The operators F_i on permutations, 132-avoiding permutations and inversions

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Abstract. We introduce the operators F_i on permutation $\pi = \pi_1 \cdots \pi_{k-1} 1 \pi_{k+1} \cdots \pi_n$ of $\{1, 2, \cdots, n\}$, where $1 \leq i \leq k-1$, i.e., define $F_i(\pi) = \pi'_1 \pi'_2 \cdots \pi'_n$ as $\pi'_j = \pi_j - 1$ for $1 \leq j \leq i$, and $\pi'_{i+1}\pi'_{i+2} \cdots \pi'_n$ has the same relative order as $\pi_{i+1}\pi_{i+2} \cdots \pi_n$. The operators F_i have many properties concerning the 132-pattern and inversions. Furthermore, we find that the operators F_i can be characterized by a series of swaps of two entries. Two applications of the operators are given. As a first application, we obtain some new objects in 132-avoiding permutations and in Dyck paths that are enumerated by the entries in Catalan's triangle. As another application, we give an algorithm to generate the set of permutations of length $n+1$ with k inversions from the set of permutations of length n with k inversions when $n \geq k+1$.

Keywords: Operator, 132-avoiding, Catalan's triangle, Inversion, Dyck path, Algorithm

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1 Introduction

The nth *Catalan number* C_n is defined by the recursion

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i},$$

with $C_0 = 1$. The Catalan numbers arise frequently in combinatorics, Stanley [19] gives over 100 objects that are counted by the Catalan numbers.

The classical *Catalan's triangle* $C(n,k)$ is defined by the recurrence relation

$$C(n,k) = C(n-1,k) + C(n,k-1), \tag{1.1}$$

n \ k	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	2							
3	1	3	5	5						
4	1	4	9	14	14					
5	1	5	14	28	42	42				
6	1	6	20	48	90	132	132			
7	1	7	27	75	165	297	429	429		
8	1	8	35	110	275	572	1001	1430	1430	
9	1	9	44	154	429	1001	2002	3432	4862	4862

Table 1 Catalan's triangle $C(n, k), 0 \leq k \leq n \leq 9$

with the boundary conditions $C(0, 0) = 1$ and $C(0, k) = 0$ for $k > 0$ or $k < 0$. An alternative recursion for the Catalan's triangle is

$$C(n, k) = C(n-1, 0) + C(n-1, 1) + \cdots + C(n-1, k). \tag{1.2}$$

The beginning of Catalan's triangle is shown in Table 1. The entries in Catalan's triangle are often called *ballot numbers*. See [18, A009766] for an overview of Catalan's triangle, as well as Barcucci and Verri [2] for earlier investigations. The Catalan numbers can always be read from Catalan's triangle by looking at the rightmost number in each row, so we have

$$C_n = C(n, n) = C(n, n-1) = C(n-1, 0) + C(n-1, 1) + \cdots + C(n-1, n-1). \tag{1.3}$$

The exact formulas for C_n and $C(n, k)$ are well-known:

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad C(n, k) = \frac{n-k+1}{n+1}\binom{n+k}{k}.$$

For now, we will give some objects that are counted by the Catalan numbers and the entries in Catalan's triangle.

Let $\mathcal{S}_n$ be the set of all permutations of $\{1, 2, \cdots, n\}$. The *reduced form* of a permutation π on a set $\{j_1, j_2, \cdots, j_r\}$, where $j_1 < j_2 < \cdots < j_r$, is the permutation in $\mathcal{S}_r$ obtained by renaming the letters of the permutation π so that j_i is renamed i for all $i \in \{1, 2, \cdots, r\}$. In other words, to find the reduced form of a permutation π on r elements, we replace the ith smallest letter of π by i, for $i = 1, 2, \cdots, r$. We denote $\text{red}(\pi)$ the reduced form of π. For example, $\text{red}(4257) = 2134$.

Let $\pi \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_r$ be two permutations. We say that π *contains* σ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that $\text{red}(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_r}) = \sigma$; in such a context σ is usually called a *pattern*. We say that π *avoids* σ, or is σ*-avoiding*, if such a subsequence does not exist. The set of all σ-avoiding permutations in $\mathcal{S}_n$ is denoted by $\mathcal{S}_n(\sigma)$, its cardinality is denoted by $S_n(\sigma)$. There are numerous results on the study of $S_n(\sigma)$, see, e.g., [4, 10, 20]. Another problem is counting the number of permutations of length n which contain exactly r σ-patterns, for $r \geq 1$. There is a larger literature devoted to it, see, e.g., [3, 14, 15].

It is well-known, see [17], that $S_n(\sigma) = C_n$ for each pattern $\sigma \in \mathcal{S}_3$. Using the reverse and complement operations, and their composition, we see that $S_n(132) = S_n(231) = S_n(213) = S_n(312)$. Also, we have $S_n(321) = S_n(123)$ by simply applying the reverse operation. From an enumerative viewpoint, there essentially are only two distinct patterns to consider, that is, $\sigma = 132$ and $\sigma = 321$.

A *right-to-left maximum* of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is an entry π_a such that $\pi_a > \pi_b$ for every $b > a$. *Right-to-left minimum* is defined accordingly. By using generating function techniques, Brändén, Claesson and Steingrímsson [6] proved that the number of permutations in $\mathcal{S}_n(132)$ with k right-to-left maxima is equal to $C(n-1, n-k)$. Desantis et al. [7] proved bijectively that the number of permutations in $\mathcal{S}_n(132)$ with the first entry being k is $C(n-1, k-1)$, Borie [5] gave this result another proof by means of an explicit bijection between 132-avoiding permutations and non-decreasing parking functions that are known to be enumerated by the entries in Catalan's triangle. Kitaev and Liese [11] studied the so-called mesh patterns, they showed that for three mesh patterns, their distributions on 132-avoiding permutations are given by the Catalan's triangle. Form (1.3) we can see that those results cited above are all the refinements of the well-known result $S_n(132) = C_n$.

Desantis et al. [7] proved bijectively that the number of permutations in $\mathcal{S}_n(321)$ with the first entry being $n-k+1$ is $C(n-1, k-1)$. Reifegerste[16] proved that the number of permutations $\pi \in \mathcal{S}_n(321)$ with $k-1$ elements $\pi_i = i+1$ is equal to $C(n-1, n-k)$. Those two results are the refinements of the well-known result $S_n(321) = C_n$.

Let us now consider another object. A *Dyck path* of length $2n$ is a path on the square lattice with steps $u = (1, 1)$ or $d = (1, -1)$ from $(0, 0)$ to $(2n, 0)$ that never falls below the x-axis. We call the steps of type u *up-steps* and those of type d we call *down-steps*. A *return* of a Dyck path is a down-step ending on the x-axis. A *peak* in a Dyck path is an occurrence of an up-step immediately followed by a down-step, i.e., the occurrence of ud. The *height* of the peak is the height of the intersection point of its two steps. It is well-known that the number of Dyck paths of length $2n$ is the nth Catalan number C_n.

Krattenthaler [12] exhibited a bijection Φ between 132-avoiding permutations of length n and Dyck paths of length $2n$. We sketch his bijection below. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a 132-avoiding permutation. We read the permutation π from left to right and successively generate a Dyck path $\Phi(\pi)$. When π_j is read, then in the path we adjoin as many up-steps as necessary, followed by a down-step from height $h_j + 1$ to height h_j (measured from the x-axis), where h_j is the number of elements in $\pi_j \cdots \pi_n$ which are larger than π_j.

Using the ballot theorem (see, e.g., [9, p. 73]), Deutsch [8] proved that the number of

Dyck paths of length $2n$ with the first (last) peak at height k is equal to $C(n-1, n-k)$. He also proved that the number of Dyck paths of length $2n$ with k returns is equal to $C(n-1, n-k)$ by exhibiting a bijection from the Dyck paths of length $2n$ with k returns to the Dyck paths of length $2n$ with the last peak at height k, Brändén, Claesson and Steingrímsson [6] provided another proof of this result by using Krattenthaler's bijection.

The approaches in the literature to deal with the problems related to 132-avoiding permutations and the Catalan's triangle are the generating function approach and the bijection approach. In this paper, we will present a new approach, that is the operator approach, to deal with such problems.

2 Outline of this paper

In this paper we introduce the operators F_i on permutation $\pi = \pi_1 \cdots \pi_{k-1} 1 \pi_{k+1} \cdots \pi_n \in \mathcal{S}_n$, where $1 \leq i \leq k-1$, i.e., define $F_i(\pi) = \pi'_1 \pi'_2 \cdots \pi'_n$ as $\pi'_j = \pi_j - 1$ for $1 \leq j \leq i$, and $\pi'_{i+1}\pi'_{i+2} \cdots \pi'_n$ has the same relative order as $\pi_{i+1}\pi_{i+2}\cdots\pi_n$. The operators F_i have many properties concerning the 132-pattern and inversions. Furthermore, we find that the operators F_i can be characterized by a series of swaps of two entries.

As a first application of the operators, we obtain some new objects in 132-avoiding permutations and in Dyck paths that are enumerated by the entries in Catalan's triangle.

