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VERIFIED ERROR BOUNDS FOR ISOLATED SINGULAR SOLUTIONS OF POLYNOMIAL SYSTEMS*

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Abstract. In this paper, we generalize the algorithm described by Rump and Graillat, as well as our previous work on verifying breadth-one singular solutions of polynomial systems, to compute verified and narrow error bounds such that a slightly perturbed system is guaranteed to possess an isolated singular solution within computed error bounds. Our verification method is based on deflation techniques using smoothing parameters. We demonstrate the performance of the algorithm for polynomial systems with singular solutions of multiplicity up to hundreds.

Key words. isolated singular solutions, polynomial systems, multiplicity structure, verification, error bounds, deflations

AMS subject classifications. 65H10, 65G20, 74G35, 68W30, 13P10

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1. Introduction.

It is a challenge to solve polynomial systems with singular solutions. In [29], Rall studied convergence properties of Newton's method for singular solutions, and many modifications of Newton's method to restore the quadratic convergence for singular solutions have been proposed in [1, 5, 6, 7, 11, 12, 13, 26, 28, 30, 31, 36, 40]. Recently, some symbolic-numeric methods have also been proposed for refining approximate isolated singular solutions to high accuracy [2, 3, 4, 9, 10, 18, 19, 20, 24, 38, 39]. In [21, 22], we described an algorithm based on the regularized Newton iterations and the computation of differential conditions satisfied at given approximate singular solutions to compute isolated singular solutions accurately to the full machine precision, when its Jacobian matrix has corank one (the breadth-one case).

Since arbitrary small perturbations of coefficients may transform an isolated singular solution into a cluster of simple roots or even make it disappear, it is more difficult to verify that a polynomial system or a nonlinear system has a multiple root if the entire computation is not performed without any rounding error [34].

In [35], by introducing a smoothing parameter, Rump and Graillat developed a verification method for computing verified and narrow error bounds, such that a slightly perturbed system is proved to possess a double root within computed error bounds. In [23], by adding a univariate polynomial in one selected variable with some smoothing parameters to one selected equation of the original system, we generalized the algorithm in [35] to compute guaranteed error bounds such that a slightly perturbed system is proved to have a breadth-one isolated singular solution within computed error bounds.

In [24], Mantzaflaris and Mourrain proposed a one-step deflation method, and by applying a well-chosen symbolic perturbation, they verified a multiple root of a nearby system with a given multiplicity structure, which depends on the accuracy of

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the given approximate multiple root. The size of the final augmented system is equal to the multiplicity times the size of the original system, which might be large (e.g., DZ1 and KSS in Table 1).

In [14], based on deflated square systems proposed by Yamamoto in [40], Kanzawa and Oishi presented a numerical method for proving the existence of "imperfect singular solutions" of nonlinear equations with guaranteed accuracy. In [40], if the second-order deflation is applied, then smoothing parameters are added not only to the original system but also independently to differential systems; see (3.14). Therefore, one can only prove the existence of an isolated solution of a slightly perturbed system which satisfies the first-order differential condition approximately.

In [8, 15, 16], Kearfott and others presented completely different and interesting methods based on verifying a nonzero topological degree to verify the existence of singular zeros of nonlinear systems.

Main contribution. Suppose we are given a system of polynomial equations and an approximate isolated singular solution. Stimulated by previous work on verifying the existence of breadth-one singular solutions [23], first we show the number of deflations used by Yamamoto to obtain that a regular and square augmented system is bounded by the depth of the singular solution. Then we show how to move the independent perturbations in the first-order differential system (3.14) appearing in [40] back to the original system. We prove that this modified deflation technique terminates after a finite number of steps bounded by the depth as well, and it returns a regular and square augmented system, which can be used to prove the existence of an isolated singular solution of a slightly perturbed polynomial system; see Theorems 3.8 and 3.9. Finally, we present an algorithm for computing verified error bounds, and a successful output of the algorithm can guarantee that there exists a unique system, which has a unique isolated singular solution within computed error bounds. The algorithm has been implemented in Maple and MATLAB, and narrow error bounds of the order of the relative rounding error are computed efficiently for examples given in literature.

Structure of the paper. Section 2 is devoted to recalling some notation and well-known facts. In section 3, we develop a novel deflation technique by adding several smoothing parameters properly to the original system, which returns a regular and square augmented system after a finite number of steps bounded by the depth. In section 4, we propose an algorithm for computing verified error bounds such that a slightly perturbed system is guaranteed to possess an isolated singular solution within computed error bounds. Some numerical results are given to demonstrate the performance of our algorithm in section 5.

2. Preliminaries.

Let $F = \{f_1, \ldots, f_n\}$ be a polynomial system, $f_i \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$, and let $I \in \mathbb{C}[\mathbf{x}]$ be the ideal generated by polynomials in F.

Definition 2.1. *An isolated solution of $F(\mathbf{x}) = \mathbf{0}$ is a point $\hat{\mathbf{x}} \in \mathbb{C}^n$ which satisfies*

$$\exists\, \varepsilon > 0 : \{\mathbf{y} \in \mathbb{C}^n : \|\mathbf{y} - \hat{\mathbf{x}}\| < \varepsilon\} \cap F^{-1}(\mathbf{0}) = \{\hat{\mathbf{x}}\}.$$

Definition 2.2. *We call $\hat{\mathbf{x}}$ a singular solution of $F(\mathbf{x}) = \mathbf{0}$ if and only if*

$$\operatorname{rank}(F_{\mathbf{x}}(\hat{\mathbf{x}})) < n, \tag{2.1}$$

where $F_{\mathbf{x}}(\mathbf{x})$ is the Jacobian matrix of $F(\mathbf{x})$ with respect to $\mathbf{x}$.

Definition 2.3. *Let $Q_{\hat{\mathbf{x}}}$ be the isolated primary component of the ideal $I = (f_1, \ldots, f_n)$ whose associate prime is $m_{\hat{\mathbf{x}}} = (x_1 - \hat{x}_1, \ldots, x_n - \hat{x}_n)$; then the multiplicity*

μ of $\hat{\mathbf{x}}$ is defined as $\mu = \dim(\mathbb{C}[\mathbf{x}]/Q_{\hat{\mathbf{x}}})$, and the index ρ of $\hat{\mathbf{x}}$ is defined as the minimal nonnegative integer ρ such that $m_{\hat{\mathbf{x}}}^{\rho} \subseteq Q_{\hat{\mathbf{x}}}$ [37].

Let $\mathbf{d}_{\hat{\mathbf{x}}}^{\boldsymbol{\alpha}} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}$ denote the differential functional defined by

$$\mathbf{d}_{\hat{\mathbf{x}}}^{\boldsymbol{\alpha}}(g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{|\boldsymbol{\alpha}|} g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(\hat{\mathbf{x}}) \quad \forall g(\mathbf{x}) \in \mathbb{C}[\mathbf{x}] \tag{2.2}$$

for a point $\hat{\mathbf{x}} \in \mathbb{C}^n$ and an array $\boldsymbol{\alpha} \in \mathbb{N}^n$. The normalized differentials have a useful property: when $\hat{\mathbf{x}} = \mathbf{0}$, we have $\mathbf{d}_{\mathbf{0}}^{\boldsymbol{\alpha}}(\mathbf{x}^{\boldsymbol{\beta}}) = 1$ if $\boldsymbol{\alpha} = \boldsymbol{\beta}$ or 0 otherwise, where $\mathbf{x}^{\boldsymbol{\beta}} = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$.

DEFINITION 2.4. *The local dual space of I at $\hat{\mathbf{x}}$ is defined as the subspace of elements of $\mathfrak{D}_{\hat{\mathbf{x}}} = \mathrm{Span}_{\mathbb{C}}\{\mathbf{d}_{\hat{\mathbf{x}}}^{\alpha}, \alpha \in \mathbb{N}^n\}$ that vanish on all elements of I,*

$$\mathcal{D}_{\hat{\mathbf{x}}} := \{\Lambda \in \mathfrak{D}_{\hat{\mathbf{x}}} \mid \Lambda(f) = 0 \ \forall f \in I\}. \tag{2.3}$$

It is clear that $\dim(\mathcal{D}_{\hat{\mathbf{x}}}) = \mu$ and the maximal degree of an element $\Lambda \in \mathcal{D}_{\hat{\mathbf{x}}}$ is equal to $\rho - 1$, which is also known as the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$.

A singular solution $\hat{\mathbf{x}}$ of a square system $F(\mathbf{x}) = \mathbf{0}$ satisfies equations

$$\begin{cases} F(\mathbf{x}) = \mathbf{0}, \\ \det(F_{\mathbf{x}}(\mathbf{x})) = 0. \end{cases} \tag{2.4}$$

The above augmented system forms the basic idea for the deflation method [26, 27, 28]. But the determinant is usually of high degree, so it is numerically unstable to evaluate the determinant of the Jacobian matrix.

