THE ROGERS–RAMANUJAN CONTINUED FRACTION AND RELATED ETA-QUOTIENT REPRESENTATIONS

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ABSTRACT. We construct eta-quotient representations of two families of q-series involving the Rogers–Ramanujan continued fraction by establishing related recurrence relations. We also display how these eta-quotient representations could be utilized to dissect certain q-series identities.

1. INTRODUCTION

Throughout, we adopt the customary q-series notation:

$$(A;q)_n := \prod_{k=0}^{n-1}(1-Aq^k),$$
$$(A;q)_\infty := \prod_{k=0}^{\infty}(1-Aq^k)$$

and

$$\left(\begin{matrix}A_1, A_2, \dots, A_n \\ B_1, B_2, \dots, B_m\end{matrix}; q\right)_\infty := \frac{(A_1;q)_\infty (A_2;q)_\infty \cdots (A_n;q)_\infty}{(B_1;q)_\infty (B_2;q)_\infty \cdots (B_m;q)_\infty}.$$

The Rogers–Ramanujan continued fraction was discovered by Rogers [16], independently by Ramanujan [14], and also independently by Schur [18]. In the literature (see, for example, [1, 6, 10]), it often refers to the generalized continued fraction

$$\frac{q^{1/5}}{1} \, + \, \frac{q}{1} \, + \, \frac{q^2}{1} \, + \, \frac{q^3}{1} \, + \cdots,$$

but in this paper we will drop off the factor of $q^{1/5}$. That is, we define

$$R(q) := \frac{1}{1} \, + \, \frac{q}{1} \, + \, \frac{q^2}{1} \, + \, \frac{q^3}{1} \, + \cdots.$$

It is known (see [12, p. 145]) that $R(q)$ can be represented as an infinite product:

$$R(q) = \left(\begin{matrix}q, q^4 \\ q^2, q^3\end{matrix}; q^5\right)_\infty.$$

2010 *Mathematics Subject Classification.* 11P84, 11A55.

Key words and phrases. Rogers–Ramanujan continued fraction, eta-quotient representations, recurrence relations.

In the past, modular equations for the Rogers–Ramanujan continued fraction have been studied extensively by many mathematicians, including Rogers and Ramanujan themselves [2, 14, 15, 17, 21]. For example, [12, Equation (40.1.10)] states that

$$\big(R(q^2)-R(q)^2\big)\big(1+qR(q)R(q^2)^2\big)=2qR(q)R(q^2)^3$$

and [12, Equation (40.1.12)] states that

$$\big(R(q^3)-R(q)^3\big)\big(1+q^2R(q)R(q^3)^3\big)=3qR(q)^2R(q^3)^2. \tag{1.1}$$

Recall that the Dedekind eta-function is defined by

$$\eta(q):=q^{\frac{1}{24}}(q;q)_\infty.$$

In this paper, we will have an investigation on eta-quotient representations of two families of q-series involving the Rogers–Ramanujan continued fraction by establishing the following recurrence relations. First, for $\alpha\in\mathbb{Z}_{\geq 0}$ and $\beta\in\mathbb{Z}$, we define

$$P(\alpha,\beta)=\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}+(-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}. \tag{1.2}$$

Theorem 1.1. *Let*

$$K=\frac{\eta(q^2)\eta(q^5)^5}{\eta(q)\eta(q^{10})^5}=q^{-1}\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{10};q^{10})_\infty^5}.$$

Then the following recurrence relations hold,

$$P(\alpha,\beta+1)=4K^{-1}P(\alpha,\beta)+P(\alpha,\beta-1) \tag{1.3}$$

and

$$P(\alpha+2,\beta)=KP(\alpha+1,\beta)+P(\alpha,\beta). \tag{1.4}$$

We also have initial values,

$$P(0,0)=2, \tag{1.5}$$

$$P(0,1)=4K^{-1}, \tag{1.6}$$

$$P(1,0)=K, \tag{1.7}$$

$$P(1,-1)=4K^{-1}-2+K. \tag{1.8}$$

Next, for $\alpha\in\mathbb{Z}_{\geq 0}$ and $\beta\in\mathbb{Z}$, we define

$$Q(\alpha,\beta)=\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}+(-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}. \tag{1.9}$$

