Symplectic critical surfaces with parallel normalized mean curvature vector in two-dimensional complex space forms

Ling He * and Jiayu Li †

Abstract. In this paper, we obtain a sufficient and necessary condition for the existence of symplectic critical surfaces with parallel normalized mean curvature vector in two-dimensional complex space forms. Explicitly, we find that there does not exist any symplectic critical surface with parallel normalized mean curvature vector in two-dimensional complex space forms of non-zero constant holomorphic sectional curvature. And there exists and only exists a two-parameters family of symplectic critical surfaces with parallel normalized mean curvature vector in two-dimensional complex plane, which are rotationally symmetric.

Keywords and Phrases. Symplectic critical surface, parallel normalized mean curvature vector, complex space form, rotationally symmetric.

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1 Introduction

Surfaces with non-zero parallel mean curvature vector in two-dimensional complex space forms are already classified when the Kähler angle is constant by Chen [1], and finally by Hirakawa [11]. If the Kähler angles of these surfaces are not constant, such surfaces are studied first by Ogata [16], later by Kenmotsu and Zhou [15], Kenmotsu ([12],[13]), Hirakawa [11], Ferreira and Tribuzy [4], and Fetcu [5]. We know that parallel mean curvature vector implies constant mean curvature and parallel normalized mean curvature vector. It is interesting and nature from the viewpoint of differential geometry to study surfaces with parallel normalized mean curvature vector. In the following, we first introduce a kind of symplectic surface, then study such surfaces with parallel normalized mean curvature vector.

Let M be a complex two-dimensional Kähler manifold. Let Σ be a compact oriented real two-dimensional Riemannian manifold and we consider an isometric immersion $x : \Sigma \to M$ from Σ into M. Let $\{e_1, e_2\}$ be an oriented orthonormal local frame field on Σ. The Kähler angle θ is a function on Σ that measures the angle of $\mathbf{J}dx(e_1)$ and $dx(e_2)$ for the Kähler metric of M, where $\mathbf{J}$ denotes the complex structure of M. This is independent

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of the choice of oriented orthonormal frames on Σ. It is said that Σ is a holomorphic curve if $\cos\theta = 1$, Σ is a Lagrangian surface if $\cos\theta = 0$ and Σ is a symplectic surface if $\cos\theta > 0$.

Let $\mathbf{H}$ be the mean curvature vector field of x, which is defined by

$$\mathbf{H} = \sum_{\alpha,i} h^{\alpha}_{ii} e_{\alpha}, \tag{1.1}$$

where h^{α}_{ij}'s are the components of the second fundamental form of x, and e_i and e_α are adapted frames along x.

A symplectic minimal surface is a critical point of the area of surfaces, which is symplectic. It may be more natural to consider directly the critical point of the functional

$$L = \int_{\Sigma} \frac{1}{\cos\theta} d\mu_{\Sigma}, \tag{1.2}$$

in the class of symplectic surfaces. The Euler-Lagrange equation of this functional is

$$\cos^3\theta \mathbf{H} = (J(J\nabla\cos\theta)^{\top})^{\perp}. \tag{1.3}$$

Such a surface is called a *symplectic critical surface*(cf.[6]).

The second author and his coauthors (cf.[6]-[10]) have obtained many interesting results about symplectic critical surface from the viewpoint of geometry analysis. In this paper we will focus on the explicit characterization of symplectic critical surface from the viewpoint of differential geometry. It follows from (1.3) that a minimal surface with constant Kähler angle is a symplectic critical surface. In this paper we mainly study symplectic critical surfaces with non-constant mean curvature and non-constant Kähler angle in two-dimensional complex space forms.

In section 2, we study the fundamental equations of symplectic critical surfaces with parallel normalized mean curvature vector in two-dimensional complex space forms. We reduced the local existence problem of such surfaces to the problem of studying a certain overdetermined system of ordinary differential equations (cf. Theorem 2.3). In section 3, we study the system under the condition $\rho = 0$ and get all solutions of the system explicitly in this case. In section 4, we study the system under the condition $\rho \neq 0$ and find there does not exist any non-trivial solution in this case (cf. Theorem 4.3). In section 5, we give some geometric results. Concretely, we find that there does not exist any symplectic critical surface with parallel normalized mean curvature vector in two-dimensional complex space forms of non-zero constant holomorphic sectional curvature (cf. Theorem 5.1). And we obtain the explicit representations of all symplectic critical surfaces with parallel normalized mean curvature vector in $\mathbb{C}^2$ (cf. Theorem 5.2).

2 The fundamental equations of symplectic critical surfaces

Suppose that M is a complex two-dimensional kähler manifold of constant holomorphic sectional curvature 4ρ. Let $\{\omega_i\}$ be a local field of unitary coframes on M, so that the kähler metric is represented by $\sum \omega_i \overline{\omega}_i$. Here and in what follows, we will agree on the following range of indices: $1 \leq i, j, k \leq 2$. We denote by ω_{ij} the unitary connection forms with respect to $\{\omega_i\}$. So we have

$$
\begin{aligned}
&d\omega_i = \sum \omega_{ij} \wedge \omega_j, \ \omega_{ij} + \overline{\omega}_{ji} = 0, \\
&d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \\
&\Omega_{ij} = -\rho \left(\omega_i \wedge \overline{\omega}_j + \delta_{ij} \sum \omega_k \wedge \overline{\omega}_k \right).
\end{aligned} \tag{2.1}
$$

We assume $\mathbf{H} \neq 0$. We can construct a unique system of global orthonormal vector fields $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ along Σ such that $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to Σ by the following: First we put $\tilde{e}_3 = -\frac{\mathbf{H}}{\|\mathbf{H}\|}$, then the normal vector field $\tilde{e}_4$ of $T^{\perp}\Sigma$ is uniquely determined by choosing it to be compatible with the fixed orientations of Σ and M. The system of vectors $\{\tilde{e}_3, \tilde{e}_4, \mathbf{J}\tilde{e}_3, \mathbf{J}\tilde{e}_4\}$ is linearly independent, because x is neither holomorphic nor anti-holomorphic. Here the angle of $\mathbf{J}\tilde{e}_4$ and $\tilde{e}_3$ is equal to the Kähler angle θ which is defined in Section 1. In fact, set

$$
\tilde{e}_1 = -\frac{\mathbf{J}\tilde{e}_4 - \langle \mathbf{J}\tilde{e}_4, \tilde{e}_3 \rangle \tilde{e}_3}{\|\mathbf{J}\tilde{e}_4 - \langle \mathbf{J}\tilde{e}_4, \tilde{e}_3 \rangle \tilde{e}_3\|}, \ \tilde{e}_2 = \frac{\mathbf{J}\tilde{e}_3 - \langle \mathbf{J}\tilde{e}_3, \tilde{e}_4 \rangle \tilde{e}_4}{\|\mathbf{J}\tilde{e}_3 - \langle \mathbf{J}\tilde{e}_3, \tilde{e}_4 \rangle \tilde{e}_4\|}.
$$

