p+\epsilon})$, and a standard telescopic argument (see Remark 2.4) gives that

$$
\begin{aligned}
&|v(x)-v(y)|\\
&\le \sum_{j=0}^{\infty}|v_{B(x,2^{-j+1}d(x,y))}-v_{B(x,2^{-j}d(x,y))}|+|v_{B(x,2d(x,y))}-v_{B(y,d(x,y))}|+\sum_{j=1}^{\infty}|v_{B(y,2^{-j+1}d(x,y))}-v_{B(y,2^{-j}d(x,y))}|\\
&\le Cd(x,y)\left(\sum_{j=0}^{\infty}2^{-j+1}\left(\fint_{B(x,2^{-j+1}d(x,y))}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}+\left(\fint_{B(x,2d(x,y))}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}\right.\\
&\qquad\left.+\sum_{j=1}^{\infty}2^{-j}\left(\fint_{B(y,2^{-j+1}d(x,y))}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}\right)\\
&\le Cd(x,y)\left\{\sum_{j=0}^{\infty}2^{-j}\left(\frac{r_B^N}{2^{-jN}d(x,y)^NV(x,r_B/8)}\right)^{1/(p+\epsilon)}\left(\int_{B(x,r_B/8)}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}\right.\\
&\qquad\qquad\left.+\sum_{j=1}^{\infty}2^{-j}\left(\frac{r_B^N}{2^{-jN}d(x,y)^NV(y,r_B/8)}\right)^{1/(p+\epsilon)}\left(\int_{B(y,r_B/8)}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}\right\}\\
&\overset{d(x,y)<r_B/100}{\le} Cd(x,y)\left\{\sum_{j=0}^{\infty}2^{-j}\left(\frac{r_B^N}{2^{-jN}d(x,y)^NV(x,r_B/8)}\right)^{1/(p+\epsilon)}\left(\int_{B(x,r_B/4)}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}\right\}\\
&\le Cr_B\left(\frac{d(x,y)}{r_B}\right)^{1-\frac{N}{p+\epsilon}}\left(\fint_{B(x,r_B/4)}|\nabla v|^{p+\epsilon}\,d\mu\right)^{1/(p+\epsilon)}\le C\left(\frac{d(x,y)}{r_B}\right)^{1-\frac{N}{p+\epsilon}}\fint_{B(x,r_B/2)}|v|\,d\mu\\
&\le C\left(\frac{d(x,y)}{r_B}\right)^{1-\frac{N}{p+\epsilon}}\fint_{2B}|v|\,d\mu.
\end{aligned}
$$

Above in the fourth inequality we used the facts $d(x,y) < r_B/100$, $V(y,r_B/8) \sim V(x,r_B/8)$ and $B(y,r_B/8) \subset B(x,r_B/4)$, and the last inequality holds since $x \in B$, $B(x,r_B/2) \subset 2B$, and $V(x,r_B/2) \sim V(\frac{1}{2}B) \sim V(2B)$. The above estimate implies (HE_α) for $\alpha = 1 - \frac{N}{p+\epsilon}$, and completes the proof. □

5 Extensions to Dirichlet metric measure spaces

In this section, we discuss extensions of main results to the setting of Dirichlet metric measure spaces. Since in a non-smooth setting, local Poincaré inequality (see Lemma 2.2) and local smoothness of harmonic functions (see Lemma 3.10) do not follow automatically from the assumptions on ends, we need to consider them as additional assumptions. However, other assumptions are the same as in the smooth settings. As the proofs are basically identical to the smooth settings (see [4, 22]), we will sketch the proofs in the section.

Let X be a locally compact, separable, metrisable, and connected space equipped with a Borel measure μ that is finite on compact sets and strictly positive on non-empty open sets. Consider a strongly local and regular Dirichlet form $\mathscr{E}$ on $L^2(X,\mu)$ with dense domain $\mathscr{D} \subset L^2(X,\mu)$ (see [27] for precise definitions). According to Beurling and Deny [8], such a form can be written as

$$\mathscr{E}(f,g) = \int_X d\Gamma(f,g)$$

for all $f, g \in \mathscr{D}$, where Γ is a measure-valued non-negative and symmetric bilinear form defined by the formula