In [19], Leykin, Verschelde, and Zhao modified (2.4) by adding new variables and new equations. Let $r = \mathrm{rank}(F_{\mathbf{x}}(\hat{\mathbf{x}}))$; with probability one, there exists a unique vector $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_1, \hat{\lambda}_2 \ldots, \hat{\lambda}_{r+1})^T$ such that $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ is an isolated solution of

$$\begin{cases} F(\mathbf{x}) = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x}) B \boldsymbol{\lambda} = \mathbf{0}, \\ \mathbf{h}^T \boldsymbol{\lambda} = 1, \end{cases} \tag{2.5}$$

where $B \in \mathbb{C}^{n \times (r+1)}$ is a random matrix, $\mathbf{h} \in \mathbb{C}^{r+1}$ is a random vector, and $\boldsymbol{\lambda} \in \mathbb{C}^{r+1}$ is a vector consisting of $r+1$ new variables $\lambda_1, \lambda_2 \ldots, \lambda_{r+1}$. If $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}})$ is still a singular solution of (2.5), the deflation is repeated. Furthermore, they proved that the number of deflations needed to derive a regular solution of an augmented system is strictly less than the multiplicity of $\hat{\mathbf{x}}$. Dayton and Zeng showed that the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$ is a tighter bound for the number of deflations [4].

Let $\mathbb{IR}$ be the set of real intervals, and let $\mathbb{IR}^n$ and $\mathbb{IR}^{n \times n}$ be the sets of real interval vectors and real interval matrices, respectively. Standard verification methods for nonlinear systems are based on the following theorem.

THEOREM 2.5 (see [17, 25, 32]). *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a system of nonlinear equations. Suppose $\tilde{\mathbf{x}} \in \mathbb{R}^n$, $\mathbf{X} \in \mathbb{IR}^n$ with $\mathbf{0} \in \mathbf{X}$ and $R \in \mathbb{R}^{n \times n}$ are given. Let $\mathbf{M} \in \mathbb{IR}^{n \times n}$ be given such that*

$$\{\nabla h_i(\mathbf{y}) : \mathbf{y} \in \tilde{\mathbf{x}} + \mathbf{X}\} \subseteq \mathbf{M}_{i,:}, i = 1, \ldots, n. \tag{2.6}$$

Denote by I_n the $n \times n$ identity matrix and assume

$$-RH(\tilde{\mathbf{x}}) + (I_n - R\mathbf{M})\mathbf{X} \subseteq \mathrm{int}(\mathbf{X}). \tag{2.7}$$

Then there is a unique $\hat{\mathbf{x}} \in \tilde{\mathbf{x}} + \mathbf{X}$ *satisfying* $H(\hat{\mathbf{x}}) = \mathbf{0}$. *Moreover, every matrix* $\widetilde{M} \in \mathbf{M}$ *is nonsingular. In particular, the Jacobian matrix* $H_{\mathbf{x}}(\hat{\mathbf{x}})$ *is nonsingular.*

Naturally the nonsingularity of the Jacobian matrix $H_{\mathbf{x}}(\hat{\mathbf{x}})$ restricts the application of Theorem 2.5 to regular solutions of square systems. Notice that Theorem 2.5 is valid over complex numbers with the necessary modifications. In the following, we propose a new deflation method which returns a regular and square augmented system, and thus Theorem 2.5 is applicable. Hence, we are able to verify the existence of an isolated singular solution of a slightly perturbed system.

3. A square and regular augmented system.

Suppose we are given a polynomial system $F = \{f_1, \ldots, f_n\}$, $f_i \in \mathbb{C}[\mathbf{x}]$, and let $\hat{\mathbf{x}} \in \mathbb{C}^n$ be an isolated solution satisfying $F(\hat{\mathbf{x}}) = \mathbf{0}$ and $\det(F_{\mathbf{x}}(\hat{\mathbf{x}})) = 0$.

The systems (2.4) and (2.5) have been used to restore the quadratic convergence of Newton's method. But notice that these augmented systems are always overdetermined, and thus Theorem 2.5 is not applicable. In [40], by introducing smoothing parameters, Yamamoto derived square augmented systems. These systems were used successfully by Kanzawa and Oishi in [14] to prove the existence of *imperfect singular solutions* of nonlinear systems. However, for isolated solutions with high singularities, the smoothing parameters are added not only to the original system but also to the differential systems independently; see (3.14). Therefore, according to (3.15), one can only prove the existence of an isolated solution of a slightly perturbed system, which satisfies the first-order differential condition approximately.

In the following, we rewrite the deflation techniques in [40] in our setting and prove that the number of deflations needed to obtain a regular system is bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$; see Theorem 3.2. Then we show how to lift the independent perturbations in the first-order differential system appearing in (3.14) back to the original system. We prove that the modified deflations will terminate after a finite number of steps bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$ as well and return a regular and square augmented system, which can be used to prove the existence of an isolated singular solution of a slightly perturbed system exactly; see Theorems 3.8 and 3.9.

3.1. The first-order deflation.

Let $\hat{\mathbf{x}} \in \mathbb{C}^n$ be an isolated singular solution of $F(\mathbf{x}) = \mathbf{0}$, and

$$\operatorname{rank}(F_{\mathbf{x}}(\hat{\mathbf{x}})) = n - d, \ (1 \le d \le n). \tag{3.1}$$

Let $\mathbf{c} = \{c_1, c_2, \ldots, c_d\}$ $(1 \le c_1 < c_2 < \cdots < c_d \le n)$ and $F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}})$ is obtained from $F_{\mathbf{x}}(\hat{\mathbf{x}})$ by deleting its $c_1, c_2, \ldots, c_d$th columns, which satisfies

$$\operatorname{rank}(F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}})) = n - d. \tag{3.2}$$

There exists a set of positive integers $\mathbf{k} = \{k_1, k_2, \ldots, k_d\}$ such that

$$\operatorname{rank}(F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}), I_{\mathbf{k}}) = n, \tag{3.3}$$

where

$$I_{\mathbf{k}} = (\mathbf{e}_{k_1}, \mathbf{e}_{k_2}, \ldots, \mathbf{e}_{k_d}), \tag{3.4}$$

and $\mathbf{e}_{k_i}$ is the k_ith unit vector of dimension n.

We introduce d smoothing parameters $\mathbf{b}_0 = (b_{0,1}, \ldots, b_{0,d})$ and consider the following square system:

$$G(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0) = \begin{cases} F(\mathbf{x}) - \sum_{i=1}^{d} b_{0,i}\mathbf{e}_{k_i} = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 = \mathbf{0}, \end{cases} \tag{3.5}$$

where $\mathbf{v}_1 \in \mathbb{C}^n$ is a vector consisting of $n-d$ variables $\boldsymbol{\lambda}_1 = (\lambda_{1,1}, \ldots, \lambda_{1,n-d})$ concatenated to a vector $(v_{c_1}, \ldots, v_{c_d}) = (1, \ldots, 1)$,

$$\mathbf{v}_1 = (\lambda_{1,1}, \ldots, \underset{c_1}{1}, \ldots, \underset{c_d}{1}, \ldots, \lambda_{1,n-d})^T \in \mathbb{C}^n.$$

It is similar to the augmented system (2.34) in [40], except that Yamamoto chose the values of $v_{c_i}, i = 1, 2, \ldots, d$, randomly. According to (3.2), the rank of $F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}})$ is $n-d$, the solution of the linear system $F_{\mathbf{x}}(\hat{\mathbf{x}})\mathbf{v}_1 = \mathbf{0}$ with the fixed value 1 at positions $c_1, \ldots, c_d$ is unique, and we denote it by $\hat{\mathbf{v}}_1$:

$$\hat{\mathbf{v}}_1 = (\hat{\lambda}_{1,1}, \ldots, \underset{c_1}{1}, \ldots, \underset{c_d}{1}, \ldots, \hat{\lambda}_{1,n-d})^T \in \mathbb{C}^n.$$

Therefore, $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$ is an isolated solution of $G(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0) = \mathbf{0}$, where

$$\hat{\boldsymbol{\lambda}}_1 = (\hat{\lambda}_{1,1}, \ldots, \hat{\lambda}_{1,n-d}).$$

If $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$ is still a singular solution, as proposed in [40], the deflation process mentioned above is repeated for the first-order deflated system G and the solution $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$.

In [40], Yamamoto did not prove explicitly the termination of the above deflation process. Motivated by the results in [4, 19], we show that the number of deflations needed to derive a regular and square augmented system is bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$.

