PROOFS OF SOME CONJECTURES ON THE RECIPROCALS OF RAMANUJAN-GORDON IDENTITIES

MIN BIAN, DAZHAO TANG, ERNEST X. W. XIA, AND FANGGANG XUE

Abstract. Recently, Lin and Wang introduced two special partition functions $RG_1(n)$ and $RG_2(n)$, the generating functions of which are the reciprocals of two identities due to Ramanujan and Gordon. They established several congruences modulo 5 and 7 for $RG_1(n)$ and $RG_2(n)$ and posed four conjectures on congruences modulo 25 for $RG_1(n)$ and $RG_2(n)$ at the end of their paper. In this paper, we confirm the four conjectures given by Lin and Wang by using Ramanujan's modular equation of fifth degree. Moreover, we also obtain new congruences modulo 25 for $RG_1(n)$ and $RG_2(n)$ based on Newman's identities. For example, we deduce that for any $n \geq 0$,

$$RG_1\left(\frac{23375n(3n+1)}{2}+974\right) \equiv RG_1\left(\frac{23375n(3n+5)}{2}+24349\right) \equiv 0 \pmod{25}.$$

1. Introduction

Recall that the well-known Jacobi triple product identity [2, p. 21, Theorem 2.8] is

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-qz;q^2)_\infty(-q/z;q^2)_\infty(q^2;q^2)_\infty, \qquad z \neq 0, \tag{1.1}$$

where here and throughout this paper, we use the following customary q-series notation:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a_1,a_2,\dots,a_n;q)_\infty = (a_1,q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty \qquad |q|<1.$$

The following three product-to-sum identities follow immediately from (1.1):

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} =(q;q)_\infty, \tag{1.2}$$

$$\sum_{n=-\infty}^{\infty} q^{n^2} =\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2}, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 11P83, 05A17.

Key words and phrases. Congruences, partitions, product-to-sum identities, Ramanujan's modular equation, Newman's identities.

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{1.4}$$

Now, the identity (1.2) is known as Euler's pentagonal number theorem [2, p. 11, Corollary 1.7]. The following famous identity is called Jacobi's identity:

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} = (q;q)_\infty^3. \tag{1.5}$$

which can also be derived from (1.1). For more details, see [4, p. 14].

The following two nice product-to-sum identities were independently discovered by Ramanujan [21, Eq. (65)] and Gordon [9]:

$$\sum_{n=-\infty}^{\infty} (6n+1) q^{n(3n+1)/2} = \frac{(q;q)_\infty^5}{(q^2;q^2)_\infty^2}, \tag{1.6}$$

$$\sum_{n=-\infty}^{\infty} (3n+1) q^{n(3n+2)} = \frac{(q;q)_\infty^2 (q^4;q^4)_\infty^2}{(q^2;q^2)_\infty}, \tag{1.7}$$

which are called Ramanujan-Gordon identities.

The reciprocals of three classical theta functions $\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n+1)/2}$, $\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}$ and $\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}$ are the generating functions of ordinary partitions, overpartitions, and partitions without repeated odd parts, respectively, which are three of the most important types of partitions. Many congruences for the three types of partition functions have been established; see for example [1, 6, 8, 13, 15, 18, 20, 23–27]. The reciprocal of $\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$ is the generating function for $p_{-3}(n)$ which enumerates the number of partitions of n in three colors. Hirschhorn [11] proved four families of congruences modulo powers of 3 for $p_{-3}(n)$, Boylan [5] and Lin [16] proved a congruence modulo 11 for $p_{-3}(n)$. Recently, Lin and Wang [17] defined two partition functions $RG_1(n)$ and $RG_2(n)$, the generating functions of which are the reciprocals of $\sum_{n=-\infty}^{\infty}(6n+1)q^{n(3n+1)/2}$ and $\sum_{n=-\infty}^{\infty}(3n+1)q^{n(3n+2)}$, respectively. Hence, the generating functions of $RG_1(n)$ and $RG_2(n)$ are

$$\sum_{n=0}^{\infty} RG_1(n) q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^5}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} RG_2(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}. \tag{1.9}$$

The partition function $RG_1(n)$ denotes the number of overpartition triples without overlined parts in the last component of n, while $RG_2(n)$ denotes the number of triples of partitions without repeated odd parts and with only even parts in the last component of n. Lin and Wang [17] also discovered several congruences modulo 5 and 7 for $RG_1(n)$

and $RG_2(n)$. At the end of their paper, Lin and Wang [17] presented four conjectures on congruences modulo 25 for $RG_1(n)$ and $RG_2(n)$.

The aim of this paper is to confirm the four conjectures on congruences modulo 25 for two special partition functions $RG_1(n)$ and $RG_2(n)$ and establish new congruences modulo 25 for $RG_1(n)$ and $RG_2(n)$ by using Newman's identities. The main results of this paper can be stated as follows.

Theorem 1.1. *For any $n \geq 0$,*

$$RG_1(125n+74) \equiv 0 \pmod{25}, \tag{1.10}$$

$$RG_1(125n+124) \equiv 0 \pmod{25}, \tag{1.11}$$

$$RG_1(625n+599) \equiv 0 \pmod{25}, \tag{1.12}$$

$$RG_2(125n+92) \equiv 0 \pmod{25}, \tag{1.13}$$

$$RG_2(125n+117) \equiv 0 \pmod{25}, \tag{1.14}$$

$$RG_2(625n+417) \equiv 0 \pmod{25}. \tag{1.15}$$

The congruences (1.10), (1.11), (1.13) and (1.14) were conjectured by Lin and Wang [17]. Based on some identities due to Newman [19], we can get the following two theorems.

