On the Number of Partitions with Designated Summands

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Abstract. Andrews, Lewis and Lovejoy introduced the partition function $PD(n)$ as the number of partitions of n with designated summands, where we assume that among parts with equal size, exactly one is designated. They proved that $PD(3n+2)$ is divisible by 3. We obtain a Ramanujan type identity for the generating function of $PD(3n+2)$ which implies the congruence of Andrews, Lewis and Lovejoy. For $PD(3n)$, Andrews, Lewis and Lovejoy showed that the generating function can be expressed as an infinite product of powers of $(1-q^{2n+1})$ times a function $F(q^2)$. We find an explicit formula for $F(q^2)$, which leads to a formula for the generating function of $PD(3n)$. We also obtain a formula for the generating function of $PD(3n+1)$. Our proofs rely on Chan's identity on Ramanujan's

Just for comparison, let us recall the notion of overpartitions. An overpartition of n is a partition of n in which the first occurrence of each part can be overlined. For example, there are fourteen overpartitions of 4:

$$\begin{array}{ccccccc} 4 & 4' & 3+1 & 3'+1 & 3+1' & 3'+1', & 2+2 \\ 2'+2 & 2+1+1 & 2'+1+1 & 2+1'+1 & 2'+1'+1, & 1+1+1+1 & 1'+1+1+1. \end{array}$$

Overpartitions have been extensively studied, and they possess many analogous properties to ordinary partitions, see, for example, [8, 9, 11, 13].

The concept of partitions with designated summands goes back to MacMahon [14]. He considered partitions with designated summands and with exactly k different sizes, see also Andrews and Rose [5]. Let $PD(n)$ denote the number of partitions of n with designated summands. Andrews, Lewis and Lovejoy [2] derived the following generating function of $PD(n)$.

Theorem 1.1. *We have*

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty (q^2;q^2)_\infty (q^3;q^3)_\infty}, \tag{1.1}$$

where $|q| < 1$ *and* $(a;q)_\infty$ *stands for the q-shifted factorial*

$$(a;q)_\infty = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

By using modular forms and q-series identities, Andrews, Lewis and Lovejoy showed that the partition function $PD(n)$ has many interesting divisibility properties. In particular, they obtained the following Ramanujan type congruence.

Theorem 1.2. ([2, Corollary 7]) *For* $n \geq 0$*, we have*

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{1.2}$$

In this paper, we obtain the following Ramanujan type identity for the generating function of $PD(3n+2)$ which implies the above congruence.

Theorem 1.3. *We have*

$$\sum_{n=0}^{\infty} PD(3n+2)q^n = 3\frac{(q^3;q^6)_\infty^3 (q^6;q^6)_\infty^6}{(q;q^2)_\infty^5 (q^2;q^2)_\infty^8}. \tag{1.3}$$

Andrews, Lewis and Lovejoy also obtained explicit formulas for the generating functions for $PD(2n)$ and $PD(2n+1)$ by using Euler's algorithm for infinite products [1, P. 98] and Sturm's criterion [16]. As for $PD(3n)$, they showed that the generating function permits the following form.

Theorem 1.4. ([2, Theorem 23]) *Define $c(n)$ by*

$$\sum_{n=0}^{\infty} PD(3n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-c(n)}, \tag{1.4}$$

then for any positive integer n,

$$\begin{aligned} c(6n+1) &= 5, \\ c(6n+3) &= 2, \\ c(6n+5) &= 5. \end{aligned}$$

Equivalently, the above theorem says that there exists a series $F(q^2)$ such that

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{F(q^2)}{(q;q^6)_\infty^5 (q^3;q^6)_\infty^2 (q^5;q^6)_\infty^5}. \tag{1.5}$$

We find an explicit formula for $F(q^2)$, that is,

$$F(q^2) = \frac{(q^4;q^4)_\infty^6 (q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^{10} (q^{12};q^{12})_\infty^2} + 3q^2 \frac{(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^6 (q^4;q^4)_\infty^2}, \tag{1.6}$$

which leads to the following generating function of $PD(3n)$.