Let $\mathbf{h} = (\underbrace{0, \ldots, 0}_{n-d}, 1)^T$, $\boldsymbol{\lambda} = (\lambda_{1,1}, \ldots, \lambda_{1,n-d}, 1)^T$ and

$$B = (\hat{\mathbf{e}}_1, \ldots, \underset{c_1}{\hat{\mathbf{e}}_{n-d+1}}, \ldots, \underset{c_d}{\hat{\mathbf{e}}_{n-d+1}}, \ldots, \hat{\mathbf{e}}_{n-d})^T \in \mathbb{C}^{n\times(n-d+1)},$$

where $\hat{\mathbf{e}}_i$ is the ith unit vector of dimension $n-d+1$. Then, the augmented system (2.5) used in [19] is equivalent to

$$\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1) = \begin{cases} F(\mathbf{x}) = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 = \mathbf{0}, \end{cases} \tag{3.6}$$

which has an isolated solution at $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$, and the Jacobian matrix of $\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1)$ at $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$ is

$$\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1) = \begin{pmatrix} F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} \\ F_{\mathbf{xx}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}) \end{pmatrix}, \tag{3.7}$$

where $\mathcal{O}_{i,j}$ denotes the $i \times j$ zero matrix and $F_{\mathbf{xx}}(\mathbf{x})$ is the Hessian matrix of $F(\mathbf{x})$. On the other hand, the Jacobian matrix of $G(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0)$ computes to

$$G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0}) = \begin{pmatrix} F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} \\ F_{\mathbf{xx}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} \end{pmatrix}. \tag{3.8}$$

LEMMA 3.1. *The null spaces of the Jacobian matrices* (3.7) *and* (3.8) *satisfy*

$$\mathrm{null}(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})) = \left\{ \begin{pmatrix} \mathbf{y} \\ \mathbf{0} \end{pmatrix} \in \mathbb{C}^{2n} \mid \mathbf{y} \in \mathrm{null}(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)) \right\}.$$

Proof. If $\mathbf{y} \in \text{null}(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1))$ then $\binom{\mathbf{y}}{\mathbf{0}} \in \text{null}(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0}))$. Suppose $\binom{\mathbf{y}}{\mathbf{z}}$ is a null vector of $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$. Corresponding to the blocks $F_{\mathbf{x}}(\hat{\mathbf{x}})$ and $\mathcal{O}_{n,n-d}$, we divide $\mathbf{y}$ into $\binom{\mathbf{y}_1}{\mathbf{y}_2}$. It follows that

$$F_{\mathbf{x}}(\hat{\mathbf{x}})\mathbf{y}_1 - I_{\mathbf{k}}\mathbf{z} = \mathbf{0}.$$

By (3.3), we have

$$\text{rank}(F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}), -I_{\mathbf{k}}) = n.$$

It is clear that $\mathbf{z}$ must be a zero vector. ∎

If $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$ is still an isolated singular solution of the deflated system (3.6), as proposed in [19], the deflation process is repeated for $\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1)$ and $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)$. As shown in [4], if the sth deflated system is singular, there exists at least one differential functional of the order $s+1$ in $\mathcal{D}_{\hat{\mathbf{x}}}$. However, the order of differential functionals in $\mathcal{D}_{\hat{\mathbf{x}}}$ is bounded by its depth, which is equal to $\rho - 1$. Therefore, after at most $\rho - 1$ steps of deflations (3.5), one will obtain a regular augmented system, i.e., the corank of the Jacobian matrix of the deflated system will be zero. As a consequence, based on Lemma 3.1, we claim the finite termination of Yamamoto's deflation method.

THEOREM 3.2. *The number of the first-order deflations* (3.5) *needed to derive a regular and square augmented system is bounded by the depth of* $\mathcal{D}_{\hat{\mathbf{x}}}$.

Proof. By Lemma 3.1, we have

$$\text{corank}(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)) = \text{corank}(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})). \tag{3.9}$$

Therefore, the smoothing parameters added in the deflated system (3.5) do not change any rank-deficient information of the Jacobian matrix of (3.6). If

$$\text{corank}(\widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1)) = \text{corank}(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})) > 0,$$

then we repeat the deflation steps for (3.5) and (3.6) accordingly. Inductively, we know that the coranks of Jacobian matrices of two different kinds of deflated systems remain equal at every step. Moreover, we have shown that, after at most $\rho - 1$ steps, the corank of the Jacobian matrix of the deflated system corresponding to (3.6) becomes zero. Therefore, the deflated system corresponding to (3.5) will also become regular after at most $\rho - 1$ steps. ∎

3.2. The second-order deflation. Suppose the Jacobian matrix $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})$ is singular, i.e.,

$$\text{rank}(G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})) = 2n - d', \ (d' \geq 1). \tag{3.10}$$

Let $\mathbf{c}' = \{c'_1, c'_2, \ldots, c'_{d'}\}$ and $\mathbf{k}' = \{k'_1, k'_2, \ldots, k'_{d'}\}$ be two sets of positive integers such that

$$\text{rank}(G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0})) = 2n - d', \tag{3.11}$$

$$\text{rank}(G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}_1, \mathbf{0}), I_{\mathbf{k}'+n}) = 2n, \tag{3.12}$$

where $G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ is a matrix obtained from $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ by deleting its $c'_1, c'_2, \ldots, c'_{d'}$th columns, and

$$I_{\mathbf{k}'+n} = \begin{pmatrix} \mathcal{O}_{n,d'} \\ I_{\mathbf{k}'} \end{pmatrix}, \; I_{\mathbf{k}'} = (\mathbf{e}_{k'_1}, \mathbf{e}_{k'_2}, \ldots, \mathbf{e}_{k'_{d'}}). \tag{3.13}$$

THEOREM 3.3. *Comparing to* $F_{\mathbf{x}}(\hat{\mathbf{x}})$, *the corank of* $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ *does not increase, i.e.,* $d' \le d$. *Moreover, we can choose* $\mathbf{c}'$ *and* $\mathbf{k}'$ *such that* $\mathbf{c}' \subseteq \mathbf{c}$, $\mathbf{k}' \subseteq \mathbf{k}$ *and satisfy* (3.11) *and* (3.12), *respectively.*

Proof. Let

$$G^{\mathbf{c}}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0}) = \begin{pmatrix} F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} \\ \star & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} \end{pmatrix}$$

be the matrix obtained from $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$ by deleting its $c_1, c_2, \ldots, c_d$th columns. By (3.2) and (3.3) we claim that

$$\operatorname{rank}(G^{\mathbf{c}}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})) = 2n - d.$$

Hence $d' \le d$. Besides, there exists a set of positive integers $\mathbf{c}' \subseteq \mathbf{c}$ such that the condition (3.11) is satisfied.

According to (3.3), it is clear that

$$\operatorname{rank}(G^{\mathbf{c}}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0}), I_{\mathbf{k}+n}) = 2n,$$

where $I_{\mathbf{k}+n} = \binom{\mathcal{O}_{n,d}}{I_{\mathbf{k}}}$. Hence, we can choose $\mathbf{k}' \subseteq \mathbf{k}$ such that condition (3.12) is satisfied. ∎

If $d' \ge 1$, then Yamamoto repeated the first-order deflation for $G(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0)$ and $(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$. By Theorem 3.3, we know that Yamamoto's second-order deflation is equivalent to adding d' smoothing parameters $\mathbf{b}_1 = (b_{1,1}, \ldots, b_{1,d'})$ to the first-order differential system $F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1$:

$$H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b}) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}, \\ G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0)\mathbf{v}_2 = \mathbf{0}, \end{cases} \tag{3.14}$$

where $\mathbf{b} = (\mathbf{b}_0, \mathbf{b}_1)$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$, and $\mathbf{v}_2 \in \mathbb{C}^{2n}$ is a vector consisting of $2n - d'$ variables $\boldsymbol{\lambda}_2 = (\lambda_{2,1}, \ldots, \lambda_{2,2n-d'})$ concatenated to a vector $(v_{c'_1}, \ldots, v_{c'_{d'}}) = (1, \ldots, 1)$,

$$\mathbf{v}_2 = (\lambda_{2,1}, \ldots, \underset{c'_1}{1}, \ldots, \underset{c'_{d'}}{1}, \ldots, \lambda_{2,2n-d'})^T \in \mathbb{C}^{2n}.$$

Let $\hat{\mathbf{v}}_2$ denote the unique solution of $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})\mathbf{v}_2 = \mathbf{0}$ with the fixed value 1 at positions $c'_1, \ldots, c'_d$:

$$\hat{\mathbf{v}}_2 = (\hat{\lambda}_{2,1}, \ldots, \underset{c'_1}{1}, \ldots, \underset{c'_{d'}}{1}, \ldots, \hat{\lambda}_{2,2n-d'})^T \in \mathbb{C}^{2n};$$

then $(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ is an isolated solution of $H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b}) = \mathbf{0}$.

