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A Berry-Esseen bound of order $\frac{1}{\sqrt{n}}$ for martingales

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Résumé. Renz [13] has established a rate of convergence $1/\sqrt{n}$ in the central limit theorem for martingales with some restrictive conditions. In the present paper a modification of the methods, developed by Bolthausen [2] and Grama and Haeusler [6], is applied for obtaining the same convergence rate for a class of more general martingales. An application to linear processes is discussed.

Keywords. Martingales, Berry-Esseen bound, Rate of convergence.

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Version française abrégée

Renz [13] a établi un taux de convergence $1/\sqrt{n}$ dans le théorème de la limite centrale pour les martingales avec certaines conditions restrictives. Dans le présent article, une modification des méthodes, développées par Bolthausen [2] et Grama et Haeusler [6], est appliquée pour obtenir le même taux de convergence pour une classe de martingales plus générales. Une application aux processus linéaires est discutée.

1. Introduction and main result

For $n \in \mathbf{N}$, let $(\xi_i, \mathcal{F}_i)_{i=0,\dots,n}$ be a finite sequence of martingale differences defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\xi_0 = 0$ and $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_n \subseteq \mathcal{F}$ are increasing σ-fields. Denote

$$X_0 = 0, \quad X_k = \sum_{i=1}^{k} \xi_i, \ k = 1, \dots, n.$$

Then $X = (X_k, \mathcal{F}_k)_{k=0,...,n}$ is a martingale. Denote by $\langle X \rangle$ the conditional variance of X:

$$\langle X \rangle_0 = 0, \quad \langle X \rangle_k = \sum_{i=1}^{k} \mathbf{E}\big[\xi_i^2 \big| \mathcal{F}_{i-1}\big], \ k = 1, ..., n.$$

Define

$$D(X_n) = \sup_{x \in \mathbf{R}} \Big| \mathbf{P}(X_n \leq x) - \Phi(x) \Big|,$$

where $\Phi(x)$ is the distribution function of the standard normal random variable. Denote by $\overset{\mathbf{P}}{\to}$ the convergence in probability as $n \to \infty$. According to the martingale central limit theorem, the "conditional Lindeberg condition"

$$\sum_{i=1}^{n} \mathbf{E}\big[\xi_i^2 \mathbf{1}_{\{|\xi_i| \geq \epsilon\}} \big| \mathcal{F}_{i-1}\big] \overset{\mathbf{P}}{\to} 0, \quad \text{for each } \epsilon > 0,$$

and the "conditional normalizing condition" $\langle X \rangle_n \overset{\mathbf{P}}{\to} 1$ together implies asymptotic normality of X_n, that is, $D(X_n) \to 0$ as $n \to \infty$.

The convergence rate of $D(X_n)$ has attracted a lot of attentions. For instance, Bolthausen [2] proved that if $|\xi_i| \leq \epsilon_n$ for a number ϵ_n and $\langle X \rangle_n = 1$ a.s., then $D(X_n) \leq c\epsilon_n^3 n \log n$, where, here and after, c is an absolute constant not depending on ϵ_n and n. El Machkouri and Ouchti [3] improved the factor $\epsilon_n^3 n \log n$ in Bolthausen's bound to $\epsilon_n \log n$ under the following more general condition

$$\mathbf{E}\big[|\xi_i|^3 \big| \mathcal{F}_{i-1}\big] \leq \epsilon_n \mathbf{E}\big[\xi_i^2 \big| \mathcal{F}_{i-1}\big] \quad a.s. \text{ for all } i = 1, 2, ..., n.$$

For more related results, we refer to Ouchti [12] and Mourrat [11]. Recently, Fan [4] proved that if there exist a positive constant ρ and a number ϵ_n, such that

$$\mathbf{E}\big[|\xi_i|^{2+\rho} \big| \mathcal{F}_{i-1}\big] \leq \epsilon_n^\rho \mathbf{E}\big[\xi_i^2 \big| \mathcal{F}_{i-1}\big] \quad a.s. \text{ for all } i = 1, 2, ..., n,$$

and $\langle X \rangle_n = 1$ a.s., then $D(X_n) \leq c_\rho \hat{\epsilon}_n$, where

$$\hat{\epsilon}_n = \begin{cases} \epsilon_n^\rho, & \text{if } \rho \in (0, 1), \\ \epsilon_n |\log \epsilon_n|, & \text{if } \rho \geq 1, \end{cases}$$

and c_ρ is a constant depending only on ρ. Fan [4] also showed that this Berry-Esseen bound is optimal. In particular, if $\epsilon_n \asymp 1/\sqrt{n}$, then we have $\epsilon_n |\log \epsilon_n| \asymp (\log n)/\sqrt{n}$. Thus, we cannot obtain the classical convergence rate $1/\sqrt{n}$ for general martingales.

However, the convergence rate $1/\sqrt{n}$ for martingales is possible to be attained with some additional restrictive conditions. For instance, Renz [13] proved that if there exists a constant $\rho > 0$ such that

$$\mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}] = 1/n, \quad \mathbf{E}[\xi_i^3 | \mathcal{F}_{i-1}] = 0 \quad \text{and} \quad \mathbf{E}\big[|\xi_i|^{3+\rho} \big| \mathcal{F}_{i-1}\big] \leq cn^{-(3+\rho)/2}, \quad \text{a.s.,} \tag{1}$$

then it holds

$$D(X_n) = O\bigg(\frac{1}{\sqrt{n}}\bigg). \tag{2}$$

He also showed that this result is not true for $\rho = 0$. More martingale Berry-Esseen bounds of convergence rate $1/\sqrt{n}$ can be found in Bolthausen [2] and Kir'yanova and Rotar [10].

In this paper we are interested in extending (2) to a class of more general martingales. The following theorem is our main result.

