STRATIFIED BUNDLES AND REPRESENTATION SPACES

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Abstract. For a given stratified bundle E on X, we construct an irreducible closed subvariety $\mathcal{N}(E)_S$ of the so called representation space $R(\mathcal{O}_{X_S}, \xi_S, P) \to S$ such that $\mathcal{N}(E)_S(\overline{\mathbb{F}}_q)$ contains a dense set of (V, β) where V is induced by a representation of $\pi_1^{\text{ét}}(X)$ and β is a frame of V at a given point (Theorem 3.7). As an application, we give a simple proof of the main theorem of [1] and its relative version (Theorem 4.2).

1. Introduction

Let X be a smooth, connected projective variety over an algebraically closed field k of characteristic $p > 0$, $\mathcal{D}_X$ be the sheaf of differential operators (in the sense of Grothendieck) and $\pi_1 = \pi_1^{\text{ét}}(X, \xi)$ be the étale fundamental group of X. For any representation $\rho : \pi_1 \to \mathrm{GL}(V)$, one can associate to ρ a $\mathcal{D}_X$-module V_ρ. Thus D. Gieseker proved the following results (see Theorem 1.10 of [4]): (i) if every $\mathcal{D}_X$-module on X is trivial, then π_1 is trivial; (ii) if all irreducible $\mathcal{D}_X$-modules are rank 1, then $[\pi_1, \pi_1]$ is a pro-p-group; (iii) if every $\mathcal{D}_X$-module is a direct sum of rank 1 $\mathcal{D}_X$-modules, then π_1 is abelian with no p-power order quotient. Following D. Gieseker, a $\mathcal{D}_X$-module E will be called a stratified bundle.

Gieseker also made the conjecture that the converses of above statements might be true. The converse of statement (i) was proved in [1], and converses of the statements (ii) and (iii) were proved in [3]. The key in these proofs is to produce a non-trivial representation of $\pi_1 = \pi_1^{\text{ét}}(X, \xi)$ from a non-trivial given stratified bundle E. An equivalent characterization of stratified bundle is that $E = (E_i)_{i \in \mathbb{N}}$ with $E_i = F_X^* E_{i+1}$ ($\forall\ i \in \mathbb{N}$) where $F_X : X \to X$ is the Frobenius map. Then it is not difficult to prove that there is an integer n_0 such that E_i ($i \geq n_0$) are p-semistable bundles with numerically trivial Chern classes.

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If the set $\Sigma = \{E_i\}_{i\geq n_0}$ of isomorphism classes of the bundles E_i is finite, then there is an F-periodic bundle E_{i_0} (i.e. there is an integer N such that $(F_X^*)^N E_{i_0} = E_{i_0}$) which induces a representation of π_1 by a theorem of Lange-Stuhler (Lemma 3.4).

When the set $\Sigma = \{E_i\}_{i\geq n_0}$ of isomorphism classes of the bundles E_i is an infinite set, a theorem of Hrushovski is used to get an F-periodic bundle on a good reduction $X_{\bar{s}}/\overline{\mathbb{F}}_q$ of X. If we have a moduli space M parametrizing **isomorphism classes** of semistable bundles, we would have a subvariety $\mathcal{N}(E) \subset M$ (by taking Zariski closure of $\Sigma = \{E_i\}_{i\geq n_0}$) such that Frobenius pullback F_X^* induces a dominant rational map $F_X^* : \mathcal{N}(E) \dashrightarrow \mathcal{N}(E)$. Then, if $k = \overline{\mathbb{F}}_q$, we find a dense set of F-periodic bundles (thus a dense set of representations of π_1) by Hrushovski's theorem. Unfortunately, we have only a moduli space M parametrizing **S-equivalence classes** of semistable bundles. Thus the approach of proving Gieseker conjecture in [1] and [3] consists of two steps: (1) prove the theorem for irreducible stratified bundles (in this case, $\Sigma = \{E_i\}_{i\geq n_0}$ consists of stable bundles), (2) studying the extensions of irreducible stratified bundles.

Let X be a projective variety over a perfect field k with a point

$$\xi : \mathrm{Spec}(k) \to X.$$

We observe in this article that for any stratified bundle $E = (E_i)_{i\in\mathbb{N}}$ of rank r there is a natural way to choose frames $\beta_i : \xi^* E_i \cong \mathcal{O}_X^{\oplus r}$ such that $(E_i, \beta_i) = F_X^*(E_{i+1}, \beta_{i+1})$ (see Lemma 3.3). Moreover, the set $R(E)_{n_0} = \{\alpha_i = (E_i, \beta_i)\}_{i\geq n_0}$ is a set of k-points of a moduli space $R(\mathcal{O}_X, \xi, P)$, which parametrizes **isomorphism classes** of (V, β) (i.e. semistable bundles V with frames β at $\xi \in X$) and was called the **Representation Space** by Simpson.

In Section 2 of this article, we generalize Simpson's construction of representation spaces $R(\mathcal{O}_X, \xi, P)$ to the case of characteristic $p > 0$ (see Theorem 2.3) and prove that Frobenius pullback F_X^* induces a rational map $f : R(\mathcal{O}_X, \xi, P) \dashrightarrow R(\mathcal{O}_X, \xi, P)$ (see Proposition 2.5). In Section 3, for a stratified bundle $E = (E_i)_{i\in\mathbf{N}}$ such that $\Sigma = \{E_i\}_{i\geq n_0}$ is an infinite set, we construct a closed subvariety $\mathcal{N}(E) \subset R(\mathcal{O}_X, \xi, P)$ such that $f : R(\mathcal{O}_X, \xi, P) \dashrightarrow R(\mathcal{O}_X, \xi, P)$ induces a dominant rational map $f^a : \mathcal{N}(E) \dashrightarrow \mathcal{N}(E)$ and $\mathcal{N}(E)(k) \cap R(E)_{n_0}$ is an infinite set (see Theorem 3.7). In Section 4, we use the construction of Section 3 to give a uniform proof (see Theorem 4.1) of the main theorem in [1], which says that there is no nontrivial stratified bundle on X if $\pi_1 = \pi_1^{\text{ét}}(X, \xi)$ is trivial. For example, when $k = \overline{\mathbb{F}}_q$, $\mathcal{N}(E)$ contains a dense set of points (V, β) such that V is induced by a representation of π_1. On the other hand, if $E = (E_i)_{i\in\mathbf{N}}$ is nontrivial, we can assume that all bundles E_i in $\Sigma = \{E_i\}_{i \geq n_0}$ are nontrivial, then the set

$$\mathcal{U} = \{\, (V, \beta) \in \mathcal{N}(E) \,|\, V \text{ is nontrivial}\}$$

is a nonempty open set, which must contain a point (V, β) such that V is induced by a representation of π_1 and we get a contradiction if π_1 is trivial. These arguments are easily applied to prove relative version of this theorem (see Theorem 4.2).

Acknowledegements: Theorem 4.2 (see [2] for an another proof) was a question that Hélène Esnault posed to me when I visited Berlin on 2013, where I proved immediately the irreducible case of Theorem 4.2 in a unpublished note (in fact, I proved the theorem for stratified bundles which are extensions of two irreducible stratified bundles). I thank her very much for the question and discussions. I would also like to thank the anonymous referees for their carefully reading and helpful comments, which improve the article very much.