Theorem 1.5. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} PD(3n)q^n = {} & \frac{1}{(q;q^6)_\infty^5 (q^3;q^6)_\infty^2 (q^5;q^6)_\infty^5} \times \\ & \left(\frac{(q^4;q^4)_\infty^6 (q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^{10} (q^{12};q^{12})_\infty^2} + 3q^2 \frac{(q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^6 (q^4;q^4)_\infty^2} \right). \end{aligned} \tag{1.7}$$

We also obtain the generating function for $PD(3n+1)$.

Theorem 1.6. *We have*

$$\sum_{n=0}^{\infty} PD(3n+1)q^n = \frac{(q^3;q^6)_\infty^3 (q^6;q^6)_\infty^6}{(q;q^2)_\infty^5 (q^2;q^2)_\infty^8} \left(4q \frac{(q;q^2)_\infty^2}{(q^3;q^6)_\infty^6} + \frac{(q^3;q^6)_\infty^3}{(q;q^2)_\infty} \right). \tag{1.8}$$

The proofs of the generating function formulas (1.3), (1.7) and (1.8) rely on Chan's identity on Ramanujan's cubic continued fraction [10] and cubic theta functions [6,12]. In Section 3, we shall give a combinatorial interpretation of the congruence $PD(3n+2) \equiv 0 \pmod 3$ by introducing a rank for partitions with designated summands.

2 Proofs

In this section, we give proofs of the generating functions for $PD(3n)$, $PD(3n+1)$ and $PD(3n+2)$. It should be noted that the generating function of $PD(3n)$ derived this way does not directly lead to a formula for $F(q^2)$. To compute $F(q^2)$, we shall make use of some identities on cubic theta functions.

Recall that Ramanujan's cubic continued fraction $\upsilon(q)$ is given by

$$\upsilon(q) := \frac{q^{\frac{1}{3}}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+ \cdots}.$$

It is known that

$$\upsilon(q) = q^{\frac{1}{3}} \frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty},$$

see Andrews and Berndt [3, P. 94]. The following identity is due to Chan [10, Eq. (13)].

Theorem 2.1. *We have*

$$\frac{1}{(q;q)_\infty (q^2;q^2)_\infty} = \frac{(q^9;q^9)^3_\infty (q^{18};q^{18})^3_\infty}{(q^3;q^3)^4_\infty (q^6;q^6)^4_\infty} \times \left\{ \left(\frac{1}{x^2(q^3)} - 2q^3 x(q^3) \right) + q \left(\frac{1}{x(q^3)} + 4q^3 x^2(q^3) \right) + 3q^2 \right\}, \tag{2.1}$$

where

$$x(q) = q^{-\frac{1}{3}} \upsilon(q) = \frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty}. \tag{2.2}$$

Proof of Theorems 1.3 and 1.6. Multiplying both sides of (2.1) by

$$\frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty},$$

we find

$$\frac{(q^6;q^6)_\infty}{(q;q)_\infty (q^2;q^2)_\infty (q^3;q^3)_\infty} = \frac{(q^9;q^9)^3_\infty (q^{18};q^{18})^3_\infty}{(q^3;q^3)^5_\infty (q^6;q^6)^3_\infty} \left\{ \left(\frac{1}{x^2(q^3)} - 2q^3 x(q^3) \right) + q \left(\frac{1}{x(q^3)} + 4q^3 x^2(q^3) \right) + 3q^2 \right\}. \tag{2.3}$$