Theorem 1.2. *Let*

$$S=\frac{\eta(q)^3\eta(q^3)^3}{\eta(q^5)^3\eta(q^{15})^3}=q^{-2}\frac{(q;q)_\infty^3(q^3;q^3)_\infty^3}{(q^5;q^5)_\infty^3(q^{15};q^{15})_\infty^3}$$

and

$$T=\frac{\eta(q^3)\eta(q^5)^5}{\eta(q)\eta(q^{15})^5}=q^{-2}\frac{(q^3;q^3)_\infty(q^5;q^5)_\infty^5}{(q;q)_\infty(q^{15};q^{15})_\infty^5}.$$

Then the following recurrence relations hold,

$$Q(\alpha,\beta+1)=\big(2+9T^{-1}\big)Q(\alpha,\beta)-Q(\alpha,\beta-1) \tag{1.10}$$

and

$$Q(\alpha+2,\beta)=\left(-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T+\frac{3}{2}\right)Q(\alpha+1,\beta)+Q(\alpha,\beta). \tag{1.11}$$

We also have initial values,

$$Q(0,0)=2, \tag{1.12}$$
$$Q(0,1)=2+9T^{-1}, \tag{1.13}$$
$$Q(1,0)=-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T+\frac{3}{2}, \tag{1.14}$$
$$Q(1,-1)=-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T-\frac{3}{2}. \tag{1.15}$$

Remark 1.3. Some of the initial values in Theorems 1.1 and 1.2 were already known in the literature. For example, to derive dissection identities of $(-q;q)_\infty$, Baruah and Begum [4, Equations (1.19)–(1.21)] proved (1.6)–(1.8). Also, (1.13) is due to Gugg [11, Theorem 5.1 (iv)]. However, the two complicated identities (1.14) and (1.15) appear to be novel.

As a by-product of Theorem 1.2, we obtain the following modular equation involving S and T.

Theorem 1.4. *We have*

$$81+144T+46T^2-16T^3+T^4-18ST-2ST^3+S^2T^2=0. \tag{1.16}$$

Finally, we remark that by (1.5) and (1.6) together with the recurrence relation (1.3), it is possible to represent $P(0,\beta)$ in terms of K for each $\beta\in\mathbb{Z}$. We also have similar representations of $P(1,\beta)$ for each $\beta\in\mathbb{Z}$. Further, the recurrence relation (1.4) reveals that for each $\alpha\geq 2$ and $\beta\in\mathbb{Z}$, we have $P(\alpha,\beta)\in\mathbb{Z}[K,K^{-1}]$. In Table 1, we list the representations of $P(\alpha,\beta)$ in terms of K with $0\leq\alpha\leq 2$ and $-3\leq\beta\leq 3$.

Similar arguments can be applied as well to $Q(\alpha,\beta)$ to show that for each $\alpha\in\mathbb{Z}_{\geq 0}$ and $\beta\in\mathbb{Z}$, $Q(\alpha,\beta)\in\mathbb{Q}[S,T,T^{-1}]$. Since such eta-quotient representations of $Q(\alpha,\beta)$ are much lengthier, we will not list them concretely like Table 1.