Suppose Theorem 2.5 is applicable to the augmented system $H(\mathbf{x},\boldsymbol{\lambda},\mathbf{b})$ and yields inclusions for $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\lambda}}$, $\hat{\mathbf{b}}_0$, and $\hat{\mathbf{b}}_1$. Thus,

$$\widetilde{F}(\hat{\mathbf{x}}) = F(\hat{\mathbf{x}}) - I_{\mathbf{k}}\hat{\mathbf{b}}_0 = \mathbf{0} \text{ and } \widetilde{F}_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 = F_{\mathbf{x}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 = I_{\mathbf{k}'}\hat{\mathbf{b}}_1, \tag{3.15}$$

where $\widetilde{F}(\mathbf{x}) = F(\mathbf{x}) - I_{\mathbf{k}}\hat{\mathbf{b}}_0$. Smoothing parameters $\hat{\mathbf{b}}_1$ might be very small but cannot be guaranteed to be zeros. Therefore, one can only prove the existence of an isolated solution $\hat{\mathbf{x}}$ of $\widetilde{F}(\mathbf{x}) = \mathbf{0}$, which satisfies the first-order differential conditions approximately.

In order to verify the existence of an isolated singular solution of a slightly perturbed system, we should add the smoothing parameters $\mathbf{b}_1$ back to the original system. Let us consider the modified system:

$$\widetilde{H}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b}) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}, \\ \widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0, \mathbf{b}_1)\mathbf{v}_2 = \mathbf{0}, \end{cases} \tag{3.16}$$

where $X_1 = (x_{c'_1}\mathbf{e}_{k'_1}, \ldots, x_{c'_{d'}}\mathbf{e}_{k'_{d'}})$ and

$$\widetilde{G}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0, \mathbf{b}_1) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}. \end{cases} \tag{3.17}$$

THEOREM 3.4. *Let*

$$\widetilde{F}(\mathbf{x}, \mathbf{b}) = F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1; \tag{3.18}$$

then we have

$$F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0} \Longleftrightarrow \widetilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{b})\mathbf{v}_1 = \mathbf{0}. \tag{3.19}$$

Proof. Recall that

$$\mathbf{b}_1 = (b_{1,1}, b_{1,2}, \ldots, b_{1,d'})^T \text{ and } \mathbf{v}_1 = (\lambda_{1,1}, \ldots, \underset{c_1}{1}, \ldots, \underset{c_d}{1}, \ldots, \lambda_{1,n-d})^T;$$

then

$$\begin{aligned} \widetilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{b})\mathbf{v}_1 &= F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - (\mathbf{0}, \cdots, \underset{c'_1}{b_{1,1}\mathbf{e}_{k'_1}}, \cdots, \underset{c'_{d'}}{b_{1,d'}\mathbf{e}_{k'_{d'}}}, \cdots, \mathbf{0})\,\mathbf{v}_1 \\ &= F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - (\mathbf{e}_{k'_1}, \ldots, \mathbf{e}_{k'_{d'}})\,\mathbf{b}_1 \text{ (since } \mathbf{c}' \subseteq \mathbf{c}) \\ &= F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1. \qquad \square \end{aligned}$$

According to Theorem 3.4, we can rewrite the system (3.16) as

$$\widetilde{H}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b}) = \begin{cases} \widetilde{F}(\mathbf{x}, \mathbf{b}) = \mathbf{0}, \\ \widetilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{b})\mathbf{v}_1 = \mathbf{0}, \\ \widetilde{G}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0, \mathbf{b}_1)\mathbf{v}_2 = \mathbf{0}. \end{cases} \tag{3.20}$$

Therefore, if we can prove that $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \hat{\mathbf{b}})$ is a regular solution of $\widetilde{H}(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b}) = \mathbf{0}$, then by Theorem 3.4, $\hat{\mathbf{x}}$ is guaranteed to be an isolated singular solution of $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) = \mathbf{0}$.

THEOREM 3.5. *The Jacobian matrices of* (3.14) *and* (3.16) *share the same null space.*

Proof. The Jacobian matrix $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \mathbf{0})$ of (3.14) computes to

$$\begin{pmatrix} F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & & \mathcal{O}_{2n,2n-d'} & & \mathcal{O}_{n,d'} \\ F_{\mathbf{xx}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & & & & -I_{\mathbf{k}'} \\ & & \mathcal{O}_{n,d} & F^{\mathbf{c}'}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & \mathcal{O}_{n,d} \\ & \star & \mathcal{O}_{n,d} & \star & F^{\mathbf{c}}_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & \mathcal{O}_{n,d} \end{pmatrix}, \tag{3.21}$$

while the Jacobian matrix $\widetilde{H}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ of (3.16) computes to

$$(3.22)\qquad \begin{pmatrix} F_{\mathbf{x}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & & \mathcal{O}_{2n,2n-d'} & & -\hat{X} \\ F_{\mathbf{xx}}(\hat{\mathbf{x}})\hat{\mathbf{v}}_1 & F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & & & & -I_{\mathbf{k}'} \\ & & \mathcal{O}_{n,d} & F_{\mathbf{x}}^{\mathbf{c}'}(\hat{\mathbf{x}}) & \mathcal{O}_{n,n-d} & -I_{\mathbf{k}} & -I_{\mathbf{k}'} \\ \multicolumn{2}{c}{\star} & \mathcal{O}_{n,d} & \star & F_{\mathbf{x}}^{\mathbf{c}}(\hat{\mathbf{x}}) & \mathcal{O}_{n,d} & \mathcal{O}_{n,d'} \end{pmatrix},$$

where the matrix $\hat{X}$ consists of vectors $\hat{x}_{\mathbf{c}'(i)}\mathbf{e}_{\mathbf{k}'(i)}$, $i=1,\ldots,d'$. Since $\mathbf{k}'\subseteq\mathbf{k}$, we can reduce the last column of the block matrix (3.22) by its third and sixth columns to get the block matrix (3.21). Therefore, two Jacobian matrices (3.21) and (3.22) are of the same corank and share the same null space. ∎

Suppose the Jacobian matrix $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ is still singular, i.e.,

$$(3.23)\qquad \operatorname{rank}(H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})) = 4n-d'',\ (d''\geq 1).$$

Let $\mathbf{c}''=\{c_1'',c_2'',\ldots,c_{d''}''\}$ and $\mathbf{k}''=\{k_1'',k_2'',\ldots,k_{d''}''\}$ be two sets of positive integers such that

$$(3.24)\qquad \operatorname{rank}(H^{\mathbf{c}''}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})) = 4n-d'',$$

$$(3.25)\qquad \operatorname{rank}(H^{\mathbf{c}''}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0}), I_{\mathbf{k}''+3n}) = 4n,$$

$H^{\mathbf{c}''}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ is a matrix obtained from $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ by deleting its $c_1'',c_2'',\ldots,c_{d''}''$th columns, and

$$(3.26)\qquad I_{\mathbf{k}''+3n} = \begin{pmatrix}\mathcal{O}_{3n,d''}\\ I_{\mathbf{k}''}\end{pmatrix},\ I_{\mathbf{k}''} = (\mathbf{e}_{k_1''},\mathbf{e}_{k_2''},\ldots,\mathbf{e}_{k'_{d''}}).$$

THEOREM 3.6. *Comparing to $G_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0})$, the corank of $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ does not increase, i.e., $d''\leq d'$. Moreover, we can choose $\mathbf{c}''$ and $\mathbf{k}''$ such that $\mathbf{c}''\subseteq\mathbf{c}'$, $\mathbf{k}''\subseteq\mathbf{k}'$ and satisfy* (3.24) *and* (3.25), *respectively.*

Proof. Similar to the proof of Theorem 3.3, let $H^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ be the matrix obtained from $H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})$ by deleting its $c_1',c_2',\ldots,c_{d'}'$th columns. By (3.11) and (3.12), we claim that

$$\operatorname{rank}(H^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0})) = 4n-d'.$$

Therefore, $d''\leq d'$, and there exists a set of positive integers $\mathbf{c}''\subseteq\mathbf{c}'$ such that the condition (3.24) is satisfied.

