First-order random coefficient INAR process with dependent counting series

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Abstract In this paper, we propose a first-order random coefficient integer-valued autoregressive process with dependent counting series. Some moments and stationary ergodicity of the process are established. The maximum likelihood estimators of the parameters of interest are presented. We conduct some simulation studies to assess the performance of our method. An example about crime data is provided for practical application.

Keywords: Asymptotic property; Dependent counting series; INAR model; Random coefficient; Thinning operator.

1 Introduction

Time series of count data have been widely studied by many authors during recent years. To describe the integer-valued structure for this kind of data, one of the most popular approach is the thinning operator based models (Weiß, 2008; Scotto et al., 2015). For example, Al-Osh and Alzaid (1987) proposed a first-order integer-valued autoregressive (INAR) model, which plays an important role in the field of integer-valued time series. Ristić et al. (2009) presented a new stationary first-order integer-valued autoregressive process with geometric marginal. Zhang et al. (2010) proposed a p-th order INAR process with signed generalized power series thinning operator. Bakouch and Ristić (2010) introduce a new first-order stationary integer-valued autoregressive process with zero truncated Poisson marginal distribution. Jazi et al. (2012) proposed a first-order integer-valued AR process with zero inflated Poisson innovations. Weiß (2015) proposed a Poisson INAR(1) model with serially dependent innovations. Nastić et al. (2016) introduced a random environment in integer-valued autoregressive process. Yang et al. (2018) and Wang et al. (2019) studied the negative binomial threshold integer-valued autoregressive process models. Besides, some random

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coefficient INAR models and related results are also published in the literature. Zheng et al. (2006, 2007) and Gomes and Castro (2009) extended the INAR models to random coefficient cases, and studied some parameter estimation topics. Zhang et al. (2011a, 2011b) studied the empirical likelihood for random coefficient INAR models. Wang and Zhang (2011) and Zhang et al. (2012) proposed some random coefficient INAR processes with signed thinning operator. Zhang and Wang (2015) considered the frequency domain analysis in random coefficient INAR(1) process. Li et al. (2018) introduced a first-order random coefficient integer-valued threshold autoregressive process. Bakouch et al. (2018) introduced a new stationary random coefficient INAR(1) process with zero-inflated geometric marginal distribution. Yu et al. (2018) proposed a class of observation-driven random coefficient integer-valued autoregressive processes based on negative binomial thinning.

The above-mentioned articles are mainly based on independent counting series. Recently, there are some articles focusing on dependent counting series for modeling integer-valued time series data. For instance, Ristić et al. (2013) proposed a geometric integer-valued autoregressive model with dependent Bernoulli counting series. Miletić Ilić (2016) and Nastić et al. (2017) introduced some geometric INAR models based on generalized binomial thinning operator with dependent counting series. Miletić Ilić et al. (2018) proposed an INAR(1) model based on a mixed dependent and independent counting series. Towards the importance of random coefficient models, it is desirable to develop some random coefficient INAR models based on dependent counting series. In this article, we extend Ristić et al. (2013)'s work to a first-order random coefficient INAR process with dependent counting series. Meanwhile, some basic statistical properties, together with the parameter estimation are presented.

The remainder of this paper is organized as follows: In Section 2, we provide the definition and statistical properties of our proposed model. In Section 3, the maximum likelihood estimators of the parameters of interest are derived. We present some simulation results to check the rationality of our method. In Section 4, we provide an application to a real data example about crime data. Some concluding remarks are given in Section 5.

2 Definition and properties of the RCINAR-D(1) process

In the literature, Ristić et al. (2013) proposed a novel thinning operator with dependent Bernoulli counting series as $\phi \odot_\theta X = \sum_{i=1}^{X} U_i$, where $U_i = (1 - V_i)W_i + V_i\xi$, $\{V_i\}$ is a sequence of i.i.d. random variables with Bernoulli(θ) distribution, $\theta \in (0, 1)$; $\{W_i\}$ and $\{\xi\}$ are i.i.d. random variables with Bernoulli(ϕ) distribution. We will extend Ristić et al. (2013)'s work by proposing a first-order random coefficient integer-valued autoregressive process with dependent counting series

(RCINAR-D(1)). Below, we present the definition and some basic statistical properties, which include the transition probabilities, moments, and ergodicity.

Definition 2.1 *The* RCINAR-D(1) *process is defined by the following recursive equation*

$$X_t = \phi_t \odot_\theta X_{t-1} + Z_t, \tag{2.1}$$

where the thinning operator $\phi_t \odot_\theta$ *is given as*

$$\phi_t \odot_\theta X_{t-1} = \sum_{i=1}^{X_{t-1}} U_{it}.$$

Here $U_{it} = (1 - V_i)W_{it} + V_i\xi_t$, $\{V_i\}$ *is a sequence of i.i.d. random variables with Bernoulli(*θ*) distribution,* $\theta \in (0, 1)$*; Given* ϕ_t, $\{W_{it}\}$ *and* $\{\xi_t\}$ *are i.i.d. random variables with Bernoulli(*ϕ_t*) distribution.* $\{\phi_t\}$ *is an i.i.d. sequence with cumulative distribution function* $P_\phi(\cdot)$ *on* $(0, 1)$*. We assume that* $\{V_i\}$, $\{W_{it}\}$ *and* $\{\xi_t\}$ *are independent for all* i *and* t*. Moreover,* $\{Z_t\}$ *is an i.i.d. nonnegative integer-valued sequence with probability mass function* $f_z(\cdot) > 0$*, and* $Cov(X_s, Z_t) = 0$ *for* $s < t$*. Let* $\phi = E(\phi_t)$, $\sigma_\phi^2 = \mathrm{Var}(\phi_t)$, $\lambda = E(Z_t)$, $\sigma_Z^2 = \mathrm{Var}(Z_t)$*, and assume that they are finite.*

Remark 1. Given ϕ_t, $\{U_{it}\}$ is a sequence of dependent random variables with Bernoulli(ϕ_t) marginal distribution, where $P(U_{it} = 1|\phi_t) = \phi_t$, $P(U_{it} = 0|\phi_t) = 1 - \phi_t$, and $Cov(U_{it}, U_{jt}|\phi_t) = \theta^2\phi_t(1 - \phi_t)$ for $i \neq j$.

