An involution on increasing trees

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Abstract. We introduce a method to construct bijections on increasing trees. Using this method, we construct an involution on increasing trees, from which we obtain the equidistribution of the statistics 'number of odd vertices' and 'number of even vertices at odd levels'. As an application, we deduce that the expected value of the number of even vertices is twice the expected value of the number of odd vertices in a random recursive tree of given size.

Keywords: Increasing tree, Random recursive tree, Involution, Bijection, Degree, Level

AMS Classification: 05A19, 05A15, 05C05

1 Introduction

An *increasing tree* on the vertex set $[\overline{n}] := \{0, 1, 2, \ldots, n\}$ is a rooted tree with vertex set $\{0, 1, 2, \ldots, n\}$ for which the labels of the vertices are increasing along any path from the root to a leaf. Denote by $\mathcal{T}_n$ the set of all increasing trees on the vertex set $[\overline{n}]$. It is known that $|\mathcal{T}_n| = n!$. By *random increasing trees* (= *random recursive trees*) on the vertex set $[\overline{n}]$, we assume that all increasing trees in $\mathcal{T}_n$ are equally likely. An alternative way of constructing a random recursive tree on the vertex set $[\overline{n}]$ is as follows. We start from a single vertex with label 0 (the 0th step); then at the ith insertion step, the new label i chooses any of the previous i vertices equally likely to be its parent, and the same procedure continues until the tree contains $n + 1$ vertices.

The *degree* of a vertex of an increasing tree is the number of its children (immediate descendants). A vertex of an increasing tree is called an *odd* (resp. *even*) vertex if it is of odd (resp. even) degree. The *level* of a vertex v, also called the *depth* of v, is measured by the number of edges lying on the unique path from the root to vertex v. In particular, the root lies on level 0. Denote by $d_T(v)$ and $l_T(v)$ the degree and the level of v in T respectively. Vertex degrees and vertex levels are two of the most important topics on increasing trees, there are numerous results on the study of them, see, e.g., [1, 3, 4, 7].

Denote by $X_n^{(d)}$ the number of vertices of degree d in a random recursive tree on the vertex set $[\overline{n}]$. Let $\mathbb{E}X_n^{(d)}$ be the expected value of $X_n^{(d)}$. Na and Rapoport [6] proved that for fixed d and large values of n,

$$\mathbb{E}X_n^{(d)} \approx \frac{n+1}{2^{d+1}}. \tag{1.1}$$

From (1.1), it seems reasonable to conjecture that the expected value of the number of even vertices is twice the expected value of the number of odd vertices in a random recursive tree on the vertex set $[\overline{n}]$ for $n \geq 2$. A motivation of this paper is to give a combinatorial proof for this conjecture.

Before we begin, let us introduce two notions. An even vertex of an increasing tree is called a *single even vertex* if it lies on an odd level. An even vertex of an increasing tree is called a *double even vertex* if it lies on an even level. Denote by O_n the number of odd vertices in a random recursive tree on the vertex set $[\overline{n}]$, denote by E_n the number of even vertices, denote by $E_{\mathrm{o},n}$ the number of single even vertices, and denote by $E_{\mathrm{e},n}$ the number of double even vertices. It is clear that $\mathbb{E}O_n + \mathbb{E}E_n = n+1$ and $\mathbb{E}E_n = \mathbb{E}E_{\mathrm{o},n} + \mathbb{E}E_{\mathrm{e},n}$.

In Section 2, we prove combinatorially that $\mathbb{E}E_{\mathrm{o},n} = \mathbb{E}E_{\mathrm{e},n}$ for $n \geq 2$. In Section 3, we introduce a method to construct bijections on increasing trees. Using this method, we construct an involution on increasing trees, and as a corollary we find that the statistics 'number of odd vertices' and 'number of single even vertices' have the same distribution on $\mathcal{T}_n$ for $n \geq 0$, implying $\mathbb{E}O_n = \mathbb{E}E_{\mathrm{o},n}$ for $n \geq 0$ of course. Then we can deduce that $\mathbb{E}E_n = 2\mathbb{E}O_n$ for $n \geq 2$ (see Corollary 3.7), which confirms the above conjecture.

2 Proof of $\mathbb{E}E_{\mathbf{o},n} = \mathbb{E}E_{\mathbf{e},n}$ for $n \geq 2$

Let $c(n,k)$ be the *signless Stirling number of the first kind*, i.e., the number of permutations of length n with exactly k cycles. It is known (see, e.g., [3, 9]) that $c(n,k)$ also counts the number of increasing trees on the vertex set $[\overline{n}]$ for which the root has k children. We start by proving a simple result.