Consider the following objects in 132-avoiding permutations (throughout this paper, we use π_t for the tth entry of π),

$$\begin{aligned}
\mathcal{C}_n(p) &:= \{\pi \in \mathcal{S}_n(132) \mid \pi_p = 1\}, \\
\mathcal{C}_n^-(k,p) &:= \{\pi \in \mathcal{S}_n(132) \mid \pi_1 = k, \pi_p = 1, \pi_n \neq n\}, \\
\mathcal{D}_{n,i}(k) &:= \{\pi \in \mathcal{S}_n(132) \mid k = \pi_1 > \pi_2 > \cdots > \pi_i\}, \\
\mathcal{D}_{n,i} &:= \{\pi \in \mathcal{S}_n(132) \mid \pi_1 > \pi_2 > \cdots > \pi_i\},
\end{aligned}$$

we prove, see Theorem 4, Theorem 5, Theorem 6, Theorem 7 respectively, that

$$\begin{aligned}
|\mathcal{C}_n(p)| &= C(n-1, p-1), \\
|\mathcal{C}_n^-(k,p)| &= C(n-2, p+k-n-2), \\
|\mathcal{D}_{n,i}(k)| &= C(n-1, k-i), \\
|\mathcal{D}_{n,i}| &= C(n, n-i).
\end{aligned}$$

We remark that $|\mathcal{C}_n^-(k,p)| = C(n-2, p+k-n-2)$ is a refinement of the result of Desantis et al. [7], stating that the number of permutations in $\mathcal{S}_n(132)$ with the first entry being k is equal to $C(n-1, k-1)$.

An *irreducible* Dyck path is a Dyck path with exactly one return. A *reducible* Dyck path is a Dyck path with at least two returns. For example, *uuddud* is reducible whereas *uududd* is irreducible. Given a peak of a Dyck path, if the down-step of this peak is immediately followed by an up-step, we call such a peak an *up-peak*. In other words, the occurrence of ud in udu is an up-peak, whereas, the occurrence of ud in udd isn't an up-peak. We obtain three new objects in Dyck paths that are enumerated by the entries in Catalan's triangle (see Theorem 8):

(i) The number of reducible Dyck paths of length $2n$ with the first peak at height k and the last peak at height p is equal to $C(n-2, n-k-p)$.

(ii) The number of Dyck paths of length $2n$ for which the first peak is at height k and the first i peaks are all up-peaks is equal to $C(n-1, n-k-i)$.

(iii) The number of Dyck paths of length $2n$ for which the first i peaks are all up-peaks is equal to $C(n, n-i-1)$.

We remark that (i) is a refinement of the result of Deutsch [8], stating that the number of Dyck paths of length $2n$ with the first (last) peak at height k is equal to $C(n-1, n-k)$.

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, we say that (π_i, π_j) is an *inversion* of π if $i < j$ but $\pi_i > \pi_j$. Denote by $\text{INV}(\pi)$ the number of inversions in π. Note that $\text{INV}(\pi)$ is nothing but the number of occurrences of the pattern 21 in π. Let $\mathcal{S}_n^k$ be the set of permutations in $\mathcal{S}_n$ with k inversions, its cardinality is denoted by S_n^k. As another application of the operators F_i, we give an algorithm to generate the set $\mathcal{S}_{n+1}^k$ from the set $\mathcal{S}_n^k$ for $n \geq k+1$. For other algorithms on permutations, see [4, Chapter 8], as well as Banderier, Baril and Moreira Dos Santos [1] for recent investigations.

Here is a guide to the sections of this paper. In Section 3, we introduce the operators F_i on permutations, then we provide some properties and a characterization of them. Some applications of the operators are given in Section 4 and Section 5.

3 Operators F_i on permutations

3.1 The definition and some properties

Definition 1. Given $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, let $p_1(\pi)$ be the position of entry 1 in π, and let $P(\pi) := p_1(\pi) - 1$. The operator F_i, $0 \leq i \leq n$, on permutation $\pi \in \mathcal{S}_n$ is defined as follows.

(1) For $i = 0$, define $F_0(\pi) = \pi$.

(2) For $i > P(\pi)$, define $F_i(\pi) = \infty$.

(3) For $1 \leq i \leq P(\pi)$, we define $F_i(\pi) = \pi'_1\pi'_2\cdots\pi'_n \in \mathcal{S}_n$ satisfying $\pi'_j = \pi_j - 1$ for $1 \leq j \leq i$ and $\text{red}(\pi'_{i+1}\pi'_{i+2}\cdots\pi'_n) = \text{red}(\pi_{i+1}\pi_{i+2}\cdots\pi_n)$.

Example 1. Let $\pi = 5321476$, we have $P(\pi) = 3$, $F_0(\pi) = 5321476$, $F_1(\pi) = 4321576$, $F_2(\pi) = 4231576$, $F_3(\pi) = 4213576$, $F_4(\pi) = F_5(\pi) = F_6(\pi) = F_7(\pi) = \infty$.

For $\pi = \pi_1\pi_2\cdots\pi_n$, we denote $\pi_{[i,j]} := \pi_i\pi_{i+1}\cdots\pi_j$, and denote $\pi_{[i,j]} - 1 := (\pi_i - 1)(\pi_{i+1} - 1)\cdots(\pi_j - 1)$. We have the following theorem.

Theorem 1. *Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $1 \le i \le P(\pi)$. Let $\pi_{n+1} = n + 1$. Assume $F_i(\pi) = \pi'_1\pi'_2\cdots\pi'_n$, we have*

(1) For $j \le i$, $\pi'_j = \pi_j - 1$.

(2) For $j \ge i + 1$, $\pi'_j = \min\{\pi_t - 1 \mid \pi_t > \pi_j, i + 1 \le t \le n + 1\}$. Therefore, $\pi'_j \ge \pi_j$ for $j \ge i + 1$.

Proof. By definition, (1) is trivially true. Now we prove (2). Let $\{\pi_{i+1} - 1, \pi_{i+2} - 1, \cdots, \pi_{n+1} - 1\} = \{a_{i+1}, a_{i+2}, \cdots, a_{n+1}\}$, where $0 = a_{i+1} < a_{i+2} < \cdots < a_n < a_{n+1} = n$. Denote $F_i(\pi) = \pi'$. By definition, we see that $\mathrm{red}(\pi'_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]} - 1)$. It is not hard to see that the underlying set of $\pi_{[i+1,n]} - 1$ is $\{a_{i+1}, a_{i+2}, a_{i+3}, \cdots, a_n\}$, and the underlying set of $\pi'_{[i+1,n]}$ is $\{a_{i+2}, a_{i+3}, \cdots, a_n, a_{n+1}\}$. So, if $\pi_j - 1 = a_r$, we have $\pi'_j = a_{r+1}$. Then, $\pi'_j = a_{r+1} = \min\{a_x \mid a_x > a_r, i+1 \le x \le n+1\} = \min\{\pi_t - 1 \mid \pi_t - 1 > \pi_j - 1, i + 1 \le t \le n + 1\} = \min\{\pi_t - 1 \mid \pi_t > \pi_j, i + 1 \le t \le n + 1\}$, completing the proof. ∎

The operators F_i have several interesting properties. Let us start with a property concerning the inversions.

Proposition 1. *If $\pi \in \mathcal{S}_n^k$, then $F_i(\pi) \in \mathcal{S}_n^{k-i}$ for $0 \le i \le P(\pi)$.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n$, $F_i(\pi) = \pi' = \pi'_1\pi'_2\cdots\pi'_n$, $\pi'' = \pi - 1 = (\pi_1 - 1)\cdots(\pi_n - 1)$. It is obvious that $\mathrm{INV}(\pi) = \mathrm{INV}(\pi'')$. By definition we see that $\pi'_{[1,i]} = \pi''_{[1,i]}$ and $\mathrm{red}(\pi'_{[i+1,n]}) = \mathrm{red}(\pi''_{[i+1,n]})$, this yields $\mathrm{INV}(\pi'_{[1,i]}) = \mathrm{INV}(\pi''_{[1,i]})$ and $\mathrm{INV}(\pi'_{[i+1,n]}) = \mathrm{INV}(\pi''_{[i+1,n]})$. Let $\{\pi_{i+1} - 1, \pi_{i+2} - 1, \cdots, \pi_n - 1\} = \{a_{i+1}, a_{i+2}, \cdots, a_n\}$, where $0 = a_{i+1} < a_{i+2} < \cdots < a_n$. It is easy to see that the underlying set of $\pi''_{[i+1,n]}$ is $\{0, a_{i+2}, a_{i+3}, \cdots, a_n\}$ and the underlying set of $\pi'_{[i+1,n]}$ is $\{a_{i+2}, a_{i+3}, \cdots, a_n, n\}$. In π'', $(\pi_1 - 1, 0), (\pi_2 - 1, 0), \cdots, (\pi_i - 1, 0)$ are inversions. Whereas, in π', $(\pi_1 - 1, n), (\pi_2 - 1, n), \cdots, (\pi_i - 1, n)$ are not inversions. This implies $\mathrm{INV}(\pi') = \mathrm{INV}(\pi'') - i = \mathrm{INV}(\pi) - i$, completing the proof. ∎

Now we are going to give some properties of the operators F_i concerning the 132-pattern.

