LOCAL WELL-POSEDNESS AND SMALL DATA SCATTERING FOR ENERGY SUPER-CRITICAL NONLINEAR WAVE EQUATIONS

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Abstract. In this work, we consider the following nonlinear wave equations

$$\partial_{tt}u - \Delta u + |u|^p u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

We prove that when $p > \frac{4}{N-2}$ and

$$3 \le N \le 9; \quad \text{or} \quad N \ge 10,\, p < \frac{N^2 - 4N + 1 - \sqrt{N^4 - 8N^3 - 14N^2 + 56N - 31}}{4(N-1)}.$$

The Cauchy problem is locally well-posed in $\dot{H}^{s_c}(\mathbb{R}^N) \times \dot{H}^{s_c-1}(\mathbb{R}^N)$ with $s_c = \frac{N}{2} - \frac{2}{p}$. Moreover, the small data theory holds under the same restriction.

1. Introduction

In this paper, we consider the Cauchy problem for the following nonlinear wave equation (NLW)

$$\begin{cases} \partial_{tt}u - \Delta u + |u|^p u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ \big(u(x,0), u_t(x,0)\big) = \big(u_0(x), u_1(x)\big). \end{cases} \tag{1.1}$$

Here $p > 0$, $u : \mathbb{R}^N \to \mathbb{R}$ is an unknown function. This equation is invariant under the scaling

$$u(t,x) \to u_\lambda(t,x) = \frac{1}{\lambda^{\frac{2}{p+1}-1}} u(\frac{x}{\lambda}, \frac{t}{\lambda}).$$

When $s_c = \frac{N}{2} - \frac{2}{p}$, we have $\|(u(0), \partial_t u(0))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} = \|(u_\lambda(0), \partial_t u_\lambda(0))\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}}$, and thus the Cauchy problem (1.1) is called $\dot{H}^{s_c}$ critical. When p is higher/lower than $\frac{4}{N-2}$, we call the nonlinearity is energy super/sub-critical.

To start with, we recall some recent related for Cauchy problem (1.1). When $p > 0$, this equation is called defocusing, and researchers have obtained some developments on the related well-posedness theories. Firstly, in the energy sub-critical case, that is, $p < p_c = \frac{4}{N-2}$, Ginibre and Velo [5] proved that (NLW) has the unique solution in energy space. The authors used a compact method established by Lions [12]. Secondly, in the energy critical case, that is, $p = p_c = \frac{4}{N-2}$, Struwe [15] proved the global existence under the radial assumption, and then Grillakis [6] proved the global existence under the general condition, and Shatah and Struwe [13] later proved the same result in other dimensional spaces. Thirdly, to the

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energy super-critical case, that is, $p > p_c = \frac{4}{N-2}$, Kenig and Merle [8] proved the global well-posedness and scattering under the condition $u \in C_t(I; \dot{H}^{s_c} \times \dot{H}^{s_c-1})$, and I is the maximal lifespan. Besides, we can see some ill-posedness results, for example, [3].

In this paper, we concentrate on studying the local well-posedness and scattering theory with small initial data for nonlinear wave equations in energy super-critical situation, that is $p > \frac{4}{N-2}$. The similar research exists in the case of nonlinear Schrödinger equations,

$$\begin{cases} iu_t + \Delta u - |u|^p u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

In the work of Killip and Visan [10], the authors proved that when

$$\frac{4}{N-2} < p < \frac{N-2-\sqrt{(N-2)^2-32}}{4},$$

the energy super-critical nonlinear Schrödinger equations is locally well-posed. The restriction on p is caused by the lack of smoothness on the nonlinearity. It is equivalent to $s_c < p+1$. However, the Strichartz estimate on linear wave flow is much more complicated than the Schrödinger flow. So far, to our knowledge, only some particular cases were proved, for instance, $N = 3, 1 < s_c < \frac{3}{2}$ in [8]. This paper is aimed at the more general result.

Now we state our main result.

Theorem 1.1. *Let $p > \frac{4}{N-2}$, $s_c = \frac{N}{2} - \frac{2}{p}$, and $(u_0, u_1) \in \dot{H}^{s_c}(\mathbb{R}^N) \times \dot{H}^{s_c-1}(\mathbb{R}^N)$. Further, assume that*

$$p < \frac{N^2-4N+1-\sqrt{N^4-8N^3-14N^2+56N-31}}{4(N-1)} \qquad \text{when} \qquad N \geq 10, \tag{1.3}$$

then the Cauchy problem (1.1) exists unique solution with the maximal lifespan $u : I \times \mathbb{R}^N \to \mathbb{R}$, and the initial data is (u_0, u_1).

Moreover, there exists $\delta_0 > 0$, and if $\|u_0\|_{\dot{H}^{s_c}(\mathbb{R}^N)} + \|u_1\|_{\dot{H}^{s_c-1}(\mathbb{R}^N)} \leq \delta_0$, there exist functions pair $(u_\pm, v_\pm) \in \dot{H}^{s_c}(\mathbb{R}^N) \times \dot{H}^{s_c-1}(\mathbb{R}^N)$, such that when $t \to \pm\infty$,

$$\left\| \begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} - \begin{bmatrix} \cos\big((t-t_0)|\nabla|\big) & |\nabla|^{-1}\sin\big((t-t_0)|\nabla|\big) \\ -|\nabla|\sin\big((t-t_0)|\nabla|\big) & \cos\big((t-t_0)|\nabla|\big) \end{bmatrix} \begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} \right\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \to 0.$$

Remark 1.2. The result on the restriction on dimension N and parameter p also appear in the following energy super-critical nonlinear Schrödinger equation (1.2). In particular, Killip and Visan[10] considered local well-posedness for Cauchy problem (1.2) in space $\dot{H}^{s_c}$ ($s_c = \frac{N}{2} - \frac{2}{p}$) under the restriction of

$$s_c < p+1, \tag{1.4}$$

that is,

$$2p^2 - (N-2)p + 4 > 0. \tag{1.5}$$

When $3 \leq N \leq 7$, the condition (1.5) is always valid for any $p > 0$. That is, there is no restriction when $3 \leq N \leq 7$.

