Stochastic Functional Hamiltonian System with Singular Coefficients

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Abstract

By Zvonkin type transforms, the existence and uniqueness of the strong solutions for a class of stochastic functional Hamiltonian systems are obtained, where the drift contains a Hölder-Dini continuous perturbation. Moreover, under some reasonable conditions, the non-explosion of the solution is proved. In addition, as applications, the Harnack and shift Harnack inequalities are derived by method of coupling by change of measure. These inequalities are new even in the case without delay and the shift Harnack inequality is also new even in the non-degenerate functional SDEs with singular drifts.

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1 Introduction

As a typical model of degenerate diffusion system, the stochastic Hamiltonian system has been investigated in [10, 23, 24, 26], see [4] for the functional version of this model with regular coefficients. Recently, Zvonkin type transforms ([29]) have been used to prove existence and uniqueness of the strong solutions fo SDEs with singular drift, see e.g. [1, 11, 14, 16, 21, 27]. In [13, 15], the first author and his co-author have investigated the non-degenerate functional SPDEs with Dini continuous drift, see also [2, 12] for the finite dimensional non-degenerate functional SDEs with integrable drifts. So far, there is few results on the degenerate functional SDEs with singular coefficients. The purpose of this paper is to deal with this problem. We will adopt the Zvonkin type transforms considered in [25] for SDEs without delay which enable us to regularize a singular perturbation without time delay. The main difficulty is to treat the delay part, which is a function on an infinite dimension space. To this end, we construct a family of homeomorphisms on $\mathscr{C}^{d_1+d_2}$ (see (2.17) below) besides the homeomorphisms on $\mathbb{R}^{d_1+d_2}$.

On the other hand, Harnack and shift Harnack inequalities have many applications. For instance, Harnack inequalities can yield strong Feller property and the uniqueness of invariant probability measure, shift Harnack inequalities implies the regularity of heat kernel with respect to Lebesgue measure, see [22, Chapter 1] for more details. For the stochastic Hamiltonian system with multiplicative noise, the coupling by change of measure is so hard to establish that there is no any result on the Harnack inequalities for SDEs of this type. Unfortunately, the Zvonkin type transforms make a stochastic Hamiltonian system even with additive noise into a new one with multiplicative noise. Thus, we can not obtain the Harnack inequalities for the original SDEs from the ones after Zvonkin type transforms as in the non-degenerate case, see [13, 15]. Instead, we adopt a new idea, i.e. directly construct the coupling by change of measure for the original SDE with singular drift. Compared with the result in [4], $\Sigma(T,h,r)$ in Harnack inequality in Theorem 3.2 contains two additional terms, i.e. the second and third term in $\Sigma(T,h,r)$, which comes from the singularity of the drift Z in (3.1).

The paper is organized as follows: In Section 2, we prove the existence and uniqueness of the solution for the stochastic functional Hamiltonian system; In Section 3, we investigate the Harnack and shift Harnack inequalities and their applications.

Throughout the paper, the letter C or c will denote a positive constant, and $C(\theta)$ or $c(\theta)$ stands for a constant depending on θ. The value of the constants may change from one appearance to another.

2 Existence and Uniqueness

Fix a constant $r>0$. For any $d\in\mathbb{N}^+$, let $\mathscr{C}^d=C([-r,0];\mathbb{R}^d)$ be equipped with the uniform norm $\|\xi\|_\infty =: \sup_{s\in[-r,0]}|\xi(s)|$. For any $f\in C([-r,\infty);\mathbb{R}^d)$, $t\geq 0$, define $f_t\in\mathscr{C}^d$ as $f_t(s)=f(t+s), s\in[-r,0]$, which is called the segment process.

Consider the following stochastic functional Hamiltonian system on $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases}\mathrm{d}X(t)=b(t,X(t))\mathrm{d}t+(0,B(t,X_t))\mathrm{d}t+(0,\sigma(t,X(t))\mathrm{d}W(t)),\\ X_0=\xi=(\xi^{(1)},\xi^{(2)})\in\mathscr{C}^{d_1+d_2},\end{cases}\tag{2.1}$$

where $W=(W(t))_{t\geq 0}$ is a d_2-dimensional standard Brownian motion with respect to a complete filtered probability space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq0},\mathbb{P})$, $b=(b^{(1)},b^{(2)}):[0,\infty)\times\mathbb{R}^{d_1+d_2}\to\mathbb{R}^{d_1+d_2}$, $B:[0,\infty)\times\mathscr{C}^{d_1+d_2}\to\mathbb{R}^{d_2}$ and $\sigma:[0,\infty)\times\mathbb{R}^{d_1+d_2}\to\mathbb{R}^{d_2}\otimes\mathbb{R}^{d_2}$ are measurable and locally bounded (bounded on bounded sets). When $B=0$ and $b=\nabla V$ for some potential V, (2.1) is called stochastic Hamiltonian system, which includes the kinetic Fokker-Planck equation as a typical example (see [20]). If $d_1 = 0$, then equation (2.1) reduces to the non-degenerate case.

In the following, we will use $\nabla^{(1)}$ and $\nabla^{(2)}$ to denote the gradient operators on the first space $\mathbb{R}^{d_1}$ and the second space $\mathbb{R}^{d_2}$ respectively. For simplicity, we denote $\nabla f(t,x) = \nabla f(t,\cdot)(x)$ for a vector valued function f defined on $[0,\infty)\times\mathbb{R}^d$. Thus, for every $(t,x)\in[0,\infty)\times\mathbb{R}^{d_1+d_2}$, $\nabla^{(2)}b^{(1)}(t,x)\in\mathbb{R}^{d_2}\otimes\mathbb{R}^{d_1}$ with $(\nabla^{(2)}b^{(1)}(t,x))h := \nabla_h^{(2)}b^{(1)}(t,x)\in\mathbb{R}^{d_1}$, $h\in\mathbb{R}^{d_2}$.

Definition 2.1. A continuous $\mathbb{R}^{d_1+d_2}$-valued process $(X(t))_{t\in[-r,\zeta)}$ is called a strong solution to (2.1) with life time ζ, if the segment process X_t is $\mathscr{F}_t$-measurable, and $\zeta>0$ is a stopping time such that $\mathbb{P}$-a.s $\limsup_{t\uparrow\zeta}|X(t)|=\infty$ holds on $\{\zeta<\infty\}$, and $\mathbb{P}$-a.s

$$\begin{aligned} X(t) = \xi(t\wedge 0) &+ \int_0^{t\vee 0}(b(s,X(s)) + (0,B(s,X_s)))\mathrm{d}s \\ &+ \int_0^{t\vee 0}(0,\sigma(s,X(s))\mathrm{d}W(s), \quad t\in[-r,\zeta). \end{aligned}$$

When $B=0$, the infinitesimal generator associated to (2.1) is

$$\mathscr{L}_t^{\Sigma,b} = \mathrm{tr}(\Sigma(t,\cdot)\cdot\nabla^{(2)}\nabla^{(2)}) + \nabla_{b(t,\cdot)}, \tag{2.2}$$

where $\Sigma(t,\cdot) := \frac{1}{2}\sigma(t,\cdot)\sigma^*(t\cdot)$, $\mathrm{tr}(\cdot)$ denotes the trace of a matrix, and $\nabla_{b(t,\cdot)}f := \langle\nabla f, b(t,\cdot)\rangle$ is the directional derivative of f along $b(t,\cdot)$ for a differentiable function on $\mathbb{R}^{d_1+d_2}$.

The following definition comes from [3] and [25].

Definition 2.2. An increasing function $\phi:[0,\infty)\to[0,\infty)$ is called a *Dini* function if

$$\int_0^1\frac{\phi(s)}{s}\mathrm{d}s<\infty.$$

A function f defined on the Euclidean space is called *Hölder − Dini* continuous of order $\alpha\in[0,1)$ if

$$|f(x)-f(y)|\le|x-y|^\alpha\phi(|x-y|),\quad |x-y|\le 1$$

holds for some Dini function ϕ, and is called *Dini − continuous* if this condition holds for $\alpha=0$.

A measurable function $\phi:[0,\infty)\to[0,\infty)$ is called a *slowly varying* function at zero (see [3]) if for any $\lambda>0$,

$$\lim_{t\to0}\frac{\phi(\lambda t)}{\phi(t)}=1.$$

Let $\mathscr{D}_0$ be the set of all Dini functions, and $\mathscr{T}_0$ the set of all slowly varying functions at zero that are bounded away from 0 and ∞ on $[\varepsilon,\infty)$ for any $\varepsilon>0$. Notice that the typical examples for functions contained in $\mathscr{D}_0\cap\mathscr{T}_0$ are $\phi(t):=(\log(1+t^{-1}))^{-\beta}$ for $\beta>1$.