Theorem 1. *Assume that there exist some numbers $\rho \in (0, +\infty), \epsilon_n \in (0, \frac{1}{2}]$ and $\delta_n \in [0, \frac{1}{2}]$ such that for all $1 \leq i \leq n$,*

$$\big|\langle X \rangle_n - 1\big| \leq \delta_n^2, \tag{3}$$

$$\mathbf{E}\big[\xi_i^3 \big| \mathcal{F}_{i-1}\big] = 0 \tag{4}$$

and

$$\mathbf{E}\big[|\xi_i|^{3+\rho}\big|\mathcal{F}_{i-1}\big] \le \epsilon_n^{1+\rho}\mathbf{E}\big[\xi_i^2\big|\mathcal{F}_{i-1}\big] \quad a.s. \tag{5}$$

Then

$$D(X_n) \le c_\rho(\epsilon_n + \delta_n),$$

where c_ρ depends only on ρ. In addition, it holds $c_\rho = O(\rho^{-1}), \rho \to 0$.

Notice that under the conditions of Renz [13], the conditions of Theorem 1 are satisfied with $\delta_n = 0$ and $\epsilon_n \asymp 1/\sqrt{n}$. Thus Theorem 1 extends Renz's result to a class of more general martingales.

Thanks to the additional condition (4), the Berry-Esseen bound (6) improves the bound of Fan [4] by replacing $\epsilon_n|\log\epsilon_n|$ with ϵ_n.

Relaxing the condition (3), we have the following analogue estimation of Fan (cf. (26) of [4]).

Theorem 2. *Assume that there exist some numbers $\rho \in (0, +\infty)$ and $\epsilon_n \in (0, \frac{1}{2}]$ such that for all $1 \le i \le n$,*

$$\mathbf{E}\big[\xi_i^3\big|\mathcal{F}_{i-1}\big] = 0$$

and

$$\mathbf{E}\big[|\xi_i|^{3+\rho}\big|\mathcal{F}_{i-1}\big] \le \epsilon_n^{1+\rho}\mathbf{E}\big[\xi_i^2\big|\mathcal{F}_{i-1}\big] \quad a.s.$$

Then, for all $p \ge 1$,

$$D(X_n) \le c_\rho\epsilon_n + c_p\left(\mathbf{E}\Big[\big|\langle X\rangle_n - 1\big|^p\Big] + \mathbf{E}\Big[\max_{1\le i\le n}|\xi_i|^{2p}\Big]\right)^{1/(2p+1)}, \tag{6}$$

where c_ρ and c_p depend only on ρ and p, respectively.

It is easy to see that when $p \to \infty$,

$$\left(\mathbf{E}\Big[\big|\langle X\rangle_n - 1\big|^p\Big]\right)^{1/(2p+1)} \to ||\langle X\rangle_n - 1||_\infty^{1/2},$$

which coincides with δ_n of Theorem 1.

2. Application

We first extend Theorem 1 to triangular arrays with infinity many terms in each line. For $n \in \mathbf{N}$, let $(\xi_{n,i}, \mathcal{F}_{n,i})_{i=-\infty}^n$ be a sequence of martingale differences defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where the adapted filtration is $\{\emptyset, \Omega\} = \mathcal{F}_{-\infty} \subset ... \subset \mathcal{F}_{n,n-1} \subset \mathcal{F}_{n,n} \subset \mathcal{F}$. Denote $X_{n,k} = \sum_{i=-\infty}^k \xi_{n,i}, k \le n$. Then $(X_{n,k}, \mathcal{F}_{n,k})_{k=-\infty}^n$ is a martingale. Let $\langle X\rangle_{n,k} = \sum_{i=-\infty}^k \mathbf{E}[\xi_{n,i}^2|\mathcal{F}_{n,i-1}], k \le n$. In particular, denote $X_n := X_{n,n}$ and $\langle X\rangle_n := \langle X\rangle_{n,n}$.

With some slight modification on the proof, Theorem 1 still holds in this new setting. Now we apply Theorem 1 with this new setting to the partial sum of linear processes. Let $(\varepsilon_i)_{i\in\mathbf{Z}}$ be a sequence of identically distributed martingale differences adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$. We consider the causal linear process in the form

$$Y_k = \sum_{j=-\infty}^{k} a_{k-j}\varepsilon_j, \tag{7}$$

where the martingale differences have finite variance and the sequence of real coefficients satisfies $\sum_{i=0}^\infty a_i^2 < \infty$. Without loss of generality, let the variance of the martingale difference to be 1. We say the linear process has long memory if $\sum_{i=0}^{\infty}|a_i| = \infty$. In this case, we assume that $a_0 = 1$ and

$$a_i = \ell(i)i^{-\alpha},\ i > 0,\ \text{with } 1/2 < \alpha < 1. \tag{8}$$

Here $\ell(\cdot)$ is a slowly varying function. On the other hand, we say the linear process has short memory if $\sum_{i=0}^{\infty}|a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i \neq 0$. The third case is $\sum_{i=0}^{\infty}|a_i| < \infty$ and $\sum_{i=0}^{\infty} a_i = 0$.

The long memory linear processes covers the well-known fractional ARIMA processes (cf. Granger and Joyeux [7]; Hosking [9]), which play an important role in financial time series modeling and application. As a special case, let $0 < d < 1/2$ and B be the backward shift operator with $B\varepsilon_k = \varepsilon_{k-1}$ and consider

$$Y_k = (1-B)^{-d}\varepsilon_k = \sum_{i=0}^{\infty} a_i\varepsilon_{k-i},\ \text{where } a_i = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}.$$

For this example we have $\lim_{n\to\infty} a_n/n^{d-1} = 1/\Gamma(d)$. Note that these processes have long memory because $\sum_{j=0}^{\infty}|a_j| = \infty$.