Observe that the left-hand side of (2.3) is the generating function for $PD(n)$. Extracting those terms involving the powers q^{3n+1} and q^{3n+2} respectively, we deduce that

$$\sum_{n=0}^{\infty} PD(3n+1) q^{3n+1} = q \frac{(q^9;q^9)^3_\infty (q^{18};q^{18})^3_\infty}{(q^3;q^3)^5_\infty (q^6;q^6)^3_\infty} \left(4q^3 x^2(q^3) + \frac{1}{x(q^3)} \right), \tag{2.4}$$

$$\sum_{n=0}^{\infty} PD(3n+2) q^{3n+2} = 3q^2 \frac{(q^9;q^9)^3_\infty (q^{18};q^{18})^3_\infty}{(q^3;q^3)^5_\infty (q^6;q^6)^3_\infty}. \tag{2.5}$$

Thus Theorem 1.3 can be deduced from (2.5) by dividing both sides by q^2 and substituting q^3 by q. Similarly, Theorem 1.6 can be deduced from (2.4) by dividing both sides by q and substituting q^3 by q. This completes the proof. ∎

If we extract the terms involving the powers q^{3n} in (2.3), and substitute q^3 by q, we get

$$\sum_{n=0}^{\infty} PD(3n)q^n = \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^5(q^2;q^2)_\infty^3}\left(-2qx(q)+\frac{1}{x^2(q)}\right). \tag{2.6}$$

It turns out that $F(q^2)$ in the generating function formula for $PD(3n)$ can be computed from (2.6) with the aid of some identities for cubic theta functions. These cubic theta functions are introduced by Borwein, Borwein and Garvan [7] and are defined by

$$\begin{aligned} a(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \\ b(q) &= \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \quad \omega = e^{2\pi i/3}, \\ c(q) &= \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}. \end{aligned}$$

Recall that

$$c(q) = 3\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}, \tag{2.7}$$

see Berndt, Bhargava and Garvan [6, Eq. (5.5)]. We shall also use the following identities for $a(q)$ and $c(q)$

$$a(q) = a(q^4) + 6q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty}, \tag{2.8}$$

$$c(q) = qc(q^4) + 3\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}, \tag{2.9}$$

$$a(q) = a(q^2) + 2q\frac{c^2(q^2)}{c(q)}. \tag{2.10}$$

Identity (2.8) for $a(q)$ and identity (2.9) for $c(q)$ are due to Hirschhorn, Garvan, and Borwein [12, Eqs.(1.36) and (1.34)]. Identity (2.10) for $a(q)$ and $c(q)$ is obtained by Berndt, Bhargava, Garvan [6, Eq. (6.3)].

We obtain the following identity on Ramanujan's cubic continued fraction $v(q)$, which is stated in terms of $x(q)$ as given by (2.2).

Lemma 2.2. *We have*

$$\frac{1}{x^2(q)} - 2qx(q) = 3q^2\frac{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^6} + \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^2}. \tag{2.11}$$

Proof. We first establish a connection between Ramanujan's cubic continued fraction $\upsilon(q)$ and the cubic theta function $c(q)$. It is easy to check that

$$\frac{1}{x^2(q)} = \frac{(q^3;q^6)_\infty^6}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty^6} \times \left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^2 = \frac{(q^2;q^2)_\infty^2}{9(q^6;q^6)_\infty^6} \times c^2(q), \tag{2.12}$$

$$2qx(q) = 2q\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3} = 2q\frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} \times \left(\frac{(q;q)_\infty}{(q^3;q^3)_\infty^3}\right) = 6q\frac{(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty} \times \frac{1}{c(q)}. \tag{2.13}$$

We now consider the 2-dissection of $1/x^2(q)$. Identity (2.9) can be viewed as the 2-dissection of $c(q)$. Hence we deduce that

$$c^2(q) = \left(q^2c^2(q^4) + 9\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4(q^{12};q^{12})_\infty^2}\right) + q\left(6c(q^4)\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\right).$$