To characterise the non-Lipschitz condition of B and σ, we introduce the class

$$\mathscr{D}_1=\Big\{\gamma\in C^1([0,\infty);(0,\infty)):\int_0^1\frac{1}{s\gamma(s)}\mathrm{d}s=\infty,\quad \liminf_{t\downarrow0}(\frac{\gamma(t)}{4}+t\gamma'(t))>0\Big\}.$$

Typical functions in $\mathscr{D}_1$ contain

$$\gamma_1(t) := \log(1+t^{-1}), \quad \gamma_2(t) := \gamma_1(t)\log\log(\mathrm{e}+t^{-1}),$$
$$\gamma_3(t) := \gamma_2(t)\log\log\log(\mathrm{e}^2+t^{-1}), \cdots.$$

We will need the following local condition (see [25, **(A)**] for details).

(A) For any $n \geq 1$, there exist a constant $C_n \in [0,\infty)$, some $\phi_n \in \mathscr{D}_0 \cap \mathscr{T}_0$ and $\gamma_n \in \mathscr{D}_1$ such that the following conditions hold for all $t \in [0,n]$:

(H1) (Hypoellipticity) For any $x \in \mathbb{R}^{d_1+d_2}$, $\sigma(t,x)$ and $[\nabla^{(2)}b^{(1)}(t,x)][\nabla^{(2)}b^{(1)}(t,x)]^*$ are invertible with

$$\sup_{|x|\leq n}\big(\|\sigma^{-1}(t,x)\| + \|\sigma(t,x)\| + |b(t,x)|\big)$$
$$+ \sup_{x\in\mathbb{R}^{d_1+d_2}, |x^{(1)}|\leq n}\bigg(\|\nabla^{(2)}b^{(1)}(t,x)\| + \Big\|\big([\nabla^{(2)}b^{(1)}(t,x)][\nabla^{(2)}b^{(1)}(t,x)]^*\big)^{-1}\Big\|\bigg) \leq C_n.$$

(H2) (Regularity of $b^{(1)}$) For any $x, y \in \mathbb{R}^{d_1+d_2}$ with $|x| \vee |y| \leq n$,

$$\begin{aligned} &\big|b^{(1)}(t,x) - b^{(1)}(t,y)\big| \leq |x^{(1)}-y^{(1)}|^{\frac{2}{3}}\phi_n(|x^{(1)}-y^{(1)}|), \text{ if } x^{(2)} = y^{(2)},\\ &\big\|\nabla^{(2)}b^{(1)}(t,x) - \nabla^{(2)}b^{(1)}(t,y)\big\| \leq \phi_n(|x^{(2)}-y^{(2)}|), \text{ if } x^{(1)} = y^{(1)}. \end{aligned} \tag{2.3}$$

(H3) (Regularity of $b^{(2)}$ and σ) Either

$$\begin{cases} |b^{(2)}(t,x) - b^{(2)}(t,y)| \leq |x^{(1)}-y^{(1)}|^{\frac{2}{3}}\phi_n(|x^{(1)}-y^{(1)}|) + \phi_n^{\frac{7}{2}}(|x^{(2)}-y^{(2)}|),\\ \|\sigma(t,x)-\sigma(t,y)\| \leq |x-y|\sqrt{\gamma_n(|x-y|^2)}, \quad |x|\vee|y| \leq n \end{cases} \tag{2.4}$$

or $\sup_{|x|\leq n}\|\nabla^{(2)}\sigma(t,x)\| \leq C_n$, and for $|x|\vee|y| \leq n$,

$$\begin{cases} |b^{(2)}(t,x) - b^{(2)}(t,y)| \leq |x^{(1)}-y^{(1)}|^{\frac{2}{3}}\phi_n(|x^{(1)}-y^{(1)}|) + \phi_n(|x^{(2)}-y^{(2)}|),\\ \|\nabla^{(2)}\sigma(t,x^{(1)},x^{(2)}) - \nabla^{(2)}\sigma(t,y^{(1)},x^{(2)})\| \leq |x^{(1)}-y^{(1)}|\sqrt{\gamma_n(|x^{(1)}-y^{(1)}|^2)},\\ \|\sigma(t,x^{(1)},x^{(2)}) - \sigma(t,y^{(1)},x^{(2)})\| \leq |x^{(1)}-y^{(1)}|\sqrt{\gamma_n(|x^{(1)}-y^{(1)}|^2)}. \end{cases} \tag{2.5}$$

(H4) (Regularity of B) For any $\xi, \eta \in \mathscr{C}^{d_1+d_2}$ with $\|\xi\|_\infty \vee \|\eta\|_\infty \leq n$,

$$|B(t,\xi)| \leq C_n, \quad |B(t,\xi)-B(t,\eta)| \leq C_n\|\xi-\eta\|_\infty\sqrt{\gamma_n(\|\xi-\eta\|_\infty^2)}. \tag{2.6}$$

2.1 Main results

The main result of this section is the following theorem.

Theorem 2.1. *(1) Assume* **(A)***. For any $\xi \in \mathscr{C}^{d_1+d_2}$, SDE* (2.1) *has a unique solution X_t^ξ up to the explosion time $\zeta(\xi) := \inf\{t \geq 0, |X^\xi(t)| = \infty\}$.*

(2) In particular, if $b(t,x)$ and $\sigma(t,x)$ do not depend on $x^{(1)}$, then the above assertion holds provided for any $n \geq 1$, there exist a constant $C_n \in [0,\infty)$, $\phi_n \in \mathscr{D}_0 \cap \mathscr{T}_0$ and $\gamma_n \in \mathscr{D}_1$ such that **(H1)**, **(H4)** *and*

$$\begin{cases} |b^{(2)}(t,x) - b^{(2)}(t,y)| + \|\nabla^{(2)} b^{(1)}(t,x) - \nabla^{(2)} b^{(1)}(t,y)\| \leq \phi_n(|x^{(2)} - y^{(2)}|), \\ \|\sigma(t,x) - \sigma(t,y)\| \leq |x^{(2)} - y^{(2)}| \sqrt{\gamma_n(|x^{(2)} - y^{(2)}|^2)} \end{cases} \tag{2.7}$$

hold for all $t, |x|, |y| \leq n$.

(3) If there exists $H \in C^2(\mathbb{R}^{d_1+d_2})$ such that

$$\begin{aligned} &H \geq 1, \quad \lim_{|x|\to\infty} H(x) = \infty, \quad \mathscr{L}_t^{\Sigma,b} H \leq \Phi(t) H, \quad |\sigma(t,\cdot)^* \nabla^{(2)} H|^2 \leq \Phi(t) H^2; \\ &\langle B(t,\xi), \nabla^{(2)} H(\xi(0))\rangle \leq \Phi(t) \|H \circ \xi\|_\infty, \quad \xi \in \mathscr{C}^{d_1+d_2}, t \geq 0 \end{aligned} \tag{2.8}$$

holds for some positive increasing function Φ, then the solution to (2.1) *is non-explosive and for any $p, T > 0$, there exists a constant $C_1(p,T,\Phi)$ depending on p, T, Φ such that*

$$\mathbb{E} \sup_{t\in[-r,T]} H^p(X^\xi(t)) \leq 2\|H \circ \xi\|_\infty^p \mathrm{e}^{C_1(p,T,\Phi)T}, \quad \xi \in \mathscr{C}^{d_1+d_2}.$$

Remark 2.2. *We should remark that Zvonkin's transform can not regularize the functional drift with singular condition. That is why we assume the functional part is regular. In fact, the finite dimensional noise W can not remove the drift B which is a function on the infinite dimensional space $\mathscr{C}^{d_1+d_2}$. More precisely, if we adopt the same trick as in Zvonkin's transform, we need to deal with the equation like*

$$\begin{aligned} &\partial_t \mathbf{u}^n(t,\xi(0)) + \mathscr{L}_t^{\Sigma^n, b^n} \mathbf{u}^n(t,\xi(0)) + b^n(t,\xi(0)) \\ &+ \nabla_{B(t,\xi)} \mathbf{u}^n(t,\xi(0)) + B(t,\xi) = \lambda \mathbf{u}^n(t,\xi(0)), \quad t \in [0,T]. \end{aligned} \tag{2.9}$$

Obvious, this equation can not be solved due to the existence of B. Thus, to remove the functional part with singular condition, we should search for some new idea instead of Zvonkin's transform for the SDEs without delay.