Theorem 1.2. *Let $c_1(n)$ be defined by*

$$\sum_{n=0}^{\infty} c_1(n)q^n := \frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty}. \tag{1.16}$$

Suppose that a is a nonnegative integer such that $c_1(a) \equiv 0 \pmod{5}$. Suppose further that $24a+19 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$ with each $\alpha_j \geq 2$ is the prime factorization of $24a+19$. Then for any $n \geq 1$,

$$RG_1\left(125an^2 + \frac{2375n^2+1}{24}\right) \equiv 0 \pmod{25}, \tag{1.17}$$

where $\gcd\left(n, 6\prod_{j=1}^{v} g_j^{\alpha_j}\right) = 1$.

Example. It is easy to check that $c_1(7) \equiv 0 \pmod{5}$. Therefore, if we set $a = 7$ in (1.17), we deduce that for any $n \geq 0$,

$$RG_1\left(\frac{23375n(3n+1)}{2} + 974\right) \equiv RG_1\left(\frac{23375n(3n+5)}{2} + 24349\right) \equiv 0 \pmod{25}.$$

Theorem 1.3. *Let $c_2(n)$ be defined by*

$$\sum_{n=0}^{\infty} c_2(n)q^n := \frac{(q;q)_\infty^{10}}{(q^2;q^2)_\infty}. \tag{1.18}$$

Suppose that a is a nonnegative integer such that $c_2(a) \equiv 0 \pmod 5$. Suppose further that $24a+8 = \prod_{i=1}^{h} r_i \prod_{j=1}^{m} s_j^{\beta_j}$ with each $\beta_j \geq 2$ is the prime factorization of $24a+8$. Then for any $n \geq 1$,

$$RG_2\left(125an^2 + \frac{125n^2+1}{3}\right) \equiv 0 \pmod{25}, \tag{1.19}$$

where $\gcd\left(n, 6\prod_{j=1}^{m} s_j^{\beta_j}\right) = 1$.

Example. One can verify that $c_2(1) \equiv 0 \pmod 5$. Thus, taking $a = 1$ in (1.19), one sees that for any $n \geq 0$,

$$RG_2\left(2000n(3n+1)+167\right) \equiv RG_2\left(2000n(3n+5)+4167\right) \equiv 0 \pmod{25}.$$

2. Preliminaries

In this section, we collect some necessary definitions and lemmas which are needed to prove the main results of this paper.

Lemma 2.1. *The following 5-dissection formulas are true:*

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left(\frac{1}{R(q^5)} - q - q^2 R(q^5)\right) \tag{2.1}$$

and

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\left(\frac{1}{R^4(q^5)} + \frac{q}{R^3(q^5)} + \frac{2q^2}{R^2(q^5)} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) + 2q^6R^2(q^5) - q^7R^3(q^5) + q^8R^4(q^5)\right), \tag{2.2}$$

where

$$R(q) := \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

The identity (2.1) was given by Ramanujan [22, p. 212]. Hirschhorn [10] gave a simple proof of (2.2) by using Jacobi's triple product identity.

Throughout this paper, for any positive integer k, define

$$\sum_{n=0}^{\infty} p_{-k}(n)q^n := \frac{1}{(q;q)_\infty^k}.$$

Lemma 2.2. *We have*

$$\sum_{n=0}^{\infty} p_{-3}(5n+2)q^n \equiv 9\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6} \pmod{25} \tag{2.3}$$

and

$$\sum_{n=0}^{\infty} p_{-6}(5n+4)q^n \equiv 15\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^{12}} \pmod{25}. \tag{2.4}$$

Proof. For a formal power series $\sum_{n=-\infty}^{\infty} s(n)q^n$, define an operator U_5 as

$$U_5\left(\sum_{n=-\infty}^{\infty} s(n)q^n\right) = \sum_{n=-\infty}^{\infty} s(5n)q^n.$$

The following identities were given by Hirschhorn and Hunt [12]:

$$U_5(\eta) = -1, \quad U_5(\eta^2) = -1, \quad U_5(\eta^3) = 5, \quad U_5(\eta^4) = -5, \tag{2.5}$$

where

$$\eta = \frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty}.$$

Furthermore, from Ramanujan's modular equation of fifth degree, we have

$$\eta^5 = -5\eta^4 - 15\eta^3 - 25\eta^2 - 25\eta + \frac{(q^5;q^5)_\infty^6}{q^5(q^{25};q^{25})_\infty^6}. \tag{2.6}$$

It follows from (2.6) that for any integer k,

$$U_5(\eta^{5+k}) = -5U_5(\eta^{4+k}) - 15U_5(\eta^{3+k}) - 25U_5(\eta^{2+k}) - 25U_5(\eta^{1+k}) + \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}U_5(\eta^k). \tag{2.7}$$