The partial sum $S_n = \sum_{k=1}^{n} Y_k$ of causal linear process (7) can be written as $S_n = \sum_{i=-\infty}^{n} b_{n,i}\varepsilon_i$, where $b_{n,i} = \sum_{j=0}^{n-i} a_j$ for $0 < i \le n$, and $b_{n,i} = \sum_{j=1-i}^{n-i} a_j$ for $i \le 0$. The variance of S_n is $B_n^2 = var(S_n) = \sum_{i=-\infty}^{n} b_{n,i}^2$. Now let $X_{n,k} = \sum_{i=-\infty}^{k} b_{n,i}\varepsilon_i/B_n$. Then $X_n = X_{n,n} = S_n/B_n$ and $\langle X\rangle_n = \sum_{i=-\infty}^{n} b_{n,i}^2\mathbf{E}[\varepsilon_i^2|\mathcal{F}_{i-1}]/B_n^2$. If we assume $|\langle X\rangle_n - 1| \le \delta_n^2$ for some $\delta_n \in [0, \frac{1}{2}]$, $\mathbf{E}[\varepsilon_i^3|\mathcal{F}_{i-1}] = 0$ and $\mathbf{E}[|\varepsilon_i|^{3+\rho}|\mathcal{F}_{i-1}] \le d_\rho^{1+\rho}\mathbf{E}[\varepsilon_i^2|\mathcal{F}_{i-1}]$ a.s. for all $i \in \mathbf{Z}$ and some constant d_ρ, then, by Theorem 1,

$$\sup_{x\in\mathbf{R}} |\mathbf{P}(S_n/B_n \le x) - \Phi(x)| \le c_\rho(\epsilon_n + \delta_n),$$

where $\epsilon_n = d_\rho \sup_{i\le n}|b_{n,i}|/B_n$.

In the case that $\sum_{i=0}^{\infty}|a_i| < \infty$, $\sup_{i\le n}|b_{n,i}| \le \sum_{i=0}^{\infty}|a_i| < \infty$ and it is well known that B_n^2 has order n. Hence ϵ_n has order $1/\sqrt{n}$ in this case. In the long memory case $\sum_{i=0}^{\infty}|a_i| = \infty$, if we assume (8), B_n^2 has order $n^{3-2\alpha}\ell^2(n)$ (e.g., Wu and Min [14]) and $\sup_{i\le n}|b_{n,i}|$ has order $n^{1-\alpha}\ell(n)$ (see Beknazaryan et al. [1] for upper bound and Fortune et al. [5] for lower bound in the case $d = 1$). Hence in this case ϵ_n also has order $1/\sqrt{n}$. In either case the Berry-Esseen bound has order $1/\sqrt{n}$ if $\delta_n = O(n^{-1/2})$. In particular, if we in addition assume that the innovations $(\varepsilon_i)_{i\in\mathbf{Z}}$ are independent, then $\delta_n = 0$ and the Berry-Esseen bound $\sup_{x\in\mathbf{R}}|\mathbf{P}(S_n/B_n \le x) - \Phi(x)|$ has order $1/\sqrt{n}$. Here the condition $\mathbf{E}[\varepsilon_i^3|\mathcal{F}_{i-1}] = 0$ is needed to have the Berry-Esseen bound of order $1/\sqrt{n}$. We cannot have this order from the result of Fan [4].

3. Proofs of theorems

3.1. *Preliminary lemmas*

In the proofs of theorems, we need the following technical lemmas. The first two lemmas can be found in Fan [4] (cf. Lemmas 3.1 and 3.2 therein).

Lemma 3. *If there exists an $s > 3$ such that*

$$\mathbf{E}[|\xi_i|^s|\mathcal{F}_{i-1}] \le \epsilon_n^{s-2}\mathbf{E}[\xi_i^2|\mathcal{F}_{i-1}],$$

then, for any $t \in [3, s)$,

$$\mathbf{E}[|\xi_i|^t|\mathcal{F}_{i-1}] \le \epsilon_n^{t-2}\mathbf{E}[\xi_i^2|\mathcal{F}_{i-1}].$$

Lemma 4. *If there exists an $s > 3$ such that*

$$\mathbf{E}[|\xi_i|^s|\mathcal{F}_{i-1}] \leq \epsilon_n^{s-2}\mathbf{E}[\xi_i^2|\mathcal{F}_{i-1}],$$

then

$$\mathbf{E}[\xi_i^2|\mathcal{F}_{i-1}] \leq \epsilon_n^2.$$

The next two technical lemmas are due to Bolthausen (cf. Lemmas 1 and 2 of [2]).

Lemma 5. *Let X and Y be random variables. Then*

$$\sup_u \Big|\mathbf{P}\big(X \leq u\big) - \Phi(u)\Big| \leq c_1 \sup_u \Big|\mathbf{P}\big(X + Y \leq u\big) - \Phi(u)\Big| + c_2 \Big|\Big|\mathbf{E}[Y^2|X]\Big|\Big|_\infty^{1/2},$$

where c_1 and c_2 are two positive constants.

Lemma 6. *Let $G(x)$ be an integrable function on $\mathbf{R}$ of bounded variation $||G||_V$, X be a random variable and $a, b \neq 0$ are real numbers. Then*

$$\mathbf{E}\Big[G\Big(\frac{X+a}{b}\Big)\Big] \leq ||G||_V \sup_u \Big|\mathbf{P}\big(X \leq u\big) - \Phi(u)\Big| + ||G||_1|b|,$$

where $||G||_1$ is the $L_1(\mathbf{R})$ norm of $G(x)$.

In the proof of Theorem 2, we also need the following lemma of El Machkouri and Ouchti [3].

Lemma 7. *Let X and Y be two random variables. Then, for $p \geq 1$,*

$$D(X+Y) \leq 2D(X) + 3\Big|\Big|\mathbf{E}\big[Y^{2p}|X\big]\Big|\Big|_1^{1/(2p+1)}. \tag{9}$$

3.2. *Proof of Theorem 1*

By Lemma 3, we only need to consider the case of $\rho \in (0, 1]$. We follow the method of Grama and Haeusler [6]. Let $T = 1 + \delta_n^2$. We introduce a modification of the conditional variance $\langle X\rangle_n$ as follows:

$$V_k = \langle X\rangle_k \mathbf{1}_{\{k<n\}} + T\mathbf{1}_{\{k=n\}}. \tag{10}$$