$$\int_X \varphi \, d\Gamma(f,g) := \frac{1}{2}\left[\mathscr{E}(f,\varphi g) + \mathscr{E}(g,\varphi f) - \mathscr{E}(fg,\varphi)\right]$$

for all $f, g \in \mathscr{D} \cap L^\infty(X,\mu)$ and $\varphi \in \mathscr{D} \cap \mathscr{C}_0(X)$. Here and in what follows, $\mathscr{C}(X)$ denotes the space of continuous functions on X and $\mathscr{C}_0(X)$ the space of functions in $\mathscr{C}(X)$ with compact support. We shall assume in addition that $\mathscr{E}$ admits a "*carré du champ*", meaning that $\Gamma(f,g)$ is absolutely continuous with respect to μ, for all $f, g \in \mathscr{D}$. In what follows, for simplicity of notation, we will denote by $\langle Df, Dg\rangle$ the energy density $\frac{d\Gamma(f,g)}{d\mu}$, and by $|Df|$ the square root of $\frac{d\Gamma(f,f)}{d\mu}$.

Since $\mathscr{E}$ is strongly local, Γ is local and satisfies the Leibniz rule and the chain rule; see [27]. Therefore we can define $\mathscr{E}(f,g)$ and $\Gamma(f,g)$ locally. Denote by $\mathscr{D}_{\rm loc}$ the collection of all $f \in L^2_{\rm loc}(X)$ for which, for each relatively compact set $K \subset X$, there exists a function $h \in \mathscr{D}$ such that $f = h$ almost everywhere on K. The intrinsic (pseudo-)distance on X associated to $\mathscr{E}$ is then defined by

$$d(x,y) := \sup\{f(x) - f(y) : \ f \in \mathscr{D}_{\rm loc} \cap \mathscr{C}(X),\ |Df| \le 1 \text{ a.e.}\}.$$

We always assume that d is indeed a distance (meaning that for $x \neq y$, $0 < d(x,y) < +\infty$) and that the topology induced by d is equivalent to the original topology on X. Moreover, we assume that (X,d) is a complete metric space.

Corresponding to such a Dirichlet form $\mathscr{E}$, there exists an operator, denoted by $\mathscr{L}$, acting on a dense domain $\mathscr{D}(\mathscr{L})$ in $L^2(X,\mu)$, $\mathscr{D}(\mathscr{L}) \subset \mathscr{D}$, such that for all $f \in \mathscr{D}(\mathscr{L})$ and each $g \in \mathscr{D}$,

$$\int_X f(x)\mathscr{L}g(x)\, d\mu(x) = \mathscr{E}(f,g).$$

The opposite of $\mathscr{L}$ is the infinitesimal generator of the heat semigroup $H_t = e^{-t\mathscr{L}}$, $t > 0$.

We assume that X is the union of a compact set X_0 and some ends $\{E_i\}_{1\le i\le k}$, $k \in \mathbb{N}$. We simply adapt all the notions from previous sections with the Laplace-Beltrami operator $\mathcal{L}$ replaced by $\mathscr{L}$, the Riemannian gradient ∇ replaced by D, and M_0 replaced by X_0; see [6, 22] for more studies in such settings.

The following result generalizes Theorem 1.5 and Theorem 1.8 to the metric setting.

Theorem 5.1. *Assume that the non-compact Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (D_N) and (RD_n) with $2 < n \le N < \infty$. Suppose that (UE), $(P_{2,\rm loc})$ and (P_2^E) hold. Let $p \in (2,n)$. Then the following statements are equivalent.*

(i) (R_p) holds;

(ii) (RH_p) holds;

(iii) (RH_p^E) and $(RH_{p,\rm loc})$ hold;

(iv) (G_p) holds.

Remark 5.2. Comparing to Theorem 1.5 and Theorem 1.8, $(P_{2,\mathrm{loc}})$ is an additional assumption. Notice that, in the smooth setting, $(P_{2,\mathrm{loc}})$ follows from (P_2^E) as in Lemma 2.2, however, in the non-smooth setting, this is not true in general. Also in the term (iii), we need $(RH_{p,\mathrm{loc}})$ additionally, since in metric setting, harmonic functions are not necessarily smooth (see [22]), and $(RH_{p,\mathrm{loc}})$ does not follow from (RH_p^E), comparing to Lemma 3.10. For instance, one can glue two Euclidean ends via a smooth part removing a suitable fractal, where the local Poincaré inequality and local smoothness of harmonic functions may not hold.

