Gradient estimates for heat kernels and harmonic functions

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Abstract. Let (X, d, μ) be a doubling metric measure space endowed with a Dirichlet form $\mathscr{E}$ deriving from a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ supports a scale-invariant L^2-Poincaré inequality. In this article, we study the following properties of harmonic functions, heat kernels and Riesz transforms for $p \in (2, \infty]$:
(i) (G_p): L^p-estimate for the gradient of the associated heat semigroup;
(ii) (RH_p): L^p-reverse Hölder inequality for the gradients of harmonic functions;
(iii) (R_p): L^p-boundedness of the Riesz transform ($p < \infty$);
(iv) (GBE): a generalised Bakry-Émery condition.
We show that, for $p \in (2, \infty)$, (i), (ii) (iii) are equivalent, while for $p = \infty$, (i), (ii), (iv) are equivalent. Moreover, some of these equivalences still hold under weaker conditions than the L^2-Poincaré inequality.
Our result gives a characterisation of Li-Yau's gradient estimate of heat kernels for $p = \infty$, while for $p \in (2, \infty)$ it is a substantial improvement as well as a generalisation of earlier results by Auscher-Coulhon-Duong-Hofmann [7] and Auscher-Coulhon [6]. Applications to isoperimetric inequalities and Sobolev inequalities are given. Our results apply to Riemannian and sub-Riemannian manifolds as well as to non-smooth spaces, and to degenerate elliptic/parabolic equations in these settings.

Contents

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1 Introduction

1.1 Background and main results

On complete Riemannian manifolds and on more general metric measure spaces endowed with a Dirichlet form, Gaussian heat kernel upper and lower estimates have been well understood since the works of Saloff-Coste [98], Grigor'yan [55], Sturm [106, 107, 108], see also [20, 14] and references therein. Together these estimates imply the doubling volume property and the Hölder regularity of the heat kernel (see [43] for a new and direct proof of the latter fact). A fundamental and non-trivial consequence of the known characterisation of these estimates, in terms of the doubling volume property and a scale-invariant L^2-Poincaré inequality (see [98, 99, 100]), is that they are stable under quasi-isometries.

By contrast, the matching upper estimate

$$(GLY_\infty) \qquad |\nabla_x h_t(x,y)| \leq \frac{C}{\sqrt{t}V(y,\sqrt{t})} \exp\left\{-c\frac{d^2(x,y)}{t}\right\}$$

(see Theorem 1.2 below) of the gradient of the heat kernel is only known to hold in very specific cases: on manifolds with non-negative Ricci curvature [85], on Lie groups with polynomial volume growth [97], and on covering manifolds with polynomial volume growth [39, 40]. There have also been many efforts to derive upper bounds of the gradient of the heat kernel by using probabilistic methods including coupling and derivation of Bismut type formulae, but only for small time (i.e. essentially local results) unless one assumes non-negativity of the curvature; see [35, 91, 93, 105, 111] and references therein.

No handy global characterisation exists for (GLY_∞) (see however [34, Theorem 4.2] in the polynomial volume growth case). Note that no equivalent property can exhibit invariance under quasi-isometry: the example of divergence form operators with bounded measurable coefficients shows that the Lipschitz character of the heat kernel is not generic and not stable under quasi-isometry. However, non-negative curvature is too restrictive a sufficient condition, since it is very unstable under perturbations of any kind. Moreover, it is desirable to find a common reason that would explain why the property holds in the above three families of examples. Such a condition was introduced in [74, Theorem 3.2] and [76, Theorem 3.1], where it is proven that a certain quantitative Lipschitz regularity of Cheeger-harmonic functions implies an upper estimate of the gradient of the heat kernel. We shall see in Section 7 that it is relatively easy to obtain such regularity of harmonic functions in the aforementioned settings.