Lemma 2.1. *Let $n \geq 2$. The number of increasing trees on the vertex set $[\overline{n}]$ with odd root is equal to the number of increasing trees on the vertex set $[\overline{n}]$ with even root.*

Proof. It is well-known (see, e.g., [2, 9]) that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1),$$

taking $x = -1$ yields (note that $n \geq 2$)

$$\sum_{k \text{ odd}} c(n,k) = \sum_{k \text{ even}} c(n,k). \tag{2.1}$$

Since the number of increasing trees on the vertex set $[\overline{n}]$ for which the root has k children is $c(n,k)$, we have the number of increasing trees on the vertex set $[\overline{n}]$ with odd root is $\sum_{k \text{ odd}} c(n,k)$, and the number of increasing trees on the vertex set $[\overline{n}]$ with even root is $\sum_{k \text{ even}} c(n,k)$. Combining this with (2.1) we complete the proof. ∎

Theorem 2.2. *Let $n \geq 2$. Then $\mathbb{E}E_{\mathrm{o},n} = \mathbb{E}E_{\mathrm{e},n}$.*

Proof. For given $T \in \mathcal{T}_n$, we interchange the two labels 0 and 1 of T, let T' be the resulting tree rooted at the vertex whose new label is 0. (Indeed, this operation is an involution, i.e., $(T')' = T$.) See Figure 1 for an example.

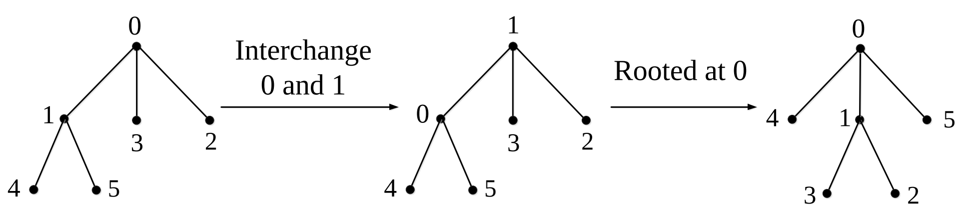

Figure 1. An example illustrating the two steps of the operation

Denote by $O(T)$ the set of odd vertices of T, denote by $E_{\mathrm{e}}(T)$ the set of double even vertices of T, and denote by $E_{\mathrm{o}}(T)$ the set of single even vertices of T. It is not hard to see the following two facts:

(A) For any vertex $i \in \{2, 3, \ldots, n\}$, $i \in E_{\mathrm{e}}(T)$ if and only if $i \in E_{\mathrm{o}}(T')$.

(B) Since $d_{T'}(1) = d_T(0) - 1$, we have $0 \notin E_{\mathrm{e}}(T)$ if and only if $1 \in E_{\mathrm{o}}(T')$.

Let $\mathcal{E}_{\rm e} := \{(i, T) : T \in \mathcal{T}_n, i \in E_{\rm e}(T)\}$ and $\mathcal{E}_{\rm o} := \{(i, T) : T \in \mathcal{T}_n, i \in E_{\rm o}(T)\}$. From (A), (B) and Lemma 2.1, we can see the following three facts respectively:

(a) $(i, T) \in \mathcal{E}_{\rm e}$ if and only if $(i, T') \in \mathcal{E}_{\rm o}$ for $i \in \{2, 3, \ldots, n\}$.

(b) $(0, T) \notin \mathcal{E}_{\rm e}$ if and only if $(1, T') \in \mathcal{E}_{\rm o}$.

(c) $|\{(0, T) : (0, T) \in \mathcal{E}_{\rm e}\}| = |\{(0, T) : (0, T) \notin \mathcal{E}_{\rm e}\}|$.

From (b) and (c) we see

$$|\{(0, T) : (0, T) \in \mathcal{E}_{\rm e}\}| = |\{(1, T') : (1, T') \in \mathcal{E}_{\rm o}\}|.$$

Combining this with (a), we get $|\mathcal{E}_{\rm e}| = |\mathcal{E}_{\rm o}|$, i.e., the total number of double even vertices of all increasing trees on the vertex set $[\overline{n}]$ is equal to the total number of single even vertices of all increasing trees on the vertex set $[\overline{n}]$, which implies $\mathbb{E}E_{{\rm o},n} = \mathbb{E}E_{{\rm e},n}$. ∎

3 An involution on increasing trees

In this section, we will introduce a method to construct bijections on increasing trees. Using this method, we will construct an involution $\phi : \mathcal{T}_n \to \mathcal{T}_n$ that maps an increasing tree with m_1 odd vertices and m_2 double even vertices to an increasing tree with m_1 single even vertices and m_2 double even vertices.