Proof of Theorem 5.1. By [22, Theorem 1.6], we have the equivalence of (RH_p) and (G_p). Moreover, the same proof of Lemma 3.9 works in the metric setting, which implies that (RH_p) is equivalent to (RH_p^E) together with $(RH_{p,\mathrm{loc}})$, for $p \in (2, n)$.

It remains to show that (R_p) is equivalent to (G_p). It holds automatically that (R_p) implies (G_p), see [4, 22] for instance. On the other hand, the same proof of Theorem 3.7 gives that (G_p) implies (R_q) for any $q \in (2, p)$. By the same proof of Lemma 3.11, one sees that there exists $\varepsilon > 0$ such that $(G_{p+\varepsilon})$ holds, which then implies (R_p), and completes the proof. □

We have the following metric version of Theorem 1.11. Recall that $N \vee 2$ stands for $\max\{N, 2\}$.

Theorem 5.3. *Assume that the non-compact Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_N) with $0 < N < \infty$. Suppose that (UE), $(P_{2,\mathrm{loc}})$ and (P_2^E) hold. Let $p \in (N \vee 2, \infty)$. Then the following statements are equivalent.*
(i) (R_p) holds;
(ii) (RH_p) holds;
(iii) (G_p) holds.

Proof. Notice that, by the same proof of Theorem 2.3, $(P_{2,\mathrm{loc}})$ and (P_2^E) imply that Poincaré inequality (P_q) holds for any $p \in (N \vee 2, \infty)$. [22, Theorem 1.9] then gives the desired conclusion. □

6 Applications

A key tool in the paper is the Gaussian upper bound of heat kernel, i.e., there exist $C, c > 0$ such that for all $t > 0$ and $x, y \in M$,

$$(UE) \qquad p_t(x,y) \le \frac{C}{V(x, \sqrt{t})} \exp\left\{-\frac{d^2(x,y)}{ct}\right\}.$$

By [28, 44, 45], it is well known that, (D) together with (UE) is equivalent to a Faber-Krahn inequality, and also equivalent to a local Sobolev inequality; see also [9]. Recent result by Grigor'yan and Saloff-Coste [29, 30] gives a very useful solution to the stability of (UE) under gluing operations. The following result follows from [30, Theorem 3.5], see also [29, Corollary 4.6].

Theorem 6.1. *Let M be a manifold with finitely many ends $\{E_i\}_{1\le i\le k}$, $k \in \mathbb{N}$. Suppose that M satisfies (D). If for each i, there exists a manifold M_i satisfying (D) and (UE), and a compact subset $K_i \subset M_i$, such that E_i is isometric to $M_i \setminus K_i$, then (UE) holds on M.*

Above by “each M_i satisfies (UE)”, we mean the heat kernel on M_i satisfies (UE), with nothing to do with the gluing manifold M.

From Theorem 6.1, we see that, if M is obtained by gluing some Riemannian manifolds with non-negative Ricci curvature, simply connected nilpotent Lie groups with polynomial growth as well as conic manifolds, together through a compact manifold smoothly, then M satisfies (UE), since (UE) holds on the aforementioned manifolds; see [1, 4, 33, 22] for instance.

As a consequence, our Theorem 1.5, Theorem 1.8 and Theorem 1.11 work, if M is obtained by gluing Riemannian manifolds with non-negative Ricci curvature, simply connected nilpotent Lie groups as well as conic manifolds, together through a compact manifold smoothly.

Another class of gluing manifolds to which our result can be applied is the manifold obtained by gluing several cocompact covering Riemannian manifolds with polynomial growth deck transformation group together. Here, a manifold $\widehat{M}$ has a cocompact covering, if there is a finitely generated discrete group G with polynomial volume growth of some order $D > 2$, that acts properly and freely on $\widehat{M}$ by isometries, such that the orbit space $M_G = \widehat{M}/G$ is a compact manifold. See [26, 22] for instance.

Note also, our results also work on the these settings with the Laplace-Beltrami operator replaced by any uniformly elliptic operators of divergence form, by Theorem 5.1 and Theorem 5.3.

Let us finish the proof of Theorem 1.14 and Corollary 1.15.