In the present paper, we first give a converse to this implication, and follow with an L^p-version of this equivalence which can be seen as an L^∞ one. An important motivation for the study of pointwise estimates of the gradient of the heat kernel is that they open up the way to the boundedness of Riesz transforms on L^p for all $p \in (1,+\infty)$ (see [7]). Further, it was discovered in [7] that the weaker L^p-version of these estimates governs the boundedness of Riesz transforms on L^p in an interval $(2, p_0)$, for $2 < p_0 < +\infty$. Details will be given below.

To summarise, we give characterisations of these pointwise and integrated estimates for the gradient of the heat kernel in terms of estimates for the gradients of harmonic functions. In other words, we eliminate time. This is a first step towards a geometric understanding of these estimates, and we expect this will enable one to treat new examples.

Let us now fix our setting. Let X be a locally compact, separable, metrisable, and connected space equipped with a Borel measure μ that is finite on compact sets and strictly positive on non-empty open sets. Consider a strongly local and regular Dirichlet form $\mathscr{E}$ on $L^2(X,\mu)$ with dense domain $\mathscr{D} \subset L^2(X,\mu)$ (see [51] or [59] for precise definitions). According to Beurling and Deny [16], such a form can be written as

$$\mathscr{E}(f,g) = \int_X d\Gamma(f,g)$$

for all $f, g \in \mathscr{D}$, where Γ is a measure-valued non-negative and symmetric bilinear form defined by the formula

$$\int_X \varphi \, d\Gamma(f,g) := \frac{1}{2}\left[\mathscr{E}(f,\varphi g) + \mathscr{E}(g,\varphi f) - \mathscr{E}(fg,\varphi)\right]$$

for all $f, g \in \mathscr{D} \cap L^\infty(X,\mu)$ and $\varphi \in \mathscr{D} \cap \mathscr{C}_0(X)$. Here and in what follows, $\mathscr{C}(X)$ denotes the space of continuous functions on X and $\mathscr{C}_0(X)$ the space of functions in $\mathscr{C}(X)$ with compact support. We shall assume in addition that $\mathscr{E}$ admits a "*carré du champ*", meaning that $\Gamma(f,g)$ is absolutely continuous with respect to μ, for all $f, g \in \mathscr{D}$. In what follows, for simplicity of notation, we will denote by $\langle \nabla f, \nabla g\rangle$ the energy density $\frac{d\Gamma(f,g)}{d\mu}$, and by $|\nabla f|$ the square root of $\frac{d\Gamma(f,f)}{d\mu}$.

Since $\mathscr{E}$ is strongly local, Γ is local and satisfies the Leibniz rule and the chain rule; see [51]. Therefore we can define $\mathscr{E}(f,g)$ and $\Gamma(f,g)$ locally. Denote by $\mathscr{D}_{\mathrm{loc}}$ the collection of all $f \in L^2_{\mathrm{loc}}(X)$ for which, for each relatively compact set $K \subset X$, there exists a function $h \in \mathscr{D}$ such that $f = h$ almost everywhere on K. The intrinsic (pseudo-)distance on X associated to $\mathscr{E}$ is then defined by

$$d(x,y) := \sup\{f(x) - f(y) : \ f \in \mathscr{D}_{\mathrm{loc}} \cap \mathscr{C}(X), \ |\nabla f| \le 1 \text{ a.e.}\}.$$

In this paper, we always assume that d is indeed a distance (meaning that for $x \neq y$, $0 < d(x,y) < +\infty$) and that the topology induced by d is equivalent to the original topology on X. Moreover, we assume that (X,d) is a complete metric space. Under this assumption, (X,d) is a geodesic length space; see for instance [106, 4, 59].

To summarise the above situation, we shall say that $(X,d,\mu,\mathscr{E})$ is a Dirichlet metric measure space endowed with a "*carré du champ*", in short a Dirichlet metric measure space.