2.2 Proof of Theorem 2.1

Firstly, for $f : \mathbb{R}^d \to \mathbb{R}$, define

$$[f]_{Lip} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|}.$$

Proof of Theorem 2.1. We first assume that **(A)** holds for some $C_n = C$, $\phi_n = \phi$ and $\gamma_n = \gamma$ independent of $n \geq 1$.

Let ρ be a non-negative smooth function with compact support in $\mathbb{R}^{d_1+d_2}$ with

$$\int_{\mathbb{R}^{d_1+d_2}} \rho(x)\mathrm{d}x = 1.$$

For any $n \in \mathbb{N}$, define $\rho_n(x) = n^{d_1+d_2}\rho(nx)$, and

$$b^n(t,\cdot) = \rho_n * b(t,\cdot), \quad \sigma^n(t,\cdot) = \rho_n * \sigma(t,\cdot). \tag{2.10}$$

By [25, Theorem 3.2, Proof of Theorem 1.1], the following equation

$$\partial_t \mathbf{u}^n(t,\cdot) + \mathscr{L}_t^{\Sigma^n, b^n}\mathbf{u}^n(t,\cdot) + b^n(t,\cdot) = \lambda \mathbf{u}^n(t,\cdot), \quad \mathbf{u}^n(T,\cdot) = 0, t \in [0,T] \tag{2.11}$$

has a unique smooth solution $\mathbf{u}^n : [0,T] \times \mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_1+d_2}$. Moreover, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{n \ge n_0, t\in[0,T]} \big(\|\nabla \mathbf{u}^n(t,\cdot)\|_\infty + \|\nabla\nabla^{(2)}\mathbf{u}^n(t,\cdot)\|_\infty\big) \le \varepsilon(\lambda) := C\int_0^T \mathrm{e}^{-\lambda s}\frac{\phi(\sqrt{s})}{s}\mathrm{d}s \tag{2.12}$$

for some constant $C > 0$. So Ascoli-Arzela's theorem implies that there exists $\mathbf{u} : [0,T] \times \mathbb{R}^{d_1+d_2} \to \mathbb{R}^{d_1+d_2}$ such that, up to a subsequence,

$$\lim_{n\to\infty} \sup_{t\in[0,T], |x|\le R} \big(|\mathbf{u}^n(t,x) - \mathbf{u}(t,x)| + |\nabla^{(2)}\mathbf{u}^n(t,x) - \nabla^{(2)}\mathbf{u}(t,x)|\big) = 0, \quad R > 0, \tag{2.13}$$

and, moreover, (2.12) and (2.13) yield that

$$\sup_{t\in[0,T]} \big([\mathbf{u}(t,\cdot)]_{Lip} + [\nabla^{(2)}\mathbf{u}(t,\cdot)]_{Lip}\big) \le \varepsilon(\lambda). \tag{2.14}$$

Then we can find a constant $\lambda_0(T,\phi) > 0$ such that for any $\lambda \ge \lambda_0(T,\phi)$,

$$\sup_{t\in[0,T]} [\mathbf{u}(t,\cdot)]_{Lip} < \frac{1}{8}. \tag{2.15}$$

From now on, we fix $\lambda = \lambda_0(T,\phi)$.

To treat the delay part, define $\mathbf{u}(s,\cdot) = \mathbf{u}(0,\cdot)$ for $s \in [-r,0]$. Let $\theta(t,x) = x + \mathbf{u}(t,x)$, $(t,x) \in [-r,T] \times \mathbb{R}^{d_1+d_2}$. Clearly, $\{\theta(t,\cdot)\}_{t\in[-r,T]}$ is a family of homeomorphisms on $\mathbb{R}^{d_1+d_2}$. For simplicity, we write $\theta^{-1}(t,x) = [\theta^{-1}(t,\cdot)](x)$, $(t,x) \in [-r,T] \times \mathbb{R}^{d_1+d_2}$. By (2.15), we have

$$\begin{aligned} &\frac{7}{8}|x-y| \le |\theta(t,x) - \theta(t,y)| \le \frac{9}{8}|x-y|, \\ &\frac{8}{9}|x-y| \le |\theta^{-1}(t,x) - \theta^{-1}(t,y)| \le \frac{8}{7}|x-y|, \quad t \in [-r,T], x,y \in \mathbb{R}^{d_1+d_2}, \end{aligned} \tag{2.16}$$

On the other hand, for any $t \in [0,T]$, define $\theta_t : \mathscr{C}^{d_1+d_2} \to \mathscr{C}^{d_1+d_2}$ as

$$(\theta_t(\xi))(s) = \theta(t+s, \xi(s)), \quad \xi \in \mathscr{C}^{d_1+d_2}, s \in [-r,0]. \tag{2.17}$$

Then (2.16) implies $\{\theta_t\}_{t\in[0,T]}$ is a family of homeomorphisms on $\mathscr{C}^{d_1+d_2}$. Moreover, it is easy to see that for any $t\in[0,T]$,

$$(\theta_t^{-1}(\xi))(s)=\theta^{-1}(t+s,\xi(s)),\quad \xi\in\mathscr{C}^{d_1+d_2}, s\in[-r,0]. \tag{2.18}$$

Furthermore, it follows from (2.16) and (2.17) that

$$\begin{aligned}\|\theta_t(\xi)-\theta_t(\eta)\|_\infty &= \sup_{s\in[-r,0]}|\theta(t+s,\xi(s))-\theta(t+s,\eta(s))|\\ &\le \frac{9}{8}\|\xi-\eta\|_\infty,\quad t\in[0,T],\xi,\eta\in\mathscr{C}^{d_1+d_2}.\end{aligned} \tag{2.19}$$

Similarly, we have

$$\|\theta_t^{-1}(\xi)-\theta_t^{-1}(\eta)\|_\infty\le\frac{8}{7}\|\xi-\eta\|_\infty,\quad t\in[0,T],\xi,\eta\in\mathscr{C}^{d_1+d_2}. \tag{2.20}$$

In a word, $\theta(t,\cdot)$, $\theta^{-1}(t,\cdot)$, θ_t and θ_t^{-1} are Lipschitiz continuous uniformly in $t\in[0,T]$.

Now, if $X(t)$ solves (2.1) up to a stopping time $\tau\le T$, then by Itô's formula and (2.11), we have $\mathbb{P}$-a.s.,

$$\begin{aligned}&X(t)+\mathbf{u}^n(t,X(t))-X(0)-\mathbf{u}^n(0,X(0))\\ &=\int_0^t[\nabla^{(2)}\mathbf{u}^n(s,X(s))B(s,X_s)+(0,B(s,X_s))]\mathrm{d}s\\ &+\int_0^t\left\{\lambda\mathbf{u}^n+\nabla\mathbf{u}^n(b-b^n)+(b-b^n)+\mathrm{tr}[(\Sigma-\Sigma^n)\nabla^{(2)}\nabla^{(2)}\mathbf{u}^n]\right\}(s,X(s))\mathrm{d}s\\ &+\int_0^t\nabla^{(2)}\mathbf{u}^n(s,X(s))\sigma(s,X(s))\mathrm{d}W(s)+\int_0^t(0,\sigma(s,X(s)))\mathrm{d}W(s),\quad t\in[0,\tau].\end{aligned} \tag{2.21}$$

So, according to (2.13), (2.14), the boundedness of B and σ and noting that $\{|b(t,\cdot)-b^n(t,\cdot)|+\|\sigma(t,\cdot)-\sigma^n(t,\cdot)\|\}_{n\ge1}$ is bounded uniformly in $t\in[0,T]$ by **(H1)** and converges to 0 as $n\to\infty$, by the dominated convergence theorem, letting $n\to\infty$, we have $\mathbb{P}$-a.s.

$$\begin{aligned}\theta(t,X(t))&=\theta(0,X(0))+\int_0^t[\nabla^{(2)}\theta(s,X(s))B(s,X_s)]\mathrm{d}s+\int_0^t\lambda\mathbf{u}(s,X(s))\mathrm{d}s\\ &\quad+\int_0^t\nabla^{(2)}\theta(s,X(s))\sigma(s,X(s))\mathrm{d}W(s),\quad t\in[0,\tau].\end{aligned} \tag{2.22}$$