Remark 2. The $\{X_t\}$ is a Markov chain on $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$with the following transition probabilities

$$P_{ij} = P(X_t = j|X_{t-1} = i) \quad = \quad \sum_{k=0}^{\min(i,j)} C_i^k f_z(j - k) \int_0^1 b_{ik}(\phi_t)dP_\phi(\phi_t), \tag{2.2}$$

where $b_{ik}(\phi_t) = (1 - \phi_t)[\phi_t(1 - \theta)]^k[1 - \phi_t(1 - \theta)]^{i-k} + \phi_t[\theta + \phi_t(1 - \theta)]^k[(1 - \phi_t)(1 - \theta)]^{i-k}$.

Below, we present some moment results on the $\{X_t\}$. Here we omit the proof details.

Proposition 2.1 *For* $t \geq 1$*, we have*

(i) $E(X_t|X_{t-1}, \phi_t) = \phi_t X_{t-1} + \lambda$.

(ii) $E(X_t|X_{t-1}) = \phi X_{t-1} + \lambda$.

(iii) If $E(X_0) = \frac{\lambda}{1-\phi}$*, then* $E(X_t) = \mu = \frac{\lambda}{1-\phi}$.

(iv) $Var(X_t|X_{t-1}, \phi_t) = \phi_t(1 - \phi_t)\theta^2 X_{t-1}^2 + \phi_t(1 - \phi_t)(1 - \theta^2)X_{t-1} + \sigma_Z^2$.

(v) $Var(X_t|X_{t-1}) = \{[\phi(1 - \phi) - \sigma_\phi^2]\theta^2 + \sigma_\phi^2\}X_{t-1}^2 + \{[\phi(1 - \phi) - \sigma_\phi^2](1 - \theta^2)\}X_{t-1} + \sigma_Z^2$.

(vi) If $Var(X_0) = \frac{c}{1-a}$*, then* $Var(X_t) = \frac{c}{1-a}$*, where* $a = \phi^2 + \tau\theta^2 + \sigma_\phi^2$, $\tau = \phi(1 - \phi) - \sigma_\phi^2$*, and* $c = (\tau\theta^2 + \sigma_\phi^2)(\frac{\lambda}{1-\phi})^2 + \tau(1 - \theta^2)\frac{\lambda}{1-\phi} + \sigma_Z^2$.

(vii) For $k \geq 1$, $\gamma_k = \phi^k Var(X_t)$*, where* $\gamma_k = E\{[X_t - E(X_t)][X_{t+k} - E(X_{t+k})]\}$.

Theorem 2.1 *The* RCINAR-D(1) *process* $\{X_t\}$ *is an ergodic Markov chain. Moreover, the stationary distribution of* $\{X_t\}$ *is given by* $\sum_{k=1}^{t-1} \phi_{k+1} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{k+1} + Z_1$*, which converges in* L_2*.*

Proof. Firstly, we prove the ergodicity of $\{X_t\}$. From (2.2) and the assumption $f_z > 0$, we can conclude that $\{X_t\}$ is an irreducible and aperiodic Markov chain. So we only need to check $\lim_{n\to\infty} P_{ij}^n \neq 0$, where $P_{ij}^n = P(X_n = j|X_0 = i)$. For this goal, we can focus on three steps.

Step 1. Repeated application of $X_t = \phi_t \odot_\theta X_{t-1} + Z_t$ with t replaced by n,

$$\begin{aligned} X_n &= \phi_n \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0 + \sum_{k=1}^{n-1} \phi_n \odot_\theta \ldots \odot_\theta \phi_{n-k+1} \odot_\theta Z_{n-k} + Z_n \\ &\overset{d}{=} \underbrace{\phi_n \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0}_{(I)} + \underbrace{\sum_{k=1}^{n-1} \phi_{k+1} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{k+1} + Z_1}_{V_n} = Y_n, \end{aligned} \tag{2.3}$$

where $\overset{d}{=}$ denotes two random variables X and Y having the same distribution, and

$$\begin{aligned} E(\phi_n \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0) &= E[E(\phi_n \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0|\phi_n)] \\ &= E[\phi_n E(\phi_{n-1} \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0)] \\ &= \phi E(\phi_{n-1} \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0) \\ &= \ldots \\ &= \phi^n E(X_0) \to 0, (as\ n \to \infty). \end{aligned}$$

By Markov inequality we can get that term (I) is $o_p(1)$.

Step 2. For any $\epsilon > 0$ and $m \in \mathbb{N}$, there exists N_0 and $n > N_0$, such that

$$\begin{aligned} & P(|Y_n - Y_{n+m}| > \epsilon) \\ &= P\left\{\left|\phi_n \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0 - \phi_{n+m} \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0 + \sum_{i=1}^{m} \phi_{n+i} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{n+i}\right| > \epsilon\right\} \\ &\leq \frac{E(|\phi_n \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0 - \phi_{n+m} \odot_\theta \ldots \odot_\theta \phi_1 \odot_\theta X_0 + \sum_{i=1}^{m} \phi_{n+i} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{n+i}|)}{\epsilon} \\ &= \frac{\phi^n(1-\phi^m)E(X_0) + \phi^{n-1}(1-\phi^{m-1})\lambda/(1-\phi)}{\epsilon} \longrightarrow 0, \text{ as } n \to \infty. \end{aligned}$$

So there exists a random variable Y, such that $Y_n \xrightarrow{P} Y$, where $\xrightarrow{P}$ denotes convergence in probability.