The domain $\mathscr{D}$ endowed with the norm $\sqrt{\|f\|_2^2 + \mathscr{E}(f,f)}$ is a Hilbert space which we denote by $W^{1,2}(X,\mu,\mathscr{E})$, in short $W^{1,2}(X)$. For an open set $U \subset X$, the local Sobolev space $W^{1,2}_{\mathrm{loc}}(U)$ is defined to be the collection of all functions f such that for any compact set $K \subset U$ there exists $F \in \mathscr{D}$ satisfying $f = F$ a.e. on K. For each $p \ge 2$, the Sobolev space $W^{1,p}(U)$ is then defined as the collection of all functions $f \in W^{1,2}_{\mathrm{loc}}(U)$ satisfying f, $|\nabla f| \in L^p(U)$; see Appendix A.1 for the existence of $|\nabla f|$. The space $W^{1,p}_0(U)$ is defined to be the closure in $W^{1,p}(X)$ of functions in $W^{1,p}(X)$ with compact support in U. Then each Lipschitz function with compact support in U belongs to $W^{1,p}_0(U)$ for any $p \in [2,\infty]$; see Appendix A.1.

Corresponding to such a Dirichlet form $\mathscr{E}$, there exists an operator denoted by $\mathcal{L}$, acting on a dense domain $\mathscr{D}(\mathcal{L})$ in $L^2(X,\mu)$, $\mathscr{D}(\mathcal{L}) \subset W^{1,2}(X)$, such that for all $f \in \mathscr{D}(\mathcal{L})$ and each $g \in W^{1,2}(X)$,

$$\int_X f(x)\mathcal{L}g(x)\, d\mu(x) = \mathscr{E}(f,g).$$

The opposite $-\mathcal{L}$ of $\mathcal{L}$ is the infinitesimal generator of the heat semigroup $H_t = e^{-t\mathcal{L}}$, $t > 0$.

Let $B(x,r)$ denote the open ball with center x and radius r with respect to the distance d, and set $CB(x,r) := B(x,Cr)$. For simplicity we write $V(x,r) := \mu(B(x,r))$ for $x \in X$ and $r > 0$. We say that the metric measure space (X,d,μ) satisfies the volume doubling property if there exists a constant $C_D > 1$ such that for every $x \in X$ and all $r > 0$,

$$(D) \qquad V(x,2r) \le C_D V(x,r).$$

If $(X,d,\mu,\mathscr{E})$ is a Dirichlet metric measure space endowed with a "*carré du champ*" and (X,d,μ) satisfies (D), we say that $(X,d,\mu,\mathscr{E})$ is a doubling Dirichlet metric measure space endowed with a "*carré du champ*", in short a doubling Dirichlet metric measure space. It easily follows from (D) that there exist $Q>0$ and $C_Q>0$ depending only on C_D such that for every $x\in X$ and all $0<r<R$,

$$V(x,R)\le C_Q\left(\frac{R}{r}\right)^Q V(x,r). \tag{D_Q}$$

Notice that (X,d,μ) satisfies (D) if and only if it satisfies (D_Q) for some $Q>0$. Moreover, since (D_Q) implies $(D_{\tilde{Q}})$ for each $\tilde{Q}>Q$, we shall assume without loss of generality that $Q\ge 2$.

One says that the local Sobolev inequality (LS_q), $q>2$, holds on $(X,d,\mu,\mathscr{E})$ if for every ball $B=B(x,r)$ and each $f\in W_0^{1,2}(B)$,

$$\left(\fint_B |f|^q\,d\mu\right)^{2/q}\le C_{LS}\left(\fint_B |f|^2\,d\mu+\frac{r^2}{V(x,r)}\mathscr{E}(f,f)\right). \tag{LS_q}$$

Under the volume doubling property (D), it is known that (LS_q), for some $q>2$, is equivalent to the assumption that the heat semigroup $H_t=e^{-t\mathcal{L}}$ has a kernel h_t, called the heat kernel, which satisfies an upper Gaussian bound

$$h_t(x,y)\le \frac{C}{V(x,\sqrt{t})}\exp\left\{-c\frac{d^2(x,y)}{t}\right\},\ \forall\, t>0, \text{for a.e. } x,y\in X, \tag{UE}$$

see [20].