Meanwhile, we know that $\operatorname{rank}(G^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda}_1,\mathbf{b}_0}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}}_1,\mathbf{0}), I_{\mathbf{k}'+n}) = 2n$; then

$$\operatorname{rank}(H^{\mathbf{c}'}_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\hat{\mathbf{x}},\hat{\boldsymbol{\lambda}},\mathbf{0}), I_{\mathbf{k}'+3n}) = 4n,$$

where $I_{\mathbf{k}'+3n} = \left(\begin{smallmatrix}\mathcal{O}_{3n,d'}\\ I_{\mathbf{k}'}\end{smallmatrix}\right)$. Therefore, we can choose $\mathbf{k}''\subseteq\mathbf{k}'$ such that condition (3.25) is satisfied. ∎

Example 3.7 (see [4, DZ1]). Consider a polynomial system

$$F = \{x_1^4 - x_2x_3x_4, x_2^4 - x_1x_3x_4, x_3^4 - x_1x_2x_4, x_4^4 - x_1x_2x_3\}.$$

The system F has $(0,0,0,0)$ as a 131-fold isolated zero. Since $F_{\mathbf{x}}(\hat{\mathbf{x}}) = \mathcal{O}_{4,4}$, we have $d = 4$, $\mathbf{c} = \mathbf{k} = \{1,2,3,4\}$, $\mathbf{v}_1 = (1,1,1,1)^T$; then

$$G(\mathbf{x}, \mathbf{b}_0) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 = \mathbf{0}, \\ 4x_1^3 - x_3x_4 - x_2x_4 - x_2x_3 = 0, \\ 4x_2^3 - x_3x_4 - x_1x_4 - x_1x_3 = 0, \\ 4x_3^3 - x_2x_4 - x_1x_4 - x_1x_2 = 0, \\ 4x_4^3 - x_2x_3 - x_1x_3 - x_1x_2 = 0. \end{cases}$$

The Jacobian matrix of $G(\mathbf{x}, \mathbf{b}_0)$ at $(\mathbf{0}, \mathbf{0})$ is

$$G_{\mathbf{x},\mathbf{b}_0}(\mathbf{0}, \mathbf{0}) = \begin{pmatrix} \mathcal{O}_{4,4} & -I_{\mathbf{k}} \\ \mathcal{O}_{4,4} & \mathcal{O}_{4,4} \end{pmatrix}.$$

Hence, $d' = 4$, $\mathbf{c}' = \mathbf{k}' = \{1,2,3,4\}$ and

$$H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b}) = \begin{cases} F(\mathbf{x}) - I_{\mathbf{k}}\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - I_{\mathbf{k}'}\mathbf{b}_1 = \mathbf{0}, \\ \widetilde{G}_{\mathbf{x},\mathbf{b}_0}(\mathbf{x}, \mathbf{b}_0, \mathbf{b}_1)\mathbf{v}_2 = \mathbf{0}, \end{cases} \tag{3.27}$$

where $\mathbf{v}_2 = (1,1,1,1,\lambda_1,\lambda_2,\lambda_3,\lambda_4)^T$, and $\widetilde{G}_{\mathbf{x},\mathbf{b}_0}(\mathbf{0},\mathbf{0},\mathbf{0})\mathbf{v}_2 = \mathbf{0}$ has a unique solution $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4) = (0,0,0,0)$. The Jacobian matrix of $H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b})$ at $(\mathbf{0},\mathbf{0},\mathbf{0})$ is

$$H_{\mathbf{x},\boldsymbol{\lambda},\mathbf{b}}(\mathbf{0},\mathbf{0},\mathbf{0}) = \begin{pmatrix} \mathcal{O}_{4,4} & -I_{\mathbf{k}} & \mathcal{O}_{4,4} & \mathcal{O}_{4,4} \\ \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & -I_{\mathbf{k}'} \\ \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & -I_{\mathbf{k}'} & -I_{\mathbf{k}'} \\ A & \mathcal{O}_{4,4} & \mathcal{O}_{4,4} & \mathcal{O}_{4,4} \end{pmatrix}, A = \begin{pmatrix} 0 & -2 & -2 & -2 \\ -2 & 0 & -2 & -2 \\ -2 & -2 & 0 & -2 \\ -2 & -2 & -2 & 0 \end{pmatrix},$$

which is nonsingular. Therefore, we obtain a regular and square augmented system $H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b})$ and a perturbed system

$$\widetilde{F}(\mathbf{x}, \mathbf{b}) = \begin{cases} x_1^4 - x_2x_3x_4 - b_1 - b_5x_1 = 0, \\ x_2^4 - x_1x_3x_4 - b_2 - b_6x_2 = 0, \\ x_3^4 - x_1x_2x_4 - b_3 - b_7x_3 = 0, \\ x_4^4 - x_1x_2x_3 - b_4 - b_8x_4 = 0. \end{cases}$$

Applying the verification method based on Theorem 2.5 to $H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{b})$, we show in section 4 that a slightly perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ for

$$|\hat{b}_i| \leq 2.9e - 323, i = 1, \ldots, 8,$$

has an isolated singular solution $\hat{\mathbf{x}}$ within

$$|\hat{x}_i| \leq 4.8e - 323, i = 1, 2, 3, 4.$$

3.3. Higher-order deflations. For higher-order deflations, we show inductively how to add new smoothing parameters properly to the original system in order to derive a regular and square augmented system, which can be used to prove the existence of an isolated singular solution of a slightly perturbed system.

Let $H^{(0)}(\mathbf{x}) = F(\mathbf{x})$; for the $(s+1)$th-order deflation, we add smoothing parameters $\mathbf{b}^{(s)} = (\mathbf{b}_0, \ldots, \mathbf{b}_s)$ and consider the square system

$$H^{(s+1)}(\mathbf{x}, \boldsymbol{\lambda}^{(s+1)}, \mathbf{b}^{(s)}) = \begin{cases} \widetilde{F}(\mathbf{x}, \mathbf{b}^{(s)}) & = \mathbf{0}, \\ \widetilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{b}^{(s)})\mathbf{v}_1 & = \mathbf{0}, \\ \vdots & \\ G^{(s)}_{\mathbf{x},\boldsymbol{\lambda}^{(s)},\mathbf{b}^{(s-1)}}(\mathbf{x}, \boldsymbol{\lambda}^{(s)}, \mathbf{b}^{(s)})\mathbf{v}_{s+1} & = \mathbf{0}. \end{cases} \tag{3.28}$$

$\boldsymbol{\lambda}^{(s+1)} = (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{s+1})$ are extra variables corresponding to vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_{s+1}\}$, $G^{(s)}(\mathbf{x}, \boldsymbol{\lambda}^{(s)}, \mathbf{b}^{(s)})$ consists of the first $2^s n$ polynomials in $H^{(s+1)}(\mathbf{x}, \boldsymbol{\lambda}^{(s+1)}, \mathbf{b}^{(s)})$,

$$\widetilde{F}(\mathbf{x}, \mathbf{b}^{(s)}) = F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - \cdots - X_s\mathbf{b}_s, \tag{3.29}$$

the matrix X_j $(0 \le j \le s)$ consists of vectors $\frac{1}{j!} \cdot x^j_{\mathbf{c}_i^{(j)}} \cdot \mathbf{e}_{\mathbf{k}_i^{(j)}}$, $i = 1, \ldots, d_j$, where $\mathbf{c}^{(j)}$ and $\mathbf{k}^{(j)}$ are two sets of positive integers selected at the jth-order deflation satisfying conditions obtained by replacing the polynomial system $F(\mathbf{x})$ in (3.2) and (3.3) by the jth augmented system $H^{(j)}(\mathbf{x}, \boldsymbol{\lambda}^{(j)}, \mathbf{b}^{(j-1)})$ and replacing $I_{\mathbf{k}}$ by $I_{\mathbf{k}^{(j)}+(2^j-1)n} = \binom{\mathcal{O}_{(2^j-1)n, d_j}}{I_{\mathbf{k}^{(j)}}}$, $I_{\mathbf{k}^{(j)}} = (\mathbf{e}_{k_1^{(j)}}, \mathbf{e}_{k_2^{(j)}}, \ldots, \mathbf{e}_{k_{d_j}^{(j)}})$, where d_j is the corank of $H^{(j)}_{\mathbf{x},\boldsymbol{\lambda}^{(j)},\mathbf{b}^{(j-1)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(j)}, \mathbf{0})$.