Then $\tilde{e}_1$ and $\tilde{e}_2$ are tangent to Σ. A straightforward calculation shows

$$
\langle \mathbf{J}\tilde{e}_4, \tilde{e}_3 \rangle = \langle \mathbf{J}\tilde{e}_1, \tilde{e}_2 \rangle = \cos\theta.
$$

It is easy to see that $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is an adapted frame on Σ in M, that is, $\tilde{e}_1$ and $\tilde{e}_2$ are sections on $T\Sigma$ and $\tilde{e}_3$ and $\tilde{e}_4$ are sections on $T^{\perp}\Sigma$. The complex structure $\mathbf{J}$ is represented under the frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ as follows:

$$
\begin{aligned}
\mathbf{J}\tilde{e}_1 &= \cos\theta \cdot \tilde{e}_2 + \sin\theta \cdot \tilde{e}_4, \\
\mathbf{J}\tilde{e}_2 &= -\cos\theta \cdot \tilde{e}_1 - \sin\theta \cdot \tilde{e}_3, \\
\mathbf{J}\tilde{e}_3 &= \sin\theta \cdot \tilde{e}_2 - \cos\theta \cdot \tilde{e}_4, \\
\mathbf{J}\tilde{e}_4 &= -\sin\theta \cdot \tilde{e}_1 + \cos\theta \cdot \tilde{e}_3.
\end{aligned}
$$

Moreover, we define vector fields e_1 and e_3 as follows:

$$
\begin{aligned}
e_1 &= \frac{\tilde{e}_1 - \mathbf{J}\tilde{e}_2}{\|\tilde{e}_1 - \mathbf{J}\tilde{e}_2\|} = \cos\frac{\theta}{2} \cdot \tilde{e}_1 + \sin\frac{\theta}{2} \cdot \tilde{e}_3, \\
e_3 &= \frac{\tilde{e}_1 + \mathbf{J}\tilde{e}_2}{\|\tilde{e}_1 + \mathbf{J}\tilde{e}_2\|} = \sin\frac{\theta}{2} \cdot \tilde{e}_1 - \cos\frac{\theta}{2} \cdot \tilde{e}_3.
\end{aligned}
$$

and put

$$
\begin{aligned}
e_2 &= \mathbf{J}e_1 = \cos\frac{\theta}{2} \cdot \tilde{e}_2 + \sin\frac{\theta}{2} \cdot \tilde{e}_4, \\
e_4 &= \mathbf{J}e_3 = -\sin\frac{\theta}{2} \cdot \tilde{e}_2 + \cos\frac{\theta}{2} \cdot \tilde{e}_4.
\end{aligned}
$$

Then $\{e_1, e_2, e_3, e_4\}$ is a $\mathbf{J}$-canonical frame along x. We extend $\{\tilde{e}_A\}$ and $\{e_A\}$ to a neighbourhood of Σ in M, where A, B and C run from 1 to 4.

Let $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$ be the dual coframes of $\{\tilde{e}_A\}$ and $\{e_A\}$ respectively. Let $\tilde{\theta}_{AB}$ and θ_{AB} be the Riemannian connection forms with respect to the canonical 1-forms $\{\tilde{\theta}_A\}$ and $\{\theta_A\}$ respectively and put

$$
\omega_j = \theta_{2j-1} + i\theta_{2j},
$$

$$\omega_{jk} = \theta_{2j-1,2k-1} + i\theta_{2j,2k-1}, \text{ where } i = \sqrt{-1}.$$

Then we have the following relations

$$\begin{aligned} \tilde{\theta}_1 + i\tilde{\theta}_2 &= \cos\frac{\theta}{2}\omega_1 + \sin\frac{\theta}{2}\overline{\omega}_2, \\ \tilde{\theta}_3 + i\tilde{\theta}_4 &= \sin\frac{\theta}{2}\omega_1 - \cos\frac{\theta}{2}\overline{\omega}_2. \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \tilde{\theta}_{12} &= i\left(\cos^2\frac{\theta}{2}\omega_{11} - \sin^2\frac{\theta}{2}\omega_{22}\right), \\ \tilde{\theta}_{34} &= i\left(\sin^2\frac{\theta}{2}\omega_{11} - \cos^2\frac{\theta}{2}\omega_{22}\right), \\ \tilde{\theta}_{13} + i\tilde{\theta}_{23} &= -\left\{\omega_{12} + \frac{1}{2}\left[d\theta - \sin\theta(\omega_{11} + \omega_{22})\right]\right\}, \\ \tilde{\theta}_{14} + i\tilde{\theta}_{24} &= i\left\{\omega_{12} - \frac{1}{2}\left[d\theta - \sin\theta(\omega_{11} + \omega_{22})\right]\right\}. \end{aligned} \tag{2.3}$$

We denote the restriction of $\{\tilde{\theta}_A\}$ to Σ by the same letters. Then we have $\tilde{\theta}_3 = 0 = \tilde{\theta}_4$ on Σ. Put

$$\phi = \tilde{\theta}_1 + i\tilde{\theta}_2,$$

the induced metric of Σ is written as

$$ds^2 = \phi\overline{\phi}.$$

By taking the exterior derivative of (2.2) restricted to Σ, we get

$$\begin{aligned} &\frac{1}{2}\left[d\theta + \sin\theta(\omega_{11} + \omega_{22})\right] = a\phi + b\overline{\phi}, \\ &\omega_{12} = b\phi + c\overline{\phi}, \end{aligned} \tag{2.4}$$

where a, b and c are complex-valued smooth functions defined locally on Σ. Let $\{h^\alpha_{ij}\}$ be the components of the second fundamental form so that $\tilde{\theta}_{i\alpha} = \sum_j h^\alpha_{ij}\tilde{\theta}_j$. By using (2.3) and (2.4), all h^α_{ij}'s can be expressed in terms of a, b and c. Indeed, we have

$$\begin{aligned} h^3_{11} &= -\frac{1}{2}\left[a + \bar{a} + 2(b + \bar{b}) + c + \bar{c}\right], \\ h^3_{12} &= \frac{i}{2}\left(-a + \bar{a} + c - \bar{c}\right), \\ h^3_{22} &= \frac{1}{2}\left[a + \bar{a} - 2(b + \bar{b}) + c + \bar{c}\right], \\ h^4_{11} &= \frac{i}{2}\left[a - \bar{a} + 2(b - \bar{b}) + c - \bar{c}\right], \\ h^4_{12} &= \frac{1}{2}\left(-a - \bar{a} + c + \bar{c}\right), \\ h^4_{22} &= \frac{i}{2}\left[-a + \bar{a} + 2(b - \bar{b}) - c + \bar{c}\right]. \end{aligned} \tag{2.5}$$