Step 3. We need to prove $\{X_t\}$ is a positive recurrent Markov chain, which means that $\lim_{n\to\infty} P_{ij}^n = P\{Y=j\} \neq 0$, for all i and j. Let $V_n = \sum_{k=1}^{n-1} \phi_{k+1} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{k+1} + Z_1$. By the above Step 2

$$\begin{aligned}
\lim_{n\to\infty} P_{ij}^n &= \lim_{n\to\infty} P(X_n = j | X_0 = i) \\
&= \lim_{n\to\infty} P(Y_n = j | X_0 = i) \\
&= \lim_{n\to\infty} P(V_n = j | X_0 = i) \\
&= \lim_{n\to\infty} P(V_n = j) \\
&= \lim_{n\to\infty} P(Y_n = j) \\
&= P(Y = j).
\end{aligned}$$

Note that $\sum_{j\in\mathbb{N}} P\{Y=j\} = \lim_{n\to\infty} \sum_{j\in\Omega} P_{ij}^n = 1$, which indicates $\lim_{n\to\infty} P_{ij}^n \neq 0$. So the process $\{X_t\}$ is positive recurrent Markov chain. Above all, we prove that $\{X_t\}$ is an irreducible aperiodic and positive recurrent (hence ergodic) Markov chain.

Secondly, we aim to prove V_n converges in L_2. Since

$$\begin{aligned}
E(|V_{n+m} - V_n|^2) &= E\left[\sum_{k=1}^{m} (\phi_{n+k} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{n+k})\right]^2 \\
&= \sum_{k=1}^{m} (\phi_{n+k} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{n+k})^2 \\
&+ 2 \sum_{1\le j<i\le m} (\phi_{n+i} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{n+i})(\phi_{n+j} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{n+j}). \qquad (2.4)
\end{aligned}$$

Let $Y_{t-1} = \phi_{t-1} \odot_\theta \ldots \odot_\theta \phi_2 \odot_\theta Z_{t-1}$, then

$$\begin{aligned}
E[(\phi_t \odot_\theta Y_{t-1})^2 | \phi_t] &= E\{E[(\phi_t \odot_\theta Y_{t-1})^2 | Y_{t-1}, \phi_t]\} \\
&= E[Var(\phi_t \odot_\theta Y_{t-1} | Y_{t-1}, \phi_t)] + E[E^2(\phi_t \odot_\theta Y_{t-1} | Y_{t-1}, \phi_t)] \\
&= \phi_t(1-\phi_t)\theta^2 E(Y_{t-1}^2) + \phi_t(1-\phi_t)(1-\theta^2)E(Y_{t-1}) + \phi_t^2 E(Y_{t-1}^2). \qquad (2.5)
\end{aligned}$$

Thus, by (2.5) we can get that

$$\begin{aligned}
E(\phi_t \odot_\theta Y_{t-1})^2 &= (\phi - \phi^2 - \sigma_\phi^2)\theta^2 E(Y_{t-1}^2) + (\phi - \phi^2 - \sigma_\phi^2)(1-\theta^2)E(Y_{t-1}) + (\phi^2 + \sigma_\phi^2)E(Y_{t-1}^2) \\
&= [\phi\theta^2 + (\phi^2 + \sigma_\phi^2)(1-\theta^2)]E(Y_{t-1}^2) + (\phi - \phi^2 - \sigma_\phi^2)(1-\theta^2)E(Y_{t-1}) \\
&= aE(Y_{t-1}^2) + bE(Y_{t-1}) \\
&= aE(Y_{t-1}^2) + b_{t-1}, \qquad (2.6)
\end{aligned}$$

where $a = \phi\theta^2 + (\phi^2 + \sigma_\phi^2)(1-\theta^2)$, $b = (\phi - \phi^2 - \sigma_\phi^2)(1-\theta^2)$, and $b_{t-1} = bE(Y_{t-1}) = b\phi^{t-2}\lambda$.

By repeated application of (2.6), it is easy to conclude that

$$\begin{aligned} E(\phi_t \odot_\theta \phi_{t-1} \odot_\theta \dots \phi_2 \odot_\theta Z_t)^2 &= aE(Y_{t-1}^2) + b_{t-1} \\ &= a^2 E(Y_{t-2}^2) + ab_{t-2} + b_{t-1} \\ &= \dots \\ &= a^{t-1}E(Z_1^2) + a^{t-2}b_1 + a^{t-3}b_2 + \dots + ab_{t-2} + b_{t-1}. \end{aligned} \tag{2.7}$$

For any $t \geq 1$ and $s \geq 1$,

$$\begin{aligned} E[(\phi_t \odot_\theta Y_{t-1})(\phi_s \odot_\theta Y_{s-1})] &= E[(\phi_t \odot_\theta Y_{t-1})(\phi_s \odot_\theta Y_{s-1})|\phi_t, \phi_s] \\ &= E[\phi_t\phi_s E(Y_{t-1}Y_{s-1})] \\ &= \phi^2 E(Y_{t-1}Y_{s-1}). \end{aligned} \tag{2.8}$$