This yields the 2-dissection of $1/x^2(q)$,

$$\begin{aligned}\frac{1}{x^2(q)} &= \left(q^2c^2(q^4)\frac{(q^2;q^2)_\infty^2}{9(q^6;q^6)_\infty^6} + \frac{(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty^6}\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4(q^{12};q^{12})_\infty^2}\right)\\ &\quad + q\left(6c(q^4)\frac{(q^2;q^2)_\infty^2}{9(q^6;q^6)_\infty^6}\frac{(q^4;q^4)_\infty^3(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty}\right)\\ &= \left(q^2\frac{(q^2;q^2)_\infty^2(q^{12};q^{12})_\infty^6}{(q^4;q^4)_\infty^2(q^6;q^6)_\infty^6} + \frac{(q^4;q^4)_\infty^6}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^2}\right)\\ &\quad + 2q\left(\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^4}\right).\end{aligned} \tag{2.14}$$

Next, we aim to derive the 2-dissection of $q/c(q)$. By (2.10), we find

$$\frac{q}{c(q)} = \frac{a(q)-a(q^2)}{2c^2(q^2)}. \tag{2.15}$$

Substituting (2.8) into (2.15), we arrive at

$$\frac{q}{c(q)} = \frac{1}{2c^2(q^2)}\left(a(q^4) + 6q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty} - a(q^2)\right). \tag{2.16}$$

Using (2.10) with q replaced by q^2, we get

$$a(q^2) - a(q^4) = 2q^2\frac{c^2(q^4)}{c(q^2)}.$$

Hence (2.16) can be written as

$$\begin{aligned}\frac{q}{c(q)} &= \frac{1}{2c^2(q^2)}\left(-2q^2\frac{c^2(q^4)}{c(q^2)} + 6q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\right)\\ &= -q^2\frac{c^2(q^4)}{c^3(q^2)} + 3q\frac{(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{c^2(q^2)(q^2;q^2)_\infty(q^6;q^6)_\infty}.\end{aligned}$$

Thus, we obtain the following 2-dissection of $2qx(q)$,

$$
\begin{aligned}
2qx(q) &= -6q^2\frac{(q^6;q^6)_\infty^3 c^2(q^4)}{(q^2;q^2)_\infty c^3(q^2)} + 18q\frac{(q^6;q^6)_\infty^3 (q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2}{c^2(q^2)(q^2;q^2)_\infty^2 (q^6;q^6)_\infty} \\
&= -2q^2\frac{(q^2;q^2)_\infty^2 (q^{12};q^{12})_\infty^6}{(q^6;q^6)_\infty^6 (q^4;q^4)_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2}{(q^6;q^6)_\infty^4}. \qquad (2.17)
\end{aligned}
$$

Subtracting (2.17) from (2.14), we obtain (2.11). This completes the proof. ∎

Proof of Theorem 1.5. By (2.6), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} PD(3n)q^n &= \frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^5 (q^2;q^2)_\infty^3}\left(-2qx(q) + \frac{1}{x^2(q)}\right) \\
&= \frac{(q^3;q^6)_\infty^3 (q^6;q^6)_\infty^6}{(q;q^2)_\infty^5 (q^2;q^2)_\infty^8}\left(-2qx(q) + \frac{1}{x^2(q)}\right) \\
&= \frac{1}{(q;q^6)_\infty^5 (q^3;q^6)_\infty^2 (q^5;q^6)_\infty^5} \times \frac{(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^8}\left(-2qx(q) + \frac{1}{x^2(q)}\right). \qquad (2.18)
\end{aligned}
$$

Applying (2.11) to (2.18), we are led to the generating function for $PD(3n)$ as given by (1.7). This completes the proof. ∎

3 A combinatorial interpretation

In this section, we give a combinatorial interpretation of the congruence $PD(3n+2) \equiv 0 \pmod 3$. In doing so, we introduce the *pd*-rank of a partition with designated summands. This rank function enables us to divide the set of partitions of $3n+2$ with designated summands into three equinumerous classes. The definition of the *pd*-rank is based on the following representation of a partition with designated summands by a pair of partitions.