Since $\mathbf{H} = -\|\mathbf{H}\|\tilde{e}_3 = (h^3_{11} + h^3_{22})\tilde{e}_3 + (h^4_{11} + h^4_{22})\tilde{e}_4$, it follows from (2.5) that

$$b = \bar{b},$$

and

$$\|\mathbf{H}\| = 4b.$$

Let K be the Gauss curvature of Σ, then

$$d\tilde{\theta}_{12} = -K\tilde{\theta}_1 \wedge \tilde{\theta}_2 = -\frac{i}{2}K\phi \wedge \overline{\phi}.$$

By taking the exterior derivative of the first formula of (2.3), using (2.1) and (2.4) we have

$$K = (1 + 3\cos^2\theta)\rho - 2(a^2 - 2b^2 + |c|^2). \tag{2.6}$$

Let K_N be the normal curvature of x defined by

$$d\tilde{\theta}_{34} = -K_N\tilde{\theta}_1 \wedge \tilde{\theta}_2 = -\frac{i}{2}K_N\phi \wedge \overline{\phi}.$$

By taking the exterior derivative of the second formula of (2.3), using (2.1) and (2.4) we have

$$K_N = 2(|a|^2 - |c|^2) - (3\cos^2\theta - 1)\rho. \tag{2.7}$$

Since

$$\begin{aligned}(\mathbf{J}\nabla\cos\theta)^\top &= (\nabla_{\tilde{e}_1}\cos\theta\cdot\mathbf{J}\tilde{e}_1 + \nabla_{\tilde{e}_2}\cos\theta\cdot\mathbf{J}\tilde{e}_2)^\top\\ &= \nabla_{\tilde{e}_1}\cos\theta\cdot\cos\theta\cdot\tilde{e}_2 - \nabla_{\tilde{e}_2}\cos\theta\cdot\cos\theta\cdot\tilde{e}_1,\end{aligned}$$

then

$$\begin{aligned}(\mathbf{J}(\mathbf{J}\nabla\cos\theta)^\top)^\perp &= \nabla_{\tilde{e}_1}\cos\theta\cdot\cos\theta\cdot(\mathbf{J}\tilde{e}_2)^\perp - \nabla_{\tilde{e}_2}\cos\theta\cdot\cos\theta\cdot(\mathbf{J}\tilde{e}_1)^\perp\\ &= -\sin\theta\cos\theta\nabla_{\tilde{e}_1}\cos\theta\cdot\tilde{e}_3 - \sin\theta\cos\theta\nabla_{\tilde{e}_2}\cos\theta\cdot\tilde{e}_4.\end{aligned}$$

Hence, in particular, $\sin\theta \neq 0$. From the symplectic critical surface equation (1.3), we get

$$4b\cos^3\theta = \sin\theta\cos\theta\nabla_{\tilde{e}_1}\cos\theta, \tag{2.8}$$

and

$$\nabla_{\tilde{e}_2}\cos\theta = 0. \tag{2.9}$$

It follows from the first formula of (2.4) that

$$d\theta = (a+b)\phi + (\overline{a}+b)\overline{\phi} = (a+\overline{a}+2b)\tilde{\theta}_1 + i(a-\overline{a})\tilde{\theta}_2. \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$a = \overline{a}, \tag{2.11}$$

which implies

$$d\theta = 2(a+b)\tilde{\theta}_1. \tag{2.12}$$

Substituting (2.12) into (2.8), we obtain

$$a = -(1 + 2\cot^2\theta)b. \tag{2.13}$$

We assume $\nabla^\perp(\frac{\mathbf{H}}{\|\mathbf{H}\|}) = 0$. Then $\tilde{e}_3$ is a parallel vector field along Σ, hence so is $\tilde{e}_4$. This implies

$$\tilde{\theta}_{34} = 0.$$

By taking the exterior derivative of (2.4) and using the above, we obtain

$$\begin{aligned}
&\tilde{\theta}_{12} = -2i(\cot\theta + \cot^3\theta)b(\phi - \overline{\phi}),\\
&d\theta = -2\cot^2\theta\, b(\phi + \overline{\phi}),\\
&db = -2\left\{(\cot\theta + 4\cot^3\theta)b^2 + \frac{3}{8}\rho\sin^3\theta\cos\theta\right\}(\phi + \overline{\phi}),\\
&dc \wedge \overline{\phi} = -2\left\{2(\cot\theta + \cot^3\theta)bc + (\cot\theta + 4\cot^3\theta)b^2 + \frac{3}{8}\rho\sin^3\theta\cos\theta\right\}\phi \wedge \overline{\phi},\\
&|c|^2 = (1 + 2\cot^2\theta)^2 b^2 - \tfrac{\rho}{2}(3\cos^2\theta - 1),\\
&\mathbf{H} = -4b\tilde{e}_3.
\end{aligned} \tag{2.14}$$

The third and fourth formulas of (2.14) are the Codazzi equations of x. The fifth formula of (2.14) follows from that the normal bundle is flat.

Set $\phi = \lambda dz$, where λ is a non-zero complex-valued function on a simply connected domain U with complex coordinate z. Then the set of the first three formulas of (2.14) is rewritten as the following system of differential equations:

$$\begin{aligned}
&\frac{\partial\lambda}{\partial\bar{z}} = 2\lambda\overline{\lambda}(\cot\theta + \cot^3\theta)b,\\
&\frac{\partial\theta}{\partial\bar{z}} = -2\overline{\lambda}\cot^2\theta\, b,\\
&\frac{\partial b}{\partial\bar{z}} = -2\overline{\lambda}\left\{(\cot\theta + 4\cot^3\theta)b^2 + \frac{3}{8}\rho\sin^3\theta\cos\theta\right\}.
\end{aligned} \tag{2.15}$$

In the following we give a lemma about the existence of isothermal coordinate.