Compared with the wave equation, the condition (1.3) is equivalent to

$$\rho < p+1 \tag{1.6}$$

for some $\rho \in [s_c - 1, s_c)$ (more precisely, $\rho = s_c - 1$ when $p > 1$ and $\rho = s_c - \frac{1}{2} - \frac{1}{N-1}$). It is strictly weaker that the condition (1.4) to the nonlinear Schrödinger equation. In particular, when $3 \leq N \leq 9$, the condition (1.6) is always valid for any $p > 0$.

The key ingredients in our proofs are presented below.

(1) Three suitable working spaces are constructed. In order to establish the uniform estimation on time T, to the different discussing cases, we establish three related working spaces. We shall prove that the estimation of each norm in X_T is closed. The selection of norms plays a significant role in this paper.

(2) Applying Leibniz and chain rule for fractional derivatives particularly for Hölder continuous function.

The classic Leibniz chain rule for fractional derivatives has the requirement on the continuous property on function, that is, $G \in C^1$. However, when $G(u) = |u|^p$, $p < 1$, we can not estimate it using the classic method because $G \notin C^1$. So some special chain rule like

$$\Big\||\nabla|^s G(u)\Big\|_{L^q} \lesssim \Big\||u|^{p-\frac{s}{\sigma}}\Big\|_{L^{q_1}} \Big\||\nabla|^\sigma u\Big\|^{\frac{s}{\sigma}}_{L^{\frac{s}{\sigma}q_2}}$$

should be employed. Such kind of chain rule can be used to handle the situation $p < 1$, and allow us to choose suitable index to apply interpolation to control the inequality so that it can reach the closed expected estimation.

The rest of the paper is organized as follows. In Section 2, we give some basic notations and some preliminary estimates that will be used throughout in our paper. In Section 3, we prove local well-posedness and small data scattering for Cauchy problem (1.1) in $\dot{H}^{s_c}(\mathbb{R}^N) \times \dot{H}^{s_c-1}(\mathbb{R}^N)$ by applying the fixed point argument.

2. Notation and Preliminary

2.1. **Notation.** We write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. The notation $a+$ denotes $a+\varepsilon$ for any small ε, and also $a-\varepsilon$ for $a-$. Denote $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $D^\alpha = (-\partial_x^2)^{\frac{\alpha}{2}}$. The Hilbert space $H^s(\mathbb{R})$ is a Banach space of elements such that $\langle \xi \rangle^s \hat{u} \in L^2(\mathbb{R})$, where $\mathscr{F}$ denotes the Fourier transform $\mathscr{F}u(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} u(x)\,\mathrm{d}x$, and equipped with the norm $\|u\|_{H^s} = \|\langle\xi\rangle^s \hat{u}(\xi)\|_{L^2}$. The critical case for H^s is $\dot{H}^s$, and equipped with the norm $\|u\|_{H^s} = \Big\||\xi|^s \hat{u}(\xi)\Big\|_{L^2}$. An usual property of the Fourier transform is the Plancherel equality, that is, $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$. We also have an embedding theorem that $\|u\|_{H^{s_1}} \lesssim \|u\|_{H^{s_2}}$ for any $s_1 \leq s_2$, $s_1, s_2 \in \mathbb{R}$. Throughout the whole paper, the letter C will denote various positive constants which are of no importance in our analysis. We use the following norms to denote the mixed spaces $L_t^q L_x^r([0,T] \times \mathbb{R})$, that is,

$$\|u\|_{L_t^q L_x^r([0,T]\times\mathbb{R})} = \Big(\int_0^T \|u\|^q_{L_x^r(\mathbb{R})}\,\mathrm{d}t\Big)^{\frac{1}{q}}$$

2.2. **Preliminary.** In this section, we state some preliminary estimates. Firstly, we recall the well-known Strichartz estimates, see [7] for example.

Definition 2.1. *A pair (q, r) of positive real numbers is said to be wave admissible if*

$$2 \leq q \leq \infty, \quad 2 \leq r < \infty \quad \text{and} \quad \frac{1}{q} \leq \frac{N-1}{2}\Big(\frac{1}{2} - \frac{1}{r}\Big). \tag{2.1}$$

Proposition 2.2. *Take two admissible pairs* (q,r) *and* (a,b). *Then for any* $I \subset \mathbb{R}$,

$$
\begin{aligned}
&\|(u,\partial_t u)\|_{C_t(I;\dot H^s\times \dot H^{s-1})} + \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^N)} \\
\lesssim &\|(u(0),\partial_t u(0))\|_{\dot H^s\times\dot H^{s-1}} + \left\| |\nabla|^{\rho}(\partial_t^2 u - \Delta u)\right\|_{L_t^{a'}L_x^{b'}(I\times\mathbb{R}^N)},
\end{aligned}
\tag{2.2}
$$

whenever

$$
\frac{1}{q}+\frac{N}{r}=\frac{N}{2}-s=\frac{1}{a'}+\frac{N}{b'}-2-\rho. \tag{2.3}
$$

The next Lemmas are the different kinds of Leibniz and chain rule for fractional derivatives. One can check [2, 9, 11] for more details.