THEOREM 3.8. *The corank d_{s+1} of $H^{(s+1)}_{\mathbf{x},\boldsymbol{\lambda}^{(s+1)},\mathbf{b}^{(s)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(s+1)}, \mathbf{0})$ does not increase and the number of deflations needed to derive a regular solution of an augmented system* (3.28) *is less than the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$, i.e., we have*

$$d_0 \ge d_1 \ge \cdots \ge d_{s+1} \ge \cdots \ge d_{\rho-1} = 0. \tag{3.30}$$

Moreover, we can choose $\mathbf{c}^{(j)}$ and $\mathbf{k}^{(j)}$ satisfying

$$\mathbf{c}^{(s)} \subseteq \cdots \subseteq \mathbf{c}^{(0)} \text{ and } \mathbf{k}^{(s)} \subseteq \cdots \subseteq \mathbf{k}^{(0)}. \tag{3.31}$$

Proof. By applying Theorems 3.3, 3.5, and 3.6 inductively, we can prove that the deflation process (3.28) produces a decreasing nonnegative-integer sequence $d_0 \ge d_1 \ge \cdots \ge d_{s+1} \ge \cdots$, which is the same as the sequence consisting of coranks of the Jacobian matrices of the augmented systems by Yamamoto's deflation method. According to Theorem 3.2, the number of first-order deflations (3.5) needed to derive a regular and square augmented system is bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$. Hence the number of modified deflations (3.28) is also bounded by the depth of $\mathcal{D}_{\hat{\mathbf{x}}}$. The proof of (3.31) is similar to the proofs of Theorems 3.3 and 3.6. □

THEOREM 3.9. *Suppose Theorem* 2.5 *is applicable to the augmented system* (3.28) *and yields inclusions for $\hat{\mathbf{x}}$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\mathbf{b}}$; then the perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ has an isolated singular solution at $\hat{\mathbf{x}}$.*

Proof. Since $(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}, \hat{\mathbf{b}})$ is a solution of the augmented system (3.28), we have

$$\widetilde{F}(\hat{\mathbf{x}}, \hat{\mathbf{b}}) = \mathbf{0} \text{ and } \widetilde{F}_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{\mathbf{b}})\hat{\mathbf{v}}_1 = \mathbf{0}, \ \hat{\mathbf{v}}_1 \ne \mathbf{0}.$$

Hence, $\hat{\mathbf{x}}$ is an isolated singular solution of the perturbed system

$$\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) = F(\mathbf{x}) - X_0\hat{\mathbf{b}}_0 - X_1\hat{\mathbf{b}}_1 - \cdots - X_s\hat{\mathbf{b}}_s. \qquad \square$$

Example 3.10 (see [4, DZ2]). Consider a polynomial system

$$F = \left\{x^4, x^2y + y^4, z + z^2 - 7x^3 - 8x^2\right\}.$$

The system F has $(0, 0, -1)$ as a 16-fold isolated zero.

The Jacobian matrix of F at $\hat{\mathbf{x}} = (0, 0, -1)$ is

$$F_{\mathbf{x}}(\hat{\mathbf{x}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ so that } d_0 = 2, \text{ and we choose } \mathbf{c}^{(0)} = \mathbf{k}^{(0)} = \{1, 2\}.$$

The first-order deflated system is

$$H^{(1)}(\mathbf{x}, \boldsymbol{\lambda}^{(1)}, \mathbf{b}^{(0)}) = \begin{cases} F(\mathbf{x}) - X_0\mathbf{b}_0 = \mathbf{0}, \\ 4x^3 = 0, \\ 2xy + x^2 + 4y^3 = 0, \\ -21x^2 - 16x + \lambda_1 + 2z\lambda_1 = 0, \end{cases}$$

where

$$X_0 = (\mathbf{e}_1, \mathbf{e}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b}_0 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \mathbf{v}_1 = (1, 1, \lambda_1)^T.$$

The Jacobian matrix of $H^{(1)}(\mathbf{x}, \boldsymbol{\lambda}_1, \mathbf{b}_0)$ at $(0, 0, -1, 0, 0, 0)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -16 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \ d_1 = 2, \ \text{and we choose } \mathbf{c}^{(1)} = \mathbf{k}^{(1)} = \{1, 2\}.$$

Therefore, we derive the second-order deflated system

$$H^{(2)}(\mathbf{x}, \boldsymbol{\lambda}^{(2)}, \mathbf{b}^{(1)}) = \begin{cases} F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - X_1'\mathbf{b}_1 = \mathbf{0}, \\ G^{(1)}_{\mathbf{x}, \boldsymbol{\lambda}^{(1)}, \mathbf{b}^{(0)}}(\mathbf{x}, \boldsymbol{\lambda}^{(1)}, \mathbf{b}^{(1)})\mathbf{v}_2 = \mathbf{0}, \end{cases}$$

where

$$X_1 = \begin{pmatrix} x & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix}, \quad \mathbf{b}_1 = \begin{pmatrix} b_3 \\ b_4 \end{pmatrix}, \quad X_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{v}_2 = (1, 1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^T.$$

Moreover, $G^{(1)}_{\mathbf{x}, \boldsymbol{\lambda}^{(1)}, \mathbf{b}^{(0)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(1)}, \mathbf{0})\mathbf{v}_2 = \mathbf{0}$ has a unique solution $\hat{\boldsymbol{\lambda}}_2 = (0, -16, 0, 0)$.

For the third-order deflation, we have $d_2 = 1$, $\mathbf{c}^{(2)} = \mathbf{k}^{(2)} = \{1\}$, so

$$H^{(3)}(\mathbf{x}, \boldsymbol{\lambda}^{(3)}, \mathbf{b}^{(2)}) = \begin{cases} F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - X_2\mathbf{b}_2 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - X_1'\mathbf{b}_1 - X_2'\mathbf{b}_2 = \mathbf{0}, \\ F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_2' - X_0\mathbf{v}_2'' - X_1'\mathbf{b}_1 - X_2'\mathbf{b}_2 = \mathbf{0}, \\ F_{\mathbf{xx}}(\mathbf{x})\mathbf{v}_1\mathbf{v}_2' + F_{\mathbf{x}}^{\mathbf{c}^{(0)}}(\mathbf{x})\lambda_3 - X_2''\mathbf{b}_2 = \mathbf{0}, \\ G^{(2)}_{\mathbf{x}, \boldsymbol{\lambda}^{(2)}, \mathbf{b}^{(1)}}(\mathbf{x}, \boldsymbol{\lambda}^{(2)}, \mathbf{b}^{(2)})\mathbf{v}_3 = \mathbf{0}, \end{cases} \tag{3.32}$$

where

$$X_2 = \begin{pmatrix} \frac{1}{2}x^2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = (b_5), \quad X_2' = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, \quad X_2'' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_2' = (1, 1, \lambda_2)^T, \quad \mathbf{v}_2'' = (\lambda_4, \lambda_5)^T, \quad \mathbf{v}_3 = (1, \lambda_6, \lambda_7, \ldots, \lambda_{16})^T.$$

Moreover, $G^{(2)}_{\mathbf{x},\boldsymbol{\lambda}^{(2)},\mathbf{b}^{(1)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(2)}, \mathbf{0})\mathbf{v}_3 = \mathbf{0}$ has a unique solution

$$\hat{\boldsymbol{\lambda}}_3 = (-2, 0, 0, 0, -16, 0, 0, -16, 0, 0, -42).$$

Finally, the Jacobian matrix $H^{(3)}_{\mathbf{x},\boldsymbol{\lambda}^{(3)},\mathbf{b}^{(2)}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\lambda}}^{(3)}, \mathbf{0})$ becomes nonsingular, and we obtain a perturbed polynomial system

$$\begin{aligned} \widetilde{F}(\mathbf{x}, \mathbf{b}) &= F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - X_2\mathbf{b}_2 \\ &= \left\{ x^4 - b_1 - b_3x - \frac{1}{2}b_5x^2, x^2y + y^4 - b_2 - b_4y, z + z^2 - 7x^3 - 8x^2 \right\}. \end{aligned} \tag{3.33}$$

Notice that

$$F_{\mathbf{x}}(\mathbf{x})\mathbf{v}_1 - X_0'\mathbf{b}_1 - X_1'\mathbf{b}_2 = \mathbf{0} \Longleftrightarrow \widetilde{F}_{\mathbf{x}}(\mathbf{x}, \mathbf{b})\mathbf{v}_1 = \mathbf{0},$$

so after applying the verification method based on Theorem 2.5 to the above augmented system (3.32), we are able to verify that a slightly perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ for

$$|\hat{b}_i| \leq 1.0e - 14, \quad i = 1, 2, \ldots, 5,$$

has an isolated singular solution $\hat{\mathbf{x}}$ within

$$|\hat{x}_i| \leq 1.0e - 14, \ i = 1, 2, \ \text{and} \ |1 + \hat{x}_3| \leq 1.0e - 14.$$

4. An algorithm for verifying multiple roots.

Based on Theorems 3.8 and 3.9, we propose below an algorithm for computing verified error bounds such that a slightly perturbed system is guaranteed to possess an isolated singular solution within the computed bounds if the algorithm is successful.

ALGORITHM 4.1. viss.