Let us first introduce a new notation for increasing trees. Given $T_{n-1} \in \mathcal{T}_{n-1}$ and $T_n \in \mathcal{T}_n$, we write

$$T_n = T_{n-1} \stackrel{v}{\leftarrow} n$$

to mean that T_n is obtained from T_{n-1} by adding a new vertex n as the child of vertex v, where $0 \le v \le n-1$. Let $\mathbf{0}$ be the unique tree in $\mathcal{T}_0$, that is the isolated vertex 0. Then for any $T_n \in \mathcal{T}_n$, we can uniquely write

$$T_n = \mathbf{0} \stackrel{v_0}{\longleftarrow} 1 \stackrel{v_1}{\longleftarrow} 2 \stackrel{v_2}{\longleftarrow} \cdots \stackrel{v_{n-1}}{\longleftarrow} n,$$

where $0 \le v_i \le i$. By definition we know that vertex $i+1$ is the child of vertex v_i in T_n. For example, let

$$T = \mathbf{0} \stackrel{0}{\leftarrow} 1 \stackrel{0}{\leftarrow} 2 \stackrel{2}{\leftarrow} 3 \stackrel{0}{\leftarrow} 4 \stackrel{2}{\leftarrow} 5 \stackrel{4}{\leftarrow} 6,$$

the diagram of T is shown in Figure 2.

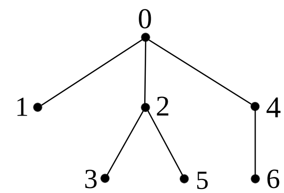

Figure 2. The diagram of $\mathbf{0} \overset{0}{\leftarrow} 1 \overset{0}{\leftarrow} 2 \overset{2}{\leftarrow} 3 \overset{0}{\leftarrow} 4 \overset{2}{\leftarrow} 5 \overset{4}{\leftarrow} 6$

Given $T_n \in \mathcal{T}_n$, assume

$$T_n = \mathbf{0} \overset{v_0}{\leftarrow} 1 \overset{v_1}{\leftarrow} 2 \overset{v_2}{\leftarrow} \cdots \overset{v_{n-1}}{\longleftarrow} n.$$

Now we are going to recursively define an increasing tree $\phi(T_n) \in \mathcal{T}_n$. Let

$$T_i = \mathbf{0} \overset{v_0}{\leftarrow} 1 \overset{v_1}{\leftarrow} 2 \overset{v_2}{\leftarrow} \cdots \overset{v_{i-1}}{\longleftarrow} i,$$

or, equivalently,

$$T_i = T_{i-1} \overset{v_{i-1}}{\longleftarrow} i.$$

Clearly, T_i is the subtree of T_n consisting of the vertices $0, 1, \ldots, i$. We associate each T_i a permutation $\sigma_i = \sigma_i(0)\sigma_i(1)\cdots\sigma_i(i)$ of $\{0, 1, \ldots, i\}$, called *the relabelled permutation of* T_i. For $0 \le i \le n$, define

$$\phi(T_i) = \mathbf{0} \overset{\sigma_0(v_0)}{\longleftarrow} 1 \overset{\sigma_1(v_1)}{\longleftarrow} 2 \overset{\sigma_2(v_2)}{\longleftarrow} \cdots \overset{\sigma_{i-1}(v_{i-1})}{\longleftarrow} i,$$

or, equivalently,

$$\phi(T_i) = \phi(T_{i-1}) \overset{\sigma_{i-1}(v_{i-1})}{\longleftarrow} i.$$

Then $\phi(T_n)$ is an increasing tree in $\mathcal{T}_n$. It is not hard to see that $\phi : \mathcal{T}_n \to \mathcal{T}_n$ is a bijection when each σ_i has been defined with respect to T_i. To see this, let us show how to recover T_n from $\phi(T_n)$ recursively. First, $\phi(T_0) = T_0 = \mathbf{0}$. Assume T_i has been recovered for some $0 \le i \le n-1$, then σ_i is determined. Assume

$$\phi(T_{i+1}) = \phi(T_i) \overset{v}{\leftarrow} i+1,$$

then T_{i+1} can be recovered as follows

$$T_{i+1} = T_i \overset{\sigma_i^{-1}(v)}{\longleftarrow} i+1.$$

This is a general method to construct bijections on increasing trees. In particular, if we define $\sigma_i = 012\cdots i$ for any T_i, then $\phi : \mathcal{T}_n \to \mathcal{T}_n$ is the identity map. In what follows, we are going to define a particular sequence $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}, \sigma_n$ such that ϕ is our desired involution.