Lemma 2.1 *Suppose Σ is a symplectic critical surface with parallel normalized mean curvature vector in M. Then there exists a complex coordinate w on a neighborhood of a point of Σ such that $\phi = \mu dw$, where μ is real-valued.*

Proof: Since θ is not constant, we claim that b is a function of θ. In fact, canceling out $(\phi + \overline{\phi})$ in the second and third formula of (2.14), we get a differential equation in b for θ. Using the claim, we write $b = b(\theta)$, and define a real-valued function

$$F(\theta) = -2\tan\theta + \cot\theta + \frac{3\rho}{8b^2}\tan\theta\sin^4\theta.$$

Taking the partial derivative of the second formula of (2.15) with respect to z and using (2.15), we have a second-order partial differential equation $\frac{\partial^2\theta}{\partial z\partial\bar{z}} - F(\theta)\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial\bar{z}} = 0$. It follows that $\frac{\partial\left(\theta_z\exp\left(-\int F(\theta)d\theta\right)\right)}{\partial\bar{z}} = 0$. Hence, there exists a holomorphic function $G(z)$ on U such that $\frac{\partial\theta}{\partial z} = G(z)\exp\left(\int F(\theta)d\theta\right)$. Setting

$$w = \int G(z)dz,\ \ \mu = -\frac{\exp\left(\int F(\theta)d\theta\right)}{2b\cot^2\theta},$$

the lemma is proved by the conjugate of the second formula of (2.15). □

Hence, for a neighbourhood U of a point of Σ, there exists an isothermal coordinate $z = u + iv$ such that

$$ds^2 = \lambda^2 dz d\bar{z},$$

where λ is a positive function defined on U, and we have

$$\phi = \lambda dz.$$

This implies that λ, θ and b are functions of single variable, and (2.15) is seen to be a system of ordinary differential equations. Consequently, if Σ is a symplectic critical surface with parallel normalized mean curvature vector in M, then there exist real-valued smooth functions of single variable λ, θ and b which are defined locally on Σ and satisfy the system of ordinary differential equations (cf.(2.16)). Moreover, by the fourth and fifth formulas of (2.14), we find that λ, θ and b are subjected to a second-order ordinary differential equation (cf.(2.17)).

Next we shall consider a converse problem to the result obtained above, that is, a local existence problem for symplectic critical surface with parallel normalized mean curvature vector in M. We need the following fundamental theorem of surfaces theory in M.

Theorem 2.2 *([3]) Let (Σ, ds^2) be a connected, simply connected two-dimensional Riemannian manifold. Given complex-valued 1-forms $\omega_1, \omega_2, \omega_{11}, \omega_{22}$ and ω_{12} defined on Σ satisfying the structure equations (2.1) and*

$$ds^2 = \omega_1 \overline{\omega}_1 + \omega_2 \overline{\omega}_2.$$

Then there exist an isometric immersion $x : \Sigma \to M$ and a unitary frame $\{E_1, E_2\}$ along x such that $\{\omega_1, \omega_2\}$ is the unitary coframe of $\{E_1, E_2\}$ and ω_{11}, ω_{22} and ω_{12} are the unitary connection forms with respect to $\{\omega_1, \omega_2\}$.

In the following we give the necessary and sufficient conditions for the existence of symplectic critical surfaces with parallel normalized mean curvature vector in M:

Theorem 2.3 *Let M be a two-dimensional complex space form of constant holomorphic sectional curvature 4ρ. If Σ is a symplectic critical surface with parallel normalized mean curvature vector in M, then there exist a system of local coordinates (u, v) on Σ and real-valued smooth functions $\lambda(u), \theta(u)$ and $b(u)$ of single variable u which are defined on an interval I of u, such that they satisfy a system of ordinary differential equations*

$$\begin{aligned} \frac{d\lambda}{du} &= 4\lambda^2(\cot\theta + \cot^3\theta)b, \ \ \lambda(u) > 0, \\ \frac{d\theta}{du} &= -4\lambda\cot^2\theta\, b, \\ \frac{db}{du} &= -4\lambda\left\{(\cot\theta + 4\cot^3\theta)b^2 + \frac{3}{8}\rho\sin^3\theta\cos\theta\right\}, \end{aligned} \tag{2.16}$$

subject to

$$\frac{d^2F}{du^2}\frac{db}{du} + \left(\frac{dF}{du}\right)^2\frac{db}{du} - \frac{dF}{du}\left(\frac{d^2b}{du^2} + \frac{d\ln\lambda^2}{du}\frac{db}{du}\right) = \frac{2}{A}\left(\frac{db}{du}\right)^3 \tag{2.17}$$

for

$$F = \ln\left\{\lambda^4 A\right\}, A = (1+2\cot^2\theta)^2 b^2 - \frac{\rho}{2}(3\cos^2\theta - 1).$$

Conversely, let $\lambda(u), \theta(u)$ and $b(u)$ be real-valued smooth functions on I, which satisfy (2.16) and (2.17). Let Σ be an open domain in $I \times (-1,1)$ of (u,v)-plane. We define a Riemannian metric on Σ by $ds^2 = \lambda(u)^2(du^2+dv^2)$. Then there exists an isometric immersion $x : \Sigma \to M$ of Σ into M such that it is a symplectic critical surface which satisfies the followings:

(1) x has parallel normalized mean curvature vector and the length of mean curvature vector is $4b$,

(2) the kähler angle of x is θ,

(3) the second fundamental form of x is explicitly written in terms of λ, θ and b.

Proof: Through the above discussions, it is enough to prove the sufficiency for the existence of symplectic critical surfaces with parallel normalized mean curvature vector in M, which is equivalent to giving the local construction of a symplectic critical surfaces with parallel normalized mean curvature vector.

Let (r,s,t) be the standard coordinate of $\mathbb{R}^3$ and $\mathbf{D}$ a domain in $\mathbb{R}^3$ such that $r>0$, $0<s<\frac{\pi}{2}$ and $t>0$. We define a $\mathbb{R}^3$-valued function $f(r,s,t)$ on $\mathbf{D}$ by

$$f(r,s,t) = \left(\begin{array}{l} r^2(\cot s + \cot^3 s)\, t \\ -r\cot^2 s \cdot t \\ -r\left\{(\cot s + 4\cot^3 s)t^2 + \frac{3}{8}\rho \sin^3 s \cos s\right\} \end{array}\right).$$

It is obvious that $f(r,s,t)$ has continuous partial derivatives on $\mathbf{D}$, so that it satisfies Lipschitz condition on $\mathbf{D}$. Hence a solution of (2.16) exists and is unique under preassigned initial conditions.