Thus, if $\{X_t\}_{t\in[0,\tau]}$ solves (2.1), then $Y_t:=\theta_t(X_t)$ solves the following SDE for $t\in[0,\tau]$:

$$\mathrm{d}Y(t)=\tilde{B}(t,Y_t)\mathrm{d}t+\tilde{b}(t,Y(t))\mathrm{d}t+\tilde{\sigma}(t,Y(t))\mathrm{d}W(t). \tag{2.23}$$

where

$$\begin{aligned}&\tilde{B}(t,\xi)=\nabla^{(2)}\theta(t,\theta^{-1}(t,\xi(0)))B(t,\theta_t^{-1}(\xi)),\quad (t,\xi)\in[0,T]\times\mathscr{C}^{d_1+d_2},\\ &\tilde{b}(t,x)=\lambda\mathbf{u}(t,\theta^{-1}(t,x)),\quad (t,x)\in[0,T]\times\mathbb{R}^{d_1+d_2},\\ &\tilde{\sigma}(t,x)=\nabla^{(2)}\theta(t,\theta^{-1}(t,x))\sigma(t,\theta^{-1}(t,x)),\quad (t,x)\in[0,T]\times\mathbb{R}^{d_1+d_2}.\end{aligned} \tag{2.24}$$

Similarly, a solution $(Y_t)_{t\in[0,\tau]}$ to (2.23) also gives a solution $X_t = \theta_t^{-1}(Y_t)$ to (2.1) for $t \in [0,\tau]$. Since by (2.14)-(2.16), (2.19), (2.20), from the condition on B and σ we see that (2.23) has a unique solution up to time T (see [19, Theorem 4.1]). Thus, (2.1) has a unique solution for $t \in [0,T]$. Moreover, by the arbitrariness of $T > 0$, we conclude that (2.1) has a non-explosive unique solution for all $t \geq 0$.

(2) Next, if σ and b do not depend on $x^{(1)}$, then so does $\mathbf{u}^n$. In this case, if **(H1)**, **(H4)** and (2.7) hold with C_n, ϕ_n and γ_n uniformly in $n \geq 1$, then by [25, (3.24)], we may repeat the above argument to prove the pathwise uniqueness.

(3) In general, by a localization argument as in [13, Proof of Theorem 2.1], we obtain the local existence and uniqueness of SDE (2.1) up to explosion time ζ. More precisely, take $\psi \in C_b^\infty([0,\infty))$ such that $0 \leq \psi \leq 1$, $\psi(u) = 1$ for $u \in [0,1]$ and $\psi(u) = 0$ for $u \in [2,\infty)$. For any $m \geq 1$, $t \in [0,\infty)$, $z \in \mathbb{R}^{d_1+d_2}$, $\xi \in \mathscr{C}^{d_1+d_2}$, let

$$\begin{aligned}
b_m^{(1)}(t,z) &= b^{(1)}(t\wedge m, z^{(1)}\psi(|z|/m), z^{(2)}),\\
b_m^{(2)}(t,z) &= b^{(2)}(t\wedge m, z\psi(|z|/m)),\\
B_m(t,\xi) &= B(t\wedge m,\xi)\psi(\|\xi\|_\infty/m),\\
\sigma_m(t,z) &= \sigma(t\wedge m, z\psi(|z|/m)).
\end{aligned}$$

If either (**A**) or **(H1)**, **(H4)** and (2.5) holds, then for any $m \geq 1$, B_m, σ_m and b_m satisfy the same assumption with some uniform C, ϕ, γ. By (1) and (2), (2.1) for B_m, σ_m and b_m in place of B, σ, b has a unique solution $X^{[m]}(t)$ starting at X_0 which is non-explosive. Let

$$\zeta_0 = 0, \quad \zeta_m = m \wedge \inf\left\{t \geq 0 : |X^{[m]}(t)| \geq m\right\}, \quad m \geq 1.$$

Since $B_m(s,\xi) = B(s,\xi)$, $\sigma_m(s,\xi(0)) = \sigma(s,\xi(0))$ and $b_m(s,\xi(0)) = b(s,\xi(0))$ hold for $s \leq m$, and $\|\xi\|_\infty \leq m$, by (1) and (2), for any $n, m \geq 1$, we have $X^{[m]}(t) = X^{[n]}(t)$ for $t \in [0, \zeta_m \wedge \zeta_n]$. In particular, ζ_m is increasing in m. Let $\zeta = \lim_{m\to\infty} \zeta_m$ and

$$X(t) = \sum_{m=1}^{\infty} 1_{[\zeta_{m-1},\zeta_m)} X^{[m]}(t), \quad t \in [0,\zeta).$$

Then it is easy to see that $X(t)_{t\in[0,\zeta)}$ is a solution to (2.1) with lifetime ζ and, due to (1) and (2), the solution is unique.

(4) Finally, we prove the non-explosion. [22, Lemma 4.4.6] gives the proof for the time-homegeneous SDEs with additive noise. For reader's convenience, we give the proof in detail.

Let X_t^ξ be the solution to (2.1) up to life time $\zeta(\xi)$. For simplicity, we denote $X_t = X_t^\xi$.

Define $\tau_n = \inf\{t \geq 0, |X(t)| \geq n\}$. By Itô's formula, and using (2.8), it is easy to see that

$$
\begin{aligned}
H^p(X(t\wedge\tau_n)) &= H^p(\xi(0)) + p\int_0^{t\wedge\tau_n} H^{p-1}(X(s))\langle B(s,X_s), \nabla^{(2)}H(X(s))\rangle \mathrm{d}s \\
&\quad + p\int_0^{t\wedge\tau_n} H^{p-1}(X(s))\mathscr{L}_s^{\Sigma,b}H(X(s))\mathrm{d}s \\
&\quad + \frac{1}{2}p(p-1)\int_0^{t\wedge\tau_n} H^{p-2}(X(s))\left|\sigma(s,X(s))^*\nabla^{(2)}H(X(s))\right|^2 \mathrm{d}s \\
&\quad + p\int_0^{t\wedge\tau_n} H^{p-1}(X(s))\langle\nabla^{(2)}H(X(s)), \sigma(s,X(s))\mathrm{d}W(s)\rangle \\
&\leq H^p(\xi(0)) + \frac{p^2+3p}{2}\int_0^{t\wedge\tau_n}\Phi(s)\|H(X_s)\|_\infty^p \mathrm{d}s \\
&\quad + p\int_0^{t\wedge\tau_n} H^{p-1}(X(s))\langle\sigma(s,X(s))^*\nabla^{(2)}H(X(s)), \mathrm{d}W(s)\rangle, \quad t\in[0,T].
\end{aligned}
\tag{2.25}
$$

Then by Burkerholder-Davis-Gundy inequality and using (2.8), we have

$$
\begin{aligned}
&p\mathbb{E}\sup_{t\in[0,v]}\left|\int_0^{t\wedge\tau_n} H^{p-1}(X(s))\langle\sigma(s,X(s))^*\nabla^{(2)}H(X(s)), \mathrm{d}W(s)\rangle\right| \\
&\leq C\mathbb{E}\left\{\int_0^{v\wedge\tau_n} p^2\Phi(s)H^{2p}(X(s))\mathrm{d}s\right\}^{\frac{1}{2}} \\
&\leq \mathbb{E}\left\{\sup_{t\in[0,v\wedge\tau_n]} H^p(X(t))\int_0^{v\wedge\tau_n} C^2p^2\Phi(s)H^p(X(s))\mathrm{d}s\right\}^{\frac{1}{2}} \\
&\leq \frac{1}{2}\mathbb{E}\sup_{t\in[0,v\wedge\tau_n]} H^p(X(t)) + \frac{1}{2}\mathbb{E}\int_0^{v\wedge\tau_n} C^2p^2\Phi(s)H^p(X(s))\mathrm{d}s, \quad v\in[0,T]
\end{aligned}
\tag{2.26}
$$

for some constant $C>0$. Combining (2.25) with (2.26), there exists a constant $C_1(p,T,\Phi)$ depending on p,T,Φ such that