2. Representation spaces and Frobenius map

Let X be an irreducible projective variety with a fixed ample line bundle $\mathcal{O}_X(1)$. For a torsion free sheaf $\mathcal{E}$ of rank $r(\mathcal{E})$ on X, $P(\mathcal{E}, m) = \chi(\mathcal{E}(m))$ is a polynomial in m (the so called Hilbert polynomial of $\mathcal{E}$) with degree $n = \dim X$.

A torsion free sheaf $\mathcal{E}$ on X is called p-semistable (resp. p-stable) if for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$, when m is large enough, we have

$$p(\mathcal{F}, m) := \frac{P(\mathcal{F}, m)}{r(\mathcal{F})} \leq \frac{P(\mathcal{E}, m)}{r(\mathcal{E})} := p(\mathcal{E}, m) \quad (\text{resp. } <).$$

Lemma 2.1. *Let $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$ be an exact sequence of torsion free sheaves, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are p-semistable with $p(\mathcal{E}_1, m) = p(\mathcal{E}_2, m)$ for all m, then $\mathcal{E}$ is p-semistable with $p(\mathcal{E}, m) = p(\mathcal{E}_1, m) = p(\mathcal{E}_2, m)$ for all m.*

Proof. It is easy to check and we omit the details. □

Let S be an affine variety over a finite field $\mathbb{F}_q$, and $X_S \to S$ be a projective, flat morphism with geometrically irreducible and reduced fibers. Fix a polynomial P of degree equal to the relative dimension $d = dim(X_S/S)$ and a relative ample line bundle $\mathcal{O}_{X_S}(1)$ on X_S. Let

$$\mathcal{Q} := \mathrm{Quot}_P(\mathcal{O}_{X_S}(-N)^{\oplus P(N)}) \to S$$

be the relative quotient scheme together with the universal quotient

$$\mathcal{O}_{X_S \times_S \mathcal{Q}}(-N)^{\oplus P(N)} \to \mathcal{F}^{univ} \to 0$$

where $\mathcal{F}^{univ}$ is $\mathcal{Q}$-flat with the fixed Hilbert polynomial P. Let

$$\pi_{\mathcal{Q}} : X_{\mathcal{Q}} := X_S \times_S \mathcal{Q} \to \mathcal{Q}$$

be the projection and $\mathcal{O}_{X_{\mathcal{Q}}}(m)$ be the pullback of $\mathcal{O}_{X_S}(m)$ (under projection $X_{\mathcal{Q}} \to X_S$). It is well-known that the determinant line bundle

$$\mathcal{L}_m = \bigwedge^{P(m)} (\pi_{\mathcal{Q}})_*(\mathcal{F}^{univ} \otimes \mathcal{O}_{X_{\mathcal{Q}}}(m))$$

of cohomology is very ample for large m, which gives a linearization of $\mathrm{SL}(P(N))$ on $\mathcal{Q}$ (see page 64 of [11] for detail). Let $\overline{Q} \subset \mathcal{Q}$ be the closure of open set of points corresponding to semistable quotients. Then the open set $\overline{Q}^{ss} \subset \mathcal{Q}$ (resp. $\overline{Q}^{s} \subset \mathcal{Q}$) of GIT semistable (resp. GIT stable) points under the action of $\mathrm{SL}(P(N))$ (respect to $\mathcal{L}_m$) is precisely the open set Q of quotients $\mathcal{O}_{X_s}(-N)^{\oplus P(N)} \to \mathcal{F}_s \to 0$ where $\mathcal{F}_s$ are p-semistable (resp. p-stable) torsion free sheaves on X_s (See Theorem 4.1 of [6] over a general base). Let

$$\varphi : Q \to M(\mathcal{O}_{X_S}, P) := Q//\mathrm{SL}(P(N)) \tag{2.1}$$

be the GIT quotient over S defined in Theorem 4 of [10]. Then

$$M(\mathcal{O}_{X_S}, P) \to S$$

is a projective scheme of finite type over S, which uniformly corepresents the functor $\mathbf{M}(\mathcal{O}_{X_S}, P) : \mathrm{Sch}/S \to \mathrm{Sets}$ defined by

$$\mathbf{M}(\mathcal{O}_{X_S}, P)(S') = \left\{ \begin{array}{c} s\text{-equivalence classes of families of } p\text{-semistable} \\ \text{sheaves on the geometric fibres of } X_{S'} \to S', \\ \text{which are flat over } S' \text{ with Hilbert polynomial } P \end{array} \right\}.$$

Definition 2.2. (1) A coherent sheaf $\mathcal{F}$ on X_S is called p-semistable with Hilbert polynomial P if it is flat over S and $\mathcal{F}_s$ are p-semistable with Hilbert polynomial P on each geometric fiber X_s of $X_S \to S$. (2) Suppose $\xi_S : S \to X_S$ is a section of $X_S \to S$, we say that $\mathcal{F}$ satisfies condition $\mathrm{LF}(\xi_S)$ if $gr(\mathcal{F}_s)$ is locally free at $\xi_S(s)$ ($\forall\ s \in S$).

Let $Q^{\mathrm{LF}(\xi_S)} \subset Q$ be the subset of Q parametrizing quotients

$$\mathcal{O}_{X_S}(-N)^{\oplus P(N)} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ satisfies condition $\mathrm{LF}(\xi_S)$. It was shown in [11] that there is an open set $M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P) \subset M(\mathcal{O}_{X_S}, P)$ such that

$$Q^{\mathrm{LF}(\xi_S)} = \varphi^{-1}(M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P))$$

and $\varphi : Q^{\mathrm{LF}(\xi_S)} \to M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P)$ is a uniform categorical quotient.

Let $\mathcal{F}^{univ}$ be the universal quotient on $X_S \times_S Q^{\mathrm{LF}(\xi_S)}$, which is locally free along the universal section $\xi_Q : Q^{\mathrm{LF}(\xi_S)} \to X_S \times_S Q^{\mathrm{LF}(\xi_S)}$, and let

$$\pi : T \to Q^{\mathrm{LF}(\xi_S)}$$

be the frame bundle of $\xi_Q^*(\mathcal{F}^{univ})$, which represents the functor that associates to any $S' \to S$ the set of all triples $(\mathcal{E}, \alpha, \beta)$, where $\mathcal{E}$ is a p-semistable torsion free sheaf of Hilbert polynomial P on $X_{S'}/S'$ satisfying condition $\mathrm{LF}(\xi_{S'})$, and α, β are isomorphisms

$$\alpha : \mathcal{O}_{S'}^{\oplus P(N)} \cong H^0(X_{S'}/S', \mathcal{E}(N)), \quad \beta : \xi_{S'}^*(\mathcal{E}) \cong \mathcal{O}_{S'}^{\oplus r}.$$

The group $\mathrm{GL}(P(N)) \times \mathrm{GL}(r)$ acts on T, compatibly with the action of $\mathrm{GL}(P(N))$ on $Q^{\mathrm{LF}(\xi_S)}$. We may choose a linearization of the action of $\mathrm{GL}(P(N))$ on $\mathcal{L}_m^b$ such that the center $G_m \subset \mathrm{GL}(P(N))$ acts trivially. Then the line bundle $\mathcal{L}_m^b$ on $Q^{\mathrm{LF}(\xi_S)}$ has a linearization with respect to the group $\mathrm{GL}(P(N)) \times \mathrm{GL}(r)$, where the second factor acts trivially. Let $\mathbf{L}$ denote the pullback of the $\mathrm{GL}(P(N)) \times \mathrm{GL}(r)$-linearized bundle $\mathcal{L}_m^b$ to T. Then we have a characteristic p analogue of a special case $(\Lambda = \mathcal{O}_{X_S})$ of Simpson's result (see Theorem 4.10 of [11]).