We say that $(X,d,\mu,\mathscr{E})$ supports a local L^p-Poincaré inequality, $p\in[2,\infty)$, if for all $r_0>0$ there exists $C_P(r_0)>0$ such that, for all $0<r<r_0$ and for every ball $B=B(x,r)$ and each $f\in W^{1,p}(B)$,

$$\fint_{B(x,r)}|f-f_B|\,d\mu\le C_P(r_0)r\left(\fint_{B(x,r)}|\nabla f|^p\,d\mu\right)^{1/p}. \tag{$P_{p,\mathrm{loc}}$}$$

Similarly, $(P_{\infty,\mathrm{loc}})$ requires for each $f\in W^{1,\infty}(B)$ that

$$\fint_{B(x,r)}|f-f_B|\,d\mu\le C_P(r_0)r\||\nabla f|\|_{L^\infty(B)}. \tag{$P_{\infty,\mathrm{loc}}$}$$

Further, if there exists a constant $C_P>0$ such that the above inequalities hold for every ball $B(x_0,r)$ and each $f\in W^{1,p}(B)$ with $C_P(r_0)$ replaced by C_P, then we say that (X,d,μ) supports a scale-invariant L^p-Poincaré inequality, (P_p), $p\in[2,\infty]$.

Obviously, inequalities (P_p) as well as $(P_{p,\mathrm{loc}})$ are weaker and weaker as p increases. Since (X,d) is geodesic, our Poincaré inequalities (P_p) and $(P_{p,\mathrm{loc}})$ have self-improving properties for $2\le p<\infty$ by [78], see Appendix A.3 for the precise statement in our setting. This fails, in general, for (P_∞) and $(P_{\infty,\mathrm{loc}})$, see [41, 42]. We also note that (P_2) together with (D) implies (LS_q) for some $q\in(2,\infty]$ but the converse is not true; see [20, 61].

By Sturm [106, 107, 108] (see Saloff-Coste [99, 98] and Grigor'yan [55] for earlier results on Riemannian manifolds), on a metric measure space (X, d, μ) endowed with a strongly local and regular Dirichlet form $\mathscr{E}$, (D) together with (P_2) are equivalent to the requirement that the heat semigroup $H_t = e^{-t\mathcal{L}}$ has a heat kernel h_t that satisfies the Li-Yau estimate

$$(LY) \qquad \frac{C^{-1}}{V(x,\sqrt{t})}\exp\left\{-\frac{d^2(x,y)}{ct}\right\} \le h_t(x,y) \le \frac{C}{V(x,\sqrt{t})}\exp\left\{-c\frac{d^2(x,y)}{t}\right\},$$

for all $t > 0$, a.e. $x, y \in X$. This estimate was originally obtained in [85] on Riemannian manifolds with non-negative Ricci curvature. Moreover, (UE) is equivalent to a parabolic Harnack inequality for solutions to the heat equation. The parabolic Harnack inequality obviously implies an elliptic Harnack inequality, which had been obtained earlier under doubling and Poincaré by Biroli and Mosco [17, 18]. Furthermore, Hebisch and Saloff-Coste [63] showed that an elliptic Harnack inequality also implies a parabolic one if one has (D) and (UE) (see also [14]).

A consequence of the elliptic Harnack inequality is that harmonic functions are Hölder in space, and a consequence of the parabolic Harnack inequality is that the heat kernel is Hölder in time and space. It follows from the above that this is the case if (D) and (P_2) hold.

However, in general, (D) and (P_2) are not sufficient for Lipschitz regularity of harmonic functions or heat kernels. This phenomenon already occurs in the case of uniformly elliptic operators of divergence form with non-smooth coefficients in Euclidean space, see for instance [21, 102]. Even in a smooth setting, additional assumptions are required in order to ensure proper pointwise estimates for gradients of harmonic functions or heat kernels.