Lemma 2.3. *Let* $G \in C^1(\mathbb{R})$, $s\in(0,1)$, $1<p\le\infty$, *and* $1<p_1,p_2,p_3,p_4\le\infty$,

$$
\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2},\quad \frac{1}{p}=\frac{1}{p_3}+\frac{1}{p_4}.
$$

Then

$$
\left\|D^s(fg)\right\|_{L^p} \lesssim \left\|D^s f\right\|_{L^{p_1}}\|g\|_{L^{p_2}} + \left\|D^s g\right\|_{L^{p_3}}\|f\|_{L^{p_4}}. \tag{2.4}
$$

Lemma 2.4. *Let* G *be a Hölder continuous function of order* $0<p<1$. *Then, for every* $0<s<p$, $1<q<\infty$, *and* $\frac{s}{p}<\sigma<1$ *we have*

$$
\left\||\nabla|^s G(u)\right\|_{L^q} \lesssim \left\||u|^{p-\frac{s}{\sigma}}\right\|_{L^{q_1}} \left\||\nabla|^{\sigma}u\right\|^{\frac{s}{\sigma}}_{L^{\frac{s}{\sigma}q_2}}, \tag{2.5}
$$

provided $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$ *and* $(1-\frac{s}{p\sigma})q_1>1$.

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We only prove the result of small data scattering, and local well-posedness can be obtained in the same manner. To this end, we split into the following two cases:

Case 1: $p\ge1$; **Case 2:** $p<1$.

In the first case, we restrict that $s_c-1<p+1$; while in the second case, we restrict that $\rho<p+1$ with $\rho=s_c-\frac{1}{2}-\frac{1}{N-1}$. First of all, we shall prove that these conditions are equivalent to (1.3). For this purpose, we consider the two cases separately.

Case 1: $p\ge1, s_c-1<p+1$. Note that combining with the definition that $s_c=\frac{N}{2}-\frac{2}{p}$, the condition $s_c-1<p+1$ is equivalent to

$$
2p^2+(4-N)p+4>0.
$$

This inequality always holds when $3\le N\le 9$. When $N>9$, it reduces to $p<\frac{N-4-\sqrt{N^2-8N-16}}{4}<1$. The latter is against the condition $p\ge1$. Hence, in the case, the condition is equivalent to

$$
3\le N\le 9,\quad p\ge 1.
$$

Case 2: $p<1, \rho<p+1$. Note that the condition $\rho<p+1$ is equivalent to

$$
2(N-1)p^2-(N^2-4N+1)p+4(N-1)>0.
$$

This inequality always holds when $3 \le N \le 8$. When $N \ge 9$, it reduces to

$$p < \frac{N^2-4N+1-\sqrt{N^4-8N^3-14N^2+56N-31}}{4(N-1)}.$$

Combining with $p<1$, it reduces to

$$p<1, \text{when } N \le 9; \quad p < \frac{N^2-4N+1-\sqrt{N^4-8N^3-14N^2+56N-31}}{4(N-1)}, \text{when } N>9.$$

Together with the cases above, we get (1.3). Now we prove Theorem 1.1 in the following two cases.

3.1. **Case 1: $p \ge 1$, $s_c - 1 < p+1$.** To consider this case, we also need to split it into the following two subcases:

$$\textbf{Subcase 1: } N>3; \quad \textbf{Subcase 2: } N=3.$$

Subcase 1: $N>3$. We define our working space as

$$\|u\|_{X_T} = \|u\|_{L_t^\infty \dot H_x^{s_c}} + \left\| |\nabla|^{s_c-1} u \right\|_{L_t^2 L_x^{\frac{2N}{N-3}}} + \|u\|_{L_t^{2p} L_x^{\frac{2Np}{3}}}. \tag{3.1}$$

For the term $\|u\|_{L_t^\infty \dot H_x^{s_c}}$, by the Strichartz estimate (2.2), we have

$$\|u\|_{L_t^\infty \dot H_x^{s_c}} \lesssim \|u_0\|_{\dot H^{s_c}} + \|u_1\|_{\dot H^{s_c-1}} + \left\| |\nabla|^{s_c-1}(|u|^p u) \right\|_{L_t^1 L_x^2}.$$

Next, we estimate on $\left\| |\nabla|^{s_c-1}(|u|^p u) \right\|_{L_t^1 L_x^2}$. We split it into two cases.

1: $s_c - 1 < 1$. By (2.4), we get

$$\begin{aligned}\left\| |\nabla|^{s_c-1}(|u|^p u) \right\|_{L_t^1 L_x^2} \lesssim & \left\| \|u\|^p_{L_x^{\frac{2Np}{3}}} \cdot \left\| |\nabla|^{s_c-1} u \right\|_{L_x^{\frac{2N}{N-3}}} \right\|_{L_t^1} \\ \lesssim & \|u\|^p_{L_t^{2p} L_x^{\frac{2Np}{3}}} \left\| |\nabla|^{s_c-1} u \right\|_{L_t^2 L_x^{\frac{2N}{N-3}}} \lesssim \|u\|^{p+1}_{X_T}.\end{aligned}$$

2: $s_c - 1 \ge 1$. Let integer part of $s_c - 1$ equals to k, that is, $[s_c-1]=k$. Then we immediately have $k \ge 1$. Set $\Lambda_k = \left\{ \partial_x^\alpha : \alpha \in \mathbb{R}^N, |\alpha| = k \right\}$. Note that we have the formula

$$\partial_{x_i}(|u|^p u) = (p+1)|u|^p \partial_{x_i} u, \quad \partial_{x_i}(|u|^p) = p|u|^{p-2} u \partial_{x_i} u,$$

then by $k < p+1$ and Riesz transformation, there exists $C_\alpha > 0$, such that

$$\begin{aligned}\left\| |\nabla|^{s_c-1}(|u|^p u) \right\|_{L_t^1 L_x^2} \lesssim & \sum_{\partial_x^\alpha \in \Lambda_k} C_\alpha \left\| |\nabla|^{s_c-1-k} \partial_x^\alpha (|u|^p u) \right\|_{L_t^1 L_x^2} \\ \lesssim & \sum_{\partial_x^{\alpha_j} \in \Lambda_k, 1 \le j \le k} \left\| |\nabla|^{s_c-1-k} (\partial_x^{\alpha_1} u \cdots \partial_x^{\alpha_k} u \cdot O(|u|^{p-k+1})) \right\|_{L_t^1 L_x^2},\end{aligned}$$