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,v\wedge\tau_n]} H^p(X(t)) &\leq 2H^p(\xi(0)) + (p^2+3p)\Phi(T)\int_0^v \mathbb{E}\sup_{q\in[-r,s\wedge\tau_n]} H^p(X(q))\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^{v\wedge\tau_n} C^2p^2\Phi(s)H^p(X(s))\mathrm{d}s \\
&\leq 2H^p(\xi(0)) + C_1(p,T,\Phi)\int_0^v \mathbb{E}\sup_{q\in[-r,s\wedge\tau_n]} H^p(X(q))\mathrm{d}s, \quad v\in[0,T].
\end{aligned}
$$

Thus we have

$$
\mathbb{E}\sup_{t\in[-r,v\wedge\tau_n]} H^p(X(t)) \leq 2\|H(\xi)\|_\infty^p + C_1(p,T,\Phi)\int_0^v \mathbb{E}\sup_{q\in[-r,s\wedge\tau_n]} H^p(X(q))\mathrm{d}s, \quad v\in[0,T].
$$

It follows from Gronwall's lemma that

$$\mathbb{E} \sup_{t\in[-r,T\wedge\tau_n]} H^p(X(t)) \leq 2\|H(\xi)\|_\infty^p \mathrm{e}^{C_1(p,T,\Phi)T}.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\mathbb{E} \sup_{t\in[-r,T]} H^p(X(t)) \leq 2\|H(\xi)\|_\infty^p \mathrm{e}^{C_1(p,T,\Phi)T}.$$

Since $\lim_{|x|\to\infty} H(x) = \infty$, it is easy to see that the solution is non-explosive. $\square$

3 Harnack and Shift Harnack Inequalities

In this section, consider the following stochastic functional Hamiltonian system on $\mathbb{R}^{m+d}$:

$$\begin{cases} \mathrm{d}X(t) = \{AX(t) + MY(t)\}\mathrm{d}t, \\ \mathrm{d}Y(t) = \{Z(X(t),Y(t)) + B(X_t,Y_t)\}\mathrm{d}t + \sigma \mathrm{d}W(t), \end{cases} \tag{3.1}$$

where $W = (W(t))_{t\geq 0}$ is an d-dimensional standard Brownian motion with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, A is an $m\times m$ matrix, M is an $m \times d$ matrix, σ is a $d\times d$ matrix, $Z : \mathbb{R}^{m+d} \to \mathbb{R}^d$, $B : \mathscr{C}^{m+d} \to \mathbb{R}^d$. When $m = 0$, we let $M = 0$ and $\mathscr{C}^{m+d} = \mathscr{C}^d := C([-r, 0]; \mathbb{R}^d)$, and then (3.1) reduces to non-degenerate functional SDEs. Throughout this section, we make the following assumptions:

(A1) (Hypoellipticity) σ is invertible and MM^* is invertible if $m > 0$.

(A2) (Regularity and growth of Z) There exists $\phi \in \mathscr{D}_0 \cap \mathscr{T}_0$ such that for any $z, \bar{z} \in \mathbb{R}^{m+d}$,

$$|Z(z) - Z(\bar{z})| \leq |z^{(1)} - \bar{z}^{(1)}|^{\frac{2}{3}} \phi(|z^{(1)} - \bar{z}^{(1)}|) + \phi(|z^{(2)} - \bar{z}^{(2)}|). \tag{3.2}$$

Moreover, there exists a constant $C > 0$ such that

$$\langle Z(z), z_2\rangle \leq C(1 + |z|^2), \quad z \in \mathbb{R}^{m+d}. \tag{3.3}$$

(A3) (Regularity of B) There exists a constant $C > 0$ such that for any $\xi, \eta \in \mathscr{C}^{m+d}$,

$$|B(\xi) - B(\eta)| \leq C\|\xi - \eta\|_\infty. \tag{3.4}$$

(A4) (Hörmander-type rank condition) There exists an integer $0 \leq k \leq m - 1$ such that

$$\mathrm{Rank}[M, AM, \cdots, A^k M] = m.$$

Remark 3.1. *Taking $H(z) = 1 + |z|^2$ in (2.8), according to Theorem 2.1 (1) and (3),* **(A1)**-**(A3)** *implies that* (3.1) *has a unique non-explosive strong solution X_t^ξ with $X_0 = \xi \in \mathscr{C}^{m+d}$. Let P_t be the associated Markov semigroup, i.e.*

$$P_t f(\xi) = \mathbb{E} f(X_t^\xi), \quad f \in \mathscr{B}_b(\mathscr{C}^{m+d}).$$

(A4) *is used to construct the coupling by change of measure for the Harnack inequalities and shift Harnack inequalities.*

3.1 Harnack inequalities

We use the coupling constructed in [4] to derive the Harnack inequalities.

Theorem 3.2. *Assume* **(A1)**-**(A4)** *and let* $T > r$. *Then for any* $\xi = (\xi_1, \xi_2), h = (h_1, h_2) \in \mathscr{C}^{m+d}$ *and positive* $f \in \mathscr{B}_b(\mathscr{C}^{m+d})$,

$$P_T \log f(\xi + h) \leq \log P_T f(\xi) + \Sigma(T, h, r),$$

and

$$(P_T f)^p(\xi + h) \leq P_T f^p(\xi) \exp\left[\frac{p}{2(p-1)}\Sigma(T, h, r)\right],$$

where

$$\begin{aligned}\Sigma(T, h, r) = {} & C(T-r)|h(0)|^2\Big(\frac{1}{(T-r)\wedge 1} + \frac{\|M\|}{(T-r)^{2(k+1)}\wedge 1}\Big)^2\\ & + CT\left|\|M\||h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}\wedge 1}\Big)\right|^{\frac{4}{3}}\phi^2\left(C\|M\||h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}\wedge 1}\Big)\right)\\ & + CT\phi^2\left(C|h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k+1}\wedge 1}\Big)\right)\\ & + CT\Big(\|h\|_\infty + \frac{\|M\|\cdot|h(0)|}{(T-r)^{2k+1}\wedge 1}\Big)^2,\end{aligned}$$

and $C > 0$ *is a constant. If* $m = 0$ *then the assertion holds for* $\|M\| = 0$. *In addition, since* $\lim_{\|h\|_\infty \to 0} \Sigma(h, T, r) = 0$, P_T *is strong Feller for any* $T > r$.

Recall that for two probability measures μ, ν on some measurable space $(E, \mathscr{F})$, the entropy and total variation norm are defined as follows:

$$\mathrm{Ent}(\nu|\mu) := \begin{cases} \int (\log \frac{\mathrm{d}\nu}{\mathrm{d}\mu})\mathrm{d}\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu,\\ \infty, & \text{otherwise;}\end{cases}$$

and

$$\|\mu - \nu\|_{var} := \sup_{A\in\mathscr{F}} |\mu(A) - \nu(A)|.$$

By Pinsker's inequality (see [7, 17]),

$$\|\mu - \nu\|_{var}^2 \leq \frac{1}{2}\mathrm{Ent}(\nu|\mu), \quad \mu, \nu \in \mathscr{P}(E), \tag{3.5}$$

here $\mathscr{P}(E)$ denotes all probability measures on $(E, \mathscr{F})$. The next corollary following from Theorem 3.2 describes the property of the transition probability, see [22, Theorem 1.4.2] for the proof.

Corollary 3.3. *Let the assumption in Theorem 3.2 hold and* $T > r$. *Then the following assertions hold.*

(i) For any $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathscr{C}^{m+d}$, $P_T(\xi, \cdot)$ is equivalent to $P_T(\eta, \cdot)$ and

$$\mathrm{Ent}(P_T(\xi,\cdot)|P_T(\eta,\cdot)) := P_T\left\{\log \frac{\mathrm{d}P_T(\xi,\cdot)}{\mathrm{d}P_T(\eta,\cdot)}\right\}(\xi) \le \Sigma(T, \xi-\eta, r),$$

which together with Pinsker's inequality (3.5) *implies that*

$$\|P_T(\xi,\cdot) - P_T(\eta,\cdot)\|_{var}^2 \le \frac{1}{2}\Sigma(T,\xi-\eta,r). \tag{3.6}$$

Moreover, for any $p > 1$,

$$P_T\left\{\left(\frac{\mathrm{d}P_T(\xi,\cdot)}{\mathrm{d}P_T(\eta,\cdot)}\right)^{\frac{1}{p-1}}\right\}(\xi) \le \exp\left[\frac{p}{2(p-1)^2}\Sigma(T,\xi-\eta,r)\right].$$

(ii) For an invariant probability measure μ of P_T, the entropy-cost inequality

$$\mu((P_T^* f)\log P_T^* f) \le \mathbb{W}_1^{\Sigma}(f\mu, \mu), \quad f \ge 0, \mu(f) = 1$$

holds for P_T^*, the adjoint operator of P_T in $L^2(\mu)$, where for any two probability measures μ_1, μ_2 on $\mathscr{C}$,

$$\mathbb{W}_1^{\Sigma}(\mu_1,\mu_2) := \inf_{\pi \in \mathbf{C}(\mu_1,\mu_2)} \int_{\mathscr{C}\times\mathscr{C}} \Sigma(T,\xi-\eta,r)\pi(\mathrm{d}\xi,\mathrm{d}\eta)$$

and $\mathbf{C}(\mu_1,\mu_2)$ is the set of all couplings of μ_1 and μ_2.