Yau's gradient estimate for positive harmonic functions (cf. Yau [114], Cheng-Yau [30]) states that on non-compact Riemannian manifolds with Ricci curvature bounded below by $-K$, $K \ge 0$, it holds that

$$(Y_\infty) \qquad \sup_{x\in B(x_0,r)} |\nabla \log u(x)| \le C\left(\frac{1}{r} + \sqrt{K}\right),$$

for every ball $B(x_0, r)$ and every positive harmonic function u on $B(x_0, 2r)$. Li-Yau's gradient estimate for heat kernels (c.f. Li and Yau [85]) on Riemannian manifolds with non-negative Ricci curvature states that

$$(GLY_\infty) \qquad |\nabla_x h_t(x,y)| \le \frac{C}{\sqrt{t}V(y,\sqrt{t})}\exp\left\{-c\frac{d^2(x,y)}{t}\right\}, \forall\, t > 0,\ x, y \in X.$$

These two gradient estimates are fundamental tools in geometric analysis and related fields, and there have been many efforts afterwards to generalise them to different settings, see for instance [34, 38, 39, 40, 44, 52, 68, 74, 83, 86, 92, 93, 97, 116, 117, 118].

Let us review some of these generalisations. Saloff-Coste [97] obtained (GLY_∞) on Lie groups with polynomial growth. Dungey [39, 40] obtained (GLY_∞) on Riemannian covering manifolds with polynomial growth. On Heisenberg type groups, Driver and Melcher [38] and Hu and Li [68] obtained a Bakry-Émery type inequality, which implies (GLY_∞). Zhang [116] obtained Yau's gradient estimate $(K = 0)$ on Riemannian manifolds of non-negative Ricci curvature modulo a small perturbation. In recent years, in a series of works [13, 52, 73, 74, 117, 118], Yau's gradient estimate for harmonic functions and Li-Yau's gradient estimate for heat kernels (and their local versions) have been further generalised to metric measure spaces and graphs satisfying suitable curvature assumptions; we refer the reader to [3, 4, 25, 26, 27, 46, 67, 87, 109, 110] for recent developments of lower Ricci curvature bounds and related calculus on metric measure spaces. Our aim in the present paper is to characterise heat kernel gradient bounds without making any curvature assumptions. One can summarise our results by saying that we reduce (GLY_∞) to a condition that is easily seen to be equivalent to (Y_∞) with $K = 0$.

The conjunction of [7, Theorem 1.4] and [34, Corollary 2.2] shows that (D) and (GLY_∞) yield the boundedness of the Riesz transform on L^p:

$$(R_p) \qquad \||\nabla \mathcal{L}^{-1/2} f|\|_p \le C\|f\|_p, \ \forall f \in L^p(X,\mu)$$

for all $p \in (1, +\infty)$. On the other hand, under (D) and (UE), (GLY_∞) is known to be equivalent to the boundedness of the gradient of the heat semigroup:

$$(G_\infty) \qquad \||\nabla H_t|\|_{\infty\to\infty} \le \frac{C}{\sqrt{t}}$$

(see [7, p.919] and [33, Theorem 4.11]). However, there are examples such as conical manifolds (cf. [82]) and uniformly elliptic operators (cf. [102] and [21]) where (R_p) only holds for p in a finite interval $(1, p_0)$, $2 < p_0 < \infty$. It was discovered in [7] that a natural substitute for (GLY_∞) or (G_∞) is the L^{p_0}-boundedness of the gradient of the heat semigroup together with the estimate

$$(G_{p_0}) \qquad \||\nabla H_t|\|_{p_0\to p_0} \le \frac{C}{\sqrt{t}},$$

$2 < p_0 < \infty$, which by [7] implies (R_p) for all $1 < p < p_0$ under (D) and (P_2). Above and in what follows, $\|\cdot\|_{p\to p}$ denotes the (sublinear or linear) operator norm from $L^p(X,\mu)$ to $L^p(X,\mu)$ for $p \in [1,\infty]$. Note conversely that (R_p) easily implies (G_p) for any $p \in (1,\infty)$. Note also that (G_p) is equivalent to the validity of the estimate $\||\nabla f|\|_p^2 \lesssim \|f\|_p\|\mathcal{L}f\|_p$ for all $f \in \mathscr{D}(\mathcal{L})$ with f, $\mathcal{L}f \in L^p(X,\mu)$ (see [34, Prop. 3.6]).