and $O(|u|^{p-k+1}))$ equals to $|u|^{p-k+1}$ or $|u|^{p-k}u$. From (2.4), we have

$$
\begin{aligned}
&\Big\||\nabla|^{s_c-1}(|u|^p u)\Big\|_{L_t^1L_x^2}\\
\lesssim &\sum_{\partial_x^{\alpha_j}\in\Lambda_k,1\le j\le k}\Big\||\nabla|^{s_c-1-k}\partial_x^{\alpha_1}u\Big\|_{L_t^{p+1}L_x^{p_1}}\Big\|\partial_x^{\alpha_2}u\Big\|_{L_t^{p+1}L_x^{p_2}}\cdots\Big\|\partial_x^{\alpha_k}u\Big\|_{L_t^{p+1}L_x^{p_k}}\|u\|^{p-k+1}_{L_t^{p+1}L_x^{p_{k+1}}}\\
&+\sum_{\partial_x^{\alpha_j}\in\Lambda_k,1\le j\le k}\Big\|\partial_x^{\widetilde{\alpha_1}}u\Big\|_{L_t^{p+1}L_x^{r_1}}\cdots\Big\|\partial_x^{\widetilde{\alpha_k}}u\Big\|_{L_t^{p+1}L_x^{r_k}}\Big\||\nabla|^{s_c-1-k}O(|u|^{p-k+1})\Big\|_{L_t^{\frac{p+1}{p-k+1}}L_x^{r_{k+1}}},
\end{aligned}
\tag{3.2}
$$

and the index satisfy

$$
\begin{aligned}
&\frac{1}{p+1}+\frac{N}{p_1}-(s_c-1-k+|\alpha_1|)=\frac{2}{p};\\
&\frac{1}{p+1}+\frac{N}{p_j}-|\alpha_j|=\frac{2}{p},\quad j=2,\cdots,k;\\
&\frac{1}{p+1}+\frac{N}{p_{k+1}}=\frac{2}{p};
\end{aligned}
$$

and

$$
\frac{1}{p+1}+\frac{N}{r_j}-|\widetilde{\alpha}_j|=\frac{2}{p},\quad j=1,\cdots,k;\qquad \frac{p-k+1}{p+1}+\frac{N}{r_{k+1}}=\frac{2}{p}(p-k+1).
$$

By interpolation, there exists $\theta_1\in(0,1)$, such that

$$
\Big\||\nabla|^{s_c-1-k}\partial_x^{\alpha_1}u\Big\|_{L_t^{p+1}L_x^{p_1}}\lesssim\Big\||\nabla|^{s_c-1}u\Big\|^{\theta_1}_{L_t^2L_x^{\frac{2N}{N-3}}}\|u\|^{1-\theta_1}_{L_t^{2p}L_x^{\frac{2Np}{3}}}\lesssim\|u\|_{X_T}.
$$

Similar interpolation as above, there exists $\theta_j\in(0,1), j=2,\cdots,k,\theta_{k+1}\in(0,1)$, such that

$$
\begin{aligned}
\Big\|\partial_x^{\alpha_j}u\Big\|_{L_t^{p+1}L_x^{p_j}}&\lesssim\Big\||\nabla|^{s_c-1}u\Big\|^{\theta_j}_{L_t^2L_x^{\frac{2N}{N-3}}}\|u\|^{1-\theta_j}_{L_t^{2p}L_x^{\frac{2Np}{3}}}\lesssim\|u\|_{X_T},\quad j=2,\cdots,k;\\
\Big\|u\Big\|_{L_t^{p+1}L_x^{p_{k+1}}}&\lesssim\Big\||\nabla|^{s_c-1}u\Big\|^{\theta_{k+1}}_{L_t^2L_x^{\frac{2N}{N-3}}}\|u\|^{1-\theta_{k+1}}_{L_t^\infty\dot H_x^{s_c}}\lesssim\|u\|_{X_T};\\
\Big\|\partial_x^{\widetilde{\alpha_j}}u\Big\|_{L_t^{p+1}L_x^{r_j}}&\lesssim\Big\||\nabla|^{s_c-1}u\Big\|^{\tau_j}_{L_t^2L_x^{\frac{2N}{N-3}}}\|u\|^{1-\tau_j}_{L_t^{2p}L_x^{\frac{2Np}{3}}}\lesssim\|u\|_{X_T},\quad j=1,\cdots,k.
\end{aligned}
$$

Next we analyse $\Big\||\nabla|^{s_c-1-k}(|u|^{p-k+1})\Big\|_{L_t^{\frac{p+1}{p-k+1}}L_x^{r_{k+1}}}$ in (3.2). By (2.5), we get

$$
\Big\||\nabla|^{s_c-1-k}(|u|^{p-k+1})\Big\|_{L_t^{\frac{p+1}{p-k+1}}L_x^{r_{k+1}}}\lesssim\Big\|u\Big\|^{p-s_c+2-}_{L_t^{p+1}L_x^{r_{k+1,1}}}\cdot\Big\||\nabla|^{1-}u\Big\|^{s_c-1-k+}_{L_t^{p+1}L_x^{r_{k+1,2}}},
$$

and $r_{k+1,1},r_{k+1,2}$ satisfy

$$
\frac{1}{p+1}+\frac{N}{r_{k+1,1}}=\frac{2}{p};\quad \frac{1}{p+1}+\frac{N}{r_{k+1,2}}-1+=\frac{2}{p}.
$$

Using interpolation again, we have

$$
\Big\||\nabla|^{s_c-1-k}(|u|^{p-k+1})\Big\|_{L_t^{\frac{p+1}{p-k+1}}L_x^{r_{k+1}}}\lesssim\|u\|^{p-k+1}_{X_T}.
$$