Input: *A square polynomial system* $F \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$, *an approximate solution* $\tilde{\mathbf{x}} \in \mathbb{C}^n$, *and a tolerance* ε.

Output: *A new polynomial system in variables* $\mathbf{x}$ *and parameters* $\mathbf{b}$

$$\widetilde{F}(\mathbf{x}, \mathbf{b}) = F(\mathbf{x}) - X_0\mathbf{b}_0 - X_1\mathbf{b}_1 - \cdots - X_s\mathbf{b}_s,$$

and two verified inclusions

$$\mathbf{X} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])^T,$$

and

$$\mathbf{B} = \left([\underline{b}_1, \overline{b}_1], \ldots, [\underline{b}_{|\mathbf{b}|}, \overline{b}_{|\mathbf{b}|}]\right)^T.$$

1. *Set* $s := 0$, $m := n$, $\widetilde{F} := F$, $G := \widetilde{F}$, $\mathbf{y} := \mathbf{x}$, *and* $\tilde{\mathbf{y}} := \tilde{\mathbf{x}}$.
2. *Compute* $d := n - \operatorname{rank}(F_{\mathbf{x}}(\tilde{\mathbf{x}}), \varepsilon)$, *select integer sets* $\mathbf{c}$ *and* $\mathbf{k}$ *satisfying* (3.2) *and* (3.3), *respectively.*
3. *Set* $\widetilde{F} := \widetilde{F} - X_s\mathbf{b}_s$, *where the matrix* X_s *consists of vectors* $\frac{1}{s!} \cdot x^s_{\mathbf{c}_i} \cdot \mathbf{e}_{\mathbf{k}_i}$, $i = 1, \ldots, d$.

(a) *If* $s \geq 1$*, then set* $G := \widetilde{F}$*; for* j *from* 1 *to* s *do*
$G := \{G, G_{\mathbf{y}}\mathbf{v}_j\}$*;* $\mathbf{y} := (\mathbf{y}, \boldsymbol{\lambda}_j, \mathbf{b}_{j-1})$.
(b) *Compute* $\tilde{\mathbf{y}} := (\tilde{\mathbf{y}}, \mathrm{LeastSquares}(G_{\mathbf{y}}(\tilde{\mathbf{y}})\mathbf{v}_{s+1} = \mathbf{0}), \mathbf{0})$.
(c) *Set* $G := \{G, G_{\mathbf{y}}\mathbf{v}_{s+1}\}$*;* $\mathbf{y} := (\mathbf{y}, \boldsymbol{\lambda}_{s+1}, \mathbf{b}_s)$*;* $m := 2m$.

4. *Compute* $d := m - \mathrm{rank}(G_{\mathbf{y}}(\tilde{\mathbf{y}}), \varepsilon)$.
   (a) *If* $d = 0$*, apply* verifynlss *to* G *and* $\tilde{\mathbf{y}}$ *to compute inclusions* $\mathbf{X}$ *and* $\mathbf{B}$ *for* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{b}}$*, return* $\widetilde{F}$*,* $\mathbf{X}$*, and* $\mathbf{B}$.
   (b) *Otherwise, select* $\mathbf{c}$*,* $\mathbf{k}$ *satisfying* (3.2), (3.3) *for the polynomial system* G*, set* $s := s + 1$*,* $\mathbf{y} := \mathbf{x}$ *and go back to step* 3.

Remark 4.2. In step 4(a) of algorithm viss, we apply INTLAB function verifynlss [33] for computing an inclusion $[\mathbf{X}, \mathbf{B}]$ for a regular solution of the polynomial system G near $\tilde{\mathbf{y}}$. If verifynlss fails, then no useful information is available and the inclusions returned by algorithm viss contain only intervals $[\mathrm{NaN}, \mathrm{NaN}]$. Otherwise, a successful output of algorithm viss contains a family of perturbed systems $\widetilde{F}(\mathbf{x}, \mathbf{b})$ to the input polynomial system $F(\mathbf{x})$ and verified error bounds $\mathbf{X}$ and $\mathbf{B}$ such that there exists a unique $\hat{\mathbf{b}} \in \mathbf{B}$ and a unique $\hat{\mathbf{x}} \in \mathbf{X}$ satisfying $\widetilde{F}(\hat{\mathbf{x}}, \hat{\mathbf{b}}) = \mathbf{0}$ and $\det(\widetilde{F}_{\mathbf{x}}(\hat{\mathbf{x}}, \hat{\mathbf{b}})) = 0$.

Remark 4.3. If the output inclusion $\mathbf{B}$ contains zero, then $F(\mathbf{x}) = \widetilde{F}(\mathbf{x}, \mathbf{0}) \in \mathfrak{F} := \{\widetilde{F}(\mathbf{x}, \mathbf{b}), \mathbf{b} \in \mathbf{B}\}$. Moreover, if the input system $F(\mathbf{x})$ does have a singular zero near $\tilde{\mathbf{x}}$, then the unique isolated singular solution $\mathbf{x}^\star$ of $F(\mathbf{x})$ is guaranteed to be within the output inclusion $\mathbf{X}$.

Example 3.1 *continued.* Given an approximate singular solution

$$\tilde{\mathbf{x}} = (0.0003445, 0.0009502, 0.0003171, 0.0006948)$$

and a tolerance $\varepsilon = 0.005$, applying algorithm viss, it returns a new polynomial system in variables $\mathbf{x}$ and parameters $\mathbf{b}$,

$$\widetilde{F}(\mathbf{x}, \mathbf{b}) = \begin{cases} x_1^4 - x_2x_3x_4 - b_1 - b_5x_1 = 0, \\ x_2^4 - x_1x_3x_4 - b_2 - b_6x_2 = 0, \\ x_3^4 - x_1x_2x_4 - b_3 - b_7x_3 = 0, \\ x_4^4 - x_1x_2x_3 - b_4 - b_8x_4 = 0, \end{cases}$$

and two verified inclusions,

$$\mathbf{X} = \begin{pmatrix} [-4.7619047619047, 0.47619047619047] \\ [-4.7619047619047, 0.47619047619047] \\ [-2.3809523809523, 0.23809523809523] \\ [-2.3809523809523, 0.23809523809523] \end{pmatrix} \cdot 1.0e - 323,$$

$$\mathbf{B} = \begin{pmatrix} [-2.8571428571428, 0.28571428571428] \\ \vdots \\ [-2.8571428571428, 0.28571428571428] \end{pmatrix} \cdot 1.0e - 323.$$

We guarantee that there exists a unique system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) \in \mathfrak{F} := \{\widetilde{F}(\mathbf{x}, \mathbf{b}), \mathbf{b} \in \mathbf{B}\}$ satisfying $|\hat{b}_i| \leq 2.9e - 323, 1 \leq i \leq 8$, which has a unique isolated singular solution $\hat{\mathbf{x}}$ within $\mathbf{X}$, i.e., $|\hat{x}_i| \leq 4.8e - 323, i = 1, 2, 3, 4$.

We notice that the input polynomial system $F(\mathbf{x}) = \widetilde{F}(\mathbf{x}, \mathbf{0}) \in \mathfrak{F}$ and its true singular zero $(0, 0, 0, 0)$ lies in $\mathbf{X}$. Hence, due to the uniqueness of $\hat{\mathbf{b}}$ and $\hat{\mathbf{x}}$, we have $\hat{b}_i = 0, 1 \leq i \leq 8$, and $\hat{x}_i = 0, i = 1, 2, 3, 4$.

Special case. The breadth-one case where the corank of the Jacobian matrix equals one occurs frequently. In the following, we show how to deal with this special case more efficiently.

We have shown in [22, Theorem 3.8] that each step of deflation described by (2.5) only reduces the multiplicity μ of the singular solution $\hat{\mathbf{x}}$ by 1. According to Theorem 3.8, the number of deflations described by (3.28) will be $\mu - 1$. Hence, algorithm viss generates an augmented regular system of size $(2^{\mu-1}n) \times (2^{\mu-1}n)$. However, in [23], we introduced a more efficient method based on the parameterized multiplicity structure to obtain a deflated regular system $G(\mathbf{x}, \mathbf{b}, \boldsymbol{\lambda})$, which is of size $(\mu n) \times (\mu n)$ and can be used to verify not only the existence of an isolated singular solution but also its multiplicity structure.