Theorem 2.3. *Every point of T is stable for the action of $\mathrm{GL}(P(N))$ with respect to the linearized line bundle $\boldsymbol{L}$, and the action of $\mathrm{GL}(P(N))$ on T is free. The geometric quotient*

$$\phi : T \to R(\mathcal{O}_{X_S}, \xi_S, P) := T//\mathrm{GL}(P(N))$$

represents a functor which associates to any $S' \to S$ the set of pairs $(\mathcal{E}, \beta)$ where $\mathcal{E}$ is a p-semistable torsion free sheaf of Hilbert polynomial P on $X_{S'}/S'$ satisfying condition $\mathrm{LF}(\xi_{S'})$, and

$$\beta : \xi_{S'}^*(\mathcal{E}) \cong \mathcal{O}_{S'}^{\oplus r}$$

is a frame. Thus $R(\mathcal{O}_{X_S}, \xi_S, P)$ is a fine moduli space. Moreover, we have the following properties:

1. *Every point of $R(\mathcal{O}_{X_S}, \xi_S, P)$ is GIT semistable under the action of $\mathrm{GL}(r)$ (respect to a $\mathcal{L}$ obtained from $\boldsymbol{L}$) and the quotient $R(\mathcal{O}_{X_S}, \xi_S, P)//\mathrm{GL}(r)$ is naturally equal to $M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P)$;*
2. *For a geometric point $\alpha = (V, \beta) \in R(\mathcal{O}_{X_S}, \xi_S, P)$, the orbit $O(\alpha)$ of $\alpha = (V, \beta)$ under $\mathrm{GL}(r)$ is closed if and only if V is a direct sum of p-stable sheaves, and $\alpha = (V, \beta)$ is a stable point if and only if V is a p-stable sheaf.*

Proof. The proof is the same with Simpson's proof in characteristic zero. For conveniences of readers, we repeat his proof here and indicate references so that it works in characteristic $p > 0$.

The projection $\pi : T \to Q^{\mathrm{LF}(\xi_S)}$ is an affine map and all points of $Q^{\mathrm{LF}(\xi_S)}$ are semistable for the action of $\mathrm{GL}(P(N))$ respect to linearized line bundle $\mathcal{L}_m^b$. Thus if $q \in T$ is any point, then there is an $\mathrm{GL}(P(N))$-invariant section $\sigma \in \mathrm{H}^0(Q^{\mathrm{LF}(\xi_S)}, \mathcal{L}_m^{ab})$ such that $(Q^{\mathrm{LF}(\xi_S)})_{\sigma\neq 0}$ is affine and $\sigma(\pi(q)) \neq 0$. Then $\pi^*(\sigma) \in \mathrm{H}^0(T, \mathbf{L}^a)$ is $\mathrm{GL}(P(N))$-invariant such that $\pi^*(\sigma)(q) \neq 0$ and $T_{\pi^*(\sigma)\neq 0} = \pi^{-1}((Q^{\mathrm{LF}(\xi_S)})_{\sigma\neq 0})$ is affine. Thus any point $q \in T$ is semistable. To prove that every point of T is stable, the key is a lemma of Simpson (Lemma 4.9 of [11]), which implies that the stabilizer of any point of T is finite and in particular orbits of all points of T have same dimension. Thus the orbit of any point of T is closed since no orbit can be contained in the closure of another orbit.

It is a general fact that there exist a geometric quotient

$$\phi : T \to R(\mathcal{O}_{X_S}, \xi_S, P) := T//\mathrm{GL}(P(N))$$

and an ample line bundle $\mathcal{L}$ on $R(\mathcal{O}_{X_S}, \xi_S, P)$ such that

$$\phi^*\mathcal{L} = \mathbf{L}^a = \pi^*\mathcal{L}_m^{ab}$$

when a is large enough. Moreover, ϕ is submersive (i.e. $U \subset R(\mathcal{O}_{X_S}, \xi_S, P)$ is open if and only if $\phi^{-1}(U) \subset T$ is open).

To show the action of $\mathrm{GL}(P(N))$ on T is free, we must show that

$$\mathrm{GL}(P(N)) \times T \to T \times_{R(\mathcal{O}_{X_S}, \xi_S, P)} T, \quad (g, q) \mapsto (g(q), q) \tag{2.2}$$

is a closed immersion. By a result of Mumford (Corollary 2.5 of Proposition 2.4 at page 55 of [9] where Proposition 2.4 is an application of Iwahori's theorem), the above morphism (2.2) is proper. Here we remark that Iwahori's theorem and Proposition 2.4 were proved in characteristic $p >$ (see Appendix to Chapter 2 of [9] at page 202). By using again Lemma 4.9 of [11], Simpson was able to show that (2.2) is an inclusion of functors. A proper map which is an inclusion of functors is a closed immersion. Thus the action of $\mathrm{GL}(P(N))$ on T is free, which implies that $\phi : T \to R(\mathcal{O}_{X_S}, \xi_S, P)$ is a principal $\mathrm{GL}(P(N))$-bundle over $R(\mathcal{O}_{X_S}, \xi_S, P)$ by Proposition 0.9 of [9].

Let $\mathcal{F}^{univ}$ be pullback of the universal quotient on $X_S \times_S Q^{\mathrm{LF}(\xi_S)}$ (under $X_S \times_S T \to X_S \times_S Q^{\mathrm{LF}(\xi_S)}$). Then the action of $\mathrm{GL}(P(N))$ on $X_S \times_S T$ lifts to an action on $\mathcal{F}^{univ}$ and

$$\mathrm{id}_{X_S} \times \phi : \; X_S \times_S T \to X_S \times_S R(\mathcal{O}_{X_S}, \xi_S, P)$$

is a principal $\mathrm{GL}(P(N))$-bundle. By Proposition 2.2 (B) of [8], the descend lemma holds in characteristic $p > 0$ if the scheme-theoretic stabilizers are linearly reductive. Thus $\mathcal{F}^{univ}$ descends to a universal p-semistable sheaf $\mathcal{F}^R$ with Hilbert polynomial P on

$$X_R := X_S \times_S R(\mathcal{O}_{X_S}, \xi_S, P) \to R := R(\mathcal{O}_{X_S}, \xi_S, P)$$

with a universal frame $\beta^R : \xi_R^*\mathcal{F}^R \cong \mathcal{O}_R^{\oplus r}$. Then, by the same arguments of Simpson, $R := R(\mathcal{O}_{X_S}, \xi_S, P)$ together with $(\mathcal{F}^R, \beta^R)$ represents the required functor.