It is easy to see that $V_0 = 0, V_n = T$, and that $(V_k, \mathcal{F}_k)_{k=0,...,n}$ is a predictable process. Set

$$\gamma = \epsilon_n + \delta_n.$$

Let c_* be some positive and sufficient large constant. Define the following non-increasing discrete time predictable process

$$A_k = c_*^2\gamma^2 + T - V_k, \quad k = 1, ..., n. \tag{11}$$

Obviously, we have $A_0 = c_*^2\gamma^2 + T$ and $A_n = c_*^2\gamma^2$. In addition, for $u, x \in \mathbf{R}$, and $y > 0$, denote

$$\Phi_u(x, y) = \Phi\Big(\frac{u - x}{\sqrt{y}}\Big). \tag{12}$$

Let $\mathcal{N} = \mathcal{N}(0,1)$ be a standard normal random variable, which is independent of X_n. Using a smoothing procedure, by Lemma 5, we deduce that

$$\begin{aligned}
\sup_u \Big|\mathbf{P}\big(X_n \le u\big) - \Phi(u)\Big| &\le c_1 \sup_u \Big|\mathbf{P}\big(X_n + c_*\gamma\mathcal{N} \le u\big) - \Phi(u)\Big| + c_2\gamma \\
&= c_1 \sup_u \Big|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \Phi(u)\Big| + c_2\gamma \\
&\le c_1 \sup_u \Big|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\Big| \\
&\quad + c_1 \sup_u \Big|\mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big] - \Phi(u)\Big| + c_2\gamma \\
&= c_1 \sup_u \Big|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\Big| \\
&\quad + c_1 \sup_u \left|\Phi\left(\frac{u}{\sqrt{c_*^2\gamma^2 + T}}\right) - \Phi(u)\right| + c_2\gamma. \qquad (13)
\end{aligned}$$

It is obvious that

$$\left|\Phi\left(\frac{u}{\sqrt{c_*^2\gamma^2 + T}}\right) - \Phi(u)\right| \le c_3 \left|\frac{1}{\sqrt{c_*^2\gamma^2 + T}} - 1\right| \le c_4\gamma. \qquad (14)$$

Returning to (13), we get

$$\sup_u \Big|\mathbf{P}\big(X_n \le u\big) - \Phi(u)\Big| \le c_1 \sup_u \Big|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\Big| + c_5\gamma. \qquad (15)$$

By a simple telescoping, we know that

$$\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big] = \mathbf{E}\Big[\textstyle\sum_{k=1}^n \Big(\Phi_u\big(X_k, A_k\big) - \Phi_u\big(X_{k-1}, A_{k-1}\big)\Big)\Big]. \qquad (16)$$

Taking into account the fact that

$$\frac{\partial^2}{\partial x^2}\Phi_u(x,y) = 2\frac{\partial}{\partial y}\Phi_u(x,y),$$

we get

$$\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big] = J_1 + J_2 - J_3, \qquad (17)$$

where

$$\begin{aligned}
J_1 &= \mathbf{E}\Bigg[\sum_{k=1}^n \bigg(\Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial x}\Phi_u(X_{k-1}, A_k)\xi_k \\
&\qquad - \frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi_u(X_{k-1}, A_k)\xi_k^2 - \frac{1}{6}\frac{\partial^3}{\partial x^3}\Phi_u(X_{k-1}, A_k)\xi_k^3\bigg)\Bigg], \qquad (18) \\
J_2 &= \frac{1}{2}\mathbf{E}\Bigg[\sum_{k=1}^n \frac{\partial^2}{\partial x^2}\Phi_u(X_{k-1}, A_k)\Big(\triangle\langle X\rangle_k - \triangle V_k\Big)\Bigg], \qquad (19) \\
J_3 &= \mathbf{E}\Bigg[\sum_{k=1}^n \bigg(\Phi_u(X_{k-1}, A_{k-1}) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial y}\Phi_u(X_{k-1}, A_k)\triangle V_k\bigg)\Bigg], \qquad (20)
\end{aligned}$$

where $\triangle\langle X\rangle_k = \langle X\rangle_k - \langle X\rangle_{k-1}$.

Now, we need to give some estimates of J_1, J_2 and J_3. To this end, we introduce some notations. Denote by ϑ_i some random variables satisfying $0 \le \vartheta_i \le 1$, which may represent different values at different places. For the rest of the paper, φ stands for the density function of the standard normal random variable.

Control of J_1 : For convenience's sake, let $T_{k-1} = \big(u - X_{k-1}\big)/\sqrt{A_k},\ k = 1, 2, ..., n$. It is easy to see that

$$\begin{aligned}
B_k =:& \Phi_u(X_k, A_k) - \Phi_u(X_{k-1}, A_k) - \frac{\partial}{\partial x}\Phi_u(X_{k-1}, A_k)\xi_k \\
&- \frac{1}{2}\frac{\partial^2}{\partial x^2}\Phi_u(X_{k-1}, A_k)\xi_k^2 - \frac{1}{6}\frac{\partial^3}{\partial x^3}\Phi_u(X_{k-1}, A_k)\xi_k^3 \\
=& \Phi\bigg(T_{k-1} - \frac{\xi_k}{\sqrt{A_k}}\bigg) - \Phi(T_{k-1}) + \Phi'(T_{k-1})\frac{\xi_k}{\sqrt{A_k}} \\
&- \frac{1}{2}\Phi''(T_{k-1})\bigg(\frac{\xi_k}{\sqrt{A_k}}\bigg)^2 + \frac{1}{6}\Phi'''(T_{k-1})\bigg(\frac{\xi_k}{\sqrt{A_k}}\bigg)^3.
\end{aligned}$$

To estimate the right hand side of the last equality, we distinguish two cases.

Case 1: $|\xi_k/\sqrt{A_k}| \leq 2 + |T_{k-1}|/2$. By a four-term Taylor expansion, it is obvious that if $|\xi_k/\sqrt{A_k}| \leq 1$, then

$$\begin{aligned}
\Big|B_k\Big| &= \bigg|\frac{1}{24}\Phi^{(4)}\bigg(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\bigg)\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^4\bigg| \\
&\leq \bigg|\Phi^{(4)}\bigg(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\bigg)\bigg|\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^{3+\rho}.
\end{aligned}$$