Proposition 2. *For any permutation π, if π contains a 132-pattern, then $F_i(\pi)$ also contains a 132-pattern, where $0 \le i \le P(\pi)$.*

Proof. Assume $\pi_{j_1}\pi_{j_2}\pi_{j_3}$ is a 132-pattern of $\pi = \pi_1\pi_2\cdots\pi_n$, i.e., $j_1 < j_2 < j_3$ and $\pi_{j_1} < \pi_{j_3} < \pi_{j_2}$. For given i, $0 \le i \le P(\pi)$, assume $F_i(\pi) = \pi'_1\pi'_2\cdots\pi'_n$, we distinguish three cases.

(i) If $i < j_1$ or $i \ge j_3$. Since $\text{red}(F_i(\pi)_{[1,i]}) = \text{red}(\pi_{[1,i]})$, $\text{red}(F_i(\pi)_{[i+1,n]}) = \text{red}(\pi_{[i+1,n]})$, then $\pi'_{j_1}\pi'_{j_2}\pi'_{j_3}$ is a 132-pattern of $F_i(\pi)$.

(ii) If $j_1 \le i < j_2$. By definition we see that $\pi'_{j_1} = \pi_{j_1} - 1$ and $\pi'_{j_3} < \pi'_{j_2}$. From Theorem 1(2) we know $\pi_{j_3} \le \pi'_{j_3}$. Thus $\pi'_{j_1} = \pi_{j_1} - 1 < \pi_{j_1} < \pi_{j_3} \le \pi'_{j_3}$, then $\pi'_{j_1} < \pi'_{j_3} < \pi'_{j_2}$, namely $\pi'_{j_1}\pi'_{j_2}\pi'_{j_3}$ is a 132-pattern of $F_i(\pi)$.

(iii) If $j_2 \le i < j_3$. By definition we have $\pi'_{j_1} = \pi_{j_1} - 1$, $\pi'_{j_2} = \pi_{j_2} - 1$. It is easy to see that in permutation $F_i(\pi)$ the entry $\pi_{j_3} - 1$ must be on the right of π'_i, then it is certainly on the right of π'_{j_2}. Therefore $\pi'_{j_1}, \pi'_{j_2}, \pi_{j_3} - 1$ is a 132-pattern of $F_i(\pi)$.

In summary, $F_i(\pi)$ contains a 132-pattern, and the proof is completed. ∎

Proposition 3. *For any permutation π, if $F_{P(\pi)}(\pi)$ contains a 132-pattern, then π must contain a 132-pattern.*

Proof. Suppose $\pi = \pi_1\pi_2\cdots\pi_n$ and $F_{P(\pi)}(\pi) = \pi' = \pi'_1\pi'_2\cdots\pi'_n$. The case of $P(\pi) = 0$, i.e., $\pi_1 = 1$, is trivially true as $F_0(\pi) = \pi$. We only consider the case of $P(\pi) > 0$ below. Assume $\pi'_{j_1}\pi'_{j_2}\pi'_{j_3}$ is a 132-pattern of π', i.e., $j_1 < j_2 < j_3$ and $\pi'_{j_1} < \pi'_{j_3} < \pi'_{j_2}$. We distinguish three cases.

(i) If $P(\pi) \ge j_3$. By definition we have $\pi_{j_1} = \pi'_{j_1} + 1$, $\pi_{j_2} = \pi'_{j_2} + 1$, $\pi_{j_3} = \pi'_{j_3} + 1$, therefore $\pi_{j_1}\pi_{j_2}\pi_{j_3}$ is a 132-pattern of π.

(ii) If $P(\pi) < j_2$. Since $\pi_{P(\pi)+1} = 1$, then $\pi_{P(\pi)+1}$ is a right-to-left minimum of π. As $\text{red}(\pi'_{[P(\pi)+1,n]}) = \text{red}(\pi_{[P(\pi)+1,n]})$, we see that $\pi'_{P(\pi)+1}$ is a right-to-left minimum of π'. Therefore $\pi'_{P(\pi)+1}\pi'_{j_2}\pi'_{j_3}$ is a 132-pattern of π', this implies $\pi_{P(\pi)+1}\pi_{j_2}\pi_{j_3}$ is a 132-pattern of π as $\text{red}(\pi'_{[P(\pi)+1,n]}) = \text{red}(\pi_{[P(\pi)+1,n]})$.

(iii) If $j_2 \le P(\pi) < j_3$. By definition we have $\pi_{j_1} = \pi'_{j_1} + 1$, $\pi_{j_2} = \pi'_{j_2} + 1$. It is easy to see that in permutation π the entry $\pi'_{j_3} + 1$ must be on the right of $\pi_{P(\pi)}$, then it is certainly on the right of π_{j_2}, therefore $\pi_{j_1}, \pi_{j_2}, \pi'_{j_3} + 1$ is a 132-pattern of π.

In summary, π contains a 132-pattern, this completes the proof. ∎

Taking a second look at Proposition 3, it is natural to ask whether the statement holds for F_i, $i < P(\pi)$. The answer is in the negative. For example, $\pi = 43512$ is 132-avoiding but $F_2(\pi) = 32514$ contains a 132-pattern. Surprisingly, adding a restriction on π will do the job, that is the content of the next proposition.

Proposition 4. *Let $\pi = \pi_1\pi_2\cdots\pi_n$ with $\pi_1 > \pi_2 > \cdots > \pi_{i+1}$, if $F_i(\pi)$ contains a 132-pattern, then π must contain a 132-pattern.*

Proof. Let $F_i(\pi) = \pi' = \pi'_1\pi'_2\cdots\pi'_n$. By definition we know that $\pi'_j = \pi_j - 1$ for $1 \le j \le i$, and $\mathrm{red}(\pi'_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]})$. Assume $\pi'_{j_1}\pi'_{j_2}\pi'_{j_3}$ is a 132-pattern of π', i.e., $j_1 < j_2 < j_3$ and $\pi'_{j_1} < \pi'_{j_3} < \pi'_{j_2}$. Since $\pi_1 > \pi_2 > \cdots > \pi_i$, we have $\pi'_1 > \pi'_2 > \cdots > \pi'_i$, this yields $j_2 \ge i+1$. We distinguish two cases.

(i) If $j_1 \ge i+1$, then $\pi_{j_1}\pi_{j_2}\pi_{j_3}$ is a 132-pattern of π as $\mathrm{red}(\pi'_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]})$.

(ii) If $j_1 \le i$, we have $\pi'_{j_1} \ge \pi'_i$. Let $\{\pi_{i+1}-1, \pi_{i+2}-1, \cdots, \pi_n-1\} = \{a_{i+1}, a_{i+2}, \cdots, a_n\}$, where $0 = a_{i+1} < a_{i+2} < \cdots < a_n$, and let $a_{n+1} = n$. Assume $\pi_{i+1} - 1 = a_r$, $\pi_{j_2} - 1 = a_s$ and $\pi_{j_3} - 1 = a_t$. From Theorem 1(2) we know that $\pi'_{i+1} = a_{r+1}$, $\pi'_{j_2} = a_{s+1}$ and $\pi'_{j_3} = a_{t+1}$. It is not hard to see

$$a_{t+1} = \pi'_{j_3} > \pi'_{j_1} \ge \pi'_i = \pi_i - 1 > \pi_{i+1} - 1 = a_r,$$

this yields $t+1 > r$. Since $\pi'_{j_2} > \pi'_{j_3} = a_{t+1} \ge a_{r+1} = \pi'_{i+1}$, we have $j_2 \ne i+1$, then $i+1 < j_2 < j_3$. We claim that $\pi_{i+1}\pi_{j_2}\pi_{j_3} = (a_r+1)(a_s+1)(a_t+1)$ is a 132-pattern of π. Since $t+1 > r$, we have $a_t \ge a_r$, then $\pi_{j_3} \ge \pi_{i+1}$, thus $\pi_{j_3} > \pi_{i+1}$ as $i+1 < j_3$. Because of $\mathrm{red}(\pi'_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]})$, $i+1 < j_2 < j_3$ and $\pi'_{j_2} > \pi'_{j_3}$, we have $\pi_{j_2} > \pi_{j_3}$, thus $\pi_{i+1} < \pi_{j_3} < \pi_{j_2}$, and our claim is true. This completes our proof. ∎

Define

$$\mathcal{D}_{n,i} := \{\pi \in \mathcal{S}_n(132) \mid \pi_1 > \pi_2 > \cdots > \pi_i\},$$

that is, $\mathcal{D}_{n,i}$ is the set of 132-avoiding permutations of length n starting with a decreasing sequence of length i. In particular, $\mathcal{D}_{n,1} = \mathcal{S}_n(132)$. Define

$$\mathcal{D}_{n,i}(k) := \{\pi \in \mathcal{S}_n(132) \mid k = \pi_1 > \pi_2 > \cdots > \pi_i\},$$

it is clear that $\mathcal{D}_{n,i} = \bigcup_{k=i}^{n} \mathcal{D}_{n,i}(k)$. We have the following result.

Proposition 5. *Let $i \le k \le n-1$, then F_i is a bijection from $\mathcal{D}_{n,i+1}(k+1)$ to $\mathcal{D}_{n,i}(k)$.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{D}_{n,i+1}(k+1)$, that is, $\pi_1 = k+1$, $\pi_1 > \pi_2 > \cdots > \pi_{i+1}$, and π is 132-avoiding. We are going to prove $F_i(\pi) = \pi'_1\pi'_2\cdots\pi'_n \in \mathcal{D}_{n,i}(k)$. By definition we have $\pi'_1 = k$ and $\pi'_1 > \pi'_2 > \cdots > \pi'_i$. By Proposition 4 we have $F_i(\pi)$ is 132-avoiding. Therefore, we have $F_i(\pi) \in \mathcal{D}_{n,i}(k)$.