Let (λ,θ,b) be a solution of (2.16) which satisfy (2.17) and we put

$$z = u + iv \text{ and } \phi = \lambda dz.$$

We define a complex-valued function c on Σ by

$$c = \sqrt{(1+2\cot^2\theta)^2 b^2 - \frac{\rho}{2}(3\cos^2\theta - 1)}\, e^{i\tau}, \tag{2.18}$$

where

$$\cos\tau = \frac{\sqrt{(1+2\cot^2\theta)^2 b^2 - \frac{\rho}{2}(3\cos^2\theta-1)}}{2}\frac{dF/du}{db/du}.$$

Then it is proved that c satisfies the fourth formula of (2.14) and $|c|^2$ satisfies the fifth one. We define $\omega_1, \omega_2, \omega_{11}, \omega_{22}$ and ω_{12} on Σ as follows:

$$\begin{aligned} \omega_1 &= \cos\frac{\theta}{2}\,\phi, \\ \omega_2 &= \sin\frac{\theta}{2}\,\overline{\phi}, \\ \omega_{11} &= -\cot\frac{\theta}{2}(1+\cot^2\theta)b(\phi - \overline{\phi}), \\ \omega_{22} &= -\tan\frac{\theta}{2}(1+\cot^2\theta)b(\phi-\overline{\phi}), \\ \omega_{12} &= -\overline{\omega}_{21} = b\phi + c\overline{\phi}. \end{aligned} \tag{2.19}$$

Note that these satisfy (2.1) because of (2.16). Therefore, by Theorem 2.2, we have an isometric immersion $x : \Sigma \to M$ such that Σ is a symplectic critical surface which has a parallel normalized mean curvature vector and θ the Kähler angle. The second fundamental form of x is explicitly written in terms of λ, θ and b by (2.5). $\square$

Remark 2.4 *Let $x : \Sigma \to M$ be a symplectic critical surface. Then the condition of parallel normalized mean curvature vector implies that the Kähler angle is not constant and the mean curvature is not constant too. Because if the Kähler angle is constant, then it follows from the second equation in (2.16) that $b = 0$, which contradicts to the supposition that the mean curvature is non-zero. If the mean curvature is constant, then from the third equation in (2.16), we have*

$$\sin\theta \neq 0,$$

and

$$(1 + 3\cos^2\theta)b^2 + \frac{3}{8}\rho\sin^6\theta = 0.$$

When $\rho \neq 0$, the above formula is impossible. When $\rho = 0$, the above formula implies that $b = 0$, which is a contradiction.

Remark 2.5 *(1) Let $x : \Sigma \to M$ be a symplectic critical surface with parallel normalized mean curvature vector. Then the curvature of the normal connection vanishes and hence we have by (2.7) or the fifth equation of (2.14),*

$$(1 + 2\cot^2\theta)^2 b^2 - \frac{\rho}{2}(3\cos^2\theta - 1) \geq 0.$$

(2) The case of

$$(1 + 2\cot^2\theta)^2 b^2 - \frac{\rho}{2}(3\cos^2\theta - 1) \equiv 0 \tag{2.20}$$

doesn't exist. In fact it follows from (2.20) that

$$b^2 = \frac{\rho}{2} \cdot \frac{3\cos^2\theta - 1}{(1 + 2\cot^2\theta)^2}.$$

When $\rho = 0$, the above formula implies that $b = 0$, which is a contradiction. When $\rho \neq 0$, substituting the above formula into the third equation of (2.16), we find it is impossible.

For later use, we change these equations (2.16)-(2.17) as follows:

Lemma 2.6 *Assume that functions $\lambda(u), \theta(u)$ and $b(u)$ satisfy (2.16). Then on the point of $(1 + 2\cot^2\theta)^2 b^2 - \frac{\rho}{2}(3\cos^2\theta - 1) \neq 0$, (2.17) is equivalent to*

$$(b^2 D_1^2 - 4|c|^2 D_2^2)\frac{db}{du} + 2b|c|^2\left(D_1\frac{dD_2}{du} - D_2\frac{dD_1}{du}\right) + D_1 D_2\left(b\frac{d|c|^2}{du} - 2|c|^2\frac{db}{du}\right) = 0, \tag{2.21}$$

where

$$\begin{aligned}
|c|^2 &= b^2(2\cot^2\theta + 1)^2 - \tfrac{\rho}{2}(3\cos^2\theta - 1),\\
D_1 &= 2b^2(2\cot^2\theta + 1)(4\cot^2\theta + 1)\\
&\quad - \rho\left[2(2\cot^2\theta - 1) + 3\cos^2\theta + \tfrac{3}{4}(\cos^2\theta + 1)^2\right],\\
D_2 &= b^2(4\cot^2\theta + 1) + \tfrac{3}{8}\rho\sin^4\theta.
\end{aligned} \tag{2.22}$$

Proof: Since

$$F = \ln(\lambda^4|c|^2).$$

Differentiating the above formula and using (2.16), we obtain

$$\frac{dF}{du} = \frac{-2}{|c|}\frac{db}{du} \cdot \frac{bD_1}{2|c|D_2}. \tag{2.23}$$

Differentiating (2.23) with respect to u, we get

$$\begin{aligned} \frac{d^2F}{du^2} &= \left(\frac{2}{|c|^2}\frac{d|c|}{du}\frac{db}{du} - \frac{2}{|c|}\frac{d^2b}{du^2}\right)\cdot\frac{bD_1}{2|c|D_2} \\ &+\frac{1}{2|c|^4D_2^2}\left[2b|c|^2\left(D_1\frac{dD_2}{du} - D_2\frac{dD_1}{du}\right) + D_1D_2\left(b\frac{d|c|^2}{du} - 2|c|^2\frac{db}{du}\right)\right]. \end{aligned} \tag{2.24}$$

Substituting (2.23) and (2.24) into (2.17), we have (2.21). Hence we finish our proof. □

Since both $\theta(u)$ and $b(u)$ are not constants, regarding θ as variable, we get from (2.16) that

$$\begin{aligned} \frac{d\lambda}{d\theta} &= -\lambda(\theta)\cdot(\tan\theta + \cot\theta), \\ \frac{db}{d\theta} &= (\tan\theta + 4\cot\theta)\cdot b(\theta) + \frac{3\rho}{8}\cdot\frac{\sin^4\theta\tan\theta}{b(\theta)}. \end{aligned} \tag{2.25}$$

The integration of the first equation of (2.25) gives us the solution of $\lambda(\theta)$ as follows:

$$\lambda(\theta) = c_1\cot\theta, \tag{2.26}$$

for any positive constant c_1.