Proof of Theorem 3.2. By the semigroup property and Jensen's inequality, we only need to consider $T - r \in (0, 1]$. For any $\eta \in \mathscr{C}^{m+d}$, let $(X^\eta(t), Y^\eta(t))$ solve (3.1) with $(X_0, Y_0) = \eta$. As in [4], for $h = (h_1, h_2) \in \mathscr{C}^{m+d}$, define

$$\tilde{\gamma}(s) := \tilde{v}(s)h_2(0) + \tilde{\alpha}(s), \quad s \in [0, T]$$

with

$$\begin{aligned}
\tilde{v}(s) &= \frac{(T-r-s)^+}{T-r},\\
\tilde{\alpha}(s) &= -\frac{s(T-r-s)^+}{(T-r)^2}M^*\mathrm{e}^{-sA^*}\\
&\qquad \tilde{Q}_{T-r}^{-1}\left(h_1(0) + \int_0^{T-r} \frac{(T-u-r)^+}{T-r}\mathrm{e}^{-uA}Mh_2(0)\mathrm{d}u\right), \quad s \in [0,T],
\end{aligned}$$

where by convention $M = 0$ (hence, $\alpha = 0$) if $m = 0$ and

$$\tilde{Q}_t := \int_0^t \frac{s(T-r-s)}{(T-r)^2}\mathrm{e}^{-sA}MM^*\mathrm{e}^{-sA^*}\mathrm{d}s, \quad t \in [0, T-r].$$

According to [18] (see also [25, Proof of Theorem 4.2(1)]), when $m \geq 1$ the matrix Q_t is invertible with

$$\|\tilde{Q}_t^{-1}\| \leq c(T-r)(t \wedge 1)^{-2(k+1)}, \quad t \in [0, T-r] \tag{3.7}$$

for some constant $c > 0$. Let $(\tilde{X}(t), \tilde{Y}(t))$ solve the equation

$$\begin{cases} \mathrm{d}\tilde{X}(t) = \{A\tilde{X}(t) + M\tilde{Y}(t)\}\mathrm{d}t, \\ \mathrm{d}\tilde{Y}(t) = \{Z(X^\xi(t), Y^\xi(t)) + B(X^\xi_t, Y^\xi_t)\}\mathrm{d}t + \sigma \mathrm{d}W(t) + \tilde{\gamma}'(t)\mathrm{d}t \end{cases} \tag{3.8}$$

with $(\tilde{X}_0, \tilde{Y}_0) = \xi + h$. Then the solution to (3.8) is non-explosive as well. Moreover, let $\Theta(s) = h(s)$ for $s \in [-r, 0]$ and

$$\Theta(s) = (\Theta_1(s), \Theta_2(s)) := \left(\mathrm{e}^{As} h_1(0) + \int_0^s \mathrm{e}^{(s-u)A} M\tilde{\gamma}(u)\mathrm{d}u,\ \tilde{\gamma}(s)\right), \quad s \in [0, T].$$

Then, for any $s \in [0, T]$,

$$\begin{aligned} &|\Theta_2'(s)| \leq C 1_{[0,T-r]}(s)|h(0)|\Big(\frac{1}{T-r} + \frac{\|M\|}{(T-r)^{2(k+1)}}\Big), \\ &|\Theta_1(s)| \leq C\|M\||h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}}\Big), \\ &|\Theta_2(s)| \leq C|h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k+1}}\Big), \\ &\|\Theta_s\|_\infty \leq C\Big(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T-r)^{2k+1}}\Big) \end{aligned} \tag{3.9}$$

for some constant $C > 0$ and

$$(\tilde{X}(s), \tilde{Y}(s)) = (X^\xi(s), Y^\xi(s)) + \Theta(s), \quad s \in [-r, T], \tag{3.10}$$

in particular, $(\tilde{X}_T, \tilde{Y}_T) = (X^\xi_T, Y^\xi_T)$. Let

$$\Phi(s) = Z(X^\xi(s), Y^\xi(s)) - Z(\tilde{X}(s), \tilde{Y}(s)) + B(X^\xi_s, Y^\xi_s) - B(\tilde{X}_s, \tilde{Y}_s) + \Theta_2'(s), \quad s \in [0, T].$$

From **(A2)**-**(A3)** and (3.9), it holds that

$$\begin{aligned} &\int_0^T |\Phi(s)|^2 \mathrm{d}s \\ &\leq C\int_0^T \Big(|\Theta_1(s)|^{\frac{2}{3}}\phi(|\Theta_1(s)|) + \phi(|\Theta_2(s)|) + \|\Theta_s\|_\infty + |\Theta_2'(s)|\Big)^2 \mathrm{d}s \\ &\leq C(T-r)|h(0)|^2\Big(\frac{1}{T-r} + \frac{\|M\|}{(T-r)^{2(k+1)}}\Big)^2 \\ &+ CT\left|\|M\||h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}}\Big)\right|^{\frac{4}{3}} \phi^2\left(C\|M\||h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}}\Big)\right) \\ &+ CT\phi^2\left(C|h(0)|\Big(1 + \frac{\|M\|}{(T-r)^{2k+1}}\Big)\right) \\ &+ CT\Big(\|h\|_\infty + \frac{\|M\| \cdot |h(0)|}{(T-r)^{2k+1}}\Big)^2. \end{aligned} \tag{3.11}$$

Girsanov's theorem implies that

$$\tilde{W} := \int_0^\cdot \sigma^{-1}\Phi(u)\mathrm{d}u + W$$

is a Brownian motion on $[0, T]$ under $\mathbb{Q}_T = R(T)\mathbb{P}$, where

$$R(T) = \exp\bigg[- \int_0^T \langle \sigma^{-1}\Phi(u), \mathrm{d}W(u)\rangle - \frac{1}{2}\int_0^T |\sigma^{-1}\Phi(u)|^2 \mathrm{d}u \bigg].$$

Then (3.8) reduces to

$$\begin{cases} \mathrm{d}\tilde{X}(t) = \{A\tilde{X}(t) + M\tilde{Y}(t)\}\mathrm{d}t, \\ \mathrm{d}\tilde{Y}(t) = \{Z(\tilde{X}(t), \tilde{Y}(t)) + B(\tilde{X}_t, \tilde{Y}_t)\}\mathrm{d}t + \sigma \mathrm{d}\tilde{W}(t) \end{cases} \tag{3.12}$$

which implies that the distribution of $(\tilde{X}_T, \tilde{Y}_T)$ under $\mathbb{Q}_T$ coincides with that of $(X_T^{\xi+h}, Y_T^{\xi+h})$ under $\mathbb{P}$.