Let us introduce briefly the method in [23] for the special case of breadth one. By adding $\mu - 1$ smoothing parameter $b_0, b_1, \ldots, b_{\mu-2}$ to a well-selected polynomial, assumed to be f_1, we derive a square augmented system

$$G(\mathbf{x}, \mathbf{b}, \boldsymbol{\lambda}) = \begin{cases} \widetilde{F}(\mathbf{x}, \mathbf{b}) &= \mathbf{0}, \\ L_1(\widetilde{F}) &= \mathbf{0}, \\ &\vdots \\ L_{\mu-1}(\widetilde{F}) &= \mathbf{0}, \end{cases}$$

where

$$\widetilde{F}(\mathbf{x}, \mathbf{b}) = \left\{ f_1(\mathbf{x}) - \sum_{\nu=0}^{\mu-2} \frac{b_\nu x_1^\nu}{\nu!}, f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}) \right\},$$

and $\{1, L_1, \ldots, L_{\mu-1}\}$ is a parameterized basis of the local dual space in variables $\boldsymbol{\lambda}$. Furthermore, we proved that if Theorem 2.5 is applicable to G and yields inclusions for $\hat{\mathbf{x}} \in \mathbb{R}^n$, $\hat{\mathbf{b}} \in \mathbb{R}^{\mu-1}$, and $\hat{\boldsymbol{\lambda}} \in \mathbb{R}^{(\mu-1)\times(n-1)}$ such that $G(\hat{\mathbf{x}}, \hat{\mathbf{b}}, \hat{\boldsymbol{\lambda}}) = \mathbf{0}$, then $\hat{\mathbf{x}}$ is a breadth-one singular solution of $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) = \mathbf{0}$ with multiplicity μ and $\{1, L_1, \ldots, L_{\mu-1}\}$ with $\boldsymbol{\lambda} = \hat{\boldsymbol{\lambda}}$ is a basis of $\mathcal{D}_{\hat{\mathbf{x}}}$.

Example 4.4 (see [35, Example 4.11]). Consider a polynomial system

$$F = \left\{ x_1^2 x_2 - x_1 x_2^2, x_1 - x_2^2 \right\}.$$

The system F has $(0, 0)$ as a 4-fold isolated zero.

We add the univariate polynomial $-b_1 - b_2 x_2 - \frac{b_3}{2} x_2^2$ to the first equation in F to obtain an augmented system

$$\begin{cases} x_1^2 x_2 - x_1 x_2^2 - b_1 - b_2 x_2 - \frac{b_3}{2} x_2^2 = 0, \\ x_1 - x_2^2 = 0, \\ 2\lambda_1 x_1 x_2 - \lambda_1 x_2^2 + x_1^2 - 2x_1 x_2 - b_2 - b_3 x_2 = 0, \\ \lambda_1 - 2x_2 = 0, \\ \lambda_1^2 x_2 + 2\lambda_1 x_1 - 2\lambda_1 x_2 + 2\lambda_2 x_1 x_2 - \lambda_2 x_2^2 - x_1 - \frac{b_3}{2} = 0, \\ \lambda_2 - 1 = 0, \\ \lambda_1^2 + 2\lambda_1 \lambda_2 x_2 - \lambda_1 + 2\lambda_2 x_1 - 2\lambda_2 x_2 + 2\lambda_3 x_1 x_2 - \lambda_3 x_2^2 = 0, \\ \lambda_3 = 0, \end{cases}$$

which is of size 8×8, while algorithm viss generates a system of size 16×16. Applying verifynlss with an initial approximation

$$(0.002, 0.003, -0.001, 0.0015, -0.002, 0.002, 1.001, -0.01),$$

we obtain verified inclusions

$$\mathbf{X} = \begin{pmatrix} [-0.00000000000001, 0.00000000000001] \\ [-0.00000000000001, 0.00000000000001] \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} [-0.00000000000001, 0.00000000000001] \\ [-0.00000000000001, 0.00000000000001] \\ [-0.00000000000001, 0.00000000000001] \end{pmatrix}. \qquad \square$$

This proves that there is a unique perturbed system $\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}})$ ($|\hat{b}_i| \leq 1.0e-14, i = 1, 2, 3$), which has a 4-fold breadth-one root $\hat{\mathbf{x}}$ within $|\hat{x}_i| \leq 1.0e-14, i = 1, 2$.

5. Experiments. In Table 1, we show the performance of algorithm viss. The experiments are done in Maple 15 for Digits := 14 and MATLAB R2011a with INTLAB_V6 under Windows 7. The first three examples, DZ1, DZ2, DZ3, are from [4] and the other examples are quoted from the PHCpack demos by Jan Verschelde. We denote n the number of polynomial equations and μ the multiplicity. The fourth column shows the decrease of the coranks. The last two columns show the quality of the interval vectors $\mathbf{X}$ and $\mathbf{B}$. We define radius($\mathbf{X}$) = $\max\{|\overline{x}_i - \underline{x}_i|/2, i = 1, \ldots, n\}$ and radius($\mathbf{B}$) = $\max\{|\overline{b}_j - \underline{b}_j|/2, j = 1, \ldots, |\mathbf{b}|\}$. Codes of algorithm viss and examples are available at http://www.mmrc.iss.ac.cn/~lzhi/ Research/hybrid/MMVISS.

Example 5.1 (see [4, DZ3]). Consider a polynomial system

$$f_1 = 14x + 33y - 3\sqrt{5}(x^2 + 4xy + 4y^2 + 2) + \sqrt{7} + x^3 + 6x^2y + 12xy^2 + 8y^3,$$
$$f_2 = 41x - 18y - \sqrt{5} + 8x^3 - 12x^2y + 6xy^2 - y^3 + 3\sqrt{7}(4xy - 4x^2 - y^2 - 2).$$

The system (f_1, f_2) has $(\frac{2\sqrt{7}}{5} + \frac{\sqrt{5}}{5}, -\frac{\sqrt{7}}{5} + \frac{2\sqrt{5}}{5})$ as a 4-fold isolated zero.

In the experiment, we round the irrational coefficients of f_1 and f_2 into 14 digits and denote the input system by $\{\tilde{f}_1, \tilde{f}_2\}$. Given an approximate singular solution $\tilde{\mathbf{x}} = (1.506, 0.366)$ and a tolerance $\varepsilon = 0.005$, applying algorithm viss to $\{\tilde{f}_1, \tilde{f}_2\}$, we obtain successfully a new polynomial system in variables $\mathbf{x}$ and parameters $\mathbf{b}$,

$$\widetilde{F}(\mathbf{x}, \mathbf{b}) = \begin{cases} \tilde{f}_1 = 0, \\ \tilde{f}_2 - b_1 - b_2 y - \frac{b_3}{2} y^2 = 0, \end{cases}$$

TABLE 1
Algorithm performance.

System	n	μ	corank($G_{\mathbf{y}}(\tilde{\mathbf{y}})$)	radius($\mathbf{X}$)	radius($\mathbf{B}$)
DZ1	4	131	$4 \to 4 \to 0$	4.8e-323	2.9e-323
DZ2	3	16	$2 \to 2 \to 1 \to 0$	1.0e-14	1.0e-14
DZ3	2	4	$1 \to 1 \to 1 \to 0$	3.5e-8	1.6e-8
cbms1	3	11	$3 \to 0$	2.4e-323	2.4e-323
cbms2	3	8	$3 \to 0$	1.0e-323	2.9e-323
mth191	3	4	$2 \to 0$	1.0e-14	1.0e-14
KSS	10	638	$9 \to 0$	1.0e-14	1.0e-14
Caprasse	4	4	$2 \to 0$	1.0e-14	1.0e-14
cyclic9	9	4	$2 \to 0$	4.9e-14	6.3e-14
RuGr09	2	4	$1 \to 1 \to 1 \to 0$	1.0e-14	1.0e-14
LiZhi12	100	3	$1 \to 1 \to 0$	1.0e-14	1.0e-14
Ojika1	2	3	$1 \to 1 \to 0$	1.0e-14	1.0e-14
Ojika2	3	2	$1 \to 0$	1.0e-14	1.0e-14
Ojika3	3	2	$1 \to 0$	1.5e-14	1.0e-14
Ojika4	3	3	$1 \to 1 \to 0$	1.0e-14	1.0e-14
Decker2	3	4	$1 \to 1 \to 1 \to 0$	1.0e-14	1.0e-14

and two verified inclusions,

$$\mathbf{X} = \begin{pmatrix} [1.50551422067815, 1.50551422777704] \\ [0.36527696237118, 0.36527696473749] \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} [-0.00000049931159, -0.00000046644645] \\ [0.00000000000408, 0.00000000000566] \\ [0.00000000000059, 0.00000000000108] \end{pmatrix}.$$

It is guaranteed that there exists a unique system

$$\widetilde{F}(\mathbf{x}, \hat{\mathbf{b}}) \in \mathfrak{F} := \left\{ \widetilde{F}(\mathbf{x}, \mathbf{b}), \mathbf{b} \in \mathbf{B} \right\},$$

which possesses a unique isolated singular solution $\hat{\mathbf{x}}$ within $\mathbf{X}$.

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