First let $\sigma_0 = \binom{0}{0}$ (the two-line notation of permutations). Assume that σ_i has been defined with respect to T_i for some $0 \le i \le n-1$. Note that $T_{i+1} = T_i \xleftarrow{v_i} i+1$, we define σ_{i+1} (with respect to T_{i+1}) by considering the following two cases. (Since $l_{T_i}(v) = l_{T_n}(v)$ and $l_{\phi(T_i)}(v) = l_{\phi(T_n)}(v)$ for all i and $0 \le v \le i$, we write $l(v)$ instead of $l_{T_i}(v)$, and write $l_\phi(v)$ instead of $l_{\phi(T_i)}(v)$.)

(i) If both $l(v_i)$ and $l_\phi(\sigma_i(v_i))$ are odd, define

$$\sigma_{i+1} = \begin{pmatrix} 0 & 1 & \cdots & i & i+1 \\ \sigma_i(0) & \sigma_i(1) & \cdots & \sigma_i(i) & i+1 \end{pmatrix}. \tag{3.1}$$

(ii) Otherwise, i.e., at least one of $l(v_i)$ or $l_\phi(\sigma_i(v_i))$ is even, define

$$\sigma_{i+1} = \begin{pmatrix} 0 & 1 & \cdots & v_i - 1 & v_i & v_i + 1 & \cdots & i & i+1 \\ \sigma_i(0) & \sigma_i(1) & \cdots & \sigma_i(v_i - 1) & i+1 & \sigma_i(v_i+1) & \cdots & \sigma_i(i) & \sigma_i(v_i) \end{pmatrix}. \tag{3.2}$$

Note that, the right-hand side of (3.2) is obtained from the right-hand side of (3.1) by swapping the letters $\sigma_i(v_i)$ and $i+1$. We call σ_i obtained from the above procedure *the relabelled permutation of T_i with respect to ϕ*. Let us look at an example.

Example 3.1. Consider the increasing tree T shown in Figure 2, that is,

$$T = \mathbf{0} \xleftarrow{0} 1 \xleftarrow{0} 2 \xleftarrow{2} 3 \xleftarrow{0} 4 \xleftarrow{2} 5 \xleftarrow{4} 6.$$

We illustrate each step of constructing $\phi(T)$ below.

(0) Let $T_0 = \phi(T_0) = \mathbf{0}$; $\sigma_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $l(0) = l_\phi(0) = 0$.

(1) Let $T_1 = \mathbf{0} \xleftarrow{0} 1$, then

$$\phi(T_1) = \mathbf{0} \xleftarrow{\sigma_0(0)} 1 = \mathbf{0} \xleftarrow{0} 1.$$

Since $l(v_0) = l(0) = 0$ is even, we have $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
$l(1) = l(0) + 1 = 1$, $l_\phi(1) = l_\phi(0) + 1 = 1$.

(2) Let $T_2 = T_1 \overset{0}{\leftarrow} 2$, then

$$\phi(T_2) = \phi(T_1) \xleftarrow{\sigma_1(0)} 2 = \phi(T_1) \overset{1}{\leftarrow} 2 = \mathbf{0} \overset{0}{\leftarrow} 1 \overset{1}{\leftarrow} 2.$$

Since $l(v_1) = l(0) = 0$ is even, we have $\sigma_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$.
$l(2) = l(0) + 1 = 1$, $l_\phi(2) = l_\phi(1) + 1 = 2$.

(3) Let $T_3 = T_2 \overset{2}{\leftarrow} 3$, then

$$\phi(T_3) = \phi(T_2) \xleftarrow{\sigma_2(2)} 3 = \phi(T_2) \overset{1}{\leftarrow} 3 = \mathbf{0} \overset{0}{\leftarrow} 1 \overset{1}{\leftarrow} 2 \overset{1}{\leftarrow} 3.$$

Since both $l(2)$ and $l_\phi(\sigma_2(2)) = l_\phi(1)$ are odd, we have $\sigma_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$.
$l(3) = l(2) + 1 = 2$, $l_\phi(3) = l_\phi(1) + 1 = 2$.