Obviously,

$$U_5(\eta^0) = 1. \tag{2.8}$$

By (2.5), (2.7), (2.8) and iterative method, we get

$$U_5(\eta^{-3}) = 3125q^3\frac{(q^5;q^5)_\infty^{18}}{(q;q)_\infty^{18}} + 375q^2\frac{(q^5;q^5)_\infty^{12}}{(q;q)_\infty^{12}} + 9q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6} \tag{2.9}$$

and

$$\begin{aligned} U_5(\eta^{-6}) =& 48828125q^6\frac{(q^5;q^5)_\infty^{36}}{(q;q)_\infty^{36}} + 11718750q^5\frac{(q^5;q^5)_\infty^{30}}{(q;q)_\infty^{30}} + 984375q^4\frac{(q^5;q^5)_\infty^{24}}{(q;q)_\infty^{24}} \\ &+ 32500q^3\frac{(q^5;q^5)_\infty^{18}}{(q;q)_\infty^{18}} + 315q^2\frac{(q^5;q^5)_\infty^{12}}{(q;q)_\infty^{12}}. \end{aligned} \tag{2.10}$$

It is easy to check that

$$U_5(\eta^{-3}) = U_5\left(q^3\frac{(q^{25};q^{25})_\infty^3}{(q;q)_\infty^3}\right) = q(q^5;q^5)_\infty^3\sum_{n=0}^{\infty} p_{-3}(5n+2)q^n \tag{2.11}$$

and

$$U_5(\eta^{-6}) = U_5\left(q^6\frac{(q^{25};q^{25})_\infty^6}{(q;q)_\infty^6}\right) = q^2(q^5;q^5)_\infty^6\sum_{n=0}^{\infty}p_{-6}(5n+4)q^n. \tag{2.12}$$

With the help of (2.9)–(2.12), we obtain (2.3) and (2.4).

Lemma 2.3. *We have*

$$\sum_{n=0}^{\infty}p_{-3}(5n+2)q^n \equiv 15q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty^9} + 9\frac{(q;q)_\infty^4(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^{10}} \pmod{25}. \tag{2.13}$$

Proof. It follows from [3, p. 262, Entry 10 (iv)] that

$$\frac{(q;q)_\infty^4}{(q^2;q^2)_\infty^2} = \frac{(q^5;q^5)_\infty^4}{(q^{10};q^{10})_\infty^2} - 4q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty(q^5;q^5)_\infty}. \tag{2.14}$$

Based on (2.14),

$$\begin{aligned}\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^5} =&\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^3}\left(\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^4(q^{10};q^{10})_\infty^2} - 4q\frac{(q^{10};q^{10})_\infty^3}{(q;q)_\infty^3(q^2;q^2)_\infty(q^5;q^5)_\infty}\right)\\ =&\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^2;q^2)_\infty^3(q^{10};q^{10})_\infty^2} - 4q\frac{(q^{10};q^{10})_\infty^3}{(q;q)_\infty(q^5;q^5)_\infty}\cdot\frac{1}{(q^2;q^2)_\infty^4}.\end{aligned} \tag{2.15}$$

On the other hand, from [3, p. 262, Entry 10(v)],

$$\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2} = \frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty} + q\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^2}, \tag{2.16}$$

which yields

$$\frac{1}{(q^2;q^2)_\infty^4} = \frac{1}{q}\left(\frac{(q^5;q^5)_\infty^2}{(q;q)_\infty^2(q^{10};q^{10})_\infty^4} - \frac{(q^5;q^5)_\infty^5}{(q;q)_\infty(q^2;q^2)_\infty^3(q^{10};q^{10})_\infty^5}\right). \tag{2.17}$$

By the binomial theorem,

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod{5}. \tag{2.18}$$

Substituting (2.17) into (2.15) and employing (2.18) yields

$$\begin{aligned}\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^5} =&5\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^2;q^2)_\infty^3(q^{10};q^{10})_\infty^2} - 4\frac{(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty}\\ \equiv&5\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^{10};q^{10})_\infty^3} - 4\frac{(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty} \pmod{25}.\end{aligned} \tag{2.19}$$

Therefore,

$$\frac{(q;q)_\infty^4(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^{10}} \equiv (q^5;q^5)_\infty(q^{10};q^{10})_\infty^2$$

$$\times\left(5\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^{10};q^{10})_\infty^3}-4\frac{(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty}\right)^2$$
$$\equiv 10\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^6}{(q;q)_\infty^5(q^{10};q^{10})_\infty^2}+16\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6}\pmod{25}. \tag{2.20}$$

from which we have

$$\frac{(q;q)_\infty^4(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^{10}}-\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6}\equiv 10\left(\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^6}{(q;q)_\infty^5(q^{10};q^{10})_\infty^2}-\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6}\right)\pmod{25}. \tag{2.21}$$

By (2.16) and (2.18),

$$\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^6}{(q;q)_\infty^5(q^{10};q^{10})_\infty^2}-\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6}\equiv -q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty^9}\pmod{5}. \tag{2.22}$$

Combining (2.21) and (2.22), we get

$$\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6}\equiv\frac{(q;q)_\infty^4(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^{10}}+10q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty^9}\pmod{25},$$

which yields

$$9\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^6}\equiv 9\frac{(q;q)_\infty^4(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^{10}}+15q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty^9}\pmod{25}. \tag{2.23}$$

The congruence (2.13) follows from (2.3) and (2.23). This completes the proof of Lemma 2.3.