Theorem 3.1. *There is a bijection Δ between the set of partitions of n with designated summands and the set of pairs of partitions (α, β) with $|\alpha| + |\beta| = n$, where α is an ordinary partition and β is a partition into parts $\not\equiv \pm 1 \pmod 6$.*

It is clear that the above theorem is a consequence of formula (1.1) for the generating function of partitions with designated summands. We shall give a combinatorial proof of this theorem which give rise to the notion of the *pd*-rank. Our construction is based on the bijective proof of MacMahon's theorem given by Andrews, Eriksson, Petrov and Romik [4].

Combinatorial proof of Theorem 3.1. Let λ be a partition of n with designated summands. We wish to construct a pair of partitions (α, β) such that $|\alpha| + |\beta| = n$, where α is an ordinary partition and β is a partition into parts $\not\equiv \pm 1 \pmod 6$.

Suppose that t is a part of λ, and suppose that t appears m_t times with the i-th part being designated. There are two cases.

- If $i = 1$, then move all the parts equal to t (including the designated part) in λ to the partition α.
- If $i \neq 1$, then move i parts equal to t in λ to γ and $(m_t - i)$ parts equal to t in λ to α.

It can be seen that each part occurs at least twice in γ . The partition β with parts $\not\equiv \pm 1 \pmod 6$ can be obtained from the partition γ with the aid of the following bijection of Andrews, Eriksson, Petrov and Romik.

First, write γ as in the form of $1^{m_1} 2^{m_2} \cdots l^{m_l}$, where m_k is the multiplicity of k. Since $m_k \neq 1$ for any k, there is a unique way to write m_k in the form $m_k = s_k + t_k$, where $s_k = 0$ or 3, and $t_k \in \{0, 2, 4, 6, 8, \ldots\}$. Now, the partition $\beta = 1^{b_1} 2^{b_2} \ldots$ is determined as follows:

$$\begin{aligned} b_{6k+1} &= 0, & b_{6k+5} &= 0, \\ b_{6k+2} &= \frac{1}{2} t_{3k+1}, & b_{6k+4} &= \frac{1}{2} t_{3k+2}, \\ b_{6k+3} &= \frac{1}{3} s_{2k+1} + t_{6k+3}, & b_{6k+6} &= \frac{1}{3} s_{2k+2} + t_{6k+6}. \end{aligned}$$

It is clear that β is a partition into parts $\not\equiv \pm 1 \pmod 6$ and the above procedure is reversible. Hence Δ is a bijection. This completes the proof. ∎

The *pd*-rank of a partition λ with designated summands can be defined in terms of the pair of partitions (α, β) under the map Δ.

Definition 3.2. *Let λ be a partition with designated summands and let $(\alpha, \beta) = \Delta(\lambda)$. Then the pd-rank of λ, denoted $r_d(\lambda)$, is defined by*

$$r_d(\lambda) = l_e(\alpha) - l_e(\beta), \tag{3.1}$$

where $l_e(\alpha)$ is the number of even parts of α and $l_e(\beta)$ is the number of even parts of β.

The following theorem shows that the *pd*-rank can be used to divide the set of partitions of $3n + 2$ with designated summands into three equinumerous classes.

Theorem 3.3. *For $i = 0, 1, 2$, let $N_d(i, 3; n)$ denote the number of partitions of n with designated summands with pd-rank congruent to $i \pmod 3$. Then we have*

$$N_d(0, 3; 3n + 2) = N_d(1, 3; 3n + 2) = N_d(2, 3; 3n + 2). \tag{3.2}$$

Proof. Let $N_d(m;n)$ denote the number of partitions of n with designated summands with pd-rank m. By the definition of the pd-rank, we see that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_d(m;n)z^m q^n = \frac{1}{(zq^2;q^2)_\infty (q;q^2)_\infty} \times \frac{1}{(z^{-1}q^2;q^2)_\infty (q^3;q^6)_\infty}. \tag{3.3}$$