(4) Let $T_4 = T_3 \overset{0}{\leftarrow} 4$, then

$$\phi(T_4) = \phi(T_3) \xleftarrow{\sigma_3(0)} 4 = \phi(T_3) \overset{2}{\leftarrow} 4 = \mathbf{0} \overset{0}{\leftarrow} 1 \overset{1}{\leftarrow} 2 \overset{1}{\leftarrow} 3 \overset{2}{\leftarrow} 4.$$

Since $l(0)$ is even, we have $\sigma_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 3 & 2 \end{pmatrix}$.
$l(4) = l(0) + 1 = 1$, $l_\phi(4) = l_\phi(2) + 1 = 3$.

(5) Let $T_5 = T_4 \overset{2}{\leftarrow} 5$, then

$$\phi(T_5) = \phi(T_4) \xleftarrow{\sigma_4(2)} 5 = \phi(T_4) \overset{1}{\leftarrow} 5 = \mathbf{0} \overset{0}{\leftarrow} 1 \overset{1}{\leftarrow} 2 \overset{1}{\leftarrow} 3 \overset{2}{\leftarrow} 4 \overset{1}{\leftarrow} 5.$$

Since both $l(2)$ and $l_\phi(\sigma_4(2)) = l_\phi(1)$ are odd, we have $\sigma_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 0 & 1 & 3 & 2 & 5 \end{pmatrix}$.
$l(5) = l(2) + 1 = 2$, $l_\phi(5) = l_\phi(1) + 1 = 2$.

(6) Let $T_6 = T_5 \overset{4}{\leftarrow} 6$, then

$$\phi(T) = \phi(T_6) = \phi(T_5) \xleftarrow{\sigma_5(4)} 6 = \phi(T_5) \overset{2}{\leftarrow} 6 = \mathbf{0} \overset{0}{\leftarrow} 1 \overset{1}{\leftarrow} 2 \overset{1}{\leftarrow} 3 \overset{2}{\leftarrow} 4 \overset{1}{\leftarrow} 5 \overset{2}{\leftarrow} 6.$$

Since $l_\phi(\sigma_5(4)) = l_\phi(5)$ is even, we have $\sigma_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 0 & 1 & 3 & 6 & 5 & 2 \end{pmatrix}$.
The diagrams of T and $\phi(T)$ are shown in Figure 3.

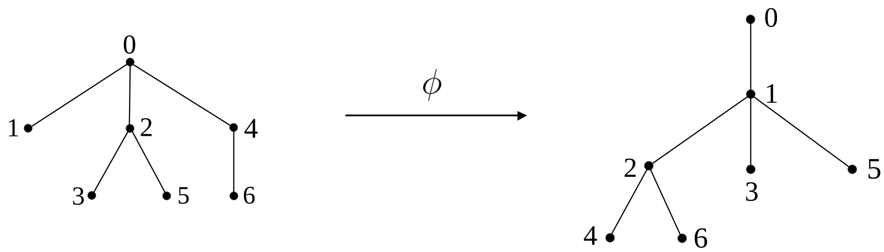

Figure 3. The diagrams of T and $\phi(T)$

The bijection $\phi : \mathcal{T}_n \to \mathcal{T}_n$ defined by the above procedure has several interesting properties. Let us start with the following lemma.

Lemma 3.2. *Let $n \geq 0$. Given $T_n \in \mathcal{T}_n$, assume $T'_n = \phi(T_n)$, let σ_n be the relabelled permutation of T_n with respect to ϕ, and let σ'_n be the relabelled permutation of T'_n with respect to ϕ. Then $\phi(T'_n) = T_n$ and $\sigma'_n = \sigma_n^{-1}$. Therefore, $\phi : \mathcal{T}_n \to \mathcal{T}_n$ is an involution.*

Proof. We use induction on n. The initial case of $n = 0$ is obvious. Assume the statement is true for n, and prove for $n+1$. Given $T_{n+1} \in \mathcal{T}_{n+1}$, assume

$$T_{n+1} = T_n \overset{v}{\leftarrow} n+1.$$

By the definition of ϕ, we have

$$T'_{n+1} = T'_n \xleftarrow{\sigma_n(v)} n+1,$$

and

$$\phi(T'_{n+1}) = \phi(T'_n) \xleftarrow{\sigma'_n(\sigma_n(v))} n+1.$$

By the induction hypothesis, we know $\phi(T'_n) = T_n$ and $\sigma'_n = \sigma_n^{-1}$, thus

$$\phi(T'_{n+1}) = \phi(T'_n) \xleftarrow{\sigma'_n(\sigma_n(v))} n+1 = T_n \xleftarrow{\sigma'_n(\sigma_n(v))} n+1 = T_n \overset{v}{\leftarrow} n+1 = T_{n+1}.$$