If $|\xi_k/\sqrt{A_k}| > 1$, by a three-term Taylor expansion, then

$$\begin{aligned}
\Big|B_k\Big| &\leq \frac{1}{2}\bigg(\bigg|\Phi'''\bigg(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\bigg)\bigg| + \Big|\Phi'''(T_{k-1})\Big|\bigg)\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^3 \\
&\leq \bigg|\Phi'''\bigg(T_{k-1} - \vartheta'\frac{\xi_k}{\sqrt{A_k}}\bigg)\bigg|\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^3 \\
&\leq \bigg|\Phi'''\bigg(T_{k-1} - \vartheta'\frac{\xi_k}{\sqrt{A_k}}\bigg)\bigg|\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^{3+\rho},
\end{aligned}$$

where

$$\vartheta' = \begin{cases} \vartheta, \text{ if } \big|\Phi'''\big(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\big)\big| \geq |\Phi'''(T_{k-1})|, \\ 0, \text{ if } \big|\Phi'''\big(T_{k-1} - \vartheta\frac{\xi_k}{\sqrt{A_k}}\big)\big| < |\Phi'''(T_{k-1})|. \end{cases}$$

Using the inequality $\max\{|\Phi'''(t)|, |\Phi''''(t)|\} \leq \varphi(t)(2 + t^4)$, we find that

$$\begin{aligned}
\Big|B_k\mathbf{1}_{\{|\xi_k/\sqrt{A_k}| \leq 2 + |T_{k-1}|/2\}}\Big| &\leq \varphi\bigg(T_{k-1} - \vartheta_1\frac{\xi_k}{\sqrt{A_k}}\bigg)\bigg(2 + \bigg(T_{k-1} - \vartheta_1\frac{\xi_k}{\sqrt{A_k}}\bigg)^4\bigg)\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^{3+\rho} \\
&\leq g_1(T_{k-1})\bigg|\frac{\xi_k}{\sqrt{A_k}}\bigg|^{3+\rho}, \qquad (21)
\end{aligned}$$

where

$$g_1(z) = \sup_{|t-z| \leq 2 + |z|/2} \varphi(t)(2 + t^4).$$

Case 2: $|\xi_k/\sqrt{A_k}| > 2 + |T_{k-1}|/2$. It is obvious that, for $|\triangle x| > 1 + |x|/2$,

$$\begin{aligned}
&\Big|\Phi(x-\triangle x) - \Phi(x) + \Phi'(x) \triangle x - \frac{1}{2}\Phi''(x)(\triangle x)^2 + \frac{1}{6}\Phi'''(x)(\triangle x)^3\Big| \\
&\le \bigg(\bigg|\frac{\Phi(x-\triangle x)-\Phi(x)}{|\triangle x|^3}\bigg| + |\Phi'(x)| + |\Phi''(x)| + |\Phi'''(x)|\bigg)|\triangle x|^3 \\
&\le \bigg(8\bigg|\frac{\Phi(x-\triangle x)-\Phi(x)}{(2+|x|)^3}\bigg| + |\Phi'(x)| + |\Phi''(x)| + |\Phi'''(x)|\bigg)|\triangle x|^3 \\
&\le \bigg(\frac{\widetilde{c}}{(2+|x|)^3} + |\Phi'(x)| + |\Phi''(x)| + |\Phi'''(x)|\bigg)|\triangle x|^3 \\
&\le \frac{\hat{c}}{(2+|x|)^3}|\triangle x|^3 \\
&\le \frac{\hat{c}}{(2+|x|)^3}|\triangle x|^{3+\rho}.
\end{aligned}$$

Hence, we have

$$\Big|B_k \mathbf{1}_{\{|\xi_k/\sqrt{A_k}|>2+|T_{k-1}|/2\}}\Big| \le g_2(T_{k-1})\Big|\frac{\xi_k}{\sqrt{A_k}}\Big|^{3+\rho}, \tag{22}$$

where

$$g_2(z) = \frac{\hat{c}}{(2+|z|)^3}.$$

Denote

$$G(z) = g_1(z) + g_2(z).$$

Combining (21) and (22) together, we get

$$|B_k| \le G(T_{k-1})\Big|\frac{\xi_k}{\sqrt{A_k}}\Big|^{3+\rho}. \tag{23}$$

Therefore,

$$\Big|J_1\Big| = \Big|\mathbf{E}\Big[\sum_{k=1}^n B_k\Big]\Big| \le \mathbf{E}\Big[\sum_{k=1}^n G(T_{k-1})\Big|\frac{\xi_k}{\sqrt{A_k}}\Big|^{3+\rho}\Big]. \tag{24}$$

Next, we consider conditional expectation of $|\xi_k|^{3+\rho}$. By condition (5), we get

$$\mathbf{E}\big[|\xi_k|^{3+\rho}\big|\mathcal{F}_{k-1}\big] \le \epsilon_n^{1+\rho} \triangle \langle X\rangle_k, \tag{25}$$

where $\triangle\langle X\rangle_k = \langle X\rangle_k - \langle X\rangle_{k-1}$ and we know that

$$\triangle \langle X\rangle_k = \triangle V_k = V_k - V_{k-1},\ 1 \le k < n,\ \triangle\langle X\rangle_n \le \triangle V_n, \tag{26}$$

then

$$\mathbf{E}\big[|\xi_k|^{3+\rho}\big|\mathcal{F}_{k-1}\big] \le \epsilon_n^{1+\rho} \triangle V_k. \tag{27}$$

By (24) and (27), we obtain

$$|J_1| \le R_1 := \epsilon_n^{1+\rho}\Big[\sum_{k=1}^n \frac{G(T_{k-1})}{A_k^{(3+\rho)/2}} \triangle V_k\Big]. \tag{28}$$

To estimate R_1, we introduce the time change τ_t as follow: for any real $t \in [0, T]$,

$$\tau_t = \min\{k \le n : V_k \ge t\}, \quad \text{where } \min\emptyset = n. \tag{29}$$

Obviously, for any $t \in [0,T]$, the stopping time τ_t is predictable. In addition, $(\sigma_k)_{k=1,\ldots,n+1}$(with $\sigma_1=0$) stands for the increasing sequence of moments when the increasing and stepwise function $\tau_t, t\in[0,T]$, has jumps. It is easy to see that $\triangle V_k = \int_{[\sigma_k,\sigma_{k+1})} dt$, and that $k=\tau_t$ for $t\in[\sigma_k,\sigma_{k+1})$. Since $\tau_T=n$, we have

$$\sum_{k=1}^{n}\frac{G(T_{k-1})}{A_k^{(3+\rho)/2}}\triangle V_k=\sum_{k=1}^{n}\int_{[\sigma_k,\sigma_{k+1})}\frac{G(T_{\tau_t-1})}{A_{\tau_t}^{(3+\rho)/2}}dt=\int_0^T\frac{G(T_{\tau_t-1})}{A_{\tau_t}^{(3+\rho)/2}}dt. \tag{30}$$