On the other hand, let $\pi' = \pi'_1\pi'_2\cdots\pi'_n \in \mathcal{D}_{n,i}(k)$, that is, $\pi'_1 = k$, $\pi'_1 > \pi'_2 > \cdots > \pi'_i$, and π' is 132-avoiding. Since $k \le n-1$, we can define $\pi := \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, such that $\pi_j = \pi'_j + 1$ for $1 \le j \le i$ and $\mathrm{red}(\pi_{[i+1,n]}) = \mathrm{red}(\pi'_{[i+1,n]})$. It is obvious that $F_i(\pi) = \pi'$. Our goal is to prove $\pi \in \mathcal{D}_{n,i+1}(k+1)$. By definition we have $\pi_1 = k+1$, $\pi_1 > \pi_2 > \cdots > \pi_i$. From Proposition 2 we know π is 132-avoiding. To achieve our goal, it is sufficient to show $\pi_i > \pi_{i+1}$. Assume the contrary, that is, $\pi_i < \pi_{i+1}$. It is not hard to see that in permutation π' the entry $\pi_{i+1} - 1$ is on the right of π'_i. From Theorem 1(2) we see that

$$\pi'_i = \pi_i - 1 < \pi_{i+1} - 1 < \pi_{i+1} \le \pi'_{i+1},$$

thus $\pi'_i\pi'_{i+1}(\pi_{i+1} - 1)$ is a 132-pattern of π', a contradiction, and the proof follows. ∎

3.2 The operator F on permutations

In this subsection, we introduce the operator F on permutations, which is a specific kind of the operators F_i.

For any permutation π, we define $F(\pi) := F_{P(\pi)}(\pi)$. Combining Proposition 2 and Proposition 3 we obtain an elementary result of the operator F.

Proposition 6. *For any permutation π, π is 132-avoiding if and only if $F(\pi)$ is 132-avoiding.*

The operator F has another simple property.

Proposition 7. *If $\pi \in \mathcal{S}_n(132)$, then the last entry of $F(\pi)$ is n.*

Proof. Let $F(\pi) = \pi' = \pi'_1\pi'_2\cdots\pi'_n$, we assume the contrary, that is $\pi'_j = n$ and $j < n$. From Proposition 6 we see that π' is 132-avoiding. By the definition of $F(\pi)$, it is easy to see that in permutation π' the entry n is on the right of $\pi'_{P(\pi)}$, i.e., $j \geq P(\pi)+1$. Because of $\mathrm{red}(\pi'_{[P(\pi)+1,n]}) = \mathrm{red}(\pi_{[P(\pi)+1,n]})$ and $\pi_{P(\pi)+1} = 1$, we have $\pi'_{P(\pi)+1}$ is a right-to-left minimum of π'. Thus $\pi'_{P(\pi)+1}, n, \pi'_n$ is a 132-pattern of π', a contradiction, this completes the proof. ∎

For any permutation $\pi \in \mathcal{S}_n$, define $F^m(\pi) = F(F^{m-1}(\pi))$, and $F^0(\pi) = \pi$. We call π a *F-sortable permutation* if $F^{k-1}(\pi) = 123\cdots n$, where k is the first entry of π.

Example 2. Let $\pi = 5321476$, it is not hard to see $F(\pi) = 4213576$, $F^2(\pi) = 3124576$, $F^3(\pi) = 2134576$, $F^4(\pi) = 1234576$. Thus π is not a F-sortable permutation.

Example 3. Let $\pi = 5321467$, it is not hard to see $F(\pi) = 4213567$, $F^2(\pi) = 3124567$, $F^3(\pi) = 2134567$, $F^4(\pi) = 1234567$. So π is a F-sortable permutation.

It is natural to ask the following question: how can we decide whether a permutation is F-sortable? Now we give an answer to this question.

Theorem 2. *A permutation is F-sortable if and only if it is 132-avoiding.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, and suppose $\pi_1 = k$. First we assume π contains a 132-pattern. From Proposition 6 we have $F^{k-1}(\pi)$ contains a 132-pattern, thus π is not a F-sortable permutation.

Now we assume π is 132-avoiding. We use induction on $\pi_1 = k$ to prove π is F-sortable. When $k = 1$, we have $\pi = 12\cdots n$ and $F^0(\pi) = \pi = 12\cdots n$, the initial case being trivial. Assume the statement is true for $k-1$, and prove it for k, $k > 1$. By Proposition 6 we have $F(\pi) = \pi'_1\pi'_2\cdots\pi'_n$ is 132-avoiding with $\pi'_1 = k-1$. Applying the induction hypothesis, we have $F^{k-1}(\pi) = F^{k-2}(F(\pi)) = 123\cdots n$, so π is F-sortable, completing the proof. ∎

3.3 A characterization of F_i in terms of a series of swaps of two entries

In this subsection, we will show that the operators F_i can be characterized by a series of swaps of two entries. In order to achieve this goal, let us first introduce the operators f_i on permutations.

Given $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, if π_i is not a right-to-left minimum of π, let $\pi_s = \max\{\pi_t \mid t > i, \pi_t < \pi_i\}$, and define

$$f_i(\pi) := \pi_1\cdots\pi_{i-1}\pi_s\pi_{i+1}\cdots\pi_{s-1}\pi_i\pi_{s+1}\cdots\pi_n,$$

in other words, $f_i(\pi)$ is obtained from π by interchanging π_i and π_s, where π_s is the maximum entry which is on the right of π_i and which is less than π_i in permutation π. We call π_s the *f_i-selected entry* of π. If π_i is a right-to-left minimum of π, we define $f_i(\pi) = \infty$. For instance, $f_1(41523) = 31524$, $f_2(41523) = \infty$, $f_3(41523) = 41325$, $f_4(41523) = \infty$, $f_5(41523) = \infty$.

Note that for any $\pi \in \mathcal{S}_n$, we have $f_n(\pi) = \infty$ since the last entry of π is a right-to-left minimum. By convention, we set $f_0(\pi) = \pi$ for each permutation π.

Two key properties of the operators f_i are given by Proposition 8 and Proposition 9.

Proposition 8. *Let $\pi \in \mathcal{S}_n$ and $1 \le i \le n$, if $f_i(\pi) \ne \infty$, then* $\mathrm{INV}(f_i(\pi)) = \mathrm{INV}(\pi) - 1$.

Proof. Let π_s be the f_i-selected entry of $\pi = \pi_1\pi_2\cdots\pi_n$, we claim that if $i < m < s$, then $\pi_m > \pi_i$ or $\pi_m < \pi_s$. Assume the contrary, i.e., $\pi_s < \pi_m < \pi_i$, then in permutation π, π_m is an entry which is on the right of π_i and which is less than π_i but $\pi_m > \pi_s$, contradicting the choice of π_s, and our claim is true. Then proving $\mathrm{INV}(f_i(\pi)) = \mathrm{INV}(\pi) - 1$ is easy. ∎

Proposition 9. *Let $\pi \in \mathcal{S}_n$ and $1 \le i \le n$, if $f_i(\pi) \ne \infty$, then* $\mathrm{red}(f_i(\pi)_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]})$.

Proof. Assume π_s is the f_i-selected entry of $\pi = \pi_1\pi_2\cdots\pi_n$. Let $\{\pi_{i+1}, \pi_{i+2}, \cdots, \pi_n\} = \{b_{i+1}, b_{i+2}, \cdots, b_n\}$, where $b_{i+1} < b_{i+2} < \cdots < b_n$. By convention, we let $b_{n+1} = n + 1$. Assume $\pi_s = b_j$, according to the choice of π_s we see that $b_j < \pi_i < b_{j+1}$, this implies $\mathrm{red}(f_i(\pi)_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]})$. ∎

Given $\pi \in \mathcal{S}_n$, for $0 \le i \le n$, we define

$$f_{(i)}(\pi) := f_i \circ f_{i-1} \cdots \circ f_2 \circ f_1 \circ f_0(\pi),$$

where $f \circ g(x)$ denotes the composition $f(g(x))$. In other words, $f_{(i)}(\pi) = f_i(f_{(i-1)}(\pi))$ for $1 \le i \le n$. We define $f_i(\infty) = \infty$ for any i. Note that $f_{(0)}(\pi) = \pi \ne \infty$ and $f_{(n)}(\pi) = \infty$, so we can define

$$p(\pi) := \max\{i \mid f_{(i)}(\pi) \ne \infty,\ 0 \le i \le n\}.$$

By definition we can see that $f_{(i)}(\pi) \ne \infty$ for $i \le p(\pi)$, and $f_{(i)}(\pi) = \infty$ for $i > p(\pi)$.

Example 4. *Let* $\pi = 45213$, $f_{(0)}(\pi) = 45213$, $f_{(1)}(\pi) = f_1(45213) = 35214$, $f_{(2)}(\pi) = f_2(35214) = 34215$, $f_{(3)}(\pi) = f_3(34215) = 34125$, $f_{(4)}(\pi) = f_4(34125) = \infty$, $f_{(5)}(\pi) = \infty$, *and clearly* $p(\pi) = 3$.