Observe that (G_2) always holds. Indeed, it follows from spectral theory that for each $f \in L^2(X,\mu)$,

$$\|\mathcal{L}H_t f\|_2 \le \frac{C}{t}\|f\|_2,$$

and hence

$$\||\nabla H_t f|\|_2^2 = \langle H_t f, \mathcal{L}H_t f\rangle \le \frac{C}{t}\|f\|_2^2,$$

i.e. (G_2) holds (here $< \cdot, \cdot >$ denotes the bracket in L^2).

By interpolation with (G_2), (G_∞) implies (G_p) for all $p \in (2,\infty)$. Finally, if $(X, d, \mu, \mathscr{E})$ satisfies (D) and (UE), in particular if it satisfies (D) and (P_2), then it follows from the above results that (GLY_∞) implies (G_p) for all $p \in (2,\infty)$.

Our main results below give a characterisation of (G_p) for each $2 < p \le \infty$ in terms of estimates for gradients of harmonic functions. For $p = \infty$, this can be seen as a gradient version of the equivalence between elliptic and parabolic Harnack inequalities under (D) and (P_2), cf. [63, 14]. Before we state these results, let us recall some terminology.

Let $\Omega \subseteq X$ be a domain. For $g \in L^2(\Omega)$, a Sobolev function $f \in W^{1,2}(\Omega)$ is called a solution to $\mathcal{L} f = g$ in Ω if

$$\int_\Omega \langle \nabla f, \nabla \varphi \rangle \, d\mu = \int_\Omega g(x)\varphi(x)\, d\mu(x), \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{1.1}$$

If $\mathcal{L} u = 0$ in Ω, then we say that u is harmonic in Ω.

Definition 1.1. *Let $(X, d, \mu, \mathscr{E})$ be a Dirichlet metric measure space and let $p \in (2, \infty)$. We say that the quantitative reverse L^p-Hölder inequality for gradients of harmonic functions holds if there exists $C > 0$ such that, for every ball B with radius r and every function u that is harmonic in $2B$,*

$$\left(\fint_B |\nabla u|^p \, d\mu \right)^{1/p} \le \frac{C}{r} \fint_{2B} |u| \, d\mu. \tag{RH_p}$$

Analogously, (RH_∞) requires that

$$\| |\nabla u| \|_{L^\infty(B)} \le \frac{C}{r} \fint_{2B} |u| \, d\mu.$$

Note that (RH_p) implies (RH_q) for $q < p$. In [74, 76], (RH_∞) was used to prove isoperimetric inequalities and gradient upper estimates for heat kernels. We shall see in Lemma 2.3 below that, under (D) and (P_2), (RH_∞) is equivalent to Yau's gradient estimate (Y_∞) with $K = 0$. See [30, 73, 114, 117] for more about (Y_∞) and (RH_∞). Actually, a more natural formulation for the reverse L^p-Hölder inequality for gradients of harmonic functions is

$$\left(\fint_B |\nabla u|^p \, d\mu \right)^{1/p} \le C \left(\fint_{2B} |\nabla u|^2 \, d\mu \right)^{1/2}, \tag{$\widetilde{RH}_p$}$$

if u is harmonic on $2B$; see [6, 102]. In general, $(\widetilde{RH}_p)$ is stronger than (RH_p). Indeed, as soon as (D) and (UE) hold, the Caccioppoli inequality (Lemma 2.4 below) together with Proposition 2.1 and a simple covering argument gives the implication $(\widetilde{RH}_p) \Longrightarrow (RH_p)$. However, (RH_p) is equivalent to $(\widetilde{RH}_p)$, if in addition one has (P_2).

We shall see in Example 4 from Section 7.1 that there exist Riemannian manifolds where (RH_p) and (G_p) hold for some $p > 2$, but $(\widetilde{RH}_p)$ does not hold. This is why we have to characterise (G_p) in terms of (RH_p) instead of $(\widetilde{RH}_p)$ in Theorem 1.6 below.