In the following we prove $\sigma'_{n+1} = \sigma_{n+1}^{-1}$. If both $l_{T_n}(v)$ and $l_{T'_n}(\sigma_n(v))$ are odd, we have

$$\sigma_{n+1} = \begin{pmatrix} 0 & 1 & \cdots & n & n+1 \\ \sigma_n(0) & \sigma_n(1) & \cdots & \sigma_n(n) & n+1 \end{pmatrix}.$$

Clearly, both $l_{T'_n}(\sigma_n(v))$ and $l_{\phi(T'_n)}(\sigma'_n(\sigma_n(v))) = l_{T_n}(v)$ are odd, then

$$\sigma'_{n+1} = \begin{pmatrix} \sigma_n(0) & \sigma_n(1) & \cdots & \sigma_n(n) & n+1 \\ 0 & 1 & \cdots & n & n+1 \end{pmatrix}.$$

In this case, we have $\sigma'_{n+1} = (\sigma_{n+1})^{-1}$. If at least one of $l_{T_n}(v)$ or $l_{T'_n}(\sigma_n(v))$ is even, then

$$\sigma_{n+1} = \begin{pmatrix} 0 & 1 & \cdots & v & \cdots & n & n+1 \\ \sigma_n(0) & \sigma_n(1) & \cdots & n+1 & \cdots & \sigma_n(n) & \sigma_n(v) \end{pmatrix}.$$

Since at least one of $l_{T'_n}(\sigma_n(v))$ or $l_{\phi(T'_n)}(\sigma'_n(\sigma_n(v))) = l_{T_n}(v)$ is even, we have

$$\sigma'_{n+1} = \begin{pmatrix} \sigma_n(0) & \sigma_n(1) & \cdots & \sigma_n(v) & \cdots & \sigma_n(n) & n+1 \\ 0 & 1 & \cdots & n+1 & \cdots & n & v \end{pmatrix}.$$

We also have $\sigma'_{n+1} = (\sigma_{n+1})^{-1}$. This completes our induction proof. ∎

Recall that, $O(T)$ denotes the set of odd vertices of T, $E_{\mathrm{e}}(T)$ denotes the set of double even vertices of T, and $E_{\mathrm{o}}(T)$ denotes the set of single even vertices of T. We have the following result.

Lemma 3.3. *Let $n \geq 0$. Given $T_n \in \mathcal{T}_n$, let σ_n be the relabelled permutation of T_n with respect to ϕ. Then for any vertex i of T_n, $0 \leq i \leq n$, we have*

(i) If $i \in O(T_n)$, then $\sigma_n(i) \in E_o(\phi(T_n))$.

(ii) If $i \in E_o(T_n)$, then $\sigma_n(i) \in O(\phi(T_n))$.

(iii) If $i \in E_e(T_n)$, then $\sigma_n(i) \in E_e(\phi(T_n))$.

Before we start our proof, let us look at an example. Consider the trees T and $\phi(T)$ shown in Figure 3. It is clear that

$$O(T) = \{0, 4\}, \quad E_{\mathrm{o}}(T) = \{1, 2\}, \quad E_{\mathrm{e}}(T) = \{3, 5, 6\},$$

and

$$O(\phi(T)) = \{0, 1\}, \quad E_{\mathrm{o}}(\phi(T)) = \{4, 6\}, \quad E_{\mathrm{e}}(\phi(T)) = \{2, 3, 5\}.$$

Let σ be the relabelled permutation of T with respect to ϕ, from Example 3.1 we see

$$\sigma = \sigma_6 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 0 & 1 & 3 & 6 & 5 & 2 \end{pmatrix}.$$

Then Lemma 3.3 is verified as

$$\begin{aligned} \sigma\left(O(T)\right) &= \{\sigma(0), \sigma(4)\} = \{4, 6\} = E_{\mathrm{o}}\left(\phi(T)\right), \\ \sigma\left(E_{\mathrm{o}}(T)\right) &= \{\sigma(1), \sigma(2)\} = \{0, 1\} = O\left(\phi(T)\right), \\ \sigma\left(E_{\mathrm{e}}(T)\right) &= \{\sigma(3), \sigma(5), \sigma(6)\} = \{3, 5, 2\} = E_{\mathrm{e}}\left(\phi(T)\right). \end{aligned}$$