Finally, we obtain

$$\left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2} \lesssim \|u\|_{X_T}^{p+1}.$$

To the term $\left\||\nabla|^{s_c-1}u\right\|_{L_t^2 L_x^{\frac{2N}{N-3}}}$ in (3.1), choosing the parameters $(a', b') = (1, 2)$, $(s, \rho) = (1, 0)$ in Strichartz's estimate (2.2), we have

$$\left\||\nabla|^{s_c-1}u\right\|_{L_t^2 L_x^{\frac{2N}{N-3}}} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2}.$$

To the term $\|u\|_{L_t^{2p} L_x^{\frac{2Np}{3}}}$ in (3.1), choosing the parameters $(a', b') = (1, 2)$, $(s, \rho) = (s_c, s_c - 1)$ in Strichartz's estimate (2.2), we have

$$\|u\|_{L_t^{2p} L_x^{\frac{2Np}{3}}} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2}.$$

We can obtain the same analyze as before. Thus we finish the proof of the first case.

Subcase 2: $N = 3$. In this subcase, we define our working space as below (use the same notation but different meaning)

$$\|u\|_{X_T} = \|u\|_{L_t^\infty \dot{H}_x^{s_c}} + \left\||\nabla|^{s_c-1}u\right\|_{L_t^{2+} L_x^{\infty-}} + \|u\|_{L_t^{2p-} L_x^{2p+}}. \tag{3.3}$$

To the term $\|u\|_{L_t^\infty \dot{H}_x^{s_c}}$, by Strichartz's estimate (2.2), we have

$$\|u\|_{L_t^\infty \dot{H}_x^{s_c}} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2}.$$

Next we estimate on $\left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2}$. We split it into two cases again.

1: $s_c - 1 < 1$. Using (2.4), we get

$$\begin{aligned}
\left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2} \lesssim & \left\| \|u\|_{L_x^{2p+}}^p \cdot \left\||\nabla|^{s_c-1}u\right\|_{L_x^{\infty-}} \right\|_{L_t^1} \\
\lesssim & \|u\|_{L_t^{2p-} L_x^{2p+}}^p \left\||\nabla|^{s_c-1}u\right\|_{L_t^{2+} L_x^{\infty-}} \\
\lesssim & \|u\|_{X_T}^{p+1}.
\end{aligned}$$

2: $s_c - 1 \geq 1$. Suppose $[s_c - 1] = \tilde{k}$, then $\tilde{k} \geq 1$. From $\tilde{k} < p + 1$ and Riesz transformation, there exists a constant $C_\beta > 0$, such that

$$\begin{aligned}
\left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2} \lesssim & \sum_{\partial_x^\beta \in \Lambda_{\tilde{k}}} C_\beta \left\||\nabla|^{s_c-1-\tilde{k}} \partial_x^\beta (|u|^p u)\right\|_{L_t^1 L_x^2} \\
\lesssim & \sum_{\partial_x^{\beta_j} \in \Lambda_{\tilde{k}}, 1 \leq j \leq \tilde{k}} \left\||\nabla|^{s_c-1-\tilde{k}} (\partial_x^{\beta_1} u \cdots \partial_x^{\beta_{\tilde{k}}} u \cdot O(|u|^{p-\tilde{k}+1}))\right\|_{L_t^1 L_x^2}.
\end{aligned}$$

By (2.4), we get

$$\begin{aligned}
&\left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2}\\
\lesssim& \sum_{\partial_x^{\beta_j}\in\Lambda_{\tilde{k}},1\le j\le \tilde{k}} \left\||\nabla|^{s_c-1-\tilde{k}}\partial_x^{\beta_1}u\right\|_{L_t^{p+1}L_x^{\tilde{p}_1}} \left\|\partial_x^{\beta_2}u\right\|_{L_t^{p+1}L_x^{\tilde{p}_2}} \cdots \left\|\partial_x^{\beta_{\tilde{k}}}u\right\|_{L_t^{p+1}L_x^{\tilde{p}_{\tilde{k}}}} \|u\|^{p-\tilde{k}+1}_{L_t^{p+1}L_x^{\tilde{p}_{\tilde{k}+1}}}\\
&+\sum_{\partial_x^{\beta_j}\in\Lambda_{\tilde{k}},1\le j\le \tilde{k}} \left\|\partial_x^{\widetilde{\beta_1}}u\right\|_{L_t^{p+1}L_x^{\tilde{r}_1}} \cdots \left\|\partial_x^{\widetilde{\beta_{\tilde{k}}}}u\right\|_{L_t^{p+1}L_x^{\tilde{r}_{\tilde{k}}}} \left\||\nabla|^{s_c-1-\tilde{k}}O(|u|^{p-\tilde{k}+1})\right\|_{L_t^{\frac{p+1}{p-\tilde{k}+1}}L_x^{\tilde{r}_{\tilde{k}+1}}},
\end{aligned}$$

and the index satisfy

$$\begin{aligned}
&\frac{1}{p+1}+\frac{3}{\tilde{p}_1}-(s_c-1-\tilde{k}+|\beta_1|)=\frac{2}{p};\\
&\frac{1}{p+1}+\frac{3}{\tilde{p}_j}-|\beta_j|=\frac{2}{p},\quad j=2,\cdots,\tilde{k};\\
&\frac{1}{p+1}+\frac{3}{\tilde{p}_{\tilde{k}+1}}=\frac{2}{p};
\end{aligned}$$

and

$$\frac{1}{p+1}+\frac{3}{\tilde{r}_j}-|\widetilde{\beta_j}|=\frac{2}{p},\quad j=1,\cdots,\tilde{k};\qquad \frac{p-\tilde{k}+1}{p+1}+\frac{3}{\tilde{r}_{\tilde{k}+1}}=\frac{2}{p}(p-\tilde{k}+1).$$