Lemma 2.4. *We have*

$$\sum_{n=0}^\infty c(5n+4)q^n\equiv 15\frac{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^2}{(q;q)_\infty^4(q^2;q^2)_\infty^4}\pmod{25}, \tag{2.24}$$

where

$$\sum_{n=0}^\infty c(n)q^n:=\frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2}. \tag{2.25}$$

Lemma 2.4 was proved by Chern and Tang [7].

3. Proof of Theorem 1.1

We first prove (1.10)–(1.12).

We can rewrite (2.14) as

$$\frac{1}{(q;q)_\infty^4}=\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}+4q\frac{(q^{10};q^{10})_\infty^5}{(q;q)_\infty^3(q^2;q^2)_\infty(q^5;q^5)_\infty^5}. \tag{3.1}$$

In view of (1.8) and (3.1),

$$\begin{aligned}\sum_{n=0}^{\infty} RG_1(n)q^n =&\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\left(\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}+4q\frac{(q^{10};q^{10})_\infty^5}{(q;q)_\infty^3(q^2;q^2)_\infty(q^5;q^5)_\infty^5}\right)\\ =&\frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty(q^5;q^5)_\infty^4}+4q\frac{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^5}{(q;q)_\infty^4(q^5;q^5)_\infty^5}\\ =&\frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty(q^5;q^5)_\infty^4}+4q\frac{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^5}\\ &\times\left(\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}+4q\frac{(q^{10};q^{10})_\infty^5}{(q;q)_\infty^3(q^2;q^2)_\infty(q^5;q^5)_\infty^5}\right)\\ =&\frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty(q^5;q^5)_\infty^4}+4q\frac{(q^{10};q^{10})_\infty^7}{(q^2;q^2)_\infty(q^5;q^5)_\infty^9}+16q^2\frac{(q^{10};q^{10})_\infty^{10}}{(q;q)_\infty^3(q^5;q^5)_\infty^{10}}. \qquad (3.2)\end{aligned}$$

If we substitute (2.2) into (3.2) and extract those terms that involve only the powers q^{5n+4}, then divide both sides by q^4 and replace q^5 by q, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} RG_1(5n+4)q^n =&5\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^5}{(q;q)_\infty^{10}}+20q\frac{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^5}{(q;q)_\infty^9}\\ &+16\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^{10}}\sum_{n=0}^{\infty}p_{-3}(5n+2)q^n. \qquad (3.3)\end{aligned}$$

Thanks to (2.13), (2.18) and (3.3),

$$\begin{aligned}\sum_{n=0}^{\infty} RG_1(5n+4)q^n \equiv&5(q^2;q^2)_\infty^2(q^5;q^5)_\infty^3+20q\frac{(q;q)_\infty(q^2;q^2)_\infty(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^2}\\ &+16\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^{10}}\left(15q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^5}{(q^2;q^2)_\infty^9}+9\frac{(q;q)_\infty^4(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^{10}}\right)\\ \equiv&5(q^2;q^2)_\infty^2(q^5;q^5)_\infty^3+19\frac{(q^5;q^5)_\infty(q^{10};q^{10})_\infty^2}{(q;q)_\infty^6}\\ &+10q\frac{(q;q)_\infty(q^2;q^2)_\infty(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^2} \pmod{25}. \qquad (3.4)\end{aligned}$$

Substituting (2.1) into (3.4), picking out the terms involving q^{5n+4}, then dividing by q^4 and replacing q^5 by q, we arrive at

$$\sum_{n=0}^{\infty} RG_1(25n+24)q^n \equiv -5(q;q)^3(q^{10};q^{10})^2+10\frac{(q^2;q^2)_\infty^5(q^5;q^5)_\infty(q^{10};q^{10})_\infty}{(q;q)^2}$$

$$
\begin{aligned}
&+19(q;q)_\infty(q^2;q^2)_\infty^2\sum_{n=0}^{\infty}p_{-6}(5n+4)q^n\\
\equiv&-5(q;q)^3(q^{10};q^{10})^2+10\frac{(q^2;q^2)_\infty^5(q^5;q^5)_\infty(q^{10};q^{10})_\infty}{(q;q)^2}\\
&+10\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^6}{(q;q)^{11}}\\
\equiv&5(q;q)^3(q^{10};q^{10})^2+10\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}{(q;q)_\infty} \pmod{25}. \quad \text{(by (2.18))}
\end{aligned} \tag{3.5}
$$

Hirschhorn and Sellers [14] proved the following congruence:

$$
(q;q)_\infty^3 \equiv (q^{10},q^{15},q^{25};q^{25})_\infty + 2q(q^5,q^{20},q^{25};q^{25})_\infty \pmod{5}. \tag{3.6}
$$

From [3, p. 49, Corollary (ii)],

$$
\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = (-q^{10},-q^{15},q^{25};q^{25})_\infty + q(-q^5,-q^{20},q^{25};q^{25})_\infty + q^3\frac{(q^{50};q^{50})_\infty^2}{(q^{25};q^{25})_\infty}. \tag{3.7}
$$

By virtue of (3.5)–(3.7), we obtain (1.10) and (1.11).

Furthermore, substituting (3.6) and (3.7) into (3.5), extracting those terms that involve only the powers q^{5n+3}, then dividing both sides by q^3, replacing q^5 by q and employing (2.18), we find that

$$
\sum_{n=0}^{\infty} RG_1(125n+99)q^n \equiv 10\frac{(q;q)_\infty^4(q^{10};q^{10})_\infty^2}{(q^5;q^5)_\infty} \equiv 10\frac{(q^{10};q^{10})_\infty^2}{(q;q)_\infty} \pmod{25}. \tag{3.8}
$$

Combining (2.2) and (3.8), we arrive at (1.12).