Applying (2.8) and taking repeated conditional expectation, for $i > j$

$$\begin{aligned} &E[(\phi_{n+i} \odot_\theta \dots \odot_\theta \phi_2 \odot_\theta Z_{n+i})(\phi_{n+j} \odot_\theta \dots \odot_\theta \phi_2 \odot_\theta Z_{n+j})] \\ &= \phi^{2(n+j-1)}\lambda E(\phi_{i-j} \odot_\theta \dots \odot_\theta \phi_2 \odot_\theta Z_{n+i}) \\ &= \phi^{2n+i+j-2}\lambda^2. \end{aligned} \tag{2.9}$$

From (2.4), (2.7) and (2.9), we have

$$\begin{aligned} &E(|V_{n+m} - V_n|^2) \\ &= \sum_{k=1}^{m} \Big[a^{n+k-1}E(Z_1^2) + a^{n+k-2}b_1 + a^{n+k-3}b_2 + \dots + ab_{n+k-2} + b_{n+k-1}\Big] + 2\sum_{1 \leq j < i \leq m} \phi^{2n+i+j-2}\lambda^2 \\ &= \underbrace{\sum_{k=1}^{m} \Big[a^{n+k-1}E(Z_1^2) + a^{n+k-2}b\lambda + a^{n+k-3}b\phi\lambda + \dots + ab\phi^{n+k-3}\lambda + b\phi^{n+k-2}\lambda\Big]}_{(II)} + 2\sum_{1 \leq j < i \leq m} \phi^{2n+i+j-2}\lambda^2. \end{aligned}$$

Since $0 < a < 1$ and $0 < \phi < 1$, we know term (II) is $o(1)$ as $n \to \infty$. Moreover,

$$\sum_{m \geq i > j \geq 1} \phi^{2n+i+j-2}\lambda^2 = \phi^{2n-2}\lambda^2 \sum_{m \geq i > j \geq 1} \phi^{i+j} \to 0, (n \to \infty).$$

By the above arguments, for all $m > 0$

$$E(|V_{n+m} - V_n|^2) \to 0, \ as \ n \to \infty.$$

This ends the proof. □

3 Estimation and simulation

Assume that $X_1, \cdots, X_n$ are strictly stationary and ergodic solutions from model (2.1). The parameters of interest are ϕ, λ, θ and σ_ϕ^2, respectively. In this section, we mainly focus on the maximum likelihood method. The performances of corresponding maximum likelihood estimators (MLE) are evaluated via numerical simulation.

3.1 Maximum likelihood estimation

Now we study the maximum likelihood (ML) estimation method for the parameters of interest. First, we are required to specify the distribution for ϕ_t. In practice, a common choice for ϕ_t is the Beta(α, β) distribution over (0,1), where its density function is

$$f(\phi_1|\alpha,\beta) = \frac{1}{B(\alpha,\beta)}(\phi_1)^{\alpha-1}(1-\phi_1)^{\beta-1} \quad \textit{with} \quad B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx.$$

In the remainder, we assume ϕ_t follows from Beta (α, β) distribution with $\phi = \frac{\alpha}{\alpha+\beta}$ and $\sigma_\phi^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$. Let $\ell(\lambda, \theta, \alpha, \beta; x_1, \ldots, x_n) = \sum_{t=1}^{n-1} \ln P(x_t, x_{t+1}; \lambda, \theta, \alpha, \beta)$, where $P(x_t, x_{t+1}; \lambda, \theta, \alpha, \beta) = P(X_{t+1} = x_{t+1}|X_t = x_t)$ is the transition probability given in (2.2). The ML estimators $\hat{\lambda}$, $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$ are given by solving the following equations

$$\begin{cases} \frac{\partial \ell(\lambda,\theta,\alpha,\beta;x_1,\ldots,x_n)}{\partial \lambda} = 0, \\ \frac{\partial \ell(\lambda,\theta,\alpha,\beta;x_1,\ldots,x_n)}{\partial \theta} = 0, \\ \frac{\partial \ell(\lambda,\theta,\alpha,\beta;x_1,\ldots,x_n)}{\partial \alpha} = 0, \\ \frac{\partial \ell(\lambda,\theta,\alpha,\beta;x_1,\ldots,x_n)}{\partial \beta} = 0. \end{cases}$$

The ML estimators $\hat{\phi}$ and $\hat{\sigma}_\phi^2$ can be obtained by the "*plug-in*" method, where

$$\hat{\phi} \;=\; \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}, \text{ and } \hat{\sigma}_\phi^2 = \frac{\hat{\alpha}\hat{\beta}}{(\hat{\alpha}+\hat{\beta})^2(\hat{\alpha}+\hat{\beta}+1)}, \tag{3.1}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the ML estimators of α and β, respectively. From the view of practical application, we can employ the R optimization function `nlminb` to get the above-mentioned MLE.

3.2 Simulation study

In this section, we conduct some simulations to verify the rationality of our method. Consider the RCINAR-D(1) model

$$X_t = \phi_t \odot_\theta X_{t-1} + Z_t, \quad t \geq 1, \tag{3.2}$$

where $\{\phi_t\}$ is an i.i.d. random sequence generated from Beta (α, β) distribution, and $\{Z_t\}$ is an i.i.d. Poission sequence with mean λ. We generate $X_1, \cdots, X_n$ from model (3.2) with the help of `R` software. For the set up of θ and λ, we consider six mechanisms: (a) $\lambda = 2, \theta = 0.3, \alpha = 2, \beta = 2$; (b) $\lambda = 2, \theta = 0.6, \alpha = 2, \beta = 2$; (c) $\lambda = 2, \theta = 0.9, \alpha = 2, \beta = 2$; (d) $\lambda = 1, \theta = 0.3, \alpha = 5, \beta = 2$; (e) $\lambda = 1, \theta = 0.6, \alpha = 5, \beta = 2$; (f) $\lambda = 1, \theta = 0.9, \alpha = 5, \beta = 2$. In Figure 1, we present some sample paths of model (3.2). To evaluate the performance of parameter estimate, we report the estimated bias (BIAS) given by the sample mean of the estimate minus the true value, and the sampling standard error (SE) of the estimate in Tables 1 and 2. The values are showed with the format (BIAS, SE). For example, $(-0.0050, 0.0616)$ means that the BIAS is -0.0050, and the SE is 0.0616. All the simulation results are based on 1000 replications with sample sizes $n = 150$, 300 and 500, respectively.