Proof of Lemma 3.3. We use induction on n. The initial case of $n = 0$ is obvious. Assume (i), (ii), (iii) are true for n, and prove for $n + 1$. Given $T_{n+1} \in \mathcal{T}_{n+1}$, assume

$$T_{n+1} = T_n \xleftarrow{v} n + 1,$$

by the definition of ϕ, we know

$$\phi(T_{n+1}) = \phi(T_n) \xleftarrow{\sigma_n(v)} n + 1.$$

Denote by $l(i)$ the level of i in T_{n+1}, and denote by $l_\phi(i)$ the level of i in $\phi(T_{n+1})$.

Using Lemma 3.2, it is easy to see that (i) and (ii) imply (iii), so we will only prove (i) and (ii). For any $0 \le i \le n + 1$ and $i \ne v, n + 1$, we have $d_{T_n}(i) = d_{T_{n+1}}(i)$, $d_{\phi(T_n)}(\sigma_n(i)) = d_{\phi(T_{n+1})}(\sigma_n(i))$, and $\sigma_{n+1}(i) = \sigma_n(i)$. Combining those facts and the induction hypothesis, we have

$$\begin{aligned} i \in O(T_{n+1}) &\Rightarrow i \in O(T_n) \\ &\Rightarrow \sigma_n(i) \in E_{\mathrm{o}}(\phi(T_n)) \\ &\Rightarrow \sigma_n(i) \in E_{\mathrm{o}}(\phi(T_{n+1})) \\ &\Rightarrow \sigma_{n+1}(i) \in E_{\mathrm{o}}(\phi(T_{n+1})). \end{aligned}$$

Thus, (i) is true for vertex i. A similar argument shows that (ii) is also true for vertex i. We only consider vertex v and vertex $n + 1$ below.

We first prove (i). Since $n+1$ is a leaf of T_{n+1}, so $n+1 \notin O(T_{n+1})$. Assume $v \in O(T_{n+1})$, our goal is to prove $\sigma_{n+1}(v) \in E_{\mathrm{o}}(\phi(T_{n+1}))$. Clearly, $v \in E_{\mathrm{o}}(T_n)$ or $v \in E_{\mathrm{e}}(T_n)$.

Case 1.1. If $v \in E_{\mathrm{o}}(T_n)$, by the induction hypothesis, we have $\sigma_n(v) \in O(\phi(T_n))$. If $l_\phi(\sigma_n(v))$ is odd, then $\sigma_n(v) \in E_{\mathrm{o}}(\phi(T_{n+1}))$. Since both $l(v)$ and $l_\phi(\sigma_n(v))$ are odd, we have $\sigma_{n+1}(v) = \sigma_n(v) \in E_{\mathrm{o}}(\phi(T_{n+1}))$. If $l_\phi(\sigma_n(v))$ is even, then $l_\phi(n+1) = l_\phi(\sigma_n(v)) + 1$ is odd, so $n + 1 \in E_{\mathrm{o}}(\phi(T_{n+1}))$. Since $l_\phi(\sigma_n(v))$ is even, we have $\sigma_{n+1}(v) = n + 1 \in E_{\mathrm{o}}(\phi(T_{n+1}))$. Therefore, in this case we always have $\sigma_{n+1}(v) \in E_{\mathrm{o}}(\phi(T_{n+1}))$.

Case 1.2. If $v \in E_{\mathrm{e}}(T_n)$, by the induction hypothesis, we have $\sigma_n(v) \in E_{\mathrm{e}}(\phi(T_n))$, so $n + 1 \in E_{\mathrm{o}}(\phi(T_{n+1}))$. Since $l(v)$ is even, we have $\sigma_{n+1}(v) = n + 1 \in E_{\mathrm{o}}(\phi(T_{n+1}))$.

We now prove (ii). It is clear that at most one of v and $n + 1$ belongs to $E_{\mathrm{o}}(T_{n+1})$. We consider two separate cases.