Let $\mathbb{H}^*:=\mathbb{H}\cup\mathbb{Q}\cup\{i\infty\}$ where $\mathbb{H}$ is the upper half complex plane. For any positive integer N, let $\Gamma_0(N)$ be the Hecke congruence subgroup of level N defined by

$$\Gamma_0(N):=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}):\ c\equiv 0 \pmod N\right\}.$$

TABLE 1. Representations of $P(\alpha,\beta)$ in $\mathbb{Z}[K,K^{-1}]$

$\beta \backslash \alpha$	0	1
-3	$-64K^{-3}-12K^{-1}$	$64K^{-3}-32K^{-2}+20K^{-1}-6+K$
-2	$16K^{-2}+2$	$-16K^{-2}+8K^{-1}-4+K$
-1	$-4K^{-1}$	$4K^{-1}-2+K$
0	2	K
1	$4K^{-1}$	$4K^{-1}+2+K$
2	$16K^{-2}+2$	$16K^{-2}+8K^{-1}+4+K$
3	$64K^{-3}+12K^{-1}$	$64K^{-3}+32K^{-2}+20K^{-1}+6+K$

$\beta \backslash \alpha$	2
-3	$-64K^{-3}+64K^{-2}-44K^{-1}+20-6K+K^2$
-2	$16K^{-2}-16K^{-1}+10-4K+K^2$
-1	$-4K^{-1}+4-2K+K^2$
0	$2+K^2$
1	$4K^{-1}+4+2K+K^2$
2	$16K^{-2}+16K^{-1}+10+4K+K^2$
3	$64K^{-3}+64K^{-2}+44K^{-1}+20+6K+K^2$

Let $K_0(N)$ denote the field of meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$. A result of Newman [13] indicates that K is in $K_0(10)$, and S and T are both in $K_0(15)$. Thus, we have the following results.

Corollary 1.5. *For any $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, $P(\alpha,\beta) \in \mathbb{Z}[K,K^{-1}]$ and therefore $P(\alpha,\beta) \in K_0(10)$.*

Corollary 1.6. *For any $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, $Q(\alpha,\beta) \in \mathbb{Q}[S,T,T^{-1}]$ and therefore $Q(\alpha,\beta) \in K_0(15)$.*

2. PROOFS

2.1. **Proofs of the recurrences.** We shall prove the following identities, from which the recurrences (1.3), (1.4), (1.10) and (1.11) follow as immediate consequences.

$$P(\alpha,\beta)P(0,1) = P(\alpha,\beta+1) - P(\alpha,\beta-1), \tag{2.1}$$

$$P(\alpha+1,\beta)P(1,0) = P(\alpha+2,\beta) - P(\alpha,\beta), \tag{2.2}$$

$$Q(\alpha,\beta)Q(0,1)=Q(\alpha,\beta+1)+Q(\alpha,\beta-1) \tag{2.3}$$

and

$$Q(\alpha+1,\beta)Q(1,0)=Q(\alpha+2,\beta)-Q(\alpha,\beta). \tag{2.4}$$

Proof of (2.1) *and* (2.2). It follows from (1.2) that

$$\begin{aligned}
&P(\alpha,\beta)P(0,1)\\
&=\left(\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}+(-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\right)\left(\frac{R(q^2)}{R(q)^2}-\frac{R(q)^2}{R(q^2)}\right)\\
&=\left(\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}\frac{R(q^2)}{R(q)^2}-(-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\frac{R(q)^2}{R(q^2)}\right)\\
&\quad-\left(\frac{1}{q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}}\frac{R(q)^2}{R(q^2)}-(-1)^{\alpha+\beta}q^\alpha R(q)^{\alpha+2\beta}R(q^2)^{2\alpha-\beta}\frac{R(q^2)}{R(q)^2}\right)\\
&=\left(\frac{1}{q^\alpha R(q)^{\alpha+2(\beta+1)}R(q^2)^{2\alpha-(\beta+1)}}+(-1)^{\alpha+(\beta+1)}q^\alpha R(q)^{\alpha+2(\beta+1)}R(q^2)^{2\alpha-(\beta+1)}\right)\\
&\quad-\left(\frac{1}{q^\alpha R(q)^{\alpha+2(\beta-1)}R(q^2)^{2\alpha-(\beta-1)}}+(-1)^{\alpha+(\beta-1)}q^\alpha R(q)^{\alpha+2(\beta-1)}R(q^2)^{2\alpha-(\beta-1)}\right)\\
&=P(\alpha,\beta+1)-P(\alpha,\beta-1),
\end{aligned}$$