Our first main result gives a characterisation of pointwise estimates for the gradient of the heat kernel.

Theorem 1.2. *Let $(X, d, \mu, \mathscr{E})$ be a non-compact doubling Dirichlet metric measure space endowed with a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ satisfies (UE) and $(P_{\infty,\mathrm{loc}})$. Then the following statements are equivalent:*

(i) (RH_∞) holds.

(ii) There exist $C, c > 0$ such that

$$(GLY_\infty) \qquad |\nabla_x h_t(x,y)| \le \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp\left\{-c\frac{d^2(x,y)}{t}\right\}$$

for all $t > 0$ and a.e. x, $y \in X$.

(iii) The gradient of the heat semigroup $|\nabla H_t|$ is bounded on $L^\infty(X)$ for each $t > 0$ with

$$(G_\infty) \qquad |||\nabla H_t|||_{\infty\to\infty} \le \frac{C}{\sqrt{t}}.$$

(iv) There exist $C, c > 0$ such that

$$(GBE) \qquad |\nabla H_t f(x)|^2 \le C H_{ct}(|\nabla f|^2)(x)$$

for every $f \in W^{1,2}(X)$, all $t > 0$ and a.e. $x \in X$.

The main novelty here is the implication $(GLY_\infty) \implies (RH_\infty)$. Indeed, $(RH_\infty) \implies (GLY_\infty)$ follows from ideas in the proof of [74, Theorem 3.2]. Moreover it is easy to see that (G_∞) is equivalent to

$$(1.2) \qquad \sup_{t>0,\ x\in M} \sqrt{t}\int_M |\nabla_x h_t(x,y)|\, d\mu(y) < +\infty;$$

therefore $(GLY_\infty) \implies (G_\infty)$ follows by integration using (D) (see [7, p.919]). Then the reasoning in [7, p.919] and [33, Theorem 4.11] yields the converse implication. The equivalence $(GLY_\infty) \iff (GBE)$ follows by a version of [7, Lemma 3.3] and [14, Theorem 3.4]. In the sequel, we shall call condition (iv) a generalised Bakry-Émery condition (GBE).

Theorem 1.2 admits a direct corollary.

Corollary 1.3. *Let $(X, d, \mu, \mathscr{E})$ be a non-compact doubling Dirichlet metric measure space endowed with a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ satisfies (P_2). Then the conditions (RH_∞), (GLY_∞), (G_∞) and (GBE) are mutually equivalent.*

In sufficiently smooth settings, the assumption $(P_{\infty,\mathrm{loc}})$ is automatically satisfied and we obtain the following.

Corollary 1.4. *Let (M, g) be a non-compact Riemannian manifold. Assume that the Dirichlet metric measure space associated to the Laplace-Beltrami operator satisfies (D) and (UE). Then the conditions (RH_∞), (GLY_∞), (G_∞) and (GBE) are mutually equivalent.*

Remark 1.5. Note that in Theorem 1.2 and Corollary 1.4, we did not require (P_2) or (LY). However, they follow as a consequence of (UE) together with (GLY_∞) or (RH_∞); cf. [34, 14].

Note that, when $C = c = 1$, (GBE) is the classical Bakry-Émery condition

$$(BE) \qquad |\nabla H_t f(x)|^2 \le H_t(|\nabla f|^2)(x),$$

which, on manifolds, is known to be equivalent to non-negativity of Ricci curvature; see [10] and also [8, 9, 113]. This equivalence has been further generalised to metric measure spaces with non-negative Ricci curvature ($RCD^*(0, N)$ spaces) in [4, 5, 46].

On Lie groups of polynomial growth, Saloff-Coste [97] obtained (GLY_∞) for the heat kernels; see also [1]. More generally, on sub-Riemannian manifolds satisfying Baudoin-Garofalo's curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \ge 0$, $\rho_2 > 0$, $\kappa \ge 0$ and $d \ge 2$, it is known that the gradient of the heat kernel satisfies the pointwise inequality (GLY_∞) (cf. [11, Theorem 4.2]). Therefore, by Theorem 1.2, we see that (RH_∞), (GBE) and (G_∞) hold on the aforementioned spaces; see Section 7 for more examples.