From the results in Tables 1-2, we can conclude that the proposed ML estimation procedure performs well for the situations considered here. Specifically, the proposed estimator seems to be unbiased, and the performance becomes better as the sample size increases. Finally, we conduct the second simulation study to assess the performance of MLE for σ_ϕ^2. Based on (3.1), we can give the BIAS and SE of the ML estimator $\hat{\sigma}_\phi^2$ in Table 3. e.g. (0.009419, 0.001314) means that the BIAS is 0.009419 and SE is 0.001314. It can be seen from the results that the MLE $\hat{\sigma}_\phi^2$ is unbiased and its SE decreases as the sample size n becoming larger.

4 Application

We consider a real application of our proposed RCINAR-D(1) model to crime data, which are extracted from *http://www.forecastingprinciples.com/Crime/crime%20data.html*. The data set consists of 108 observations, starting in January 1990 and ending in December 1998 . Here we denote the time series data as $X_1, \cdots, X_{108}$. The plots of sample path, autocorrelation function (ACF) and partial autocorrelation function (PACF) are presented in Figure 2. From which we can see that X_t may come from an AR(1)-type process. For this crime data, the mean is 6.9813 and variance is 32.0562 (strong overdispersion). In Figure 3, we report the histogram of these crime data. Below, we consider four candidate models to fit this count time series data.

Model I. INAR(1) model (Al-Osh and Alzaid 1987)

$$X_t = \phi \circ X_{t-1} + Z_t,$$

where $\phi \in (0, 1)$, and $\phi \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} B_i$ with $P(B_i = 1) = 1 - P(B_i = 0) = \phi$; Z_t follows from the negative binomial distribution $NB(p, r)$ with $P(Z_t = k) = C_{k+r-1}^{r-1} p^r (1-p)^k$, $k = 0, 1, 2, 3, \ldots$. $E(Z_t) = \lambda = r(1-p)/p$, for $p \in (0, 1)$ and $r = 1, 2, 3, \ldots$.

Model II. RCINAR(1) model (Zheng et al. 2007)

$$X_t = \phi_t \circ X_{t-1} + Z_t,$$

where $\{\phi_t\}$ is from $Beta(\alpha, \beta)$ distribution; $\phi_t \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} B_{it}$ with $P(B_{it} = 1) = 1 - P(B_{it} = 0) = \phi_t$; Z_t follows from the negative binomial distribution $NB(p, r)$.

Model III. INAR-D(1) model (Ristić et al. 2013)

$$X_t = \phi \circ_\theta X_{t-1} + Z_t,$$

where $\phi \odot_\theta X_{t-1} = \sum_{i=1}^{X_{t-1}} U_i$, and $U_i = (1 - V_i)W_i + V_i\xi$; $\{V_i\}$ is a sequence of i.i.d. random variables with Bernoulli(θ) distribution, $\theta \in (0, 1)$; $\{W_i\}$ and $\{\xi\}$ are i.i.d. random variables with Bernoulli(ϕ) distribution; Z_t follows from the negative binomial distribution $NB(p, r)$.

Model IV. RCINAR-D(1) model (2.1)

$$X_t = \phi_t \odot_\theta X_{t-1} + Z_t,$$

where $\{\phi_t\}$ is from $Beta(\alpha, \beta)$ distribution, and Z_t follows from the negative binomial distribution $NB(p, r)$.

In Table 4, we report the MLE of model parameters, the standard error (SE) of MLE, the AIC (Akaike information criteria). The Pearson residuals (PR; Harvey and Fernandes, 1989) are defined as $R_t = \frac{X_t - \hat{E}(X_t|X_{t-1})}{\sqrt{V\hat{a}r(X_t|X_{t-1})}}$, where $\hat{E}(X_t|X_{t-1}) = \hat{\phi}X_{t-1} + \hat{\lambda}$, $V\hat{a}r(X_t|X_{t-1}) = \{[\hat{\phi}(1 - \hat{\phi}) - \hat{\sigma}_\phi^2]\hat{\theta}^2 + \hat{\sigma}_\phi^2\}X_{t-1}^2 + \{[\hat{\phi}(1 - \hat{\phi}) - \hat{\sigma}_\phi^2](1 - \hat{\theta}^2)\}X_{t-1} + \hat{\sigma}_Z^2$, and $\hat{\sigma}_Z^2 = \frac{\hat{r}(1-\hat{p})}{\hat{p}^2}$. We also give the mean of Pearson residuals (MPR) in Table 4. It can be seen from the results that our proposed RCINAR-D(1) model has the smallest AIC and absolute MPR. Specifically, the mean and variance of PR for our model are -0.0583 and 0.9651, respectively. To further check the model adequacy, we report the ACF of PR with the RCINAR-D(1) model in Figure 4. In a word, it may be reasonable to use the proposed RCINAR-D(1) model for the analysis of this crime data in practice.