To prove (1), recall $\mathcal{L}$ is the ample line bundle on $R(\mathcal{O}_{X_S}, \xi_S, P)$ such that $\phi^*(\mathcal{L}) = \mathbf{L}^a = \pi^*\mathcal{L}_m^{ab}$, we will prove that for any point $q \in T$ the point $\phi(q) \in R(\mathcal{O}_{X_S}, \xi_S, P)$ is GIT semistable under the action of $\mathrm{GL}(r)$ (respect to a $\mathcal{L}$). As above, there is an $\sigma \in \mathrm{H}^0(Q^{\mathrm{LF}(\xi_S)}, \mathcal{L}_m^{ab})$ such that $(Q^{\mathrm{LF}(\xi_S)})_{\sigma\neq 0}$ is affine and $\sigma(\pi(q)) \neq 0$. Then $\pi^*(\sigma) \in \mathrm{H}^0(T, \mathbf{L}^a)$ is $\mathrm{GL}(P(N))$-invariant such that $\pi^*(\sigma)(q) \neq 0$ and $T_{\pi^*(\sigma)\neq 0}$ is affine. Let $\tau \in \mathrm{H}^0(R(\mathcal{O}_{X_S}, \xi_S, P), \mathcal{L}^a)$ be the section such that $\phi^*(\tau) = \pi^*\sigma$. Then $\tau(\phi(q)) \neq 0$ and $R(\mathcal{O}_{X_S}, \xi_S, P)_{\tau\neq 0} = T_{\pi^*(\sigma)\neq 0}//\mathrm{GL}(P(N))$ is affine. On the other hand, $\pi^*(\sigma)$ is $\mathrm{GL}(r)$-invariant since $\pi : T \to Q^{\mathrm{LF}(\xi_S)}$ is a principal $\mathrm{GL}(r)$-bundle, which implies that τ is $\mathrm{GL}(r)$-invariant and $\phi(q)$ is semistable under the action of $\mathrm{GL}(r)$ respect to $\mathcal{L}$. Let

$$\psi : R(\mathcal{O}_{X_S}, \xi_S, P) \to \mathbf{M} := R(\mathcal{O}_{X_S}, \xi_S, P)//\mathrm{GL}(r).$$

Then both $T \xrightarrow{\phi} R(\mathcal{O}_{X_S}, \xi_S, P) \xrightarrow{\psi} \mathbf{M}$ and

$$T \xrightarrow{\pi} Q^{\mathrm{LF}(\xi_S)} \xrightarrow{\varphi} M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P)$$

are categorical quotients of T by $\mathrm{GL}(P(N)) \times \mathrm{GL}(r)$. Thus $\mathbf{M}$ is naturally equal to $M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P)$ and we have the commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\phi} & R(\mathcal{O}_{X_S}, \xi_S, P) \\ {\scriptstyle\pi}\Big\downarrow & & {\scriptstyle\psi}\Big\downarrow \\ Q^{\mathrm{LF}(\xi_S)} & \xrightarrow{\varphi} & M^{\mathrm{LF}(\xi_S)}(\mathcal{O}_{X_S}, P). \end{array}$$

To prove (2) of the theorem, let

$$q = (\mathcal{O}_{X_{\bar{s}}}(-N)^{\oplus P(N)} \xrightarrow{q} V \to 0, \beta) \in T$$

such that $\phi(q) = \alpha = (V, \beta) \in R(\mathcal{O}_{X_S}, \xi_S, P)$ and

$$q' := \pi(q) = (\mathcal{O}_{X_{\bar{s}}}(-N)^{\oplus P(N)} \xrightarrow{q} V \to 0) \in Q^{\mathrm{LF}(\xi_S)}.$$

Let $O_{\mathrm{GL}(r)}(\alpha) \subset R(\mathcal{O}_{X_S}, \xi_S, P)$ (resp. $O_{\mathrm{GL}(P(N))}(q') \subset Q^{\mathrm{LF}(\xi_S)}$) be the orbit of α (resp. q') under $\mathrm{GL}(r)$ (resp. $\mathrm{GL}(P(N))$). Then

$$\phi^{-1}(O_{\mathrm{GL}(r)}(\alpha)) = O_{\mathrm{GL}(P(N))\times\mathrm{GL}(r)}(q) = \pi^{-1}(O_{\mathrm{GL}(P(N))}(q'))$$

since $T \xrightarrow{\phi} R(\mathcal{O}_{X_S}, \xi_S, P)$ (resp. $T \xrightarrow{\pi} Q^{\mathrm{LF}(\xi_S)}$) is a principal $\mathrm{GL}(P(N))$-bundle (resp. a principal $\mathrm{GL}(r)$-bundle), where

$$O_{\mathrm{GL}(P(N))\times\mathrm{GL}(r)}(q) \subset T$$

is the orbit of $q \in T$ under $\mathrm{GL}(P(N)) \times \mathrm{GL}(r)$. Thus

$$O_{\mathrm{GL}(r)}(\alpha) \subset R(\mathcal{O}_{X_S}, \xi_S, P)$$

is closed if and only if $O_{\mathrm{GL}(P(N))}(q') \subset Q^{\mathrm{LF}(\xi_S)}$ is closed. But

$$O_{\mathrm{GL}(P(N))}(q') \subset Q^{\mathrm{LF}(\xi_S)}$$

is closed if and only if V is a direct sum of p-stable sheaves. On the other hand, the group of automorphisms of determinant one of such a direct sum is finite if and only if the sum has exactly one p-stable component. Thus $\alpha = (V, \beta) \in R(\mathcal{O}_{X_S}, \xi_S, P)$ has closed orbit and finite stabilizer in $\mathrm{SL}(r)$ (i.e. a stable point) if and only if V is a p-stable sheaf. Here we use a fact that $\mathrm{Aut}(V)$ is naturally isomorphic to the stabilizer of $\alpha = (V, \beta)$. □

Remarks 2.4. (1) According to Simpson, the moduli spaces

$$R(\mathcal{O}_{X_S}, \xi_S, P) \to S$$

are called **Representation spaces**.

(2) Let X be a smooth, connected projective variety over a perfect field k of characteristic $p > 0$ with a given point

$$\xi : \mathrm{Spec}(k) \to X.$$

There exist an $\mathbb{F}_q$-algebra $A \subset k$ of finite type and a scheme

$$X_S \to S = \mathrm{Spec}(A)$$

of finite type over S such that its base change under $\mathrm{Spec}(k) \to S$ (induced by $A \subset k$) is isomorphic to $X \to \mathrm{Spec}(k)$, which is called a model of $X \to \mathrm{Spec}(k)$. We can choose S such that $X_S \to S$ is a smooth projective flat morphism and $\xi : \mathrm{Spec}(k) \to X$ extends to a section $\xi_S : S \to X_S$. Then the representation space $R(\mathcal{O}_{X_S}, \xi_S, P) \to S$, we will use in this article, is the case when $P(m) = \chi(\mathcal{O}_X(m)^{\oplus r})$.

Proposition 2.5. *There exists a rational map*

$$f : R(\mathcal{O}_{X_S}, \xi_S, P) \dashrightarrow R(\mathcal{O}_{X_S}, \xi_S, P)$$

over S satisfying the following conditions:

(1) $\forall\, \alpha = (E, \beta) \in R(\mathcal{O}_X, \xi, P) = R(\mathcal{O}_{X_S}, \xi_S, P) \times_S \mathrm{Spec}(k)$, *$f$ is well-defined at α if and only if $F_X^* E$ is p-semistable, where $F_X : X \to X$ is the (absolute) Frobenius map. In this case,*

$$f(\alpha) = (F_X^* E, F_k^* \beta).$$

(2) $\forall$ *geometric closed point* $\alpha_{\bar{s}} = (\mathcal{E}_{\bar{s}}, \beta_{\bar{s}}) \in R(\mathcal{O}_{X_S}, \xi_S, P)$, f *is well-defined at* $\alpha_{\bar{s}}$ *if and only if* $F^*_{X_{\bar{s}}}\mathcal{E}_{\bar{s}}$ *is p-semistable, where* $X_{\bar{s}}$ *is a geometric closed fiber of* $X_S \to S$. *In this case,*

$$f(\alpha_{\bar{s}}) = (F^*_{X_{\bar{s}}}\mathcal{E}_{\bar{s}}, F^*_{k(\bar{s})}\beta_{\bar{s}}).$$