On the other hand, by Young's inequality,

$$\begin{aligned} P_T \log f(\xi + h) &= \mathbb{E}^{\mathbb{Q}_T} \log f((\tilde{X}_T, \tilde{Y}_T)) \\ &= \mathbb{E}^{\mathbb{Q}_T} \log f((X_T^\xi, Y_T^\xi)) \le \log P_T f(\xi) + \mathbb{E}R(T)\log R(T), \end{aligned}$$

and by Hölder inequality,

$$\begin{aligned} P_T f(\xi + h) &= \mathbb{E}^{\mathbb{Q}_T} f((\tilde{X}_T, \tilde{Y}_T)) \\ &= \mathbb{E}^{\mathbb{Q}_T} f((X_T^\xi, Y_T^\xi)) \le (P_T f^p(\xi))^{\frac{1}{p}} \{\mathbb{E}R(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}}. \end{aligned}$$

Since $\tilde{W}$ is a Brownian motion under $\mathbb{Q}_T$, by the definition of $R(T)$, it is easy to see that

$$\mathbb{E}R(T)\log R(T) = \mathbb{E}^{\mathbb{Q}_T} \log R(T) = \frac{1}{2}\mathbb{E}^{\mathbb{Q}_T} \int_0^T |\sigma^{-1}\Phi(u)|^2 \mathrm{d}u \le \Sigma(h, T, r),$$

and

$$\begin{aligned} &\mathbb{E}R(T)^{\frac{p}{p-1}} \\ &\le \mathbb{E}\bigg\{ \exp\bigg[- \frac{p}{p-1}\int_0^T \langle \sigma^{-1}\Phi(u), \mathrm{d}W(u)\rangle - \frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^s |\sigma^{-1}\Phi(u)|^2\mathrm{d}u \bigg] \\ &\qquad \times \exp\bigg[\frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^s |\sigma^{-1}\Phi(u)|^2\mathrm{d}u - \frac{1}{2}\frac{p}{p-1}\int_0^s |\sigma^{-1}\Phi(u)|^2\mathrm{d}u \bigg] \bigg\} \\ &\le \operatorname*{ess\,sup}_\Omega \exp\bigg\{ \frac{p}{2(p-1)^2}\int_0^T |\sigma^{-1}\Phi(u)|^2\mathrm{d}u \bigg\}. \end{aligned}$$

Combining (3.11), we derive the Harnack inequality. Finally, since $\lim_{\|h\|_\infty \to 0} \Sigma(h, T, r) = 0$, the strong Feller of P_T for $T > r$ follows from [22, Theorem 1.4.1 (1)]. □

3.2 Shift Harnack Inequalities

The following result provides shift Harnack inequalities $P_T, T > r$, and the coupling by change of measure is new.

Theorem 3.4. *Assume* **(A1)-(A4)** *and let* $T > r$. *Then for any* $\xi = (\xi_1, \xi_2) \in \mathscr{C}^{m+d}$, $\eta = (\eta_1, \eta_2) \in C^1([-r, 0]; \mathbb{R}^{m+d})$ *satisfying*

$$\int_{-r}^{0} |\eta_2'(s)|^2 \mathrm{d}s < \infty,$$

and

$$\mathrm{e}^{-sA}\eta_1(s) - \mathrm{e}^{rA}\eta_1(-r) = \int_{-r}^{s} \mathrm{e}^{-uA} M \eta_2(u)\mathrm{d}u, \quad s \in [-r, 0], \tag{3.13}$$

which is equivalent to

$$\mathrm{e}^{-sA}\eta_1(s-T) - \mathrm{e}^{(r-T)A}\eta_1(-r) = \int_{T-r}^{s} \mathrm{e}^{-uA} M \eta_2(u-T)\mathrm{d}u, \quad s \in [T-r, T], \tag{3.14}$$

and any positive $f \in \mathscr{B}_b(\mathscr{C}^{m+d})$,

$$P_T \log f(\xi) \le (\log P_T f(\eta + \cdot))(\xi) + \beta(T, \eta, r),$$

and

$$(P_T f)^p(\xi) \le (P_T f^p(\eta + \cdot))(\xi) \exp\left[\frac{p}{2(p-1)}\beta(T, \eta, r)\right]$$

where

$$\begin{aligned}
\beta(T, \eta, r) = {} & C(T-r)|\eta(-r)|^2 \Big(\frac{1}{(T-r)\wedge 1} + \frac{\|M\|}{(T-r)^{2(k+1)} \wedge 1}\Big)^2 + C\int_{-r}^{0} |\eta_2'(s)|^2 \mathrm{d}s \\
& + T\left|C\|M\||\eta(-r)|\Big(1 + \frac{\|M\|}{(T-r)^{2k} \wedge 1}\Big) + C\|\eta_1\|_\infty\right|^{\frac{4}{3}} \\
& \quad \times \phi^2\left(C\|M\||\eta(-r)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}\wedge 1}\Big) + C\|\eta_1\|_\infty\right) \\
& + CT\phi^2\left(C|\eta(-r)|\Big(1 + \frac{\|M\|}{(T-r)^{2k+1}\wedge 1}\Big) + C\|\eta_2\|_\infty\right) \\
& + CT\Big(\|\eta\|_\infty + \frac{\|M\| \cdot |\eta(-r)|}{(T-r)^{2k+1} \wedge 1}\Big)^2,
\end{aligned}$$

and $C > 0$ *is a constant. If* $m = 0$ *then the assertion holds for* $\|M\| = 0$.

The following corollary is a direct conclusion of Theorem 3.4 in the case $r = 0$, see [22, Theorem 1.4.3 (2)].

Corollary 3.5. *If $r=0$, then under the assumption of Theorem 3.4, for any $T>0$, P_T has transition density $p_T(x,y)$ with respect to the Lebesgue measure such that*

$$\int_{\mathbb{R}^{m+d}} p_T(x,y)^{\frac{p}{p-1}}\mathrm{d}y \le \frac{1}{(\int_{\mathbb{R}^{m+d}} \mathrm{e}^{-\frac{p}{2(p-1)}\beta(T,\eta)}\mathrm{d}\eta)^{\frac{1}{p-1}}}$$

for any $p>1$, where

$$\begin{aligned}\beta(T,\eta) &= C(T\wedge 1)|\eta|^2\Big(\frac{1}{T\wedge 1}+\frac{\|M\|}{(T\wedge 1)^{2(k+1)}}\Big)^2\\ &\quad + T\left|C\|M\||\eta|\Big(1+\frac{\|M\|}{(T\wedge 1)^{2k}}\Big)+C|\eta_1|\right|^{\frac{4}{3}}\\ &\qquad \times \phi^2\left(C\|M\||\eta|\Big(1+\frac{\|M\|}{(T\wedge 1)^{2k}}\Big)+C|\eta_1|\right)\\ &\quad + CT\phi^2\left(C|\eta|\Big(1+\frac{\|M\|}{(T\wedge 1)^{2k+1}}\Big)+C|\eta_2|\right)\\ &\quad + CT\Big(|\eta|+\frac{\|M\|\cdot|\eta|}{(T\wedge 1)^{2k+1}}\Big)^2.\end{aligned}$$

If $m=0$ then the assertion holds for $\|M\|=0$.

Proof of Theorem 3.4. Again by the semigroup property and Jensen's inequality, we only need to consider $T-r\in(0,1]$. Define

$$\gamma(s) := \begin{cases} v(s)\eta_2(-r)+\alpha(s), & \text{if } s\in[-r,T-r],\\ \eta_2(s-T), & \text{if } s\in(T-r,T],\end{cases}$$

with

$$\begin{aligned} v(s) &= \frac{s^+}{T-r},\\ \alpha(s) &= \frac{s^+(T-r-s)}{(T-r)^2}M^*\mathrm{e}^{(T-r-s)A^*}\\ &\qquad Q_{T-r}^{-1}\Big(\eta_1(-r)-\int_0^{T-r}\frac{u}{T-r}\mathrm{e}^{(T-r-u)A}M\eta_2(-r)\mathrm{d}u\Big),\quad s\in[-r,T-r],\end{aligned}$$

where by convention $M=0$ (hence, $\alpha=0$) if $m=0$ and

$$Q_t := \int_0^t \frac{s(T-r-s)}{(T-r)^2}\mathrm{e}^{(T-r-s)A}MM^*\mathrm{e}^{(T-r-s)A^*}\mathrm{d}s,\quad t\in[0,T-r].$$

According to [18] (see also [25, Proof of Theorem 4.2(1)]), when $m\ge 1$ the matrix Q_t is invertible with

$$\|Q_t^{-1}\| \le c(T-r)(t\wedge 1)^{-2(k+1)},\quad t\in[0,T-r] \tag{3.15}$$

for some constant $c > 0$.