The operators $f_{(i)}$ on permutations have the following two properties, they can be easily proved by Proposition 8 and Proposition 9 respectively.

Proposition 10. *Let* $\pi \in \mathcal{S}_n^k$, *we have* $f_{(i)}(\pi) \in \mathcal{S}_n^{k-i}$ *for* $0 \le i \le p(\pi)$.

Proposition 11. *Let* $\pi \in \mathcal{S}_n$, *we have* $\mathrm{red}(f_{(i)}(\pi)_{[i+1,n]}) = \mathrm{red}(\pi_{[i+1,n]})$ *for* $0 \le i \le p(\pi)$.

Now we are in a position to give a characterization of the operator F_i by a series of swaps of two entries, that is the content of the following theorem.

Theorem 3. *Let* $\pi \in \mathcal{S}_n$, *we have* $P(\pi) = p(\pi)$, *and* $F_i(\pi) = f_{(i)}(\pi)$ *for* $0 \le i \le P(\pi)$.

Proof. If $P(\pi) = 0$, i.e., $\pi_1 = 1$, we have $P(\pi) = p(\pi) = 0$, and $F_0(\pi) = f_{(0)}(\pi) = \pi$, the theorem is proved. We assume $P(\pi) > 0$ below.

First we proceed to prove the statement that for $0 \le i \le P(\pi)$, we have $f_{(i)}(\pi) \ne \infty$ and $F_i(\pi) = f_{(i)}(\pi)$. We use induction on i to prove the statement. When $i = 0$, it is trivially true. Assume our statement is true for i, and prove it for $i+1$, where $0 \le i \le P(\pi) - 1$. Given $\pi = \pi_1\pi_2\cdots\pi_n$, by the induction hypothesis we have $F_i(\pi) = f_{(i)}(\pi) := \pi' = \pi'_1\pi'_2\cdots\pi'_n$. Let $\{\pi_{i+1}-1, \pi_{i+2}-1, \cdots, \pi_n-1\} = \{a_{i+1}, a_{i+2}, \cdots, a_n\}$, where $0 = a_{i+1} < a_{i+2} < \cdots < a_n$, and let $a_{n+1} = n$. Suppose $\pi_{i+1} - 1 = a_m$, from Theorem 1(2) we see $\pi'_{i+1} = a_{m+1}$. Since $i \le P(\pi) - 1$, we have $a_m \ne 0$. Since $\{\pi'_{i+1}, \pi'_{i+2}, \cdots, \pi'_n\} = \{a_{i+2}, \cdots, a_n, a_{n+1}\}$, we have $\{\pi'_{i+2}, \pi'_{i+3}, \cdots, \pi'_n\} = \{a_{i+2}, \cdots, a_m, a_{m+2}, \cdots, a_{n+1}\}$, then in permutation π', a_m is the maximum entry which is on the right of π'_{i+1} and which is less than $\pi'_{i+1} = a_{m+1}$, i.e., the f_{i+1}-selected entry of π' is $a_m = \pi_{i+1} - 1$, this yields $f_{(i+1)}(\pi)_{[1,i+1]} = F_{i+1}(\pi)_{[1,i+1]}$. From Proposition 11, we have $\mathrm{red}(f_{(i+1)}(\pi)_{[i+2,n]}) = \mathrm{red}(\pi_{[i+2,n]}) = \mathrm{red}(F_{i+1}(\pi)_{[i+2,n]})$, combining this with $f_{(i+1)}(\pi)_{[1,i+1]} = F_{i+1}(\pi)_{[1,i+1]}$ leads to $f_{(i+1)}(\pi) = F_{i+1}(\pi)$, completing the induction proof.

Now we proceed to prove $p(\pi) = P(\pi)$. From the above argument we find $p(\pi) \ge P(\pi)$, and $F_{P(\pi)}(\pi) = f_{(P(\pi))}(\pi) := \pi'$. Since 1 is in the position $P(\pi)+1$ of π, and 1 is a right-to-left minimum of π, we have the $(P(\pi)+1)$-th entry of π' is a right-to-left minimum of π' as $\mathrm{red}(\pi'_{[P(\pi)+1,n]}) = \mathrm{red}(\pi_{[P(\pi)+1,n]})$. Thus, $f_{(P(\pi)+1)}(\pi) = f_{P(\pi)+1}(\pi') = \infty$, therefore, $p(\pi) = P(\pi)$, and the proof follows. ∎

4 Application I: Catalan's triangle in 132-avoiding permutations and Dyck paths

In this section we are going to use the operators introduced in the previous section to obtain some new objects in 132-avoiding permutations and Dyck paths that are enumerated by the entries in Catalan's triangle.

Given $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n(132)$, assume $F(\pi) = \pi'_1\pi'_2\cdots\pi'_{n-1}n$ (by Proposition 7), we define $\phi(\pi) := \pi'_1\pi'_2\cdots\pi'_{n-1}$, i.e., $\phi(\pi)$ is obtained from $F(\pi)$ by deleting the entry n. It is straightforward to see the following result from Proposition 6.

Lemma 1. *Let $\pi \in \mathcal{S}_n(132)$, we have $\phi(\pi) \in \mathcal{S}_{n-1}(132)$.*

Define

$$\mathcal{C}_n(p) := \{\pi \in \mathcal{S}_n(132) \mid \pi_p = 1\}.$$

Lemma 2. *We have ϕ is a bijection from $\mathcal{C}_n(p)$ to $\bigcup_{i\le p}\mathcal{C}_{n-1}(i)$.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{C}_n(p)$, that is, π is 132-avoiding and $\pi_p = 1$. We first prove $\phi(\pi) \in \bigcup_{i\le p}\mathcal{C}_{n-1}(i)$. Suppose $\phi(\pi) = \pi'_1\pi'_2\cdots\pi'_{n-1}$, by Lemma 1 we have $\phi(\pi) \in \mathcal{S}_{n-1}(132)$. In order to prove $\phi(\pi) \in \bigcup_{i\le p}\mathcal{C}_{n-1}(i)$, it suffices to show $p_1(\phi(\pi)) \le p_1(\pi) = p$. Since $\pi_p = 1$, it is obvious that $\pi_j > 1$ for $j \ge p+1$. From Theorem 1(2) we see that $\pi'_j \ge \pi_j > 1$ for $j \ge p+1$, this implies $p_1(\phi(\pi)) \le p$.

On the other hand, let $\pi' = \pi'_1\pi'_2\cdots\pi'_{n-1} \in \mathcal{C}_{n-1}(i)$, $i \le p$. Since π' avoids 132-pattern and $\pi'_i = 1$, we have $\pi'_i\pi'_{i+1}\cdots\pi'_{n-1}$ is an increasing sequence, so π'_p is a right-to-left minimum of π' as $i \le p$. Let $\pi'' = \pi'_1\pi'_2\cdots\pi'_{n-1}n$, we can see that π'' is 132-avoiding and π'_p is a right-to-left minimum of π''. Define $\pi := \pi_1\cdots\pi_{p-1}\pi_p\cdots\pi_n \in \mathcal{S}_n$ so that $\pi_1\cdots\pi_{p-1} = (\pi'_1+1)\cdots(\pi'_{p-1}+1)$ and $\mathrm{red}(\pi_{[p,n]}) = \mathrm{red}(\pi''_{[p,n]})$. It is easy to see that $F(\pi) = \pi''$ and $\phi(\pi) = \pi'$. Our goal is to prove $\pi \in \mathcal{C}_n(p)$. Since $F(\pi) = \pi''$ is 132-avoiding, we have π is 132-avoiding by Proposition 6. Since π'_p is a right-to-left minimum of π'', then π_p is a right-to-left minimum of π. Combining this with the fact $\pi_j \ge 2$ for $j \le p-1$, we have $\pi_p = 1$, this proves $\pi \in \mathcal{C}_n(p)$. Therefore ϕ has an inverse and must be a bijection. ∎

Theorem 4. $|\mathcal{C}_n(p)| = C(n-1, p-1)$.

Proof. We use induction on n. When $n = 1$, there is only one choice for p, i.e., $p = 1$. Clearly $\mathcal{C}_n(p) = \{1\}$ and $C(n-1, p-1) = C(0,0) = 1$, so our theorem is true for $n = 1$. Now assume our theorem is true for $n-1$, and prove it for n. By Lemma 2 and the induction hypothesis, we have

$$\begin{aligned} |\mathcal{C}_n(p)| &= \sum_{i\le p} |\mathcal{C}_{n-1}(i)| \\ &= \sum_{i\le p} C(n-2, i-1) \\ &= C(n-1, p-1), \qquad (\text{ by } (1.2)\ ) \end{aligned}$$

completing the induction proof. ∎

Define

$$\widetilde{\mathcal{C}}_n(k) := \{\pi \in \mathcal{S}_n(132) \mid \pi_1 = k\}.$$

It is not hard to see that, π avoids 132-pattern if and only if π^{-1}, the inverse permutation of π, avoids 132-pattern. The following result of Desantis et al. [7] that we have already mentioned in the introduction can be obtained from Theorem 4 immediately.