from which we arrive at (2.1). Also, (2.2) follows by a similar argument. □

Proof of (2.3) *and* (2.4). It follows from (1.9) that

$$\begin{aligned}
&Q(\alpha,\beta)Q(0,1)\\
&=\left(\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}+(-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\right)\left(\frac{R(q^3)}{R(q)^3}+\frac{R(q)^3}{R(q^3)}\right)\\
&=\left(\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}\frac{R(q^3)}{R(q)^3}+(-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\frac{R(q)^3}{R(q^3)}\right)\\
&\quad+\left(\frac{1}{q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}}\frac{R(q)^3}{R(q^3)}+(-1)^{\alpha}q^\alpha R(q)^{2\alpha+3\beta}R(q^3)^{\alpha-\beta}\frac{R(q^3)}{R(q)^3}\right)\\
&=\left(\frac{1}{q^\alpha R(q)^{2\alpha+3(\beta+1)}R(q^3)^{\alpha-(\beta+1)}}+(-1)^{\alpha}q^\alpha R(q)^{2\alpha+3(\beta+1)}R(q^3)^{\alpha-(\beta+1)}\right)\\
&\quad+\left(\frac{1}{q^\alpha R(q)^{2\alpha+3(\beta-1)}R(q^3)^{\alpha-(\beta-1)}}+(-1)^{\alpha}q^\alpha R(q)^{2\alpha+3(\beta-1)}R(q^3)^{\alpha-(\beta-1)}\right)\\
&=Q(\alpha,\beta+1)+Q(\alpha,\beta-1).
\end{aligned}$$

This therefore proves (2.3). Likewise, one may derive (2.4) by the same procedure. □

2.2. **Proofs of (1.14) and (1.15).** As we have commented in Remark 1.3, the only (and true) difficulty at this place is proving (1.14) and (1.15). Let us begin with an interesting relation between $Q(1,0)$ and $Q(1,-1)$.

Lemma 2.1. *We have*

$$Q(1,0)-Q(1,-1)=3. \tag{2.5}$$

Proof. Notice that

$$\begin{aligned} Q(1,0)-Q(1,-1) &= \left(\frac{1}{qR(q)^2R(q^3)}-qR(q)^2R(q^3)\right)-\left(\frac{R(q)}{qR(q^3)^2}-\frac{qR(q^3)^2}{R(q)}\right) \\ &= \frac{\big(R(q^3)-R(q)^3\big)\big(1+q^2R(q)R(q^3)^3\big)}{qR(q)^2R(q^3)^2}. \end{aligned}$$

Thanks to the modular equation(1.1), we arrive at (2.5). □

Lemma 2.1 implies that if one of (1.14) and (1.15) is proved, then the other follows automatically.

Now recall that $K_0(N)$ is the field of meromorphic functions on the compact Riemann surface $\Gamma_0(N)\backslash\mathbb{H}^*$. Further, for $f(\tau)\in K_0(N)$ with Fourier expansion

$$f(\tau)=\sum_{n=n_0}^{\infty}a_nq^n,$$

we define the U-operator by

$$U(f)=\sum_{n=\lceil\frac{n_0}{5}\rceil}^{\infty}a_{5n}q^n. \tag{2.6}$$

Then a standard result [3, pp. 80–82] states that for any positive integer N, if $f\in K_0(5N)$, we have $U(f)\in K_0(N)$.

For notational convenience, let us write

$$E(q)=(q;q)_\infty.$$

Our proof of (1.14) relies on a surprisingly neat 5-dissection identity as follows.