Now, we proceed to prove (1.13)–(1.15).

Replacing q by $-q$ in (1.9) and utilizing

$$
(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty}, \tag{3.9}
$$

we arrive at

$$
\sum_{n=0}^{\infty}(-1)^n RG_2(n)q^n = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^5}. \tag{3.10}
$$

Define

$$
\sum_{n=0}^{\infty} a(n)q^n = 5\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^{10};q^{10})_\infty^3} \tag{3.11}
$$

and

$$\sum_{n=0}^{\infty} b(n)q^n = -4\frac{(q^5;q^5)_\infty}{(q;q)_\infty^3(q^{10};q^{10})_\infty}. \tag{3.12}$$

Combining (2.19) and (3.10)–(3.12) yields

$$(-1)^n RG_2(n) \equiv a(n) + b(n) \pmod{25}. \tag{3.13}$$

In view of (2.16) and (3.11),

$$\begin{aligned}\sum_{n=0}^{\infty} a(n)q^n =&5\frac{(q^5;q^5)_\infty^4}{(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^3}\left(\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty}+q\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^2}\right)\\ =&5\frac{(q^5;q^5)_\infty^7}{(q;q)_\infty(q^2;q^2)_\infty(q^{10};q^{10})_\infty^4}+5q\frac{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^2}\\ \equiv&5\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^2}\frac{(q;q)_\infty(q^5;q^5)_\infty^7}{(q^{10};q^{10})_\infty^5}+5q(q^2;q^2)_\infty^3(q^5;q^5)_\infty^2\\ =&5\frac{(q;q)_\infty(q^5;q^5)_\infty^7}{(q^{10};q^{10})_\infty^5}\left(\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^3}{(q;q)_\infty(q^{10};q^{10})_\infty}+q\frac{(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty^2}\right)+5q(q^2;q^2)_\infty^3(q^5;q^5)_\infty^2\\ =&5\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^{10}}{(q^{10};q^{10})_\infty^6}+5q\frac{(q;q)_\infty(q^5;q^5)_\infty 5}{(q^{10};q^{10})_\infty}+5q(q^2;q^2)_\infty^3(q^5;q^5)_\infty^2 \pmod{25}.\end{aligned} \tag{3.14}$$

Substituting (2.1) and (3.6) into (3.14), then picking out the terms involving q^{5n+2}, then dividing by q^2 and replacing q^5 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(5n+2)q^n \equiv&-5\frac{(q;q)_\infty^{10}(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^6}-5\frac{(q;q)_\infty^5(q^5;q^5)_\infty}{(q^2;q^2)_\infty}\\ \equiv&15\frac{(q^5;q^5)_\infty^2}{(q^2;q^2)_\infty} \pmod{25}. \quad (\text{by } (2.18))\end{aligned} \tag{3.15}$$

It follows from (2.2) and (3.15) that for any $n \geq 0$

$$a(25n+17) \equiv 0 \pmod{25}. \tag{3.16}$$

It follows from (3.12) that

$$\sum_{n=0}^{\infty} b(5n+2)q^n = -4\frac{(q;q)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty} p_{-3}(5n+2)q^n \pmod{25}. \tag{3.17}$$

Substituting (2.3) into (3.17) yields

$$\sum_{n=0}^{\infty} b(5n+2)q^n \equiv 14\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^5(q^2;q^2)_\infty} \pmod{25}. \tag{3.18}$$

By (3.1),

$$\frac{1}{(q;q)_\infty^6}=\frac{1}{(q;q)_\infty^2}\left(\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2(q^5;q^5)_\infty^4}+4q\frac{(q^{10};q^{10})_\infty^5}{(q;q)_\infty^3(q^2;q^2)_\infty(q^5;q^5)_\infty^5}\right),$$

which yields

$$4\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^5(q^2;q^2)_\infty}=-\frac{1}{q}\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^3}+\frac{1}{q}\frac{(q^5;q^5)_\infty^8}{(q;q)_\infty^6(q^{10};q^{10})_\infty^5}$$

and

$$\frac{(q^5;q^5)_\infty^3}{(q;q)_\infty^5(q^2;q^2)_\infty}\equiv\frac{6}{q}\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^3}-\frac{6}{q}\frac{(q^5;q^5)_\infty^8}{(q;q)_\infty^6(q^{10};q^{10})_\infty^5}\pmod{25}.\tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\sum_{n=0}^{\infty}b(5n+2)q^n\equiv\frac{9}{q}\frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^2(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^3}-\frac{9}{q}\frac{(q^5;q^5)_\infty^8}{(q;q)_\infty^6(q^{10};q^{10})_\infty^5}\pmod{25},$$

which implies

$$\sum_{n=0}^{\infty}b(25n+17)q^n\equiv 9\frac{(q;q)_\infty^4}{(q^2;q^2)_\infty^3}\sum_{n=0}^{\infty}c(5n+4)q^n-9\frac{(q;q)_\infty^8}{(q^2;q^2)_\infty^5}\sum_{n=0}^{\infty}p_{-6}(5n+4)q^n\pmod{25},\tag{3.20}$$

where $c(n)$ is defined by (2.25).