Let $x = \sin^2\theta$ denote new variable. Then the second equation of (2.25) implies that

$$\frac{db}{dx} = b(x)\cdot\frac{4-3x}{2x(1-x)} + \frac{3\rho}{16}\cdot\frac{x^2}{b(x)(1-x)}, \tag{2.27}$$

and from (2.21) that

$$\rho\cdot\sum_{k=0}^{3} h_{3-k}(x)\rho^{3-k}b(x)^{2k} = 0 \tag{2.28}$$

with

$$\begin{aligned} h_0(x) &= 16(x-1)(x-2)^3, \\ h_1(x) &= -\frac{1}{4}x^2(3x^5 - 148x^3 + 464x^2 - 448x + 128), \\ h_2(x) &= \frac{1}{32}x^5(9x^5 - 36x^4 - 6x^3 - 96x^2 + 264x - 128), \\ h_3(x) &= \frac{3}{64}x^8(3x-2)(3x^2-2). \end{aligned} \tag{2.29}$$

We assume that (2.27) and (2.28) hold on some interval $(0 \le)x_1 < x < x_2(\le 1)$.

3 Analysis of the overdetermined system: $\rho = 0$ case

When $\rho = 0$, we get all solutions of the system (2.16)-(2.17) as follows.

Lemma 3.1 *Assume that $\rho = 0$. Then all solutions of the system (2.16)-(2.17) are given by*

$$\begin{aligned} \lambda(\theta) &= c_1 \cot\theta, \\ b(\theta) &= c_2 \sin^3\theta \tan\theta, \end{aligned} \tag{3.1}$$

for any positive constants c_1 and c_2.

Proof: In this case, the second equation of (2.25) reduces to

$$\frac{db}{d\theta} = (\tan\theta + 4\cot\theta) \cdot b(\theta).$$

The integration of the above equation gives us the solution of $b(\theta)$. Combining (2.26), we get (3.1). Furthermore, since $\rho = 0$, then (2.28) is automatically satisfied. Hence we finish our proof. □

4 Analysis of the overdetermined system: $\rho \neq 0$ case

In this section we study the system (2.16)-(2.17) with $\rho \neq 0$. We will prove that the system has no solution of $\theta(u)$ being nonconstant.

If there are nonconstant functions $\lambda(u), \theta(u), b(u)$ such that they satisfy the system (2.16)-(2.17), then they must satisfy (2.27) and (2.28) too. Since $\rho \neq 0$, then (2.28) reduces to

$$\sum_{k=0}^{3} h_{3-k}(x)\rho^{3-k} b(x)^{2k} = 0 \tag{4.1}$$

with $h_i(x)$ $(i = 0, 1, 2, 3)$ being given by (2.29). We assume that $I = (x_1, x_2)$ is the maximal interval of the existence of solution of (2.27) and (4.1).

Lemma 4.1 *For $\rho > 0$, we have $I = (0, \sqrt{\frac{2}{3}})$ or $(\sqrt{\frac{2}{3}}, 1)$. For $\rho < 0$, we have $I = (0, 1)$.*

Proof: Claim. For $\rho > 0$, $b(x) = 0$ on the interval $(0, 1)$ if and only if $x = \sqrt{\frac{2}{3}}$. For $\rho < 0$, $b(x)$ is non-zero everywhere on $(0, 1)$.

Proof of Claim. If $b(x) = 0$ for $x \in (0, 1)$, then by (4.1) we have $h_3(x) = 0$ for $x \in (0, 1)$, which implies $x = \frac{2}{3}$ or $x = \sqrt{\frac{2}{3}}$ by (2.29). Conversely, substituting $x = \frac{2}{3}$ and $x = \sqrt{\frac{2}{3}}$ into (4.1) respectively, we get

$$\begin{aligned} & b\left(\frac{2}{3}\right)^2 \cdot \left[144 \cdot b\left(\frac{2}{3}\right)^4 + 10\rho \cdot b\left(\frac{2}{3}\right)^2 - \frac{1}{9}\rho^2\right] = 0, \\ & b\left(\sqrt{\frac{2}{3}}\right)^2 \cdot \left[a_0 \cdot b\left(\sqrt{\frac{2}{3}}\right)^4 + a_1\rho \cdot b\left(\sqrt{\frac{2}{3}}\right)^2 + a_2\rho^2\right] = 0, \end{aligned} \tag{4.2}$$

where a_0, a_1, a_2 are fixed positive numbers.

If $\rho > 0$, then $a_0 \cdot b\left(\sqrt{\frac{2}{3}}\right)^4 + a_1\rho \cdot b\left(\sqrt{\frac{2}{3}}\right)^2 + a_2\rho^2 > 0$, which implies $b\left(\sqrt{\frac{2}{3}}\right) = 0$ by (4.2). But $b\left(\frac{2}{3}\right)^2 = \frac{-5+\sqrt{41}}{144} \cdot \rho \neq 0$.

If $\rho < 0$, then from (4.2) we get $b\left(\frac{2}{3}\right)^2 = \frac{5+\sqrt{41}}{144} \cdot (-\rho) \neq 0$ and $b\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0} \cdot (-\rho) \neq 0$. This proves the Claim.