Proof. Let $(\mathcal{F}^R, \beta^R)$ be the universal object on $X_R := X_S \times_S R(\mathcal{O}_{X_S}, \xi_S, P)$ where $\beta^R : \xi_R^*\mathcal{F}^R \cong \mathcal{O}^{\oplus r}_{R(\mathcal{O}_{X_S}, \xi_S, P)}$ is the universal frame of $\xi_R^*\mathcal{F}^R$. Let

$$F : X_R \to X'_R$$

denote the relative Frobenius over $R := R(\mathcal{O}_{X_S}, \xi_S, P)$. Consider

$$\begin{tikzcd} X_R \arrow[r, "F"] \arrow[rr, bend left, "F_{X_R}"] \arrow[dr, "p_R"'] & X'_R \arrow[r] \arrow[d, "p'_R"] & X_R \arrow[d, "p_R"] \\ & R \arrow[r, "F_R"] & R. \end{tikzcd} \tag{2.3}$$

Then the pullback $(F^*_{X_R}\mathcal{F}^R, F^*_R\beta^R)$ of $(\mathcal{F}^R, \beta^R)$, where

$$F^*_R\beta^R : \xi^*_R(F^*_{X_R}\mathcal{F}^R) = F^*_R(\xi^*_R\mathcal{F}^R) \xrightarrow{F^*_R(\beta^R)} F^*_R(\mathcal{O}^{\oplus r}_R) = \mathcal{O}^{\oplus r}_R,$$

defines the rational map

$$f : R(\mathcal{O}_{X_S}, \xi_S, P) \dashrightarrow R(\mathcal{O}_{X_S}, \xi_S, P). \tag{2.4}$$

It is well-defined on an open subscheme $R_0 \subset R(\mathcal{O}_{X_S}, \xi_S, P)$ such that

$$F^*_{X_R}\mathcal{F}^R$$

is a family of p-semistable sheaves on $X_S \times_S R_0$.

To see that the rational map (2.4) satisfies the requirements (1) and (2) in the proposition, it is enough to make the following remak. For any point $t \to R$, let $s \to S$ be its image under $R \to S$, and X_s be the fiber of $X_S \to S$ at $s \to S$. Then the diagram (2.3) specializes to

$$\begin{tikzcd} X_s \times t \arrow[r, "F_t"] \arrow[rr, bend left, "F_{X_s \times t}"] \arrow[dr] & (X_s \times t)' \arrow[r] \arrow[d] & X_s \times t \arrow[d] \\ & t \arrow[r] & t. \end{tikzcd}$$

□

3. Stratified bundles and Representation spaces

Let k be a perfect field of characteristic $p > 0$, and X a smooth connected projective variety over k. A stratified bundle (or $\mathcal{D}$-module) on X is by definition a coherent $\mathcal{O}_X$-module $\mathcal{E}$ with a homomorphism

$$\nabla : \mathcal{D}_X \to \mathcal{E}nd_k(\mathcal{E})$$

of $\mathcal{O}_X$-algebras, where $\mathcal{D}_X$ is the sheaf of differential operators acting on the structure sheaf of X. By a theorem of Katz (cf. [4, Theorem 1.3]), it is equivalent to the following definition.

Definition 3.1. A stratified bundle on X is a sequence of bundles

$$E = \{E_0, E_1, E_2, \cdots, \sigma_0, \sigma_1, \ldots\} = \{E_i, \sigma_i\}_{i\in\mathbb{N}}$$

where $\sigma_i : F_X^* E_{i+1} \to E_i$ is a $\mathcal{O}_X$-linear isomorphism, and $F_X : X \to X$ is the absolute Frobenius.

A morphism $\alpha = \{\alpha_i\} : \{E_i, \sigma_i\} \to \{F_i, \tau_i\}$ between two stratified bundles is a sequence of morphisms $\alpha_i : E_i \to F_i$ of $\mathcal{O}_X$-modules such that

$$\begin{tikzcd} F_X^* E_{i+1} \arrow[r, "F_X^*\alpha_{i+1}"] \arrow[d, "\sigma_i"'] & F_X^* F_{i+1} \arrow[d, "\tau_i"] \\ E_i \arrow[r, "\alpha_i"] & F_i \end{tikzcd}$$

is commutative. The category $\mathbf{str}(X)$ of stratified bundles is abelian, rigid, monoidal. We will drop the isomorphisms σ_i and will use the notation $E = (E_i)_{i\in\mathbb{N}}$ and in particular $E(n_0) = (E_i)_{i\geq n_0}$ is also a stratified bundle.

Lemma 3.2. *Let $E = (E_i)_{i\in\mathbb{N}}$ be a stratified bundle. Then there is an n_0 such that E_i are p-semistable of $p(E_i, m) = p(\mathcal{O}_X, m)$ for all $i \geq n_0$.*

Proof. It is known that $p(E_i, m) = p(\mathcal{O}_X, m)$ for all $i \geq 0$ (see Corollary 2.2 of [1]). By Proposition 2.3 of [1], there is an $n_0 > 0$ such that the stratified bundle $E(n_0) = (E_i)_{i\geq n_0}$ is a successive extension of stratified bundles $U = (U_i)_{i\in\mathbb{N}}$ with the property that all U_i for $i \in \mathbb{N}$ are p-stable bundles with $p(U_i, m) = p(\mathcal{O}_X, m)$. Then, by Lemma 2.1, all E_i for $i \geq n_0$ are p-semistable of $p(E_i, m) = p(\mathcal{O}_X, m)$. $\square$

Lemma 3.3. *Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$ with a fixed rational point $\xi : \mathrm{Spec}(k) \to X$. Let $F_X : X \to X$ be the Frobenius map, and V, V' be vector bundles satisfying $V = F_X^*(V')$. Then, for any frame $\beta : \xi^* V \cong \mathcal{O}_{\mathrm{Spec}(k)}^{\oplus r}$, there* *is a unique frame* $\beta' : \xi^* V' \cong \mathcal{O}^{\oplus r}_{\mathrm{Spec}(k)}$ *such that* $\beta = F_k^*(\beta')$ *where* $F_k : \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ *is the Frobenius morphism and*

$$F_k^*(\beta') : \xi^* V = F_k^*(\xi^* V') \xrightarrow{F_k^*(\beta')} F_k^*(\mathcal{O}^{\oplus r}_{\mathrm{Spec}(k)}) = \mathcal{O}^{\oplus r}_{\mathrm{Spec}(k)}.$$

Proof. It is clearly a local question and we can assume $X = \mathrm{Spec}(A)$, $\xi^* V = V \otimes_A k = V/mV$, $\xi^* V' = V' \otimes_A k = V'/mV'$ (where $k = A/m$, $m \subset A$ is a maximal ideal) and $\xi^* V = F_k^* \xi^* V' = \xi^* V' \otimes_{k^p} k$. The frame β is uniquely determined by a base $\beta_1, \cdots, \beta_r \in \xi^* V$ of $\xi^* V$. Since k is perfect, there are uniquely $\beta'_1, \cdots, \beta'_r \in \xi^* V'$ such that $\beta_i = \beta'_i \otimes_{k^p} 1$ ($1 \le i \le r$). Then $\beta'_1, \cdots, \beta'_r \in \xi^* V'$ are clearly linear independent (thus it is a base of $\xi^* V'$), which defines a frame $\beta' : \xi^* V' \cong \mathcal{O}^{\oplus r}_{\mathrm{Spec}(k)}$ satisfying $F_k^*(\beta') = \beta$. □

For a given stratified bundle $E = (E_i)_{i \in \mathbb{N}}$ on X, we can fix a frame

$$\beta_1 : \xi^* E_1 \cong \mathcal{O}^{\oplus r}_{\mathrm{Spec}(k)}.$$

By Lemma 3.2 and Lemma 3.3, we get a set of k-points

$$R(E)_{n_0} := \{\alpha_i = (E_i, \beta_i)\}_{i \ge n_0} \subset R(\mathcal{O}_X, \xi, P)(k) \tag{3.1}$$

with $F_X^*(\alpha_{i+1}) = \alpha_i$. To produce a representation of $\pi_1 = \pi_1^{\text{ét}}(X, a)$ from the stratified bundle E, the following two results are important.