Based on (2.4), (2.24) and (3.20),

$$\begin{aligned}\sum_{n=0}^{\infty}b(25n+17)q^n\equiv&10\frac{(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^7}-10\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^4(q^2;q^2)_\infty^5}\\ \equiv&10(q^2;q^2)_\infty^3(q^5;q^5)_\infty^2-10(q;q)_\infty\frac{(q^{25};q^{25})_\infty}{(q^{10};q^{10})_\infty}\pmod{25}.\end{aligned}\tag{3.21}$$

It follows from (2.1), (3.6) and (3.21) that for any $n\geq 0$

$$b(125n+92)\equiv b(125n+117)\equiv 0\pmod{25}.\tag{3.22}$$

The congruences (1.13) and (1.14) follow from (3.13), (3.16) and (3.22).

Furthermore, if we substitute (2.1) and (3.6) into (3.21) and pick out the terms involving q^{5n+1}, then divide by q and replace q^5 by q, we deduce that

$$\sum_{n=0}^{\infty}b(125n+42)q^n\equiv 10\frac{(q^5;q^5)_\infty^2}{(q^2;q^2)_\infty}\pmod{25}.\tag{3.23}$$

By (2.2) and (3.23),

$$b(625n+417)\equiv 0\pmod{25}.\tag{3.24}$$

With the aid of (3.13), (3.16) and (3.24), we obtain (1.15). This finishes the proof of Theorem 1.1.

4. Proofs of Theorems 1.2 and 1.3

We need to the following two lemmas.

Lemma 4.1. *Let $c_1(n)$ be defined by (1.16) and suppose that a is a nonnegative integer such that $c_1(a) \equiv 0 \pmod 5$. Suppose further that $24a+19 = \prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$ with each $\alpha_j \geq 2$ is the prime factorization of $24a+19$. Then for any $n \geq 1$,*

$$c_1\left(an^2 + \frac{19(n^2-1)}{24}\right) \equiv 0 \pmod 5, \tag{4.1}$$

where $\gcd\left(n, 6\prod_{j=1}^{v} g_j^{\alpha_j}\right) = 1$.

Proof. We prove Lemma 4.1 by induction on the total number of prime factors of n. Let $c_1(n)$ be defined by (1.16). If $n=1$ (n has no prime factors), then (4.1) states $c_1(a) \equiv 0 \pmod 5$, which is true by hypothesis. Let $p \geq 5$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)_L$ is defined by

$$\left(\frac{a}{p}\right)_L := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0, & \text{if } p \mid a, \\ -1, & \text{if } a \text{ is a nonquadratic residue modulo } p. \end{cases}$$

Newman [19] proved that for any $n \geq 0$,

$$c_1\left(p^2 n + \frac{19(p^2-1)}{24}\right) = \chi(n) c_1(n) - p^7 c_1\left(\frac{n - \frac{19(p^2-1)}{24}}{p^2}\right), \tag{4.2}$$

where

$$\chi(n) = p^7 d - p^3 \left(\frac{2n+1-\frac{7(p^2-1)}{12}}{p}\right)_L, \tag{4.3}$$

and d is a constant.

To obtain (4.1), we also need to prove that $\chi(n)$ is an integer. Taking $n=0$ in (4.2) and using the facts that $c_1\left(\frac{-\frac{19(p^2-1)}{24}}{p^2}\right) = 0$ and $c_1(0)=1$, we have

$$\chi(0) = c_1\left(\frac{19(p^2-1)}{24}\right). \tag{4.4}$$

Setting $n=0$ in (4.3) and utilizing (4.4), we deduce that

$$p^7 d = c_1\left(\frac{19(p^2-1)}{24}\right) + p^3 \left(\frac{1-\frac{7(p^2-1)}{12}}{p}\right)_L. \tag{4.5}$$

Substituting (4.5) into (4.3) yields

$$\chi(n) = c_1\left(\frac{19(p^2-1)}{24}\right) + p^3\left(\frac{1-\frac{7(p^2-1)}{12}}{p}\right)_L - p^3\left(\frac{2n+1-\frac{7(p^2-1)}{12}}{p}\right)_L. \tag{4.6}$$

Suppose further that $24a+19=\prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}$ with each $\alpha_j \geq 2$ is the prime factorization of $24a+19$. Let $p_1 \geq 5$ be a prime with $\gcd\left(p_1, \prod_{j=1}^{v} g_j^{\alpha_j}\right)=1$. Replacing (n,p) by (a,p_1) in (4.2) and employing the hypothesis that $c_1(a)\equiv 0 \pmod 5$ and the fact that $\chi(a)$ is an integer, we find that

$$c_1\left(ap_1^2+\frac{19(p_1^2-1)}{24}\right) \equiv -p^7 c_1\left(\frac{a-\frac{19(p_1^2-1)}{24}}{p_1^2}\right) \pmod 5. \tag{4.7}$$

Note that

$$\frac{a-\frac{19(p_1^2-1)}{24}}{p_1^2} = \frac{24a+19-19p_1^2}{24p_1^2} = \frac{\prod_{i=1}^{u} f_i \prod_{j=1}^{v} g_j^{\alpha_j}-19p_1^2}{24p_1^2}$$

is not an integer since $\gcd\left(p_1, \prod_{j=1}^{v} g_j^{\alpha_j}\right)=1$. Therefore,

$$c_1\left(\frac{a-\frac{19(p_1^2-1)}{24}}{p_1^2}\right)=0. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$c_1\left(ap_1^2+\frac{19(p_1^2-1)}{24}\right)\equiv 0 \pmod 5.$$

Therefore, (4.1) holds when $n=p_1$ (n has only one prime factor). Suppose that (4.1) is true for all integers with not more than k prime factors. In order to prove Theorem 1.2, it suffices to prove that (4.1) is true when n has $k+1$ prime factors. We write n as $n=p_1p_2\cdots p_kp_{k+1}$ with $5\leq p_1\leq p_2\leq\cdots\leq p_k\leq p_{k+1}$ and $\gcd\left(p_1\cdots p_{k-1}p_kp_{k+1}, \prod_{j=1}^{v} g_j^{\alpha_j}\right)=1$.