Proof of Lemma 4.1. For $\rho > 0$, suppose $0 < x_1 < \sqrt{\frac{2}{3}}$, then from (2.29) that $\lim_{x\to x_1} h_3(x)$ exists and the limit is a non-zero finite number. Then by the formula (4.1), we can see that $\lim_{x\to x_1} b(x)$ exists and the limit is also a non-zero finite number. Since $0 < x_1 < \sqrt{\frac{2}{3}}$ and $b(x_1) \neq 0$, there exists a finite limit of $b'(x)$ as x tends to x_1 in (2.27). Hence, $b = b(x)$ can be extended to $x \leq x_1$ by (2.27). This contradicts the definition of x_1. Therefore we must have $x_1 = 0$. Similarly we get $x_2 = \sqrt{\frac{2}{3}}$ or $x_1 = \sqrt{\frac{2}{3}}$, $x_2 = 1$. So for $\rho > 0$, we have $I = (0, \sqrt{\frac{2}{3}})$ or $(\sqrt{\frac{2}{3}}, 1)$. If $\rho < 0$, through the similar discussion as the case of $\rho > 0$, we have $I = (0, 1)$. This proves Lemma 4.1. □

In the following we prove:

Lemma 4.2 *For any $\rho \neq 0$, the system (2.16)-(2.17) does not possess any solution such that $\theta(u)$ is nonconstant.*

Proof: From the discussion before, we only need to show that there do not exist any nonconstant solutions of (2.27) with (4.1) on I. Assume that there exists a nonconstant solution $b = b(x)$ of (2.27) with (4.1) on I. Set $\sigma = b^2$. Since $b \neq 0$ on I, then (2.27) and (4.1) become to the following system:

$$\begin{aligned} &\sigma' = 2f_0\sigma + 2g_0\rho, \\ &h_0\sigma^3 + h_1\rho\sigma^2 + h_2\rho^2\sigma + h_3\rho^3 = 0, \end{aligned} \tag{4.3}$$

where h_i $(i = 0, 1, 2, 3)$ are given by (2.29) and

$$f_0 = \frac{4 - 3x}{2x(1-x)}, \ g_0 = \frac{3x^2}{16(1-x)}.$$

Differentiating the second equation of (4.3) on x and using the first equation of (4.3), we have

$$(h_0' + 6h_0f_0)\sigma^3 + (h_1' + 4h_1f_0 + 6h_0g_0)\rho\sigma^2 + (h_2' + 2h_2f_0 + 4h_1g_0)\rho^2\sigma + (h_3' + 2h_2g_0)\rho^3 = 0. \tag{4.4}$$

Eliminating the term of σ^3 from (4.4) and the second equation of (4.3), by $\rho \neq 0$ we get

$$\mu_0\sigma^2 + \mu_1\rho\sigma + \mu_2\rho^2 = 0, \tag{4.5}$$

where

$$\begin{aligned} \mu_0 &= h_0'h_1 - h_0h_1' + 2h_0h_1f_0 - 6h_0^2g_0, \\ \mu_1 &= h_0'h_2 - h_0h_2' + 4h_0h_2f_0 - 4h_0h_1g_0, \\ \mu_2 &= h_0'h_3 - h_0h_3' + 6h_0h_3f_0 - 2h_0h_2g_0. \end{aligned}$$

Differentiating (4.5) on x and using the first equation of (4.3), we have

$$(\mu_0' + 4\mu_0 f_0)\sigma^2 + (\mu_1' + 2\mu_1 f_0 + 4\mu_0 g_0)\rho\sigma + (\mu_2' + 2\mu_1 g_0)\rho^2 = 0. \tag{4.6}$$

Eliminating the term of σ^2 from (4.5) and (4.6), by $\rho \neq 0$ we get

$$\zeta_0\sigma + \zeta_1\rho = 0, \tag{4.7}$$

where

$$\zeta_0 = \mu_0'\mu_1 - \mu_0\mu_1' + 2\mu_0\mu_1 f_0 - 4\mu_0^2 g_0,$$
$$\zeta_1 = \mu_0'\mu_2 - \mu_0\mu_2' + 4\mu_0\mu_2 f_0 - 2\mu_0\mu_1 g_0.$$

Differentiating (4.7) on x and using the first equation of (4.3) again, we have

$$(\zeta_0' + 4\zeta_0 f_0)\sigma + (\zeta_1' + 2\zeta_0 g_0)\rho = 0. \tag{4.8}$$

Eliminating the term of σ from (4.7) and (4.8), by $\rho \neq 0$ we get

$$\zeta_0'\zeta_1 - \zeta_0\zeta_1' + 2\zeta_0\zeta_1 f_0 - 2\zeta_0^2 g_0 = 0. \tag{4.9}$$

A straightforward computation by Mathematica of Wolfram shows that the left hand side of (4.9) is a polynomial of degree 39 with respect to x. So it is impossible for $x \in I$ no matter that $\rho > 0$ or $\rho < 0$. Hence we finish our proof. $\square$

Combining Lemma 3.1 and Lemma 4.2, we have the following result:

Theorem 4.3 *The system of (2.16)-(2.17) has a solution such that $\theta(u)$ is nonconstant if and only if $\rho = 0$. Moreover, in the case of $\rho = 0$, all solutions are given by (3.1).*

5 Geometric results

In this section, we show geometric results obtained by the application of Theorem 4.3. First, we state our result in the case of $\rho \neq 0$:

Theorem 5.1 *Assume that M is a two-dimensional complex space form of non-zero constant holomorphic sectional curvature. Then there does not exist any symplectic critical surface with parallel normalized mean curvature vector in M.*

Next, we will find all symplectic critical surfaces in $\mathbb{C}^2$ such that the normalized mean curvature vectors are parallel.

Let $x : \Sigma \to \mathbb{C}^2$ be an isometric immersion such that Σ is a symplectic critical surface with parallel normalized mean curvature vector. By Lemma 3.1, (2.26) and (2.18), the complex-valued 1-forms ω_j and ω_{jk} of the immersion x are fixed using two constants. So, there exists and only exists a two-parameters family of symplectic critical surfaces with parallel normalized mean curvature vector in $\mathbb{C}^2$. Now we get the explicit representations of such surfaces.

There exists a unitary frame $\{Z_1, Z_2\}$ along x such that $\{\omega_1, \omega_2\}$ is the unitary coframe of $\{Z_1, Z_2\}$ and ω_{jk}'s are the unitary connection forms with respect to $\{\omega_1, \omega_2\}$.