Lemma 2.2. *We have*

$$U\left(\frac{E(q^3)^2}{E(q)}\right)=\frac{E(q^3)^3E(q^5)^2}{E(q)^3E(q^{15})}. \tag{2.7}$$

Proof. It follows from Newman [13] that

$$q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3}\in K_0(15)$$

and

$$q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2} \in K_0(75).$$

If we compare the Fourier expansions of

$$q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3} \quad \text{and} \quad U\left(q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\right),$$

which are both in $K_0(15)$, it can be observed that

$$U\left(q^{-5}\frac{E(q^3)^2E(q^{25})}{E(q)E(q^{75})^2}\right) = q^{-1}\frac{E(q^3)^3E(q^5)^3}{E(q)^3E(q^{15})^3},$$

from which (2.7) follows. □

Now we are in the position of proving (1.14). Recall that the 5-dissection formulas for $E(q)$ and $1/E(q)$ (see [12, Equations (8.1.1) and (8.4.4)]) read respectively as follows,

$$E(q) = E(q^{25})\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right) \tag{2.8}$$

and

$$\frac{1}{E(q)} = \frac{E(q^{25})^5}{E(q^5)^6}\left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \right.$$
$$\left. - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right). \tag{2.9}$$

Therefore,

$$\frac{E(q^3)^2}{E(q)} = \frac{E(q^{25})^5E(q^{75})^2}{E(q^5)^6}$$
$$\times\left(\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5)\right.$$
$$\left. + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right)\left(\frac{1}{R(q^{15})} - q^3 - q^6R(q^{15})\right)^2,$$

from which we extract

$$U\left(\frac{E(q^3)^2}{E(q)}\right) = \frac{q^2E(q^5)^5E(q^{15})^2}{E(q)^6}\left(\left(\frac{1}{q^2R(q)^4R(q^3)^2} + q^2R(q)^4R(q^3)^2\right)\right.$$
$$- 4\left(\frac{1}{qR(q)^2R(q^3)} - qR(q)^2R(q^3)\right) - 3\left(\frac{R(q)}{qR(q^3)^2} - \frac{qR(q^3)^2}{R(q)}\right)$$
$$\left. + 2\left(\frac{R(q^3)}{R(q)^3} + \frac{R(q)^3}{R(q^3)}\right) - 5\right).$$

Thus,

$$\frac{E(q^3)^3E(q^5)^2}{E(q)^3E(q^{15})} = \frac{q^2E(q^5)^5E(q^{15})^2}{E(q)^6}\big(Q(2,0)-4Q(1,0)-3Q(1,-1)+2Q(0,1)-5\big),$$

that is,

$$S = Q(2,0)-4Q(1,0)-3Q(1,-1)+2Q(0,1)-5.$$

It follows from (2.4) and (1.12) that

$$Q(2,0) = Q(1,0)^2+Q(0,0) = Q(1,0)^2+2$$

and from (2.4) and (2.12) that

$$Q(1,-1)Q(1,0) = Q(2,-1)-Q(0,-1) = -9T^{-1}-4+T.$$

Also, (2.5) states that

$$Q(1,0)-Q(1,-1) = 3.$$

Therefore,

$$\begin{aligned}
S &= \big(Q(1,0)^2+2\big)-4Q(1,0)-3Q(1,-1)+2Q(0,1)-5\\
&= Q(1,0)\big(Q(1,-1)+3\big)-4Q(1,0)-3\big(Q(1,0)-3\big)+2Q(0,1)-3\\
&= -4Q(1,0)+Q(1,0)Q(1,-1)+2Q(0,1)+6\\
&= -4Q(1,0)+\big(-9T^{-1}-4+T\big)+2\big(2+9T^{-1}\big)+6\\
&= -4Q(1,0)+9T^{-1}+6+T,
\end{aligned}$$

from which (1.14) follows. Further, (1.15) follows from (1.14) and (2.5).