Let $a_t=c_*^2\gamma^2+T-t$. Because of $\triangle V_{\tau_t}\le 2\epsilon_n^2+2\delta_n^2$ (cf. Lemma 4), we know that

$$t\le V_{\tau_t}=V_{\tau_t-1}+\triangle V_{\tau_t}\le t+2\epsilon_n^2+2\delta_n^2,\quad t\in[0,T]. \tag{31}$$

Assume $c_*\ge 2$, then we have

$$\frac{1}{2}a_t\le A_{\tau_t}=c_*^2\gamma^2+T-V_{\tau_t}\le a_t,\quad t\in[0,T]. \tag{32}$$

Note that $G(z)$ is symmetric and is non-increasing in $z\ge 0$. The last bound implies that

$$R_1\le 2^{(3+\rho)/2}\epsilon_n^{1+\rho}\int_0^T\frac{1}{a_t^{(3+\rho)/2}}\mathbf{E}\Big[G\Big(\frac{u-X_{\tau_t-1}}{a_t^{1/2}}\Big)\Big]dt. \tag{33}$$

Note also that $G(z)$ is a symmetric integrable function of bounded variation. By Lemma 6, it is obvious that

$$\mathbf{E}\Big[G\Big(\frac{u-X_{\tau_t-1}}{a_t^{1/2}}\Big)\Big]\le c_6\sup_z\Big|\mathbf{P}\big(X_{\tau_t-1}\le z\big)-\Phi(z)\Big|+c_7\sqrt{a_t}. \tag{34}$$

Because of $c_*\ge 2, V_{\tau_t-1}=V_{\tau_t}-\triangle V_{\tau_t},\ V_{\tau_t}\ge t$ and $\triangle V_{\tau_t}\le 2\epsilon_n^2+2\delta_n^2$, we obtain

$$V_n-V_{\tau_t-1}=V_n-V_{\tau_t}+\triangle V_{\tau_t}\le 2\epsilon_n^2+2\delta_n^2+T-t\le a_t. \tag{35}$$

Therefore

$$\begin{aligned}\mathbf{E}\Big[\big(X_n-X_{\tau_t-1}\big)^2\Big|\mathcal{F}_{\tau_t-1}\Big]&=\mathbf{E}\Big[\sum_{k=\tau_t}^{n}\mathbf{E}\big[\xi_k^2\big|\mathcal{F}_{k-1}\big]\Big|\mathcal{F}_{\tau_t-1}\Big]\\&=\mathbf{E}\big[\langle X\rangle_n-\langle X\rangle_{\tau_t-1}\big|\mathcal{F}_{\tau_t-1}\big]\\&\le\mathbf{E}[V_n-V_{\tau_t-1}|\mathcal{F}_{\tau_t-1}]\\&\le a_t.\end{aligned}$$

Then, by Lemma 5, we deduce that for any $t\in[0,T]$,

$$\sup_z\big|\mathbf{P}\big(X_{\tau_t-1}\le z\big)-\Phi(z)\big|\le c_8\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+c_9\sqrt{a_t}. \tag{36}$$

Combining (28), (33), (34) and (36) together, we get

$$|J_1|\le c_{10}\epsilon_n^{1+\rho}\int_0^T\frac{1}{a_t^{(3+\rho)/2}}dt\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+c_{11}\epsilon_n^{1+\rho}\int_0^T\frac{1}{a_t^{1+\rho/2}}dt. \tag{37}$$

Taking some elementary computations, it follows that

$$\int_0^T\frac{1}{a_t^{(3+\rho)/2}}dt=\int_0^T\frac{1}{(c_*^2\gamma^2+T-t)^{(3+\rho)/2}}dt\le\frac{2}{c_*^{1+\rho}(1+\rho)\gamma^{1+\rho}} \tag{38}$$

and

$$\int_0^T\frac{1}{a_t^{1+\rho/2}}dt=\int_0^T\frac{1}{(c_*^2\gamma^2+T-t)^{1+\rho/2}}dt\le\frac{2}{c_*^{\rho}\rho\gamma^{\rho}}. \tag{39}$$

This yields

$$\big|J_1\big|\le\frac{c_{12}}{c_*^{1+\rho}}\sup_z\big|\mathbf{P}\big(X_n\le z\big)-\Phi(z)\big|+\frac{c_{\rho,1}\epsilon_n}{\rho}. \tag{40}$$

Control of J_2 : Since $0 \leq \triangle V_k - \triangle \langle X \rangle_k \leq 2\delta^2 \mathbf{1}_{\{k=n\}}$, we have

$$|J_2| \leq \mathbf{E}\Big[\frac{1}{2A_n}\big|\varphi'(T_{n-1})(\triangle V_n - \triangle \langle X \rangle_n)\big|\Big].$$

Denote $\widetilde{G}(z) = \sup_{|z-t|\leq 1} |\varphi'(t)|$, and then $|\varphi'(z)| \leq \widetilde{G}(z)$ for any real z. Since $A_n = c_*^2\gamma^2$, then we get the following estimation:

$$|J_2| \leq \frac{1}{c_*^2}\mathbf{E}\big[\widetilde{G}(T_{n-1})\big].$$

Note that $\widetilde{G}$ is non-increasing in $z \geq 0$, and thus it has bounded variation on $\mathbf{R}$. By Lemma 6, we get

$$|J_2| \leq \frac{c_{13}}{c_*^2}\sup_z \big|\mathbf{P}\big(X_{n-1} \leq z\big) - \Phi(z)\big| + c_{*,2}(\epsilon_n + \delta_n). \tag{41}$$

Then, by Lemma 5, we deduce that

$$\sup_z \big|\mathbf{P}\big(X_{n-1} \leq z\big) - \Phi(z)\big| \leq c_{14}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + c_{15}\epsilon_n. \tag{42}$$