5 Concluding remarks

In this article, we have proposed a first-order random coefficient integer-valued autoregressive process with dependent counting series. Some moments and stationary ergodicity of the proposed process were provided. For the parameter estimation, we used the ML method to estimate the parameters of interest. Some simulations and a real data application were provided to illustrate the usefulness of our method.

There exist several topics for future research by extending our proposed RCINAR-D(1) process. First, we can introduce a p-th random coefficient integer-valued autoregressive process with dependent counting series by extending our model to high-order case. Second, we can propose a RCINAR-D(1) model with zero-inflated innovations (Jazi et al. 2012). Third, the construction of bivariate INAR model has attracted much interest recently (Jowaheer et al. 2018), it is an interesting direction to study a bivariate random coefficient integer-valued autoregressive process with dependent counting series.

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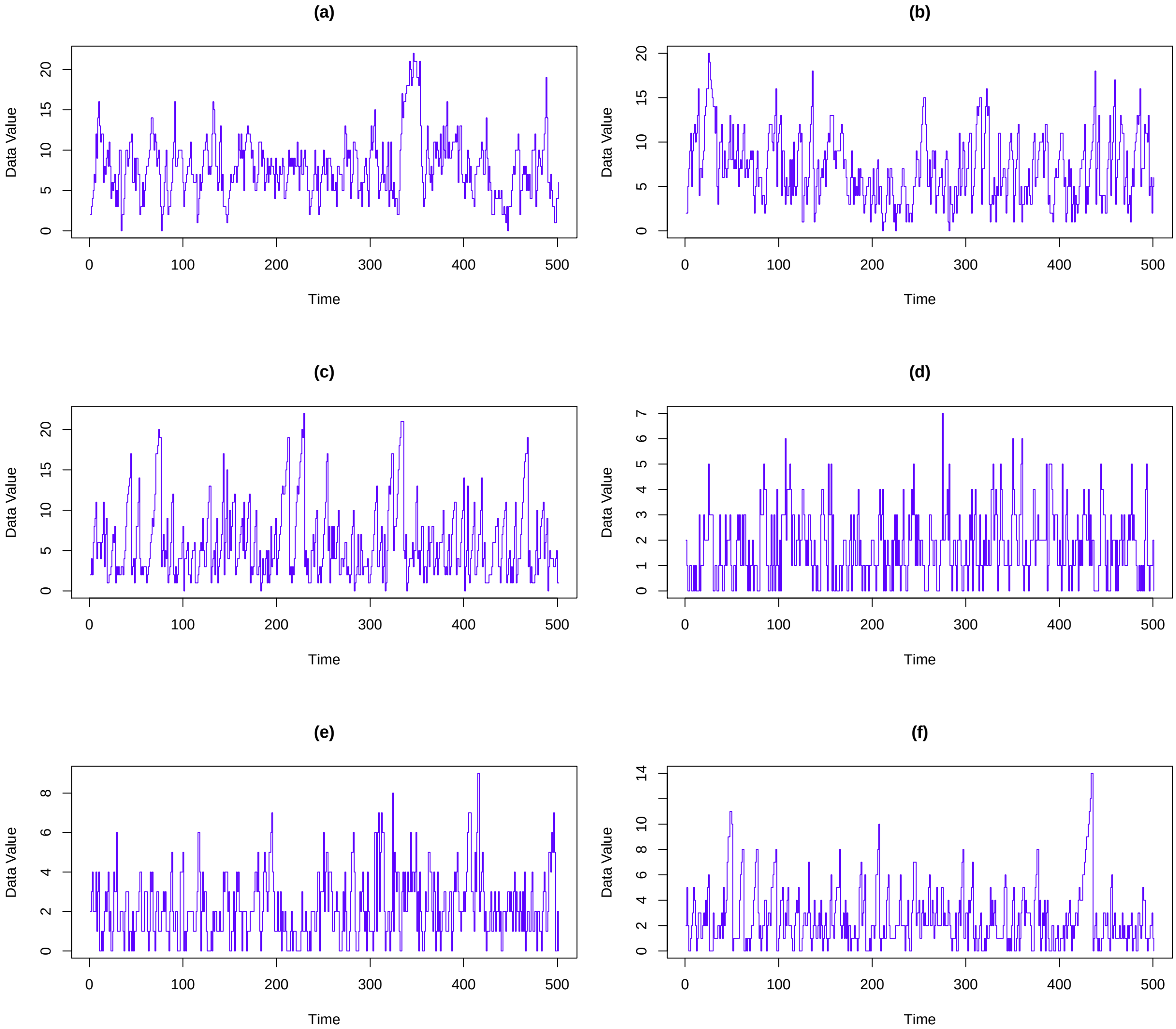

Figure 1. Sample path for (a) $\lambda = 2, \theta = 0.3, \alpha = 5, \beta = 2$; (b) $\lambda = 2, \theta = 0.6, \alpha = 5, \beta = 2$; (c) $\lambda = 2, \theta = 0.9, \alpha = 5, \beta = 2$; (d) $\lambda = 1, \theta = 0.3, \alpha = 2, \beta = 2$; (e) $\lambda = 1, \theta = 0.6, \alpha = 2, \beta = 2$; (f) $\lambda = 1, \theta = 0.9, \alpha = 2, \beta = 2$.