Lemma 3.4 (Lange-Stuhler, [7]). *Let X be a smooth projective variety over k of* $\mathrm{char}(k) = p > 0$, *$F_X : X \to X$ be the Frobenius map. If there is a vector bundle $\mathcal{E}$ on X and an integer $m > 0$ such that*

$$(F_X^m)^* \mathcal{E} \cong \mathcal{E}$$

then there exists a geometrically connected étale finite cover

$$\sigma : Z \to X$$

such that $\sigma^* \mathcal{E} \cong \mathcal{O}_Z^{\oplus \mathrm{rk}(\mathcal{E})}$. This gives a representation

$$\pi_1^{\acute{e}t}(X \otimes_k \bar{k}, a) \to GL(V)$$

whose associated bundle is $\mathcal{E} \otimes_k \bar{k}$ on $X \otimes_k \bar{k}$.

To find Frobenius periodic bundles from a given stratified bundle, the key tool is a theorem of Hrushovski. In fact, we only need a special case of his theorem [5, Corollary 1.2] (see also [13] for a proof in algebraic geometry).

Theorem 3.5 (Corollary of twisted Lang-Weil estimate, [5, Corollary 1.2]). *Let Y be an affine variety over $\mathbb{F}_q$, and let*

$$\Gamma \subset (Y \times_{\mathbb{F}_q} Y) \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$$

be an irreducible subvariety over $\overline{\mathbb{F}}_q$. *Assume the two projections* $\Gamma \to Y$ *are dominant. Then, for any closed subvariety* $W \subsetneq Y$, *there exists* $x \in Y(\overline{\mathbb{F}}_q)$ *such that* $(x, x^{q^m}) \in \Gamma$ *and* $x \notin W$ *for large enough natural number* m.

However, we will use Hrushovski's theorem in following formulation.

Lemma 3.6 (Corollary of Hrushovski's theorem)**.** *Let* Y *be a variety over* $\bar{\mathbb{F}}_p$ *and* $f : Y \dashrightarrow Y$ *a dominant rational map. Then the subset* $\{\, x \in Y(\bar{\mathbb{F}}_p) \mid \exists\, b,\ f^b(x) = x \,\} \subset Y(\bar{\mathbb{F}}_p)$ *is dense in* Y.

Proof. We prove that any nontrivial affine open set of Y contains a periodic point of f. Replace Y by the affine open set, we can assume that $Y \subset \mathbb{A}^n_{\mathbb{F}_q}$ is an affine variety over $\mathbb{F}_q$ and $f : Y \dashrightarrow Y$ is also defined over $\mathbb{F}_q$. Let $\Gamma = \overline{graph(f)} \subset Y \times Y$ and $W \subset Y$ where f is not well-defined. Let $f = (f_1, ..., f_n)$ be defined by the rational functions $f_i \in \mathbb{F}_q(Y)$ on Y. Then, by Theorem 3.5, there is a point $x = (x_1, ..., x_n) \in Y(\overline{\mathbb{F}}_q)$ such that $x \notin W$ and $(x, x^{q^m}) \in \Gamma$ where

$$x^{q^m} := (x_1^{q^m}, x_2^{q^m}, ..., x_n^{q^m}),$$

which implies that $f(x) = (x_1^{q^m}, ..., x_n^{q^m}) = x^{q^m}$. On the other hand, since f is defined by rational functions $f_i \in \mathbb{F}_q(Y)$, we have

$$f(x^{q^a}) = (f_1(x^{q^a}), ..., f_n(x^{q^a})) = (f_1(x)^{q^a}, ..., f_n(x)^{q^a}) := f(x)^{q^a}$$

for any integer $a > 0$. Thus

$$f^b(x) = f^{b-1}(x^{q^m}) = f^{b-2}(x^{q^{2m}}) = \cdots = x^{q^{bm}} = x$$

when b is large enough since $x = (x_1, ..., x_n) \in Y(\overline{\mathbb{F}}_q)$. □

Theorem 3.7. *Let* X *be a smooth projective variety over a perfect field* k *of characteristic* $p > 0$ *with a fixed point* $\xi : \mathrm{Spec}(k) \to X$. *Let* $E = (E_i)_{i \in \mathbb{N}}$ *be a stratified bundle on* X *such that*

$$\Sigma = \{\, E_i \,\}_{i \in \mathbb{N}}$$

is an infinite set. Then there exist a choice of S *and an irreducible closed subset* $\mathcal{N}(E)_S \subset R(\mathcal{O}_{X_S}, \xi_S, P)$ *such that*

(1) $\{\, V \in \Sigma \,|\, (V, \beta) \in \mathcal{N}(E)_S(k) \} \subset \Sigma$ *is an infinite subset, where* $\mathcal{N}(E)_S(k)$ *denote the set of* k*-points of* $\mathcal{N}(E)_S$.
(2) $\mathcal{N}(E)_S$ *contains a dense subset of points* $\alpha_{\bar{s}} = (V_{\bar{s}}, \beta_{\bar{s}})$, *where* $V_{\bar{s}}$ *are vector bundles on geometric fibers* $X_{\bar{s}}$ *associated to representations* $\rho_{\bar{s}} : \pi_1^{\acute{e}t}(X_{\bar{s}}, \xi_{\bar{s}}) \to GL(r, k(\bar{s}))$.

Proof. For the given stratified bundle $E = (E_i)_{i \in \mathbb{N}}$, without loss of generality, we assume that all bundles E_i ($\in \mathbb{N}$) are p-semistable. Fix a frame $\beta_1 : \xi^* E_1 \cong \mathcal{O}_{\mathrm{Spec}(k)}^{\oplus r}$, by Lemma 3.3, the stratified bundle $E = (E_i)_{i \in \mathbb{N}}$ gives an infinite set of points

$$R(E) = \{\, Q_i = (E_i, \beta_i) \in R(\mathcal{O}_X, \xi, P) \,|\, F_X^*(Q_{i+1}) = Q_i \,\}$$

and subsets $R(E)_n := \{\, Q_i \in R(E) \,\}_{i \geq n}$ ($n \in \mathbb{N}$), which satisfy

$$R(E) = R(E)_1 \supseteq R(E)_2 \supseteq \cdots \supseteq R(E)_n \supseteq R(E)_{n+1} \supseteq \cdots \tag{3.2}$$

and $F_X^*(R(E)_{n+1}) = R(E)_n$. Let $\mathcal{Z}_n = \overline{R(E)_n} \subset R(\mathcal{O}_X, \xi, P)$ be the Zariski closure of $R(E)_n$. Then, by (3.2), we have

$$\mathcal{Z}_1 \supseteq \mathcal{Z}_2 \supseteq \cdots \supseteq \mathcal{Z}_n \supseteq \mathcal{Z}_{n+1} \supseteq \cdots$$

which implies that there is $n_0 > 0$ such that $\mathcal{Z}_n = \mathcal{Z}_{n_0}$ ($n \geq n_0$). Let

$$\mathcal{Z} = \bigcap_{i=1}^{\infty} \mathcal{Z}_i \subset R(\mathcal{O}_X, \xi, P).$$