Corollary 1 (Desantis et al. [7])**.** $|\widetilde{\mathcal{C}}_n(k)| = C(n-1, k-1)$.

Theorem 4 and Corollary 1 tell us that both the distribution of the first entry and the position of the minimum in 132-avoiding permutations of given length are *Catalan's distribution*.

Example 5. We list below all permutations in $\mathcal{S}_5(132)$.

$$\begin{aligned}\mathcal{S}_5(132) \;=\; &\{12345, 21345, 23145, 23415, 23451, 31245, 32145, 32415, 32451,\\ &34125, 34215, 34251, 34512, 34521, 41235, 42135, 42315, 43125,\\ &43215, 42351, 43521, 43251, 45231, 45321, 43512, 45213, 45312,\\ &45123, 51234, 52134, 52314, 52341, 53124, 53214, 53241, 53412,\\ &53421, 54123, 54213, 54231, 54312, 54321\}.\end{aligned}$$

There are $1, 4, 9, 14, 14$ permutations with first entries being $1, 2, 3, 4, 5$ respectively in $\mathcal{S}_5(132)$, there are $1, 4, 9, 14, 14$ permutations for which the positions of 1 are $1, 2, 3, 4, 5$ respectively in $\mathcal{S}_5(132)$. The numbers $1, 4, 9, 14, 14$ are the fifth row ($n = 4$) of Catalan's triangle (see Table 1).

Define

$$\mathcal{C}_n(k, p) := \{\pi \in \mathcal{S}_n(132) \mid \pi_1 = k, \pi_p = 1\},$$

that is,

$$\mathcal{C}_n(k, p) = \widetilde{\mathcal{C}}_n(k) \cap \mathcal{C}_n(p).$$

Since both $\widetilde{\mathcal{C}}_n(k)$ and $\mathcal{C}_n(p)$ are enumerated by the entries in Catalan's triangle, it is nature to ask whether $\mathcal{C}_n(k, p)$ is enumerated by the entries in Catalan's triangle as well. The answer to the question is in the negative. But adding a restriction on the set $\mathcal{C}_n(k, p)$ will do the job. Define

$$\mathcal{C}_n^-(k, p) := \{\pi \in \mathcal{S}_n(132) \mid \pi_1 = k, \pi_p = 1, \pi_n \neq n\},$$

we have the following theorem.

Theorem 5. *$|\mathcal{C}_n^-(k, p)| = C(n-2, k+p-n-2)$ for $n \geq 2$.*

In order to prove the theorem, we need two lemmas.

Lemma 3. *Let $2 \leq k \leq n$, we have ϕ is a bijection from $\mathcal{C}_n^-(k, n)$ to $\widetilde{\mathcal{C}}_{n-1}(k-1)$, and therefore $|\mathcal{C}_n^-(k, n)| = C(n-2, k-2)$.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{C}_n^-(k,n)$, i.e., $\pi_1 = k, \pi_n = 1$ and π is 132-avoiding. Since $\phi(\pi) \in \mathcal{S}_{n-1}(132)$ and the first entry of $\phi(\pi)$ is $k-1$, we have $\phi(\pi) \in \widetilde{\mathcal{C}}_{n-1}(k-1)$. It is not hard to see ϕ has an inverse and must be a bijection. ∎

Lemma 4. *Let $2 \le k \le n$ and $p < n$, we have ϕ is a bijection from $\mathcal{C}_n^-(k,p)$ to $\bigcup_{i\le p}\mathcal{C}_{n-1}^-(k-1,i)$, and therefore $|\mathcal{C}_n^-(k,p)| = \sum_{i\le p}|\mathcal{C}_{n-1}^-(k-1,i)|$.*

Proof. Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{C}_n^-(k,p)$, assume $\phi(\pi) = \pi_1'\pi_2'\cdots\pi_{n-1}'$. We are going to show $\phi(\pi) \in \bigcup_{i\le p}\mathcal{C}_{n-1}^-(k-1,i)$. To this end, it will suffice to show the following four facts:

(i) $\phi(\pi)$ is 132-avoiding;

(ii) $\pi_1' = \pi_1 - 1$;

(iii) $p_1(\pi') \le p_1(\pi) = p$;

(iv) $\pi_{n-1}' \ne n-1$.

(i) and (ii) are clear. The proof of (iii) is the same as that in the proof of Lemma 2. Now we prove (iv). Assume $\pi_j = n$, where $j \ne n$. We claim that $j \le P(\pi)$. Otherwise, if $j > P(\pi)$, then $1, n, \pi_n$ is a 132-pattern of π, a contradiction, and our claim is true. By our claim we have $\pi_j' = n-1$. Since $j \le P(\pi) = p-1 < n-1$, then $\pi_{n-1}' \ne n-1$, completing the proof of (iv). By a same argument as that in the proof of Lemma 2, we see that ϕ has an inverse and must be a bijection. ∎

Proof of Theorem 5. We first prove our theorem for the case of $k=1$. In this case, we have $p=1$, thus $C(n-2, k+p-n-2) = C(n-2,-n) = 0$. Clearly, $12\cdots n$ is the only 132-avoiding permutation for which the first entry is 1, then we have $\mathcal{C}_n^-(1,1) = \emptyset$ as $12\cdots n \notin \mathcal{C}_n^-(1,1)$. Therefore, our theorem is true for $k=1$.

Let us prove our theorem by induction on n. When $n=2$, if $k=1$, we have already proved it. If $k=2$, there is only one choice for p, i.e., $p=2$. In this case, we have $C(n-2, p+k-n-2) = C(0,0) = 1$, and it is easy to see $\mathcal{C}_2^-(2,2) = \{21\}$, so our theorem is true for $k=2$, and the initial case of $n=2$ is true. Now assume our theorem is true for $n-1$, and prove it for n. Since we have already proved it for $k=1$, we assume $k \ge 2$ below. If $p=n$, by Lemma 3, we have $|\mathcal{C}_n^-(k,n)| = C(n-2,k-2)$, completing the induction proof. If $p<n$, by Lemma 4 and the induction hypothesis, we have

$$\begin{aligned} |\mathcal{C}_n^-(k,p)| &= \sum_{i\le p}|\mathcal{C}_{n-1}^-(k-1,i)| \\ &= \sum_{i\le p} C(n-3, k+i-n-2) \\ &= C(n-2, k+p-n-2), \qquad (\text{ by } (1.2)\ ) \end{aligned}$$

completing the induction proof. ∎

Remark 1. Theorem 5 tells us that, for given k the distribution of the position of 1 in

$$\widetilde{\mathcal{C}}_n^-(k) := \{\pi \in \mathcal{S}_n(132) \mid \pi_1 = k, \pi_n \neq n\}$$

is Catalan's distribution. Similarly, for given p the distribution of the first entry in

$$\mathcal{C}_n^-(p) := \{\pi \in \mathcal{S}_n(132) \mid \pi_p = 1, \pi_n \neq n\}$$

is Catalan's distribution as well.

So far we have given some applications of the operator F, now let us give an application of the operators F_i. Recall that, for $i \le k \le n$,

$$\mathcal{D}_{n,i}(k) = \{\pi \in \mathcal{S}_n(132) \mid k = \pi_1 > \pi_2 > \cdots > \pi_i\},$$

by using the operators F_i we can obtain the cardinality of $\mathcal{D}_{n,i}(k)$, which is given by the following theorem.

Theorem 6. $|\mathcal{D}_{n,i}(k)| = C(n-1, k-i)$.

Proof. By Proposition 5, we see that F_{i-1} is a bijection from $\mathcal{D}_{n,i}(k)$ to $\mathcal{D}_{n,i-1}(k-1)$, F_{i-2} is a bijection from $\mathcal{D}_{n,i-1}(k-1)$ to $\mathcal{D}_{n,i-2}(k-2)$, $\cdots$, F_1 is a bijection from $\mathcal{D}_{n,2}(k+2-i)$ to $\mathcal{D}_{n,1}(k+1-i)$, note that $\mathcal{D}_{n,1}(k+1-i) = \widetilde{\mathcal{C}}_n(k+1-i)$. Define $H_{i-1} := F_1 \circ F_2 \circ \cdots \circ F_{i-1}$, thus H_{i-1} is a bijection form $\mathcal{D}_{n,i}(k)$ to $\widetilde{\mathcal{C}}_n(k+1-i)$. Combining this with Corollary 1, we find $|\mathcal{D}_{n,i}(k)| = |\widetilde{\mathcal{C}}_n(k+1-i)| = C(n-1, k-i)$. ∎

Recall that,

$$\mathcal{D}_{n,i} = \{\pi \in \mathcal{S}_n(132) \mid \pi_1 > \pi_2 > \cdots > \pi_i\},$$

the cardinality of $\mathcal{D}_{n,i}$ is given by the following theorem.

Theorem 7. $|\mathcal{D}_{n,i}| = C(n, n-i)$.

Proof. From Theorem 6 and the fact $\mathcal{D}_{n,i} = \bigcup_{k=i}^{n} \mathcal{D}_{n,i}(k)$, we have

$$|\mathcal{D}_{n,i}| = \sum_{k=i}^{n} |\mathcal{D}_{n,i}(k)| = \sum_{k=i}^{n} C(n-1, k-i) = C(n, n-i),$$

as desired. ∎

Remark 2. From Theorem 7 we see $|\mathcal{S}_n(132)| = |\mathcal{D}_{n,1}| = C(n, n-1) = C_n$. Therefore, Theorem 7 is a refinement of the fact that $|\mathcal{S}_n(132)| = C_n$.