2.3. **Proof of Theorem 1.4.** It follows from (1.12), (1.13) and the recurrence relation (1.10) that

$$\begin{aligned}
Q(0,-1) &= \big(2+9T^{-1}\big)Q(0,0)-Q(0,1)\\
&= 2+9T^{-1}.
\end{aligned}\tag{2.10}$$

Therefore, by (1.11), (1.15) and (2.10), we have

$$\begin{aligned}
Q(2,-1) &= \left(-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T+\frac{3}{2}\right)Q(1,-1)+Q(0,-1)\\
&= \left(-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T+\frac{3}{2}\right)\left(-\frac{1}{4}S+\frac{9}{4}T^{-1}+\frac{1}{4}T-\frac{3}{2}\right)+\big(2+9T^{-1}\big).
\end{aligned}\tag{2.11}$$

On the other hand, Gugg [11, Theorem 5.1 (v)] proved that

$$Q(2,-1) = -2+T.\tag{2.12}$$

We therefore arrive at Theorem 1.4 by equating (2.11) and (2.12).

3. Applications

In this section, we will explain how to take advantage of the eta-quotientrepresentations of $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ to prove q-series identities and congruences.

First, applications of Theorem 1.1 were extensively used in several recent work. For example, in [9], the authors used Theorem 1.1 to give an elementary proof of congruences modulo 25 for broken k-diamond partitions that were first discovered in [19, Theorem 2]. Also, by Theorem 1.1, the second author [20] derived several congruences modulo 25 for the 5 dots bracelet partition function. Further, the first author and Hirschhorn [8] utilized the eta-quotient representations of $P(\alpha,\beta)$ to establish an elementary proof of an infinite family of congruences modulo powers of 5 for partitions into distinct parts. A similar treatment was used for 1-shell totally symmetric plane partitions [7]. The interested reader may refer to [7, Section 2.1] for a detailed account of such applications.

For applications of the eta-quotient representations of $Q(\alpha,\beta)$, we prove the following q-series identity as an illustration.

Theorem 3.1. *Let the U-operator be as in (2.6). Then,*

$$U\left(q^{-2}\frac{E(q^3)^3}{E(q)^2}\right)=\frac{5}{4}q^{-1}\frac{E(q^3)^4E(q^5)^{12}}{E(q)^{10}E(q^{15})^5}-\frac{5}{4}q^{-1}\frac{E(q^3)^6E(q^5)^4}{E(q)^6E(q^{15})^3}+\frac{5}{2}q\frac{E(q^3)^3E(q^5)^7}{E(q)^9}-\frac{495}{4}q^3\frac{E(q^3)^2E(q^5)^2E(q^{15})^5}{E(q)^8}. \tag{3.1}$$

In [22], Zhang and Shi showed that if we expand the sixth order mock theta function

$$\beta(q)=\sum_{n=0}^{\infty}\frac{q^{3n^2+3n+1}}{(q;q^3)_{n+1}(q^2;q^3)_{n+1}}=:\sum_{n=0}^{\infty}p_\beta(n)q^n,$$

then

$$\sum_{n=0}^{\infty}p_\beta(3n+1)q^n=\frac{E(q^3)^3}{E(q)^2},$$

from which Zhang and Shi deduced that

$$p_\beta(15n+7)\equiv 0 \pmod 5. \tag{3.2}$$

One shall see that (3.1) is a strengthening of (3.2).

Proof of Theorem 3.1. Substituting the 5-dissection identities of $E(q)$ and $1/E(q)$, that is, (2.8) and (2.9), into $E(q^3)^3/E(q)^2$, and applying the U-operator, we have

$$U\left(q^{-2}\frac{E(q^3)^3}{E(q)^2}\right)=\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}U(\Pi),$$

where

$$\Pi=q^{-2}\left(\frac{1}{R(q^{15})}-q^3-q^6R(q^{15})\right)^3$$

$$\times\left(\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4-3q^5R(q^5)\right.$$
$$\left.+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\right)^2.$$