Then the rational map $F_X^* : R(\mathcal{O}_X, \xi, P) \dashrightarrow R(\mathcal{O}_X, \xi, P)$ induce a dominant rational map $F_X^* : \mathcal{Z} \dashrightarrow \mathcal{Z}$. Thus there is an irreducible component $\mathcal{N}(E) \subset \mathcal{Z} \subset R(\mathcal{O}_X, \xi, P)$ such that

- $\{\, V \in \Sigma \,|\, (V, \beta) \in \mathcal{N}(E)(k)\} \subset \Sigma$ is an infinite subset;
- there is an integer $a > 0$ such that $(F_X^*)^a : \mathcal{Z} \dashrightarrow \mathcal{Z}$ induces a dominant rational map $(F_X^*)^a : \mathcal{N}(E) \dashrightarrow \mathcal{N}(E)$.

Choose a smooth, geometrically irreducible affine variety S over a finite field $\mathbb{F}_q$ with rational function field $\mathbb{F}_q(S) \subset k$ such that $\mathcal{N}(E)$, $(F_X^*)^a$ are defined over S and there exist a smooth model $X_S \to S$ of $X \to \mathrm{Spec}(k)$ and a section $\xi_S : S \to X_S$ extending $\xi \in X(k)$. Let

$$R(\mathcal{O}_{X_S}, \xi_S, P) \to S$$

be the representation space constructed in Theorem 2.3. Then

$$R(\mathcal{O}_X, \xi, P) = R(\mathcal{O}_{X_S}, \xi_S, P) \times_S \mathrm{Spec}(k)$$

and the subvariety $\mathcal{N}(E) \subset R(\mathcal{O}_{X_S}, \xi_S, P) \times_S \mathrm{Spec}(k)$ is defined over S by the choice of S. Thus there exists a closed subvariety

$$\mathcal{N}(E)_S \subset R(\mathcal{O}_{X_S}, \xi_S, P) \tag{3.3}$$

such that $\mathcal{N}(E) = \mathcal{N}(E)_S \times_S \mathrm{Spec}(k)$, which implies that

$$\mathcal{N}(E)_S(k) = \mathcal{N}(E)(k).$$

This proves (1) of the theorem.

To show statement (2) of the theorem, recall the rational map

$$f : R(\mathcal{O}_{X_S}, \xi_S, P) \dashrightarrow R(\mathcal{O}_{X_S}, \xi_S, P)$$

constructed in Proposition 2.5 and $F_X^* = f \otimes k$ is induced by

$$f : R(\mathcal{O}_{X_S}, \xi_S, P) \dashrightarrow R(\mathcal{O}_{X_S}, \xi_S, P)$$

under the base change

$$R(\mathcal{O}_{X_S}, \xi_S, P) \times_S \mathrm{Spec}(k) \xrightarrow{f\otimes k} R(\mathcal{O}_{X_S}, \xi_S, P) \times_S \mathrm{Spec}(k).$$

Then $f^a \otimes k : \mathcal{N}(E)_S \times_S \mathrm{Spec}(k) \dashrightarrow \mathcal{N}(E)_S \times_S \mathrm{Spec}(k)$ is a dominant rational map, which implies (by shrinking S if necessary) that

$$f^a : \mathcal{N}(E)_S \dashrightarrow \mathcal{N}(E)_S$$

is a dominant rational map over $\mathbb{F}_q$. By Lemma 3.6, the subset

$$\Gamma = \{\, \alpha_{\bar{s}} \in \mathcal{N}(E)_S(\bar{\mathbb{F}}_q) \mid \exists\, m,\ f^m(\alpha_{\bar{s}}) = \alpha_{\bar{s}} \,\}$$

of f^a-periodic points is dense in $\mathcal{N}(E)_S$. By Lemma 3.4, if a point $\alpha_{\bar{s}} = (V_{\bar{s}}, \beta_{\bar{s}}) \in \Gamma$, $V_{\bar{s}}$ is a vector bundle on geometric fiber $X_{\bar{s}}$ associated to a representation $\rho_{\bar{s}} : \pi_1^{\text{ét}}(X_{\bar{s}}, \xi_{\bar{s}}) \to GL(r, k(\bar{s}))$.

□

4. An application of Representation spaces

In this section, we present an application of our Theorem 3.7 by giving a proof of relative version of Gieseker's problem. To warm up, we prove firstly the main theorem of [1] via representation spaces.

Theorem 4.1 (Esnault-Mehta, [1, Theorem 3.15])**.** *Let X be a smooth connected projective variety defined over a perfect field k of characteristic $p > 0$ with a k-rational point $\xi \in X(k)$. If $\pi_1^{\acute{e}t}(X_{\bar{k}}, \xi) = \{1\}$, there is no nontrivial stratified bundle on X.*

Proof. We prove it by contradiction. If there is a nontrivial stratified bundle $E = (E_i)_{i\in\mathbb{N}}$ on X, without loss of generality, we assume that all $E_i \in \Sigma = \{E_i\}_{i\in\mathbb{N}}$ are nontrivial bundles. In fact, the subset

$$C = \{\, j \in \mathbb{N} \mid E_j \cong \mathcal{O}_X^{\oplus r} \}$$

must be a finite set (otherwise, for any $i \in \mathbb{N}$, there is an $j \in C$ such that $i < j$, which implies that $E_i \cong F^{(j-i)*}E_j$ is trivial and we obtain a contradiction since $E = (E_i)_{i\in\mathbb{N}}$ is a nontrivial stratified bundle). Drop a finite number of E_j, we can assume that all E_i are nontrivial.

If Σ is a finite set, there is an $E_{i_0} \in \Sigma$ such that $(F_X^*)^a E_{i_0} = E_{i_0}$ for some integer $a > 0$. By Lemma 3.4, there is a nontrivial geometrically connected étale finite cover $\sigma : Z \to X$ such that $\sigma^* E_{i_0} \cong \mathcal{O}_Z^{\oplus r}$, which is a contradiction with $\pi_1^{\text{ét}}(X_{\bar{k}}, \xi) = \{1\}$.