Using interpolation, there exists $\tilde{\theta}_j,\tilde{\tau}_j\in(0,1), j=1,\cdots,\tilde{k}$, $\tilde{\theta}_{\tilde{k}+1}\in(0,1)$, such that

$$\begin{aligned}
\left\||\nabla|^{s_c-1-\tilde{k}}\partial_x^{\beta_1}u\right\|_{L_t^{p+1}L_x^{\tilde{p}_1}} &\lesssim \left\||\nabla|^{s_c-1}u\right\|^{\tilde{\theta}_1}_{L_t^{2+}L_x^{\infty-}} \|u\|^{1-\tilde{\theta}_1}_{L_t^{2p-}L_x^{2p+}} \lesssim \|u\|_{X_T};\\
\left\|\partial_x^{\beta_j}u\right\|_{L_t^{p+1}L_x^{\tilde{p}_j}} &\lesssim \left\||\nabla|^{s_c-1}u\right\|^{\tilde{\theta}_j}_{L_t^{2+}L_x^{\infty-}} \|u\|^{1-\tilde{\theta}_j}_{L_t^{2p-}L_x^{2p+}} \lesssim \|u\|_{X_T},\quad j=2,\cdots,\tilde{k};\\
\|u\|_{L_t^{p+1}L_x^{\tilde{p}_{k+1}}} &\lesssim \left\||\nabla|^{s_c-1}u\right\|^{\theta_{\tilde{k}+1}}_{L_t^{2+}L_x^{\infty-}} \|u\|^{1-\theta_{\tilde{k}+1}}_{L_t^{\infty}\dot{H}_x^{s_c}} \lesssim \|u\|_{X_T};\\
\left\|\partial_x^{\widetilde{\beta_j}}u\right\|_{L_t^{p+1}L_x^{\tilde{r}_j}} &\lesssim \left\||\nabla|^{s_c-1}u\right\|^{\tilde{\tau}_j}_{L_t^{2+}L_x^{\infty-}} \|u\|^{1-\tilde{\tau}_j}_{L_t^{2p-}L_x^{2p+}} \lesssim \|u\|_{X_T},\quad j=1,\cdots,\tilde{k}.
\end{aligned}$$

Next we analyse $\left\||\nabla|^{s_c-1-\tilde{k}}(|u|^{p-\tilde{k}+1})\right\|_{L_t^{\frac{p+1}{p-\tilde{k}+1}}L_x^{\tilde{r}_{\tilde{k}+1}}}$. Using (2.5), we get

$$\left\||\nabla|^{s_c-1-\tilde{k}}(|u|^{p-\tilde{k}+1})\right\|_{L_t^{\frac{p+1}{p-\tilde{k}+1}}L_x^{\tilde{r}_{\tilde{k}+1}}} \lesssim \|u\|^{p-s_c+2-}_{L_t^{p+1}L_x^{\tilde{r}_{\tilde{k}+1,1}}}\cdot \left\||\nabla|^{1-}u\right\|^{s_c-1-\tilde{k}+}_{L_t^{p+1}L_x^{\tilde{r}_{\tilde{k}+1,2}}},$$

and $\tilde{r}_{\tilde{k}+1,1},\tilde{r}_{\tilde{k}+1,2}$ satisfy

$$\frac{1}{p+1}+\frac{3}{\tilde{r}_{\tilde{k}+1,1}}=\frac{2}{p};\quad \frac{1}{p+1}+\frac{3}{\tilde{r}_{\tilde{k}+1,2}}-1+=\frac{2}{p}.$$

Using interpolation again, we have

$$\left\||\nabla|^{s_c-1-\tilde{k}}(|u|^{p-\tilde{k}+1})\right\|_{L_t^{\frac{p+1}{p-\tilde{k}+1}}L_x^{\tilde{r}_{\tilde{k}+1}}} \lesssim \|u\|^{p-\tilde{k}+1}_{X_T}.$$

Finally, we obtain

$$\left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2} \lesssim \|u\|_{X_T}^{p+1}.$$

To the term $\left\||\nabla|^{s_c-1}u\right\|_{L_t^{2+}L_x^{\infty-}}$ in (3.3), set $(a', b') = (1, 2)$, $(s, \rho) = (1, 0)$ in (2.3), by strichartz's estimate (2.2), we have

$$\left\||\nabla|^{s_c-1}u\right\|_{L_t^{2+}L_x^{\infty-}} \lesssim \|u_0\|_{\dot H^{s_c}} + \|u_1\|_{\dot H^{s_c-1}} + \left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2}.$$

To the term $\|u\|_{L_t^{2p-}L_x^{2p+}}$, set $(a', b') = (1, 2)$, $(s, \rho) = (s_c, s_c - 1)$ in (2.3), by strichartz's estimate (2.2), we have

$$\|u\|_{L_t^{2p-}L_x^{2p+}} \lesssim \|u_0\|_{\dot H^{s_c}} + \|u_1\|_{\dot H^{s_c-1}} + \left\||\nabla|^{s_c-1}(|u|^p u)\right\|_{L_t^1 L_x^2},$$

Using the same argument as above, we get our desirable result.

3.2. **Case 2: $p < 1$, $\rho < p + 1$.** Note that in this case, it should be $N > 6$. We also use the same notation below to define our working space

$$\|u\|_{X_T} = \|u\|_{L_t^\infty \dot H_x^{s_c}} + \left\||\nabla|^\rho u\right\|_{L_t^2 L_x^{\frac{2(N-1)}{N-3}}} + \|u\|_{L_t^2 L_x^{\frac{2Np}{4-p}}}. \tag{3.4}$$

To the term $\|u\|_{L_t^\infty \dot H_x^{s_c}}$, let $P = \frac{2N(N-1)}{N(N-3)+(4-p)(N-1)}$, and by Strichartz's estimate (2.2), we have

$$\|u\|_{L_t^\infty \dot H_x^{s_c}} \lesssim \|u_0\|_{\dot H^{s_c}} + \|u_1\|_{\dot H^{s_c-1}} + \left\||\nabla|^\rho(|u|^p u)\right\|_{L_t^{\frac{2}{p+1}} L_x^P}.$$

Next we estimate on $\left\||\nabla|^\rho(|u|^p u)\right\|_{L_t^{\frac{2}{p+1}} L_x^P}$. We split it into two cases again.