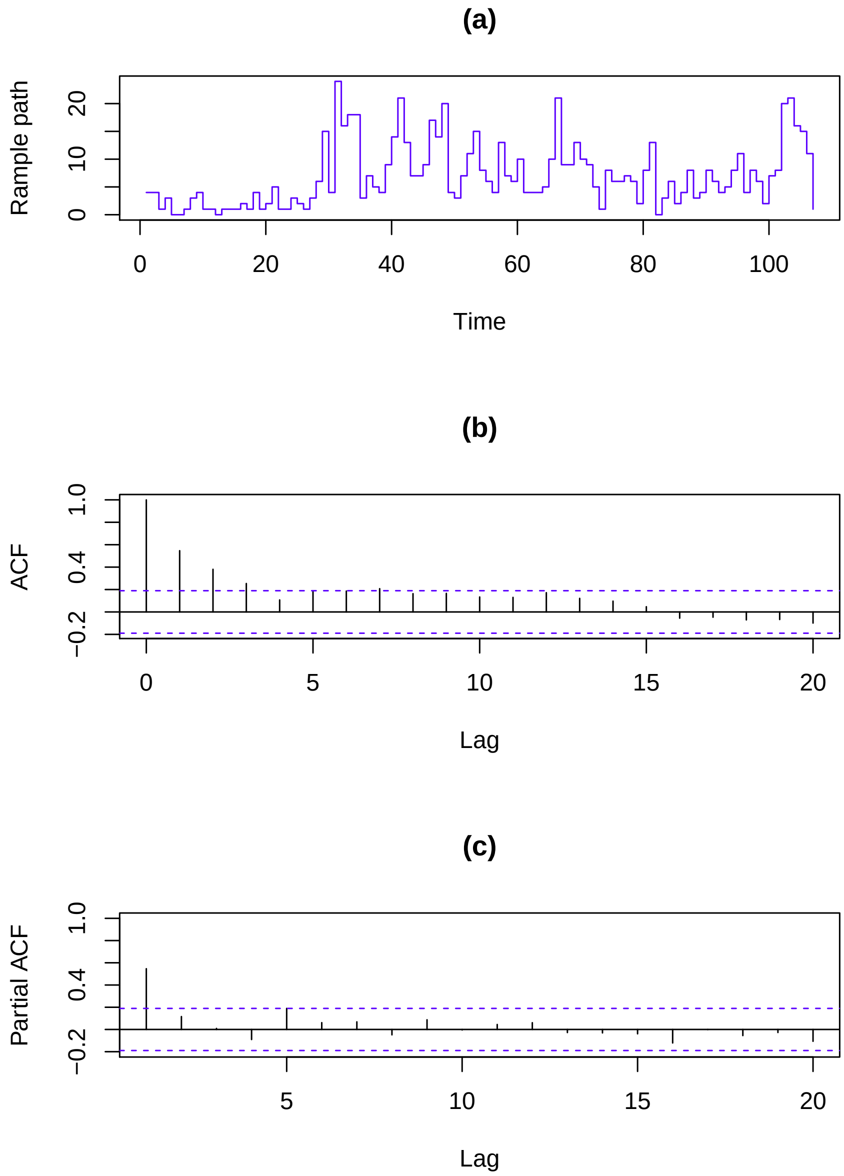

Figure 2. (a) Sample path for the crime data; (b) The ACF of crime data ; (c) The PACF of crime data.

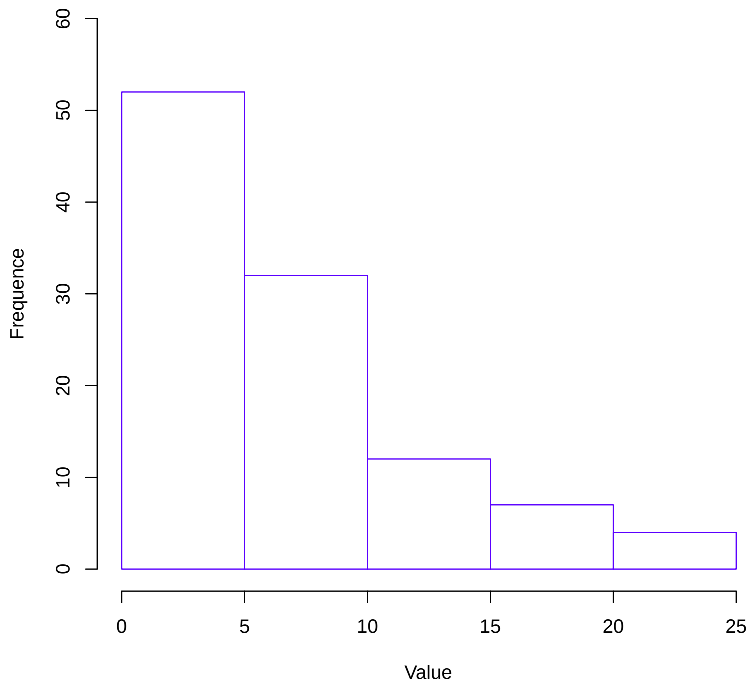

Figure 3. The histgram of crime data

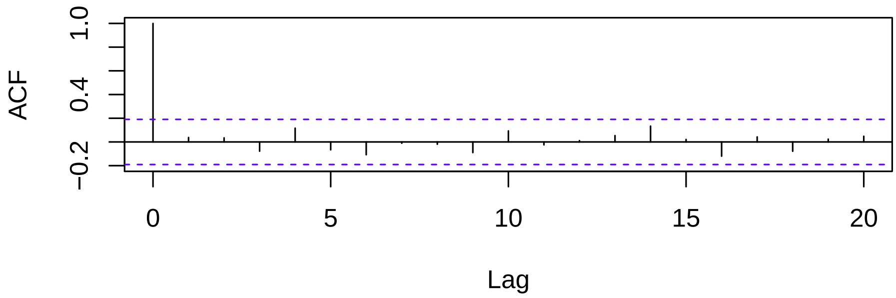

Figure 4. The ACF of R_t with RCINAR-D(1) model.