Setting $z = \zeta = e^{\frac{2\pi i}{3}}$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_d(m;n)\zeta^m q^n &= \sum_{n=0}^{\infty}\sum_{i=0}^{2} N_d(i,3;n)\zeta^i q^n \\ &= \frac{1}{(\zeta q^2;q^2)_\infty (q;q^2)_\infty (\zeta^{-1}q^2;q^2)_\infty (q^3;q^6)_\infty} \\ &= \frac{(-q^3;q^3)_\infty}{(q;q^2)_\infty (\zeta q^2;q^2)_\infty (\zeta^{-1}q^2;q^2)_\infty}. \end{aligned} \tag{3.4}$$

Multiplying the right hand side of (3.4) by

$$\frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty},$$

and noting that

$$(1-x)(1-x\zeta)(1-x\zeta^2) = 1 - x^3,$$

we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{i=0}^{2} N_d(i,3;n)\zeta^i q^n &= \frac{(-q^3;q^3)_\infty}{(q;q^2)_\infty (\zeta q^2;q^2)_\infty (\zeta^{-1}q^2;q^2)_\infty} \times \frac{(q^2;q^2)_\infty}{(q^2;q^2)_\infty} \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \times \frac{(-q^3;q^3)_\infty}{(q^6;q^6)_\infty}.\end{aligned}$$

By Gauss's identity [1, P. 23]

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \sum_{n=0}^{\infty} q^{\binom{n+1}{2}},$$

we get

$$\sum_{n=0}^{\infty}\sum_{i=0}^{2} N_d(i,3;n)\zeta^i q^n = \frac{(-q^3;q^3)_\infty}{(q^6;q^6)_\infty} \sum_{n=0}^{\infty} q^{\binom{n+1}{2}}. \tag{3.5}$$

Since

$$\binom{n+1}{2} \equiv 0 \text{ or } 1 \pmod 3,$$

the coefficient of q^{3n+2} in (3.5) is zero. It follows that

$$N_d(0,3;3n+2) + N_d(1,3;3n+2)\zeta + N_d(1,3;3n+2)\zeta^2 = 0.$$

Since the minimal polynomial of ζ is $1 + x + x^2$, we conclude that

$$N_d(0,3;3n+2) = N_d(1,3;3n+2) = N_d(2,3;3n+2).$$

This completes the proof. ∎

For example, for $n = 5$, we have $PD(5) = 15$. The fifteen partitions of 5 with designated summands, the corresponding pairs of partitions, and the pd-ranks modulo 3 are listed in Table 3.1. It can be checked that

$$N_d(0,3;5) = N_d(1,3;5) = N_d(2,3;5) = 5.$$

λ	$(\alpha,\beta) = \Delta(\lambda)$	$r_d(\lambda) \pmod 3$
$5'$	$(5,\ \emptyset)$	0
$4' + 1'$	$(4+1,\ \emptyset)$	1
$3' + 2'$	$(3+2,\ \emptyset)$	1
$3' + 1' + 1$	$(3+1+1,\ \emptyset)$	0
$3' + 1 + 1'$	$(3,\ 2)$	2
$2' + 2 + 1'$	$(2+2+1,\ \emptyset)$	2
$2 + 2' + 1'$	$(1,\ 4)$	2
$2' + 1' + 1 + 1$	$(2+1+1+1,\ \emptyset)$	1
$2' + 1 + 1' + 1$	$(2+1,\ 2)$	0
$2' + 1 + 1 + 1'$	$(2,\ 3)$	1
$1' + 1 + 1 + 1 + 1$	$(1+1+1+1+1,\ \emptyset)$	0
$1 + 1' + 1 + 1 + 1$	$(1+1+1,\ 2)$	2
$1 + 1 + 1' + 1 + 1$	$(1+1,\ 3)$	0
$1 + 1 + 1 + 1' + 1$	$(1,\ 2+2)$	1
$1 + 1 + 1 + 1 + 1'$	$(\emptyset,\ 3+2)$	2

Table 3.1: The case for $n = 5$.

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