By hypothesis, (4.1) is true for all integers with not more than k prime factors. Therefore,

$$c_1\left(ap_1^2p_2^2\cdots p_{k-1}^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2-1)}{24}\right)\equiv 0 \pmod 5 \tag{4.9}$$

and

$$c_1\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-1)}{24}\right)\equiv 0 \pmod 5. \tag{4.10}$$

Replacing n by $ap_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-1)}{24}$ and replacing p by p_{k+1} in (4.2), then utilizing (4.10) and the fact that $\chi\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-1)}{24}\right)$ is an integer, we deduce that

$$c_1\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2-1)}{24}\right)$$

$$\equiv -p_{k+1}^7 c_1\left(\frac{ap_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-p_{k+1}^2)}{24}}{p_{k+1}^2}\right) \pmod 5. \tag{4.11}$$

Now, we break our proof into two cases: $p_{k+1}=p_k$ and $p_{k+1}>p_k$. If $p_{k+1}=p_k$, in view of (4.9), we can rewrite (4.11) as

$$\begin{aligned} &c_1\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2-1)}{24}\right)\\ &\equiv -p_{k+1}^7c_1\left(ap_1^2p_2^2\cdots p_{k-1}^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2-1)}{24}\right)\equiv 0 \pmod 5. \end{aligned} \tag{4.12}$$

If $p_{k+1}>p_k$, then $p_{k+1}\not\in\{p_1,p_2,\ldots,p_k\}$. It should be note that

$$\begin{aligned} \frac{ap_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-p_{k+1}^2)}{24}}{p_{k+1}^2} &=\frac{(24a+19)p_1^2p_2^2\cdots p_{k-1}^2p_k^2-19p_{k+1}^2}{24p_{k+1}^2}\\ &=\frac{p_1^2p_2^2\cdots p_{k-1}^2p_k^2\prod_{i=1}^u f_i\prod_{j=1}^v g_j^{\alpha_j}-19p_{k+1}^2}{24p_{k+1}^2} \end{aligned}$$

is not an integer since $\gcd\left(p_{k+1},\prod_{j=1}^v g_j^{\alpha_j}\right)=1$. Thus,

$$c_1\left(\frac{ap_1^2p_2^2\cdots p_{k-1}^2p_k^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2-p_{k+1}^2)}{24}}{p_{k+1}^2}\right)=0. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) yields

$$c_1\left(ap_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2+\frac{19(p_1^2p_2^2\cdots p_{k-1}^2p_k^2p_{k+1}^2-1)}{24}\right)\equiv 0 \pmod 5. \tag{4.14}$$

Therefore, for any case, (4.1) is true when $n=p_1p_2\cdots p_kp_{k+1}$. Lemma 4.1 is proved by induction.

Using Newman's identity on $c_2(n)$ given by Newman [19] and the same method for proving Lemma 4.1, we can prove the following lemma. Since the process is similarly, we omit the details.

Lemma 4.2. *Let $c_2(n)$ be defined by (1.18) and suppose that a is a nonnegative integer such that $c_2(a)\equiv 0 \pmod 5$. Suppose further that $24a+8=\prod_{i=1}^h r_i\prod_{j=1}^m s_j^{\beta_j}$ with each $\beta_j\geq 2$ is the prime factorization of $24a+8$. Then for any $n\geq 1$,*

$$c_2\left(an^2+\frac{n^2-1}{3}\right)\equiv 0 \pmod 5, \tag{4.15}$$

where $\gcd\left(n,6\prod_{j=1}^m s_j^{\beta_j}\right)=1$.

To conclude this section, we present the proofs of Theorems 1.2 and 1.3.

By (2.18) and (3.8),

$$\sum_{n=0}^{\infty} RG_1(125n+99)q^n \equiv 10\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty} \pmod{25}. \tag{4.16}$$

In view of (1.16) and (4.16),

$$RG_1(125n+99) \equiv 10c_1(n) \pmod{25}. \tag{4.17}$$

The congruence (1.17) follows from (4.1) and (4.17).

It follows from (2.18) and (3.23) that

$$\sum_{n=0}^{\infty} b(125n+42)q^n \equiv 10\frac{(q;q)_\infty^{10}}{(q^2;q^2)_\infty} \pmod{25}. \tag{4.18}$$

In view of (3.13), (3.16) and (4.18),

$$\sum_{n=0}^{\infty}(-1)^n RG_2(125n+42)q^n \equiv 10\frac{(q;q)_\infty^{10}}{(q^2;q^2)_\infty} \pmod{25}. \tag{4.19}$$

The congruences (1.18) and (4.19) imply that for any $n \geq 0$,

$$(-1)^n RG_2(125n+42) \equiv 10c_2(n) \pmod{25}. \tag{4.20}$$

Combining (4.15) and (4.20), we arrive at (1.19). This completes the proof.