We know that

$$dZ_j = \sum_{k=1}^{2} \overline{\omega}_{jk} Z_k,\ 1 \leq j \leq 2 \tag{5.1}$$

and

$$dx = \cos\frac{\theta}{2}\,\phi \cdot Z_1 + \sin\frac{\theta}{2}\,\overline{\phi} \cdot Z_2. \tag{5.2}$$

Let $\lambda(\theta)$ and $b(\theta)$ be the functions of (3.1). Put $z = u(\theta) + iv$ and $\phi = \lambda(\theta)dz$. By (2.18), (2.22) and (2.23), we set

$$c = -(1 + 2\cot^2\theta)b.$$

Then (2.19) implies that

$$\overline{\omega}_{11} = f_1(\theta)dv,\ \overline{\omega}_{22} = f_2(\theta)dv,\ \overline{\omega}_{12} = f_3(\theta)d\theta + f_4(\theta)dv,$$

where

$$\begin{aligned} f_1(\theta) &= 4ic_1c_2\cos^2\frac{\theta}{2}, \\ f_2(\theta) &= 4ic_1c_2\sin^2\frac{\theta}{2}, \\ f_3(\theta) &= \frac{1}{2}, \\ f_4(\theta) &= -2ic_1c_2\sin\theta. \end{aligned}$$

From (5.1) and the equations above, we have 2×2-matrix differential equations

$$\begin{aligned} \frac{\partial}{\partial\theta}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} 0 & f_3(\theta) \\ -\bar{f}_3(\theta) & 0 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \\ \frac{\partial}{\partial v}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} f_1(\theta) & f_4(\theta) \\ -\bar{f}_4(\theta) & f_2(\theta) \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}. \end{aligned} \tag{5.3}$$

We solve these as follows:

By direct computation, we know that the eigenvalues of the matrix of coefficients of the second equation in (5.3) are 0 and $4ic_1c_2$, which are independent of the variable θ. Take a matrix $T(\theta)$ which diagonalizes the matrix of coefficients. Then the equations (5.3) are written as

$$\begin{aligned} \frac{\partial}{\partial\theta}T(\theta)^{-1}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \frac{\partial}{\partial v}T(\theta)^{-1}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 4ic_1c_2 \end{pmatrix}T(\theta)^{-1}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \end{aligned}$$

where

$$T(\theta) = \begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}.$$

By integration of the above equations, we can see that there exist constant vectors $\mathbf{v}_1, \mathbf{v}_2$ such that

$$\begin{pmatrix} Z_1(\theta, v) \\ Z_2(\theta, v) \end{pmatrix} = \begin{pmatrix} \sin\frac{\theta}{2} \cdot \mathbf{v}_1 - \cos\frac{\theta}{2} \cdot e^{4ic_1c_2v}\mathbf{v}_2 \\ \cos\frac{\theta}{2} \cdot \mathbf{v}_1 + \sin\frac{\theta}{2} \cdot e^{4ic_1c_2v}\mathbf{v}_2 \end{pmatrix}.$$

Since $\{Z_1(\theta, v), Z_2(\theta, v)\}$ is the unitary frame field, we can put

$$\mathbf{v}_1 = \mathbf{e}_1, \ \mathbf{v}_2 = \mathbf{e}_2,$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbb{C}^2$.

The differential equation (5.2) is equivalent to

$$\begin{aligned} \frac{\partial x}{\partial u} &= \lambda(u) \left\{ \sin\theta(u) \cdot \mathbf{e}_1 - \cos\theta(u) \cdot e^{4ic_1c_2v}\mathbf{e}_2 \right\}, \\ \frac{\partial x}{\partial v} &= -i\lambda(u)e^{4ic_1c_2v}\mathbf{e}_2. \end{aligned} \tag{5.4}$$

We integrate the second equation in (5.4) for v. After that, using the first equation in (5.4),(2.26),(2.16) and a parallel translation of $\mathbb{C}^2$, we get an expression of the immersion $x : \Sigma \to \mathbb{C}^2$ by

$$x(u, v) = \left(F_1(u), \ F_2(u)e^{4ic_1c_2v}\right) \in \mathbb{C}^2, \tag{5.5}$$

where $F_1(u) = \frac{1}{4c_2} \ln\cot\theta(u)$ and $F_2(u) = \frac{-1}{4c_2} \cot\theta(u)$.

These surfaces are rotationally symmetric and are just the ones given by the second author and his coauthors (cf. [8],§6). From the above discussion, we have

Theorem 5.2 *There exists and only exists a two-parameters family of symplectic critical surfaces with parallel normalized mean curvature vector in $\mathbb{C}^2$, which is rotationally symmetric and congruent to* (5.5).

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References

[1] B.Y. Chen, *Special slant surfaces and a basic inequality*, Result. Math., 33(1998), 65-78.

[2] S.S. Chern and J. Wolfson, *Minimal surfaces by moving frames*, Am. J. Math., 105(1983), 59-83.

[3] J.H. Eschenburg, I.V. Guadalupe and R.A. Tribuzy, *The fundamental equations of minimal surface in CP^2*, Math. Ann., 270(1985), 571-598.

[4] M.J. Ferreia and R.A. Tribuzy, *Parallel mean curvature surfaces in symmetric spaces*, Ark. Mat., 52(2014), 93-98.

[5] D. Fetcu, *Surfaces with parallel mean curvature vector in complex space forms*, J. Diff. Geom., 91(2012), 215-232.

[6] X. Han, J. Li, *Symplectic critical surfaces in Kaehler surfaces*, J. Eur. Math. Soc., 12(2) (2010), 505-527.

[7] X. Han, J. Li, *The second variation of the functional L of symplectic critical surfaces in Kähler surfaces*, Commun. Math. Stat., 2(3-4)(2014), 311C330.

[8] X. Han, J. Li, and J. Sun, *The symplectic critical surfaces in a Kaehler surface*, Springe Proc. Math. Stat., 154(2016), 185-193.

[9] X. Han, J. Li, and J. Sun, *The deformation of symplectic critical surfaces in a Kaehler surface-II-compactness*, Calc. Var. PDE, 3(2017), 56-84.

[10] X. Han, J. Li, and J. Sun, *The deformation of symplectic critical surfaces in a Kähler surface-I*, Int. Math. Res. Not., 20(2018), 6290-6328.

[11] S. Hirakawa, *Constant Gaussian curvature surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Geom. Dedicata, 118(2006), 229-244.

[12] K. Kenmotsu, *Complete parallel mean curvature surfaces in two-dimensional complex space forms"*, arXiv:1712.07757v1, 2017.

[13] K. Kenmotsu, *Correction to "The classification of the surfaces with parallel mean curvature vector in two-dimensional complex space forms"*, Amer. J. Math., 138(2016), 395-402.

[14] K. Kenmotsu and T. Ogata, *Correction to "Surfaces with parallel mean curvature vector in* CP^2*"*, Kodai Math. J., 38(2015), 687-689.

[15] K. Kenmotsu and D. Zhou, *The classification of the surfaces with parallel mean curvature vector in two-dimensional complex space forms*, Amer. J. Math., 122(2000), 295-317.

[16] T. Ogata, *Surfaces with parallel mean curvature vector in* CP^2, Kodai Math. J., 90(1995), 397-407.