1: $\rho < 1$. Using (2.4), we get

$$\begin{aligned}\left\||\nabla|^\rho(|u|^p u)\right\|_{L_t^{\frac{2}{p+1}} L_x^P} &\lesssim \left\| \|u\|_{L_x^{\frac{2Np}{4-p}}}^p \cdot \left\||\nabla|^\rho u\right\|_{L_x^{\frac{2(N-1)}{N-3}}} \right\|_{L_t^{\frac{2}{p+1}}} \\ &\lesssim \|u\|_{L_t^2 L_x^{\frac{2Np}{4-p}}}^p \left\||\nabla|^\rho u\right\|_{L_t^2 L_x^{\frac{2(N-1)}{N-3}}} \lesssim \|u\|_{X_T}^{p+1}.\end{aligned}$$

2: $\rho \geq 1$. Suppose $[\rho] = l$, then $l \geq 1$. From $l < p + 1$ and Riesz transformation, there exists a constant $C_\gamma > 0$, such that

$$\begin{aligned}\left\||\nabla|^\rho(|u|^p u)\right\|_{L_t^{\frac{2}{p+1}} L_x^P} &\lesssim \sum_{\partial_x^\gamma \in \Lambda_l} C_\gamma \left\||\nabla|^{\rho-l}\partial_x^\gamma(|u|^p u)\right\|_{L_t^{\frac{2}{p+1}} L_x^P} \\ &\lesssim \sum_{\partial_x^{\gamma_j} \in \Lambda_l, 1\leq j\leq l} \left\||\nabla|^{\rho-l}(\partial_x^{\gamma_1}u \cdots \partial_x^{\gamma_l}u \cdot O(|u|^{p-l+1}))\right\|_{L_t^{\frac{2}{p+1}} L_x^P}.\end{aligned}$$

By (2.4), we get

$$\begin{aligned}
&\Big\||\nabla|^\rho(|u|^p u)\Big\|_{L_t^{\frac{2}{p+1}}L_x^P}\\
\lesssim &\sum_{\partial_x^{\gamma_j}\in\Lambda_l,1\le j\le l}\Big\||\nabla|^{\rho-l}\partial_x^{\gamma_1}u\Big\|_{L_t^2L_x^{q_1}}\Big\|\partial_x^{\gamma_2}u\Big\|_{L_t^2L_x^{q_2}}\cdots\Big\|\partial_x^{\gamma_l}u\Big\|_{L_t^2L_x^{q_l}}\|u\|_{L_t^2L_x^{q_{l+1}}}^{p-l+1}\\
&+\sum_{\partial_x^{\gamma_j}\in\Lambda_l,1\le j\le l}\Big\|\partial_x^{\gamma_1}u\Big\|_{L_t^2L_x^{h_1}}\cdots\Big\|\partial_x^{\gamma_l}u\Big\|_{L_t^2L_x^{h_l}}\Big\||\nabla|^{\rho-l}(|u|^{p-l+1})\Big\|_{L_t^{\frac{2}{p-l+1}}L_x^{h_{l+1}}},
\end{aligned}\tag{3.5}$$

and the index satisfy

$$\begin{aligned}
&\frac{1}{2}+\frac{N}{q_1}-(\rho-l+|\gamma_1|)=\frac{2}{p};\\
&\frac{1}{2}+\frac{N}{q_j}-|\gamma_j|=\frac{2}{p},\quad j=2,\cdots,l;\\
&\frac{1}{2}+\frac{N}{q_{l+1}}=\frac{2}{p};
\end{aligned}$$

and

$$\frac{1}{2}+\frac{N}{h_j}-|\gamma_j|=\frac{2}{p},\quad j=1,\cdots,k;\qquad \frac{p-l+1}{2}+\frac{N}{h_{l+1}}=\frac{2}{p}(p-l+1).$$

Using interpolation, there exists $\tilde{\tilde{\theta}}_j,\tilde{\tilde{\tau}}_j\in(0,1), j=1,\cdots,l,\ \tilde{\tilde{\theta}}_{l+1}\in(0,1)$, such that

$$\begin{aligned}
\Big\||\nabla|^{\rho-l}\partial_x^{\gamma_1}u\Big\|_{L_t^2L_x^{q_1}}&\lesssim\Big\||\nabla|^\rho u\Big\|_{L_t^2L_x^{\frac{2(N-1)}{N-3}}}^{\tilde{\tilde{\theta}}_1}\|u\|_{L_t^2L_x^{\frac{2Np}{4-p}}}^{1-\tilde{\tilde{\theta}}_1}\lesssim\|u\|_{X_T};\\
\Big\|\partial_x^{\gamma_j}u\Big\|_{L_t^2L_x^{q_j}}&\lesssim\Big\||\nabla|^\rho u\Big\|_{L_t^2L_x^{\frac{2(N-1)}{N-3}}}^{\tilde{\tilde{\theta}}_j}\|u\|_{L_t^2L_x^{\frac{2Np}{4-p}}}^{1-\tilde{\tilde{\theta}}_j}\lesssim\|u\|_{X_T},\quad j=2,\cdots,l;\\
\|u\|_{L_t^2L_x^{q_{l+1}}}&\lesssim\Big\||\nabla|^\rho u\Big\|_{L_t^2L_x^{\frac{2(N-1)}{N-3}}}^{\tilde{\tilde{\theta}}_{l+1}}\|u\|_{L_t^\infty\dot{H}_x^{s_c}}^{1-\tilde{\tilde{\theta}}_{l+1}}\lesssim\|u\|_{X_T};\\
\Big\|\partial_x^{\gamma_j}u\Big\|_{L_t^2L_x^{h_j}}&\lesssim\Big\||\nabla|^\rho u\Big\|_{L_t^2L_x^{\frac{2(N-1)}{N-3}}}^{\tilde{\tilde{\tau}}_j}\|u\|_{L_t^2L_x^{\frac{2Np}{4-p}}}^{1-\tilde{\tilde{\tau}}_j}\lesssim\|u\|_{X_T},\quad j=1,\cdots,k.
\end{aligned}$$