Proof of Theorem 1.14. Notice that (P_2^E) holds automatically, as each M_i satisfies (P_2^E). Moreover, (UE) follows from Theorem 6.1 since M_i supports (UE).

For each $p \in (2, n)$, since (R_p) implies (RH_p) on each M_i by Theorem 1.5, we see that (RH_p^E) holds on M and the conclusion follows from Theorem 1.8. □

Proof of Corollary 1.15. Since (D) plus (P_2) imply (UE) and (P_2^E), we see that Theorem 1.14 applies.

(i) By [3, Theorem 0.4], for each M_i, there exists $\epsilon_i > 0$ such that the Riesz transform is bounded on $L^{2+\epsilon_i}(M_i)$. This implies that there exists $\epsilon > 0$, possibly smaller than ϵ_i, $1 \le i \le k$, such that $2 + \epsilon < n$ and $(R_{2+\epsilon})$ holds on each M_i. By Theorem 1.14 we see that $(R_{2+\epsilon})$ holds.

(ii) Since (R_p) holds on each M_i, we apply Theorem 1.14 to conclude that (R_p) holds on M. □

Consider an n-dimensional conic manifold $C(X)$ with compact basis X, $C(X) := \mathbb{R}^+ \times X$, where the metric is given by $dr^2 + r^2 d_X$. Let λ_1 be the smallest nonzero eigenvalue of the Laplacian on the basis X. By Li [38], the Riesz transform is bounded on $L^p(C(X))$ for all $p \in (1, p_0)$ and not bounded for $p \ge p_0$, where

$$p_0 := n\left(\frac{n}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1}\right)^{-1}$$

if $\lambda_1 < n - 1$ and $p_0 = \infty$ otherwise; see also [4]. The following question was also asked in [13].

Question 6.2 (Open Problem 8.1 [13]). *Is a result similar to H.-Q. Li's valid for smooth manifolds with one conic or asymptotically conic end? What happens for several conic ends?*

Guillarmou and Hassell [32] had solved the above question, which was recovered by recent work of Carron [12]. Our result also gives a new proof to the above question.

Let us explain how the proof works.

Notice that the measure satisfies $V(x,r) \sim r^n$ for each $x \in M$ and each $r > 0$, where $n \geq 2$. Suppose that the manifold has at least two conic ends. If $n \geq 3$, then Corollary 1.10 applies to show that the Riesz transform is bounded on $L^p(M)$, for any $p \in (1,n)$, while [13] (see Theorem 1.2) already implies the Riesz transform cannot bounded for any $p \geq n$ if $n \geq 3$. If $n = 2$, (R_p) holds for any $p \in (1,2]$ by [20], and is not bounded for any $p > 2$ by applying Corollary 1.12 and using the fact that there exists a non-constant harmonic function of logarithmic growth (cf. [12, Section 7]).

If the manifold has only one conic end, our Theorem 4.2 and Corollary 1.12 apply since the Ricci curvature satisfies

$$Ric_M(x) \geq -\frac{C_M}{[d(x,x_M)+1]^2}$$

for some $C_M > 0$. The existence of harmonic functions of sub-linear growth, and the elliptic Hölder regularity of harmonic functions can be found in [18] and also [12, Section 7].

Let $(\widetilde{M}, g_0)$ be a simply connected nilpotent Lie group of dimension $n > 2$ (endowed with a left-invariant metric), and ν be the homogenous dimension of $\widetilde{M}$, i.e. for some $o \in \widetilde{M}$

$$\nu := \lim_{R\to\infty} \frac{\log V(o,R)}{\log R}.$$

Notice that $\nu \geq n > 2$. Let (M,g) be a manifold obtained by gluing $k > 1$ copies of $(\widetilde{M}, g_0)$. Carron-Coulhon-Hassell [13] showed that (R_p) does not hold if $p \geq \nu$, and they asked

Question 6.3 (Open Problem 8.3 [13])**.** *Show that the Riesz transform on (M,g) is bounded on L^p for $p \in (1,\nu)$.*

Carron [11] had solved the question. Our Corollary 1.15 also provides a proof, by noticing that (D_ν) and (RD_ν) hold on M, and on a Lie group of polynomial growth (P_2) holds and (R_p) holds for all $p \in (1,\infty)$; see [1, 22, 49].

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