We expand the products in Π and find that $U(\Pi)$ has terms

$$\begin{aligned}U(\Pi)&=\frac{5}{R(q)^6R(q^3)^3}-\frac{60q}{R(q)^4R(q^3)^2}+\frac{20q}{R(q)R(q^3)^3}+\frac{50q^2}{R(q)^5}\\&\quad+\frac{60q^2R(q)}{R(q^3)^2}+\frac{20q^2R(q)^4}{R(q^3)^3}-\frac{15q^3R(q^3)^2}{R(q)^6}+75q^3\\&\quad-\frac{15q^3R(q)^6}{R(q^3)^2}+\frac{20q^4R(q)^3}{R(q^3)^4}-\frac{60q^4R(q^3)^2}{R(q)}-50q^4R(q)^5\\&\quad+20q^5R(q)R(q^3)^3-60q^5R(q)^4R(q^3)^2-5q^6R(q)^6R(q^3)^3.\end{aligned}$$

In light of the definition (1.9), grouping the first and last terms gives $5q^3Q(3,0)$, and likewise, grouping the second and second last terms gives $-60q^3Q(2,0)$. Thus, we find that

$$\begin{aligned}U(\Pi)&=5q^3\big(Q(3,0)-12Q(2,0)+4Q(2,-1)+10Q(1,1)\\&\quad+12Q(1,-1)+4Q(1,-2)-3Q(0,2)+15\big),\end{aligned}$$

from which we conclude that

$$\begin{aligned}U\left(q^{-2}\frac{E(q^3)^3}{E(q)^2}\right)&=5q^3\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}\big(Q(3,0)-12Q(2,0)+4Q(2,-1)\\&\quad+10Q(1,1)+12Q(1,-1)+4Q(1,-2)-3Q(0,2)+15\big).\end{aligned}$$

If we apply Theorem 1.2 to write each summand $Q(\cdot,\cdot)$ in terms of S and T, then

$$U\left(q^{-2}\frac{E(q^3)^3}{E(q)^2}\right)=\frac{5q^3}{64T^3}\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}\times A(q),$$

where

$$\begin{aligned}A(q)&=729+1458T+783T^2+92T^3+23T^4-14T^5+T^6\\&\quad-243ST-1764ST^2-50ST^3+28ST^4-3ST^5\\&\quad+27S^2T^2-14S^2T^3+3S^2T^4-S^3T^3.\end{aligned}$$

Recalling (1.16),

$$81+144T+46T^2-16T^3+T^4-18ST-2ST^3+S^2T^2=0,$$

we have,

$$\begin{aligned}A(q)&=A(q)-\big(81+144T+46T^2-16T^3+T^4-18ST-2ST^3+S^2T^2\big)\\&\quad\times\big(T^2+2T+9-ST\big)\end{aligned}$$

$$= -1584ST^2 + 32ST^3 + 16ST^4 - 16S^2T^3.$$

It follows that

$$U\left(q^{-2}\frac{E(q^3)^3}{E(q)^2}\right) = 5q^3\frac{E(q^5)^{10}E(q^{15})^3}{E(q)^{12}}\left(-\frac{99}{4}ST^{-1} + \frac{1}{2}S + \frac{1}{4}ST - \frac{1}{4}S^2\right).$$

This is exactly (3.1). □

Remark 3.2. In a private communication with Nayandeep Deka Baruah, we were informed that Baruah, Begum and Das [5] recently derived a handful of dissection identities for several partition functions. For instance, they showed that

$$U\left(q^{-1}\frac{1}{E(q)E(q^3)}\right) = \frac{E(q^5)^5}{E(q)^6E(q^{15})} + 10q\frac{E(q^5)^{10}}{E(q)^7E(q^3)^5} + q^2\frac{E(q^{15})^5}{E(q^3)^6E(q^5)}$$
$$+ 45q^3\frac{E(q^5)^5E(q^{15})^5}{E(q)^6E(q^3)^6} - 90q^5\frac{E(q^{15})^{10}}{E(q)^5E(q^3)^7}.$$

We remark that these identities could also be shown with the assistance of Theorem 1.2 and (1.16).

Acknowledgement

The authors would like to thank Julia Q. D. Du for her helpful comments and suggestions on a preliminary draft of this paper. The second author was supported by the Postdoctoral Science Foundation of China (No. 2019M661005) and the Fundamental Research Funds for the Central Universities (No. 2018CDXYST0024).

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