Case 2.1. If $v \in E_{\mathrm{o}}(T_{n+1})$, our goal is to prove $\sigma_{n+1}(v) \in O(\phi(T_{n+1}))$. Since $v \in E_{\mathrm{o}}(T_{n+1})$, we have $v \in O(T_n)$ and $l(v)$ is odd. By the induction hypothesis, we have $\sigma_n(v) \in E_{\mathrm{o}}(\phi(T_n))$, then $\sigma_n(v) \in O(\phi(T_{n+1}))$. Since both $l(v)$ and $l_\phi(\sigma_n(v))$ are odd, we have $\sigma_{n+1}(v) = \sigma_n(v) \in O(\phi(T_{n+1}))$.

Case 2.2. If $n+1 \in E_{\mathrm{o}}(T_{n+1})$, then $l(v)$ is even, thus, we have $\sigma_{n+1}(n+1) = \sigma_n(v)$. Our goal is to prove $\sigma_{n+1}(n+1) = \sigma_n(v) \in O(\phi(T_{n+1}))$. Since $l(v)$ is even, we have $v \in O(T_n)$ or $v \in E_{\mathrm{e}}(T_n)$. If $v \in O(T_n)$, by the induction hypothesis, we have $\sigma_n(v) \in E_{\mathrm{o}}(\phi(T_n))$, then $\sigma_{n+1}(n+1) = \sigma_n(v) \in O(\phi(T_{n+1}))$. If $v \in E_{\mathrm{e}}(T_n)$, by the induction hypothesis, we have $\sigma_n(v) \in E_{\mathrm{e}}(\phi(T_n))$, we also have $\sigma_{n+1}(n+1) = \sigma_n(v) \in O(\phi(T_{n+1}))$.

The induction proof is completed. ∎

Combining Lemma 3.2 and 3.3, we obtain the main theorem of this section.

Theorem 3.4. *Let $n \geq 0$. Then $\phi : \mathcal{T}_n \to \mathcal{T}_n$ is an involution that maps an increasing tree with m_1 odd vertices and m_2 double even vertices to an increasing tree with m_1 single even vertices and m_2 double even vertices.*

It is straightforward to see the following corollary from Theorem 3.4.

Corollary 3.5. *Let $n \geq 0$. Then the number of increasing trees on the vertex set $[\overline{n}]$ with m_1 odd vertices and m_2 double even vertices is equal to the number of increasing trees on the vertex set $[\overline{n}]$ with m_1 single even vertices and m_2 double even vertices.*

Summing over all m_2 in Corollary 3.5 for fixed m_1, we obtain the following corollary.

Corollary 3.6. *Let $n \geq 0$. Then the number of increasing trees on the vertex set $[\overline{n}]$ with m_1 odd vertices is equal to the number of increasing trees on the vertex set $[\overline{n}]$ with m_1 single even vertices.*

From Corollary 3.6 and Theorem 2.2, we immediately obtain the following result, which confirms our conjecture in the introduction.

Corollary 3.7. *Let $n \geq 2$. Then $\mathbb{E}O_n = \mathbb{E}E_{\mathrm{o},n} = \mathbb{E}E_{\mathrm{e},n}$. Therefore, $\mathbb{E}E_n = 2\mathbb{E}O_n$ for $n \geq 2$.*

Let $\mathcal{F}_n$ be the increasing trees on the vertex set $[\overline{n}]$ with no double even vertex. From Corollary 3.6 we find that the statistics 'number of odd vertices' and 'number of even vertices' have the same distribution on $\mathcal{F}_n$.

Let $E(n)$ be the nth *Euler number*, i.e., the number of alternating (down-up) permutations of length n. The following two formulas are well-known:

$$\sec x = \sum_{n=0}^{\infty} E(2n)\frac{x^{2n}}{(2n)!} \quad \text{and} \quad \tan x = \sum_{n=0}^{\infty} E(2n+1)\frac{x^{2n+1}}{(2n+1)!}.$$

For this reason $E(2n)$ is called the nth *secant number* and $E(2n+1)$ is called the nth *tangent number*, see [8]. An *even tree* is an increasing tree such that every vertex is an even vertex. (Such a tree must have an odd number of vertices.) Let $\mathcal{E}_{2n}$ be the set of even trees on the vertex set $\{0, 1, 2, \ldots, 2n\}$. It is known [5, 9] that $|\mathcal{E}_{2n}| = E(2n)$. Combining this with Corollary 3.6, we obtain the following result.

Corollary 3.8. *Let $n \geq 0$. Then the number of increasing trees on the vertex set $\{0, 1, 2, \ldots, 2n\}$ with no single even vertex is equal to the nth secant number $E(2n)$.*

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