Table 1. Bias and SE of the MLE with $\alpha = 2$ and $\beta = 2$.

Parameters	Sample size	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\theta}$
$\lambda = 1, \theta = 0.3$	$n = 150$	$(0.0008, 0.0580)$	$(-0.0003, 0.1296)$	$(0.0049, 0.2200)$
	$n = 300$	$(-0.0042, 0.0433)$	$(0.0033, 0.0925)$	$(0.0195, 0.1763)$
	$n = 500$	$(-0.0009, 0.0344)$	$(-0.0007, 0.0692)$	$(0.0358, 0.1418)$
$\lambda = 1, \theta = 0.6$	$n = 150$	$(-0.0050, 0.0616)$	$(0.0132, 0.1194)$	$(-0.0154, 0.1679)$
	$n = 300$	$(-0.0029, 0.0449)$	$(0.0038, 0.0865)$	$(0.0186, 0.1064)$
	$n = 500$	$(-0.0051, 0.0349)$	$(0.0056, 0.0679)$	$(0.0186, 0.077)$
$\lambda = 1, \theta = 0.9$	$n = 150$	$(-0.0063, 0.0623)$	$(0.0044, 0.1058)$	$(-0.0043, 0.0887)$
	$n = 300$	$(-0.0038, 0.0442)$	$(0.0046, 0.0731)$	$(0.0048, 0.0580)$
	$n = 500$	$(-0.0002, 0.0344)$	$(0.0004, 0.0585)$	$(0.0050, 0.0405)$

Table 2. Bias and SE of the MLE with $\alpha = 5$ and $\beta = 2$.

Parameters	Sample size	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\theta}$
$\lambda = 2, \theta = 0.3$	$n = 150$	$(-0.0097, 0.0396)$	$(0.0532, 0.2432)$	$(-0.0002, 0.1191)$
	$n = 300$	$(-0.0048, 0.0270)$	$(0.0314, 0.1671)$	$(0.0183, 0.0791)$
	$n = 500$	$(-0.0041, 0.0211)$	$(0.0242, 0.1254)$	$(0.0284, 0.0611)$
$\lambda = 2, \theta = 0.6$	$n = 150$	$(-0.0035, 0.0385)$	$(0.0246, 0.1915)$	$(0.0105, 0.0562)$
	$n = 300$	$(-0.0038, 0.0282)$	$(0.0252, 0.1329)$	$(0.0095, 0.0398)$
	$n = 500$	$(-0.0040, 0.0211)$	$(0.0174, 0.1018)$	$(0.0112, 0.0298)$
$\lambda = 2, \theta = 0.9$	$n = 150$	$(-0.0044, 0.0417)$	$(0.0007, 0.1363)$	$(0.0049, 0.0346)$
	$n = 300$	$(-0.0026, 0.0309)$	$(0.0038, 0.0993)$	$(0.0045, 0.0230)$
	$n = 500$	$(-0.0019, 0.0232)$	$(0.0035, 0.0783)$	$(0.0025, 0.0187)$

Table 3. Bias and SE of the MLE for σ_ϕ^2 with $\theta = 0.9, \lambda = 2$.

Parameters	$n = 150$	$n = 300$	$n = 500$
$\alpha = 3, \beta = 6$	$(0.009419, 0.001314)$	$(0.009408, 0.000498)$	$(0.009452, 0.000237)$
$\alpha = 4, \beta = 6$	$(0.009850, 0.002774)$	$(0.009784, 0.001980)$	$(0.009683, 0.001446)$
$\alpha = 2, \beta = 9$	$(-0.002754, 0.004505)$	$(-0.003065, 0.003575)$	$(-0.003329, 0.002952)$
$\alpha = 5, \beta = 3$	$(-0.002899, 0.006084)$	$(-0.002635, 0.005203)$	$(-0.002581, 0.004826)$
$\alpha = 3, \beta = 8$	$(-0.002066, 0.005772)$	$(-0.002002, 0.004495)$	$(-0.001986, 0.003364)$

Table 4. Estimation results for the crime data‡.

Model	MLE	SE	AIC	MPR
INAR(1)	$\hat{\phi} = 0.2827$	0.0057		
	$\hat{p} = 0.2366$	0.0049		
	$\hat{\lambda} = 5.1657$	0.0608		
	$\hat{r} = 1.5028$	0.0337	597.98	0.1456
RCINAR(1)	$\hat{\alpha} = 2.6685$	0.0734		
	$\hat{\beta} = 2.6570$	0.0812		
	$\hat{p} = 0.2366$	0.0049		
	$\hat{r} = 1.1405$	0.0312		
	$\hat{\lambda} = 3.6797$	0.0481		
	$\hat{\sigma}_{\phi}^2 = 0.0395$	$(-)$	596.48	0.1008
INAR-D(1)	$\hat{\phi} = 0.4375$	0.0066		
	$\hat{p} = 0.2584$	0.0054		
	$\hat{r} = 1.3750$	0.0405		
	$\hat{\lambda} = 3.9453$	0.0537		
	$\hat{\theta} = 0.5302$	0.0080	586.93	0.1645
RCINAR-D(1)	$\hat{\alpha} = 1.1305$	0.0636		
	$\hat{\beta} = 1.2202$	0.0729		
	$\hat{p} = 0.2746$	0.0513		
	$\hat{r} = 1.4032$	0.0441		
	$\hat{\lambda} = 3.7065$	0.0513		
	$\hat{\theta} = 0.2137$	0.0128		
	$\hat{\sigma}_{\phi}^2 = 0.0745$	$(-)$	586.25	-0.0583

‡ The notation $(-)$ denotes corresponding term is not available.