Next, we construct couplings. For fixed $\xi = (\xi_1, \xi_2) \in \mathscr{C}^{m+d}$, let $(X(t), Y(t))$ solve (3.1) with $(X_0, Y_0) = \xi$; and let $(\bar{X}(t), \bar{Y}(t))$ solve the equation

$$
\begin{cases} \mathrm{d}\bar{X}(t) = \{A\bar{X}(t) + M\bar{Y}(t)\}\mathrm{d}t, \\ \mathrm{d}\bar{Y}(t) = \{Z(X(t), Y(t)) + B(X_t, Y_t)\}\mathrm{d}t + \sigma \mathrm{d}W(t) + \gamma'(t)\mathrm{d}t \end{cases} \tag{3.16}
$$

with $(\bar{X}_0, \bar{Y}_0) = \xi$. Then the solution to (3.16) is non-explosive as well. Moreover,

$$
(\bar{X}(s), \bar{Y}(s)) = (X(s), Y(s)) + \Gamma(s), \quad s \in [-r, T] \tag{3.17}
$$

holds for

$$
\Gamma(s) = (\Gamma_1(s), \Gamma_2(s)) := \left(\int_0^s \mathrm{e}^{(s-u)A} M\gamma(u)\mathrm{d}u, \ \gamma(s) \right), \quad s \in [-r, T].
$$

Noting that

$$
\int_0^{T-r} \mathrm{e}^{(T-r-s)A} M\gamma(s)\mathrm{d}s = \eta_1(-r), \quad t \geq T - r \tag{3.18}
$$

from the definition of γ, v and α, we derive from (3.13) and (3.18) that

$$
(\bar{X}_T, \bar{Y}_T) = (X_T + \eta_1, Y_T + \eta_2)
$$

since $\gamma(s) = \eta_2(s - T), s \in [T - r, T]$. By (3.15) and the definitions of α and v, there exists a constant $C > 0$ such that for any $s \in [0, T]$,

$$
\begin{aligned}
&|\gamma'(s)| \leq C1_{[0,T-r]}(s)|\eta(-r)|\Big(\frac{1}{T-r} + \frac{\|M\|}{(T-r)^{2(k+1)}}\Big) + C1_{[T-r,T]}(s)|\eta_2'(s-T)|, \\
&|\Gamma_1(s)| \leq C\|M\||\eta(-r)|\Big(1 + \frac{\|M\|}{(T-r)^{2k}}\Big) + C\|\eta_1\|_\infty, \\
&|\Gamma_2(s)| \leq C|\eta(-r)|\Big(1 + \frac{\|M\|}{(T-r)^{2k+1}}\Big) + C\|\eta_2\|_\infty, \\
&\|\Gamma_s\|_\infty \leq C\Big(\|\eta\|_\infty + \frac{\|M\| \cdot |\eta(-r)|}{(T-r)^{2k+1}}\Big).
\end{aligned} \tag{3.19}
$$

Let

$$
\bar{\Phi}(s) = Z(X(s), Y(s)) - Z(\bar{X}(s), \bar{Y}(s)) + B(X_s, Y_s) - B(\bar{X}_s, \bar{Y}_s) + \gamma'(s).
$$

From **(A2)**-**(A3)** and (3.19), it holds

$$
\begin{aligned}
\int_0^T |\bar{\Phi}(s)|^2 \mathrm{d}s &\le C \int_0^T \Big(|\Gamma_1(s)|^{\frac{2}{3}} \phi(|\Gamma_1(s)|) + \phi(|\Gamma_2(s)|) + \|\Gamma_s\|_\infty + |\gamma'(s)| \Big)^2 \mathrm{d}s \\
&\le C(T-r)|\eta(-r)|^2 \Big(\frac{1}{T-r} + \frac{\|M\|}{(T-r)^{2(k+1)}} \Big)^2 + C \int_{-r}^0 |\eta_2'(s)|^2 \mathrm{d}s \\
&+ T \left| C\|M\| |\eta(-r)| \Big(1 + \frac{\|M\|}{(T-r)^{2k}} \Big) + C\|\eta_1\|_\infty \right|^{\frac{4}{3}} \\
&\qquad \times \phi^2 \left(C\|M\| |\eta(-r)| \Big(1 + \frac{\|M\|}{(T-r)^{2k}} \Big) + C\|\eta_1\|_\infty \right) \\
&+ CT\phi^2 \left(C|\eta(-r)| \Big(1 + \frac{\|M\|}{(T-r)^{2k+1}} \Big) + C\|\eta_2\|_\infty \right) \\
&+ CT \Big(\|\eta\|_\infty + \frac{\|M\| \cdot |\eta(-r)|}{(T-r)^{2k+1}} \Big)^2 .
\end{aligned}
\tag{3.20}
$$

Set

$$
\bar{R}(s) = \exp\Big[- \int_0^s \langle \sigma^{-1} \bar{\Phi}(u), \mathrm{d}W(u) \rangle - \frac{1}{2} \int_0^s |\sigma^{-1} \bar{\Phi}(u)|^2 \mathrm{d}u \Big],
$$

and

$$
\bar{W}(s) = W(s) + \int_0^s \sigma^{-1} \bar{\Phi}(u) \mathrm{d}u.
$$

Girsanov's theorem implies that $\bar{W}$ is a Brownnian motion on $[0,T]$ under $\bar{\mathbb{Q}}_T = \bar{R}(T)\mathbb{P}$. Then (3.16) reduces to

$$
\begin{cases}
\mathrm{d}\bar{X}(t) = \{A\bar{X}(t) + M\bar{Y}(t)\}\mathrm{d}t, \\
\mathrm{d}\bar{Y}(t) = \{Z(\bar{X}(t), \bar{Y}(t)) + B(\bar{X}_t, \bar{Y}_t)\}\mathrm{d}t + \sigma \mathrm{d}\bar{W}(t).
\end{cases}
\tag{3.21}
$$

Thus, the distribution of $(\bar{X}_T, \bar{Y}_T)$ under $\bar{\mathbb{Q}}_T$ coincides with that of (X_T, Y_T) under $\mathbb{P}$.

On the other hand, by Young's inequality,

$$
\begin{aligned}
P_T \log f(\xi) &= \mathbb{E}^{\bar{\mathbb{Q}}_T} \log f((\bar{X}_T, \bar{Y}_T)) \\
&= \mathbb{E}^{\bar{\mathbb{Q}}_T} \log f((X_T + \eta_1, Y_T + \eta_2)) \\
&\le \log P_T f(\cdot + \eta)(\xi) + \mathbb{E}\bar{R}(T) \log \bar{R}(T),
\end{aligned}
$$

and by Hölder inequality,

$$
\begin{aligned}
P_T f(\xi) &= \mathbb{E}^{\bar{\mathbb{Q}}_T} f((\bar{X}_T, \bar{Y}_T)) \\
&= \mathbb{E}^{\bar{\mathbb{Q}}_T} f((X_T + \eta_1, Y_T + \eta_2)) \le (P_T f^p(\cdot + \eta))^{\frac{1}{p}}(\xi) \{\mathbb{E}\bar{R}(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}}.
\end{aligned}
$$

Since $\bar{W}$ is a Brownian motion under $\bar{\mathbb{Q}}_T$, by the definition of $\bar{R}(T)$, it is easy to see that

$$
\mathbb{E}\bar{R}(T) \log \bar{R}(T) = \mathbb{E}^{\bar{\mathbb{Q}}_T} \log \bar{R}(T) = \frac{1}{2} \mathbb{E} \int_0^T |\sigma^{-1} \bar{\Phi}(u)|^2 \mathrm{d}u \le \beta(T, \eta, r),
$$

and

$$
\begin{aligned}
&\mathbb{E}\bar{R}(T)^{\frac{p}{p-1}}\\
&\leq \mathbb{E}\Bigg\{ \exp\left[-\frac{p}{p-1}\int_0^T \langle (\sigma^{-1}\bar{\Phi}(u), \mathrm{d}W(u)\rangle - \frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^s |\sigma^{-1}\bar{\Phi}(u)|^2\mathrm{d}u \right]\\
&\qquad\qquad \times \exp\left[\frac{1}{2}\frac{p^2}{(p-1)^2}\int_0^s |\sigma^{-1}\bar{\Phi}(u)|^2\mathrm{d}u - \frac{1}{2}\frac{p}{p-1}\int_0^s |\sigma^{-1}\bar{\Phi}(u)|^2\mathrm{d}u\right]\Bigg\}\\
&\leq \operatorname*{ess\,sup}_{\Omega}\exp\left\{\frac{p}{2(p-1)^2}\int_0^T |\sigma^{-1}\bar{\Phi}(u)|^2\mathrm{d}u\right\}.
\end{aligned}
$$

Combining (3.20), the shift Harncak inequality holds. □

Remark 3.6. *In fact, from the construction of the coupling by change of measure, we only use the weak existence and uniqueness of* (3.1)*. Thus, we may obtain Harnack and shift Harnack inequalities under weaker conditions, see [5, 6, 8, 9, 28] for conditions on the weak uniqueness of SDEs with degenerate noise.*

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