We conclude this section with four objects in Dyck paths that are enumerated by the entries in Catalan's triangle.

Theorem 8. (1) *The number of Dyck paths of length $2n$ with the last peak at height k is equal to $C(n-1,n-k)$.*

(2) *The number of reducible Dyck paths of length $2n$ with the first peak at height k and the last peak at height p is equal to $C(n-2,n-k-p)$.*

(3) *The number of Dyck paths of length $2n$ for which the first peak is at height k and the first i peaks are all up-peaks is equal to $C(n-1,n-k-i)$.*

(4) *The number of Dyck paths of length $2n$ for which the first i peaks are all up-peaks is equal to $C(n,n-i-1)$.*

Proof. It is not hard to see that under the Krattenthaler's bijection (see Introduction), (1), (2), (3), (4) of Theorem 8 coincide with Theorem 4, Theorem 5, Theorem 6, Theorem 7 respectively. ∎

Remark 3. We point out that, (1) of Theorem 8 is due to Deutsch [8] that we have mentioned in Introduction. While (2), (3), (4) of Theorem 8 seem to be new.

5 Application II: an algorithm to generate $\mathcal{S}_{n+1}^k$ from $\mathcal{S}_n^k$ for $n \geq k+1$

Recall that $\mathcal{S}_n^k$ is the set of all permutations on the set $\{1,2,\cdots,n\}$ with k inversions, in this section we are going to give an algorithm to generate the set $\mathcal{S}_{n+1}^k$ from the set $\mathcal{S}_n^k$ for $n \geq k+1$.

To state our result, we need the following notations. Let $\pi = \pi_1\pi_2\cdots\pi_n$, we denote $\pi+1 := (\pi_1+1)(\pi_2+1)\cdots(\pi_n+1)$. Let $\mathcal{P}$ be a set of some permutations, define

$$\begin{aligned}\mathcal{P}(+) &:= \{\omega \in \mathcal{P} \mid \text{the minimum of } \omega \text{ precedes the maximum of } \omega\},\\ \mathcal{P}(-) &:= \{\omega \in \mathcal{P} \mid \text{the maximum of } \omega \text{ precedes the minimum of } \omega\}.\end{aligned}$$

It is obvious that $\mathcal{P} = \mathcal{P}(+) \cup \mathcal{P}(-)$, and $\mathcal{P}(+) \cap \mathcal{P}(-) = \emptyset$.

Given $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, for $1 \leq i \leq n$, we define $I_i(\pi)$ to be the permutation obtained from π by inserting the entry 0 after the ith entry of π, i.e., $I_i(\pi) = \pi_1\cdots\pi_i 0 \pi_{i+1}\cdots\pi_n$; for $i=0$, we define $I_0(\pi) = 0\pi_1\pi_2\cdots\pi_n$. It is obvious that for $\pi \in \mathcal{S}_n$, $I_i(\pi)$ is a permutation of $\{0,1,\cdots,n\}$. Let $\widetilde{I}_i(\pi) := I_i(\pi)+1$. Thus, for any $\pi \in \mathcal{S}_n$, we have $\widetilde{I}_i(\pi) \in \mathcal{S}_{n+1}$. Define

$$\mathcal{F}_\pi := \{\widetilde{I}_i(F_i(\pi)) \mid 0 \leq i \leq P(\pi)\}.$$

Now we are ready to state and prove the main result of this section.

Theorem 9. *For $0 \leq k \leq \binom{n}{2}$, we have $\mathcal{S}_{n+1}^k(+) = \bigcup_{\pi \in \mathcal{S}_n^k} \mathcal{F}_\pi$, where $\mathcal{F}_\pi \cap \mathcal{F}_{\pi'} = \emptyset$ for $\pi \neq \pi'$.*

Proof. Let $\pi \in \mathcal{S}_n^k$ and let $0 \le i \le P(\pi)$, from Proposition 1 we know $F_i(\pi) \in \mathcal{S}_n^{k-i}$, thus $\widetilde{I}_i(F_i(\pi)) \in \mathcal{S}_{n+1}^k$. By the definition of F_i, we see that in permutation $F_i(\pi)$, the entry n is on the right of the ith entry, this implies 0 precedes n in $I_i(F_i(\pi))$, equivalently, 1 precedes $n+1$ in $\widetilde{I}_i(F_i(\pi))$, so $\widetilde{I}_i(F_i(\pi)) \in \mathcal{S}_{n+1}^k(+)$. This yields $\bigcup_{\pi\in\mathcal{S}_n^k} \mathcal{F}_\pi \subseteq \mathcal{S}_{n+1}^k(+)$.

On the other hand, let $\widetilde{\pi} \in \mathcal{S}_{n+1}^k(+)$, assume $\widetilde{\pi}-1 = \pi_1\pi_2\cdots\pi_i 0 \pi_{i+1}\cdots\pi_n$ and assume $\pi_k = n$, where $k \ge i+1$. Let $\pi = \pi_1\pi_2\cdots\pi_n$, then $\pi \in \mathcal{S}_n^{k-i}$. Define $\pi' := \pi'_1\pi'_2\cdot\cdots\cdot\pi'_n \in \mathcal{S}_n$ so that $\pi'_j = \pi_j + 1$, $1 \le j \le i$, and $\text{red}(\pi'_{[i+1,n]}) = \text{red}(\pi_{[i+1,n]})$. ($\pi'$ is well-defined since $k \ge i+1$). It is easy to see that $F_i(\pi') = \pi$, and $\widetilde{I}_i(F_i(\pi')) = \widetilde{\pi}$. Since $\pi \in \mathcal{S}_n^{k-i}$, we have $\pi' \in \mathcal{S}_n^k$ by Proposition 1. From the above argument we see that $\mathcal{S}_{n+1}^k(+) \subseteq \bigcup_{\pi\in\mathcal{S}_n^k} \mathcal{F}_\pi$. Combining this with the result of the previous paragraph, we have $\mathcal{S}_{n+1}^k(+) = \bigcup_{\pi\in\mathcal{S}_n^k} \mathcal{F}_\pi$.

Now we prove $\mathcal{F}_\pi \cap \mathcal{F}_{\pi'} = \emptyset$ for $\pi \ne \pi'$. Assume the contrary, suppose $\mathcal{F}_\pi \cap \mathcal{F}_{\pi'} = \omega$, and suppose $\omega = \widetilde{I}_j(F_j(\pi)) = \widetilde{I}_k(F_k(\pi'))$. It is clear that $j = k$, then $F_j(\pi) = F_j(\pi')$, this implies $\pi = \pi'$, a contradiction, and the proof follows. ∎

Corollary 2. *For $n \ge k+1$, we have $\mathcal{S}_{n+1}^k = \bigcup_{\pi\in\mathcal{S}_n^k} \mathcal{F}_\pi$, where $\mathcal{F}_\pi \cap \mathcal{F}_{\pi'} = \emptyset$ for $\pi \ne \pi'$.*

Proof. We proceed to prove $\mathcal{S}_{n+1}^k(-) = \emptyset$. Otherwise, take $\omega \in \mathcal{S}_{n+1}^k(-)$, so 1 is on the right of $n+1$, this yields $\text{INV}(\omega) \ge n > k$, a contradiction. Therefore we have $\mathcal{S}_{n+1}^k = \mathcal{S}_{n+1}^k(+)$, and the proof follows from Theorem 9 immediately. ∎

In order to give an algorithm to generate the set $\mathcal{S}_{n+1}^k(+)$ from the set $\mathcal{S}_n^k$, it is better to use the operators $f_{(i)}$ which are equivalent to the operators F_i by Theorem 3.

An algorithm to generate the set $\mathcal{S}_{n+1}^k(+)$ from the set $\mathcal{S}_n^k$.

Input: The set $\mathcal{S}_n^k$.

Output: The set $\mathcal{S}_{n+1}^k(+)$.

Step 0. Set $\mathcal{S}_{n+1}^k(+) = \emptyset$.

Step 1. If $\mathcal{S}_n^k = \emptyset$, stop; If $\mathcal{S}_n^k \ne \emptyset$, take $\pi \in \mathcal{S}_n^k$, set $\mathcal{S}_n^k = \mathcal{S}_n^k - \{\pi\}$, and set $i = 0$.

Step 2. If $i > P(\pi)$, return to Step 1; If $i \le P(\pi)$, set $\pi = f_i(\pi)$, then set $\mathcal{S}_{n+1}^k(+) = \mathcal{S}_{n+1}^k(+) \cup \widetilde{I}_i(\pi)$, and set $i = i+1$, return to Step 2.