This yields

$$|J_2| \leq \frac{c_{16}}{c_*^2}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + c_{\rho,2}(\epsilon_n + \delta_n). \tag{43}$$

Control of J_3. By a two-term Taylor expansion, it follows that

$$|J_3| = \frac{1}{8}\mathbf{E}\Bigg[\sum_{k=1}^n \frac{1}{(A_k - \vartheta_k \triangle A_k)^2}\varphi'''\bigg(\frac{u - X_{k-1}}{\sqrt{A_k - \vartheta_k \triangle A_k}}\bigg)(\triangle A_k)^2\Bigg].$$

Note that $c_* \geq 2, \triangle A_k \leq 0$ and, by Lemma 4, $|\triangle A_k| = \triangle V_k \leq 2\epsilon_n^2 + 2\delta_n^2$. We obtain

$$A_k \leq A_k - \vartheta_k \triangle A_k \leq c_*^2\gamma^2 + T - V_k + 2\epsilon_n^2 + 2\delta_n^2 \leq 2A_k. \tag{44}$$

Denote $\widehat{G}(z) = \sup_{|t-z|\leq 2} |\varphi'''(t)|$. Then $\widehat{G}(z)$ is symmetric, and is non-increasing in $z \geq 0$. Using (44), we get

$$|J_3| \leq (2\epsilon_n^2 + 2\delta_n^2)\mathbf{E}\Bigg[\sum_{k=1}^n \frac{1}{A_k^2}\widehat{G}\bigg(\frac{T_{k-1}}{\sqrt{2}}\bigg) \triangle V_k\Bigg]. \tag{45}$$

By an argument similar to that of (40), we get

$$\begin{aligned} |J_3| &\leq \frac{c_{17}(2\epsilon_n^2 + 2\delta_n^2)}{c_*^2\gamma^2}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + \frac{2c_{18}(2\epsilon_n^2 + 2\delta_n^2)}{c_*\gamma} \\ &\leq \frac{c_{19}}{c_*^2}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + \frac{4c_{18}(\epsilon_n + \delta_n)^2}{c_*\gamma} \\ &\leq \frac{c_{19}}{c_*^2}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + c_{\rho,3}(\epsilon_n + \delta_n). \end{aligned} \tag{46}$$

Combining (17), (40), (43) and (46) together, we get

$$\Big|\mathbf{E}\big[\Phi_u\big(X_n, A_n\big)\big] - \mathbf{E}\big[\Phi_u\big(X_0, A_0\big)\big]\Big| \leq \frac{c_{20}}{c_*^{1+\rho}}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + \frac{\hat{c}_\rho}{\rho}(\epsilon_n + \delta_n),$$

By (15), we know that

$$\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| \leq \frac{c_{21}}{c_*^{1+\rho}}\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| + \frac{\tilde{c}_\rho}{\rho}(\epsilon_n + \delta_n),$$

from which, choosing $c_*^{1+\rho} = \max\{2c_{21}, 2^{1+\rho}\}$, we get

$$\sup_z \big|\mathbf{P}\big(X_n \leq z\big) - \Phi(z)\big| \leq \frac{2\tilde{c}_\rho(\epsilon_n + \delta_n)}{\rho}. \tag{47}$$

3.3. *Proof of Theorem 2*

Following the method of Bolthausen [2], we enlarge the sequence $(\xi_i, \mathcal{F}_i)_{1\le i\le n}$ to $\big(\hat{\xi}_i, \hat{\mathcal{F}}_i\big)_{1\le i\le N}$ such that $\langle \hat{X}\rangle_N := \sum_{i=1}^N \mathbf{E}\big[\hat{\xi}_i^2|\hat{\mathcal{F}}_{i-1}\big] = 1$ a.s., and then apply Theorem 1 to the enlarged sequence. Consider the stopping time

$$\tau = \sup\{k \le n : \langle X\rangle_k \le 1\}. \tag{48}$$

Assume that $0 \le \varepsilon \le \epsilon_n$. Let $r = \Big\lfloor \frac{1-\langle X\rangle_\tau}{\varepsilon^2}\Big\rfloor$, where $\lfloor x\rfloor$ denotes the "integer part" of x. It is easy to see that $r \le \Big\lfloor \frac{1}{\varepsilon^2}\Big\rfloor$. Set $N = n + r + 1$. Let $(\zeta_i)_{i\ge 1}$ be a sequence of independent Rademacher random variables, which is independent of the martingale differences $(\xi_i)_{1\le i\le n}$. Consider the random variables $\big(\hat{\xi}_i, \hat{\mathcal{F}}_i\big)_{1\le i\le N}$ defined as follows:

$$\hat{\xi}_i = \begin{cases} \xi_i \quad \text{a.s.,} & \text{if } i \le \tau, \\ \varepsilon\zeta_i \quad \text{a.s.,} & \text{if } \tau+1 \le i \le \tau + r, \\ \big(1-\langle X\rangle_\tau - r\varepsilon^2\big)^{1/2}\zeta_i \quad \text{a.s.,} & \text{if } i = \tau+r+1, \\ 0 \quad \text{a.s.,} & \text{if } \tau+r+1 \le i \le N, \end{cases}$$

and $\hat{\mathcal{F}}_i = \sigma\big(\hat{\xi}_1, \hat{\xi}_2, ..., \hat{\xi}_i\big)$.