Next we analyse $\Big\||\nabla|^{\rho-l}(|u|^{p-l+1})\Big\|_{L_t^{\frac{2}{p-l+1}}L_x^{h_{l+1}}}$ in (3.5). By (2.5), we have

$$\Big\||\nabla|^{\rho-l}(|u|^{p-l+1})\Big\|_{L_t^{\frac{2}{p-l+1}}L_x^{h_{l+1}}}\lesssim\|u\|_{L_t^2L_x^{h_{l+1,1}}}^{p-\rho+1-}\cdot\Big\||\nabla|^{1-}u\Big\|_{L_t^2L_x^{h_{l+1,2}}}^{\rho-l+},$$

and $h_{l+1,1},h_{l+1,2}$ satisfy

$$\frac{1}{2}+\frac{N}{h_{l+1,1}}=\frac{2}{p};\quad \frac{1}{2}+\frac{N}{h_{l+1,2}}-1+=\frac{2}{p}.$$

Using interpolation again, we get

$$\Big\||\nabla|^{\rho-l}(|u|^{p-l+1})\Big\|_{L_t^{\frac{2}{p-l+1}}L_x^{h_{l+1}}}\lesssim\|u\|_{X_T}^{p-l+1}.$$

Finally, we get

$$\left\| |\nabla|^{\rho}(|u|^{p}u) \right\|_{L_t^{\frac{2}{p+1}} L_x^{P}} \lesssim \|u\|_{X_T}^{p+1}.$$

To the term $\left\| |\nabla|^{\rho} u \right\|_{L_t^2 L_x^{\frac{2(N-1)}{N-3}}}$ in (3.4), set $(a', b') = (\frac{2}{p+1}, P)$, $(s, \rho) = (\frac{N+1}{2(N-1)}, -\frac{N+1}{2(N-1)})$ in (2.3). By Strichartz's estimate (2.2), we have

$$\left\| |\nabla|^{\rho} u \right\|_{L_t^2 L_x^{\frac{2(N-1)}{N-3}}} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \left\| |\nabla|^{\rho}(|u|^{p}u) \right\|_{L_t^{\frac{2}{p+1}} L_x^{P}}.$$

To the term $\|u\|_{L_t^2 L_x^{\frac{2Np}{4-p}}}$, set $(a', b') = (\frac{2}{p+1}, P)$, $(s, \rho) = (s_c, s_c - \frac{N+1}{2(N-1)})$ in (2.3). By Strichartz's estimate (2.2), we have

$$\|u\|_{L_t^2 L_x^{\frac{2Np}{4-p}}} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \left\| |\nabla|^{\rho}(|u|^{p}u) \right\|_{L_t^{\frac{2}{p+1}} L_x^{P}}.$$

Then we obtain the same analyze as above and finish the proof of this section.

Combining the three cases above, we get

$$\|u\|_{X_T} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}} + \|u\|_{X_T}^{p+1},$$

uniformly on T. Therefore, we have $\|u\|_{X_\infty} \lesssim \|u_0\|_{\dot{H}^{s_c}} + \|u_1\|_{\dot{H}^{s_c-1}}$.

In particular, we prove the Strichartz estimates below.
When $p \geq 1$, $3 < N \leq 9$,

$$\left\| |\nabla|^{s_c-1} u \right\|_{L_t^2 L_x^{\frac{2N}{N-3}}} + \|u\|_{L_t^{2p} L_x^{\frac{2Np}{3}}} < \infty;$$

when $p \geq 1$, $N = 3$,

$$\left\| |\nabla|^{s_c-1} u \right\|_{L_t^{2+} L_x^{\infty-}} + \|u\|_{L_t^{2p-} L_x^{2p+}} < \infty;$$

when $p < 1, \rho < p + 1$,

$$\left\| |\nabla|^{\rho} u \right\|_{L_t^2 L_x^{\frac{2(N-1)}{N-3}}} + \|u\|_{L_t^2 L_x^{\frac{2Np}{4-p}}} < \infty.$$

So we choose scattering state as

$$\begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \int_0^{+\infty} \begin{bmatrix} |\nabla|^{-1} \sin\big(-s|\nabla|\big) \\ \cos\big(-s|\nabla|\big) \end{bmatrix} \big(|u(s)|^{p} u(s)\big)\, \mathrm{d}s.$$

From the Strichartz estimate above (the standard process can be found in [1]), we have that when $t \to \pm\infty$,

$$\left\| \begin{bmatrix} u(t) \\ u_t(t) \end{bmatrix} - \begin{bmatrix} \cos\big((t-t_0)|\nabla|\big) & |\nabla|^{-1} \sin\big((t-t_0)|\nabla|\big) \\ -|\nabla| \sin\big((t-t_0)|\nabla|\big) & \cos\big((t-t_0)|\nabla|\big) \end{bmatrix} \begin{bmatrix} u_\pm \\ v_\pm \end{bmatrix} \right\|_{\dot{H}^{s_c} \times \dot{H}^{s_c-1}} \to 0.$$

This finishes the proof of Theorem 1.1.

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