Proof. By Theorem 3 and Theorem 9, we have

$$\mathcal{S}_{n+1}^k(+) = \bigcup_{\pi\in\mathcal{S}_n^k} \mathcal{F}_\pi = \bigcup_{\pi\in\mathcal{S}_n^k} \bigcup_{i=0}^{P(\pi)} \{\widetilde{I}_i(f_{(i)}(\pi))\}.$$

It is clear that Step 2 corresponds $\bigcup_{i=0}^{P(\pi)} \{\widetilde{I}_i(f_{(i)}(\pi))\}$ for given $\pi \in \mathcal{S}_n^k$, Step 1 means that π ranges over all permutations in $\mathcal{S}_n^k$. ∎

Remark 4. From Corollary 2 we can see that when $n \geq k+1$ the set $\mathcal{S}_{n+1}^k(+)$ we obtained by the above algorithm is actually the set $\mathcal{S}_{n+1}^k$.

Example 6. Let $n = 4, k = 2$, it is easy to verify $\mathcal{S}_4^2 = \{3124, 2314, 2143, 1342, 1423\}$. In Figure 1, we illustrate the above algorithm to generate the set $\mathcal{S}_5^2(+) = \mathcal{F}_{3124} \cup \mathcal{F}_{2314} \cup \mathcal{F}_{2143} \cup \mathcal{F}_{1342} \cup \mathcal{F}_{1423}$. Since $n \geq k+1$, by Corollary 2 we have $\mathcal{S}_5^2 = \mathcal{S}_5^2(+) = \{$14235, 31245, 13425, 21435, 23145, 13254, 21354, 12453, 12534$\}$.

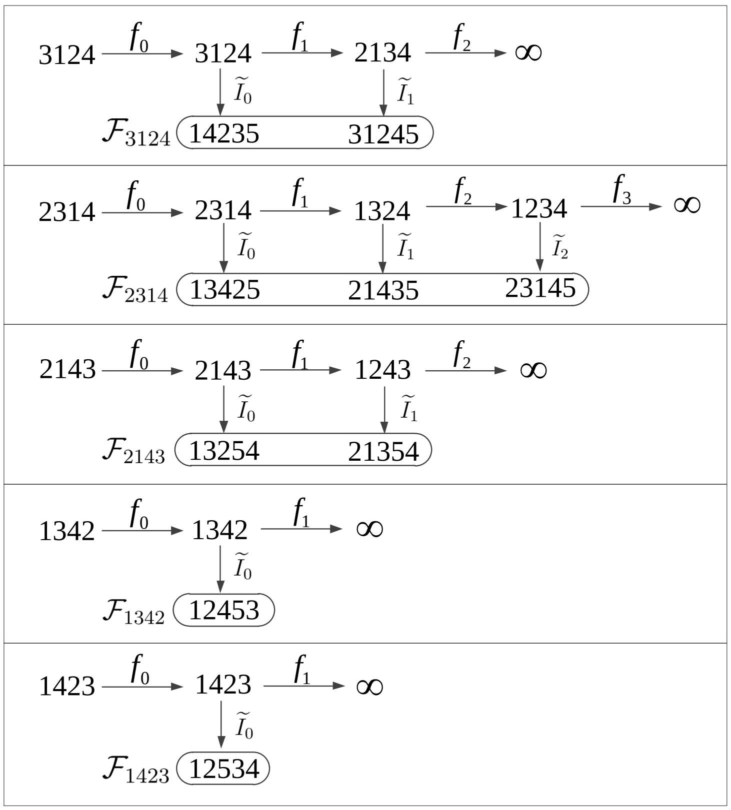

Figure 1. An example illustrating our algorithm

Recall that, $p_1(\pi)$ is the position of 1 in π, i.e., $p_1(\pi) = P(\pi) + 1$. It is easy to see that $|\mathcal{F}_\pi| = P(\pi) + 1 = p_1(\pi)$. From Theorem 9 we have the following corollary directly.

Corollary 3. *For* $0 \leq k \leq \binom{n}{2}$, *we have*

$$S_{n+1}^k(+) = \sum_{\pi \in \mathcal{S}_n^k} p_1(\pi),$$

where $S_{n+1}^k(+)$ *is the cardinality of* $\mathcal{S}_{n+1}^k(+)$. *In particular, for* $n \geq k+1$, *we have*

$$S_{n+1}^k = \sum_{\pi \in \mathcal{S}_n^k} p_1(\pi).$$

Now let us generalize the case of $n \geq k+1$ of Corollary 3 by using our algorithm m times, $m \geq 1$.

Theorem 10. *Let $n \geq k+1$, $m \geq 1$, we have*

$$S^k_{n+m} = \sum_{\pi \in \mathcal{S}^k_n} \binom{p_1(\pi)+m-1}{m}.$$

Proof. Given $\pi \in \mathcal{S}^k_n$, let $M_1(\pi) = \mathcal{F}_\pi \subseteq \mathcal{S}^k_{n+1}$, $M_2(\pi) = \{\mathcal{F}_{\pi'} \mid \pi' \in M_1(\pi)\} \subseteq \mathcal{S}^k_{n+2}, \cdots$, $M_m(\pi) = \{\mathcal{F}_{\pi'} \mid \pi' \in M_{m-1}(\pi)\} \subseteq \mathcal{S}^k_{n+m}$. We claim that there are $\binom{p_1(\pi)+m-1-i}{m-1}$ permutations in $M_m(\pi)$ for which the entry 1 is in position i, where $1 \leq i \leq p_1(\pi)$. We use induction on m. When $m=1$, $M_1(\pi) = \mathcal{F}_\pi$, it is clear that there is exactly one permutation in $M_1(\pi)$ for which the entry 1 is in position i, $1 \leq i \leq p_1(\pi)$. Note that when $m=1$, $\binom{p_1(\pi)+m-1-i}{m-1} = 1$ for $1 \leq i \leq p_1(\pi)$. Thus our claim is true for $m=1$. Suppose our claim is true for m and prove it for $m+1$. Note that $M_{m+1}(\pi) = \{\mathcal{F}_{\pi'} \mid \pi' \in M_m(\pi)\} \subseteq \mathcal{S}^k_{n+m+1}$, it is not hard to see that the number of permutations in $M_{m+1}(\pi)$ for which the entry 1 is in position i is equal to the number of permutations in $M_m(\pi)$ for which the entry 1 is in position at least i. Combining this with the induction hypothesis, we find that there are

$$\binom{p_1(\pi)+m-1-i}{m-1} + \binom{p_1(\pi)+m-1-(i+1)}{m-1} + \cdots + \binom{m-1}{m-1}$$

permutations in $M_{m+1}(\pi)$ for which the entry 1 is in position i. It is easy to see that the above summation is $\binom{p_1(\pi)+m-i}{m}$, and the induction proof of our claim is completed. From our claim, we find

$$|M_m(\pi)| = \sum_{i=1}^{p_1(\pi)} \binom{p_1(\pi)+m-1-i}{m-1} = \binom{p_1(\pi)+m-1}{m},$$

thus we have

$$S^k_{n+m} = \sum_{\pi \in \mathcal{S}^k_n} |M_m(\pi)| = \sum_{\pi \in \mathcal{S}^k_n} \binom{p_1(\pi)+m-1}{m}.$$

∎

Example 7. Let $n=4, k=2$, we know that $\mathcal{S}^2_4 = \{3124, 2314, 2143, 1342, 1423\}$. Obviously, $p_1(3124)=2$, $p_1(2314)=3$, $p_1(2143)=2$, $p_1(1342)=1$, $p_1(1423)=1$. By Theorem 10 we have $S^2_5 = 2+3+2+1+1 = 9$, $S^2_6 = 3+6+3+1+1 = 14$, $S^2_7 = 4+10+4+1+1 = 20$.

We point out that, if we know the set $\mathcal{S}^k_n$, $n \geq k+1$, using our algorithm m times we can generate the set $\mathcal{S}^k_{n+m}$, and we can know its cardinality S^k_{n+m} from Theorem 10.

We conclude this section by discussing the time complexity of our algorithm for fixed k and $n \geq k+1$. The following result due to Margolius [13] gives an asymptotic formula for S^k_n when $n \geq k$.

Lemma 5 (Margolius). *Let $n \geq k$, we have*

$$S_n^k = \frac{Q2^{k-1}}{\sqrt{k\pi}} 2^n (1 + O(n^{-1})),$$

where $Q = \prod_{j=1}^{\infty}(1 - \frac{1}{2^j}) \approx 0.2887880951$.

Given $\pi \in \mathcal{S}_n^k$, for any i, $1 \leq i \leq P(\pi) < n$, it is clear that the time complexity of finding the f_i-selected entry of π is $O(n)$. Thus, the time complexity of generating the set $\bigcup_{i=0}^{P(\pi)}\{\widetilde{I}_i(f_{(i)}(\pi))\}$ is at most $O(n^2)$, combining this with Lemma 5, we see that the time complexity of our algorithm to generate the set $\mathcal{S}_{n+1}^k$ from the set $\mathcal{S}_n^k$ is at most $O(2^n n^2)$, where k is fixed and $n \geq k+1$.

Note that, a natural algorithm to generate the set $\mathcal{S}_{n+1}^k$ is using the brute force $\binom{n+1}{2}$ way of counting the number of inversions for each permutation in $\mathcal{S}_{n+1}^k$ and then checking to see if they are equal to k, the time complexity of this algorithm is $O\big((n+1)!n^2\big)$. Then we can see that, for the case of $n \geq k+1$, if we know the set $\mathcal{S}_n^k$, using our algorithm will reduce the time complexity $O\big((n+1)!n^2\big)$ to $O(2^n n^2)$.

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