Clearly, $\big(\hat{\xi}_i, \hat{\mathcal{F}}_i\big)_{1\le i\le N}$ still forms a martingale difference sequence with respect to the enlarged filtration. Then $\hat{X}_k = \sum_{i=1}^k \hat{\xi}_i,\ k = 0, ..., N$, with $\hat{X}_0 = 0$, is also a martingale. Moreover, it holds that $\langle \hat{X}\rangle_N = 1$, $\mathbf{E}\big[\hat{\xi}_i^3\big|\hat{\mathcal{F}}_{i-1}\big] = 0$ and

$$\mathbf{E}\big[\big|\hat{\xi}_i\big|^{3+\rho}\big|\hat{\mathcal{F}}_{i-1}\big] \le \epsilon_n^{1+\rho}\mathbf{E}\big[\hat{\xi}_i^2\big|\hat{\mathcal{F}}_{i-1}\big], \qquad a.s.$$

By Theorem 1, we have

$$D\big(\hat{X}_N\big) \le \frac{c_\rho \epsilon_n}{\rho}. \tag{49}$$

Using Lemma 7, we obtain that

$$\begin{aligned} D(X_n) &\le 2D\big(\hat{X}_N\big) + 3\Big\|\mathbf{E}\big[\big|X_n - \hat{X}_N\big|^{2p}\big|\hat{X}_N\big]\Big\|_1^{1/(2p+1)} \\ &\le \frac{2c_\rho\epsilon_n}{\rho} + 3\Big(\mathbf{E}\big[\big|\hat{X}_N - X_n\big|^{2p}\big]\Big)^{1/(2p+1)}. \end{aligned} \tag{50}$$

Since τ is a stopping time and

$$\hat{X}_N - X_n = \sum_{i=\tau+1}^{N} \big(\hat{\xi}_i - \xi_i\big), \qquad \text{where put } \xi_i = 0 \text{ for } i > n, \tag{51}$$

$(\hat{\xi}_i - \xi_i, \hat{\mathcal{F}}_i)_{i\ge\tau+1}$ still forms a martingale difference sequence. Applying Theorem 2.11 of Hall and Heyde [8], we get

$$\begin{aligned} \mathbf{E}\Big[\big|\hat{X}_N - X_n\big|^{2p}\Big] &\le \mathbf{E}\Big[\max_{\tau+1\le i\le N}\big|\hat{X}_i - X_i\big|^{2p}\Big] \\ &\le c_p\Bigg(\mathbf{E}\Big[\Big|\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\hat{\xi}_i-\xi_i\big)^2\big|\hat{\mathcal{F}}_{i-1}\big]\Big|^p\ \Big] + \mathbf{E}\Big[\max_{\tau+1\le i\le N}\big|\hat{\xi}_i - \xi_i\big|^{2p}\Big]\Bigg). \end{aligned} \tag{52}$$

As $\mathbf{E}[\xi_i\hat{\xi}_i|\hat{\mathcal{F}}_{i-1}] = 0$ for all $i \ge \tau+1$, we have

$$\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\hat{\xi}_i-\xi_i\big)^2\big|\hat{\mathcal{F}}_{i-1}\big] = \sum_{i=\tau+1}^{N}\mathbf{E}\big[\hat{\xi}_i^2\big|\hat{\mathcal{F}}_{i-1}\big] + \sum_{i=\tau+1}^{n}\mathbf{E}\big[\xi_i^2\big|\hat{\mathcal{F}}_{i-1}\big] = 1 - 2\langle X\rangle_\tau + \langle X\rangle_n.$$

Noting that $1-\mathbf{E}[\xi_{\tau+1}^2|\mathcal{F}_\tau] \leq \langle X\rangle_\tau$. Consequently, using the inequality $|a+b|^p \leq 2^{p-1}\left(|a|^p+|b|^p\right), p\geq 1$, and Jensen's inequality, we derive that

$$\begin{aligned}\Big|\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\hat{\xi}_i-\xi_i\big)^2\big|\hat{\mathcal{F}}_{i-1}\big]\Big|^p &\leq \Big|\langle X\rangle_n - 1 + 2\mathbf{E}\big[\xi_{\tau+1}^2\big|\mathcal{F}_\tau\big]\Big|^p \\ &\leq 2^{2p-1}\bigg(\big|\langle X\rangle_n-1\big|^p+\Big|\mathbf{E}\big[\xi_{\tau+1}^2\big|\mathcal{F}_\tau\big]\Big|^p\bigg) \\ &\leq 2^{2p-1}\Big(\big|\langle X\rangle_n-1\big|^p+\mathbf{E}\big[|\xi_{\tau+1}|^{2p}\big|\mathcal{F}_\tau\big]\Big). \qquad (53)\end{aligned}$$

Taking expectations on both sides of the last inequality, we deduce that

$$\begin{aligned}\mathbf{E}\Big[\Big|\sum_{i=\tau+1}^{N}\mathbf{E}\big[\big(\hat{\xi}_i-\xi_i\big)^2\big|\hat{\mathcal{F}}_{i-1}\big]\Big|^p\,\Big] &\leq 2^{2p-1}\bigg(\mathbf{E}\Big[\big|\langle X\rangle_n-1\big|^p\Big]+\mathbf{E}\big[|\xi_{\tau+1}|^{2p}\big]\bigg) \\ &\leq 2^{2p-1}\bigg(\mathbf{E}\Big[\big|\langle X\rangle_n-1\big|^p\Big]+\mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big]\bigg). \qquad (54)\end{aligned}$$

Similarly, using the inequality $|a+b|^p \leq 2^{p-1}\left(|a|^p+|b|^p\right), p\geq 1$,

$$\begin{aligned}\mathbf{E}\Big[\max_{\tau+1\leq i\leq N}\big|\hat{\xi}_i-\xi_i\big|^{2p}\Big] &\leq 2^{2p-1}\mathbf{E}\Big[\max_{\tau+1\leq i\leq N}\Big(|\xi_i|^{2p}+\big|\hat{\xi}_i\big|^{2p}\Big)\Big] \\ &\leq 2^{2p-1}\bigg(\mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big]+\varepsilon^{2p}\bigg). \qquad (55)\end{aligned}$$

Combining (52), (54) and (55) together, we obtain

$$\mathbf{E}\Big[\big|\hat{X}_N-X_n\big|^{2p}\Big] \leq \hat{c}_p\bigg(\mathbf{E}\Big[\big|\langle X\rangle_n-1\big|^p\Big]+\mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big]+\varepsilon^{2p}\bigg). \qquad (56)$$

Finally, applying the last inequality to (50) and let $\varepsilon\to 0$, then we have

$$D(X_n) \leq \tilde{c}_\rho\frac{\epsilon_n}{\rho}+\tilde{c}_p\bigg(\mathbf{E}\Big[\big|\langle X\rangle_n-1\big|^p\Big]+\mathbf{E}\Big[\max_{1\leq i\leq n}|\xi_i|^{2p}\Big]\bigg)^{1/(2p+1)}.$$

This completes the proof of Theorem 2.

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