If $\Sigma = \{E_i\}_{i\in\mathbb{N}}$ is an infinite set, let $\mathcal{N}(E)_S \subset R(\mathcal{O}_{X_S}, \xi_S, P)$ be the closed subset constructed in Theorem 3.7 and

$$\mathbf{B} = \{\, (V, \beta\,) \in \mathcal{N}(E)_S \mid V \text{ is trivial} \} \subset R(\mathcal{O}_{X_S}, \xi_S, P).$$

By (2) of Theorem 2.3, $\mathbf{B}$ is a closed subset. By the assumption that $E = (E_i)_{i\in\mathbb{N}}$ is a nontrivial stratified bundle, the open set

$$U = \mathcal{N}(E)_S \setminus \mathbf{B}$$

is non-empty. Then, by (2) of Theorem 3.7, there is a point

$$\alpha_{\bar{s}} = (V_{\bar{s}}, \beta_{\bar{s}}) \in U$$

such that $V_{\bar{s}}$ is a vector bundle on geometric fiber $X_{\bar{s}}$ associated to a representation $\rho_{\bar{s}} : \pi_1^{\text{ét}}(X_{\bar{s}}, \xi_{\bar{s}}) \to GL(r, k(\bar{s}))$, which must be nontrivial by definition of U. This is a contradiction with $\pi_1^{\text{ét}}(X_{\bar{k}}, \xi) = \{1\}$ since the specialization homomorphism $\pi_1^{\text{ét}}(X_{\bar{k}}, \xi) \to \pi_1^{\text{ét}}(X_{\bar{s}}, \xi_{\bar{s}})$ is surjective ([12, Exposé X, Théorème 3.8]). $\square$

Theorem 4.2. *Let $f : Y \to X$ be a morphism of smooth projective varieties over a perfect field k of characteristic $p > 0$, $\xi' \in Y(k)$ and $\xi \in X(k)$ be k-points such that $f(\xi') = \xi$. If the homomorphism*

$$f_* : \pi_1^{\text{ét}}(Y_{\bar{k}}, \xi') \to \pi_1^{\text{ét}}(X_{\bar{k}}, \xi)$$

is trivial, then for any stratified bundle E on X, f^*E is trivial.

Proof. We prove the theorem by contradiction. If there is a stratified bundle $E = (E_i)_{i\in\mathbb{N}}$ on X such that f^*E is nontrivial, without loss of generality, we assume that all $f^*E_i \in \Sigma(f^*E) = \{f^*E_i\}_{i\in\mathbb{N}}$ are nontrivial bundles.

If $\Sigma = \{E_i\}_{i\in\mathbb{N}}$ is a finite set, there is an $E_{i_0} \in \Sigma(E)$ such that for any $j > i_0$ there is a $1 \leq j_0 \leq i_0$ such that $E_j = E_{j_0}$ that implies

$$(F_X^*)^{j-j_0} E_j = E_{j_0} = E_j.$$

Thus E_j is induced by a representation of $\pi_1^{\text{ét}}(X_{\bar{k}}, \xi)$ by Lemma 3.4. Then f^*E_j is trivial, which implies that all f^*E_i are trivial, a contradiction with our assumption.

If $\Sigma = \{E_i\}_{i\in\mathbb{N}}$ is an infinite set, without loss of generality, we assume that all $f^*E_i \in \Sigma(f^*E) = \{f^*E_i\}_{i\in\mathbb{N}}$ are p-semistable bundles on Y of Hilbert polynomial P'. Let $\mathcal{N}(E)_S \subset R(\mathcal{O}_{X_S}, \xi_S, P)$ be the closed subset constructed in Theorem 3.7 and

$$f_S^* : R(\mathcal{O}_{X_S}, \xi_S, P) \dashrightarrow R(\mathcal{O}_{Y_S}, \xi'_S, P')$$

be the rational map that sends a point $\alpha_{\bar{s}} = (V_{\bar{s}}, \beta_{\bar{s}}) \in R(\mathcal{O}_{X_S}, \xi_S, P)$ to a point $f_S^*(\alpha_{\bar{s}}) = (f_{\bar{s}}^*(V_{\bar{s}}), f_{\bar{s}}^*(\beta_{\bar{s}})) \in R(\mathcal{O}_{Y_S}, \xi'_S, P')$ when $f_{\bar{s}}^*(V_{\bar{s}})$ is p-semistable on $Y_{\bar{s}}$, where $f_S : Y_S \to X_S$ is a model of $f : Y \to X$ and $f_{\bar{s}} = f_S \otimes k(\bar{s}) : Y_{\bar{s}} = Y_S \otimes k(\bar{s}) \to X_S \otimes k(\bar{s}) = X_{\bar{s}}$ is the induced morphism on geometric fibers. Consider the open set

$$\mathcal{U} = \{\, (V, \beta\,) \in R(\mathcal{O}_{Y_S}, \xi'_S, P') \,|\, V \text{ is not trivial} \,\} \subset R(\mathcal{O}_{Y_S}, \xi'_S, P'),$$

which is open by (2) of Theorem 2.3, we have a rational map

$$f_S^*: \mathcal{N}(E)_S \dashrightarrow \mathcal{U}. \tag{4.1}$$

Let $W \subset \mathcal{N}(E)_S$ be the open set where the rational map (4.1) is well defined. Then $\{\, V \in \Sigma \,|\, (V, \beta) \in W\} \subset \Sigma$ is an infinite set since all f^*E_i ($\forall\, E_i \in \Sigma$) are nontrivial p-semistable bundles on Y. But, by (2) of Theorem 3.7, W contains a point $\alpha_{\bar{s}} = (V_{\bar{s}}, \beta_{\bar{s}})$ where $V_{\bar{s}}$ is a vector bundle on geometric fiber $X_{\bar{s}}$ associated to a representation

$$\rho_{\bar{s}} : \pi_1^{\text{ét}}(X_{\bar{s}}, \xi_{\bar{s}}) \to GL(r, k(\bar{s})).$$

Then $f_{\bar{s}}^*(V_{\bar{s}})$ must be trivial by the condition of the theorem. Thus the rational map (4.1) is not well-defined at $\alpha_{\bar{s}} = (V_{\bar{s}}, \beta_{\bar{s}}) \in W$, which is a contradiction. □

References

[1] H. Esnault and V. Mehta: *Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles*, Invent. math. **181** (2010), 449–465.

[2] H. Esnault and V. Srinivas: *A relative version of Gieseker's problem on stratifications in characteristic $p > 0$* , Int. Math. Res. Not., https://doi.org/10.1093/imrn/rnx281.

[3] H. Esnault and X. Sun: *Stratified bundles and étale fundamental group* , Ann. Sc. Norm. Super. Pisa Cl. Sci. **5** Vol. XIII (2014), 795-812.

[4] D. Gieseker: *Flat vector bundles and the fundamental group in non-zero characteristics*, Ann. Sc. Norm. Super. Pisa, **4** Ser. 2 (1), (1975), 1-31.

[5] E. Hrushovski: *The Elementary Theory of the Frobenius Automorphism*, arXiv: math/0406514v1 [math.LO], 2004.

[6] A. Langer: *Moduli spaces of sheaves in mixed characteristic*, Duke Math., vol **124** (2004), 571–586.

[7] H. Lange and U. Stuhler: *Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe*, Math. Z., **156**, (1977), 73-83.

[8] V. B. Mehta and T. R. Ramadas: *Moduli of vector bundles, Frobenius splitting, and invariant theory*, Ann. of Math. **144** (1996), 269–313.

[9] D. Mumford, J. Fogarty and F. Kirwan: *Geometric Invariant Theory*, Third Enlarged Edition. Ergebniss der Mathematik und ihrer Grenzgebiete, **34** (1994).

[10] C.S.Seshadri: *Geometric reductivity over arbitrary base*, Adv. Math. **26** (1977), 225-274.

[11] C.-T. Simpson: *Moduli of representations of the fundamental groups of a smooth projective variety I*, Publ. Math. I. H. E. S. **79**, (1994), 47–129.

[12] SGA1.: *Revêtements étales et groupe fondamental*, SGA 1.

[13] Y. Varshavsky: *Intersection of a correspondence with a graph of Frobenius*, J. Algebraic Geometry, vol **27** (2018), 1–20

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