Acknowledgments. Min Bian, Ernest X. W. Xia and Fanggang Xue were supported by the National Natural Science Foundation of China (No. 11571143) and the Nature Funds for Distinguished Young Scientists of Jiangsu Province (No. BK20180044). Dazhao Tang was supported by the Fundamental Research Funds for the Central Universities (No. 2018CDXYST0024) and the Postdoctoral Science Foundation of China (No. 2019M661005). All authors would like to acknowledge the helpful comments and suggestions made by the referee.

References

[1] S. Ahlgren and M. Boylan, Arithmetic properties of the partition function, *Invent. Math.* **153** (2003), 487–502. 2

[2] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, MA, 1976; reissued: Cambridge University Press, Cambridge, 1998. 1, 2

[3] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991. 6, 9

[4] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, American Mathematical Society, Providence, RI, 2006. 2

[5] M. Boylan, Exceptional congruences for powers of the partition functions. *Acta Arith.* **111** (2004) 187–203. 2

[6] W. Y. C. Chen and E. X. W. Xia, Proof of a conjecture of Hirschhorn and Sellers on overpartitions, *Acta Arith.* **163** (2014) 59–69. 2

[7] S. Chern and D. Tang, Elementary proof of congruences modulo 25 for broken k-diamond partitions, arXiv:1807.01890v1, (2018). 7
[8] D. Q. J. Dou and B. L. S. Lin, New Ramanujan type congruences modulo 5 for overpartitions, *Ramanujan J.* **44** (2017) 401–410. 2
[9] B. Gordon, Some identities in combinatorial analysis, *Quart. J. Math.* **12** (1961) 285–290. 2
[10] M. D. Hirschhorn, Ramanujan's "most beautiful identity", *Amer. Math. Monthly* **118** (2011) 839–845. 4
[11] M. D. Hirschhorn, Partitions in 3 colours, *Ramanujan J.* **45** (2018) 399–411. 2
[12] M. D. Hirschhorn and D. C. Hunt, A simple proof of the Ramanujan conjecture for powers of 5, *J. Reine Angew. Math.* **326** (1981) 1–17. 5
[13] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of partitions with odd parts distinct, *Ramanujan J.* **22** (2010) 273–284. 2
[14] M. D. Hirschhorn and J. A. Sellers, Arithmetic properties of 1-shell totally symmetric plane partitions, *Bull. Aust. Math. Soc.* **89** (2014) 473–478. 9
[15] X. Huang and O. X. M. Yao, Proof of a conjecture on congruence modulo 243 for overpartitions, *Period. Math. Hung.* **79** (2019) 227–235. 2
[16] B. L. S. Lin, Ramanujan-style proof of $p_{-3}(11n+7) \equiv 0 \pmod{11}$, *Ramanujan J.* **42** (2017) 223–231. 2
[17] B. L. S. Lin and A. Y. Z. Wang, Congruences for the reciprocals of the Ramanujan-Gordon idenities, *Colloq. Math.* **154** (2018) 137–148. 2, 3
[18] K. Mahlburg, The overpartition function modulo small powers of 2, *Discrete Math.* **286** (2004), 263–267. 2
[19] M. Newman, Modular forms whose coefficients possess multiplicative properties, II, *Ann. Math.* **75** (1962) 242–250. 3, 12, 14
[20] K. Ono, Distribution of the partition function modulo m, *Ann. Math.* **151** (2000) 293–307. 2
[21] S. Ramanujan, On certain arithmetical functions, Trans. *Cambridge Philos. Soc.* **22** (1916) 159–184. 2
[22] S. Ramanujan, *Collected Papers*, G.H. Hardy, P.V. Seshu Aiyar, B.M.Wilson, eds., AMS Chelsea, Providence, RI, 2000. 4
[23] E. X. W. Xia, Congruences modulo 9 and 27 for overpartitions, *Ramanujan J.* **42** (2017) 301–323. 2
[24] E. X. W. Xia, Congruences modulo 5 for the number of spin characters of the double covers of the symmetric and alternating groups. *Acta Arith.* **187** (2019) 255–269. 2
[25] E. X. W. Xia and Y. Zhang, Proofs of some conjectures of Sun on the relations between sums of squares and sums of triangular numbers. *Int. J. Number Theory* **15** (2019) 189–212. 2
[26] F. Xue and O. X. M. Yao, Explicit congruencesmodulo 2048 for overpartitions. *Ramanujan J.* To appear. https://doi.org/10.1007/s11139-019-00204-6. 2
[27] O. X. M. Yao, Congruences modulo 64 and 1024 for overpartitions, *Ramanujan J.* **46** (2018) 1–18. 2

(Min Bian) Department of Mathematics, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R. China
E-mail address: bianminjs@163.com

(Dazhao Tang) Center for Applied Mathematics, Tianjin University, Tianjin 300072, P.R. China
E-mail address: dazhaotang@sina.com

(Ernest X. W. Xia) Department of Mathematics, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R. China
E-mail address: ernestxwxia@163.com

(Fanggang Xue) Department of Mathematics, Jiangsu University, Zhenjiang, Jiangsu, 212013, P.R. China
E-mail address: xfg950118@126.com