As far as L^p-estimates for the gradient of the heat kernel are concerned, we have the following characterisation.

Theorem 1.6. *Let $(X, d, \mu, \mathscr{E})$ be a non-compact doubling Dirichlet metric measure space endowed with a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ satisfies (UE) and $(P_{2,\mathrm{loc}})$. Let $p \in (2, \infty)$. Then the following statements are equivalent:*

(i) (RH_p) holds.

(ii) There exists $\gamma > 0$ such that

$$(GLY_p) \qquad \int_X |\nabla_x h_t(x,y)|^p \exp\left\{\gamma d^2(x,y)/t\right\} d\mu(x) \le \frac{C}{t^{p/2} V(y, \sqrt{t})^{p-1}}$$

for all $t > 0$ and a.e. $y \in X$.

(iii) The gradient of the heat semigroup, $|\nabla H_t|$, is bounded on $L^p(X, \mu)$ for each $t > 0$ with

$$(G_p) \qquad \||\nabla H_t|\|_{p \to p} \le \frac{C}{\sqrt{t}}.$$

Note that in Theorem 1.6 it is enough to assume $(P_{2,\mathrm{loc}})$ instead of the much stronger global condition (P_2): a Riemannian manifold that is the union of a compact part and a finite number of Euclidean ends is a typical example satisfying (UE), $(P_{2,\mathrm{loc}})$, but *not* (P_2); see [23, 32]. On the other hand, since (P_2) implies $(P_{2,\mathrm{loc}})$ and (UE), we have the following corollary.

Corollary 1.7. *Let $(X, d, \mu, \mathscr{E})$ be a non-compact doubling Dirichlet metric measure space endowed with a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ satisfies (P_2). Let $p \in (2, \infty)$. Then the conditions (RH_p), (GLY_p), and (G_p) are mutually equivalent.*

Remark 1.8. (i) Note that for $p = 2$ all the conditions (i), (ii), (iii) in Theorem 1.6 hold. This is obvious for (i) and we already observed that this is also the case for (iii). Finally, (ii) follows from [56], also see [32, Lemma 2.3].

(ii) Also note that the limit case $p = \infty$ of Theorem 1.6 is nothing but Theorem 1.2.

(iii) Theorem 1.6 actually holds with $(P_{2,\rm loc})$ replaced by the weaker condition $(P_{p,\rm loc})$. However, by [14, Theorem 6.3] together with [15, Corollary 3.8], one has that, (UE) and $(P_{p,\rm loc})$ together with (RH_p) or (G_p) imply $(P_{2,\rm loc})$.

(iv) Finally, note that under (D) and (P_2), there always exists $\varepsilon > 0$ such that (RH_p), hence (GLY_p) and (G_p), hold for $2 < p < 2 + \varepsilon$; see [6, Section 2.1] and Lemma 5.2 below.

To the best of our knowledge, Theorem 1.2 and Theorem 1.6 are new even on Riemannian manifolds. Since our assumptions are quite mild, our setting includes Riemannian metric measure spaces, sub-Riemannian manifolds, and degenerate elliptic/parabolic equations in these settings; see the final section.

Regarding the proofs, the main difficulties and novelties appear in the proof of "$(RH_p) \Longrightarrow (GLY_p)$" for $p \in (2,\infty)$, and in "$(G_p) \Longrightarrow (RH_p)$" for $p \in (2,\infty]$.

A version of the implication $(RH_\infty) \Longrightarrow (GLY_\infty)$ was proven in [74, Theorem 3.2] via quantitative regularity estimates for solutions to the Poisson equation in [76, Theorem 3.1], under (D) and (P_2). In the present work, we replace the assumptions (D) and (P_2) there by the slightly weaker combination (D), (UE) and $(P_{\infty,\rm loc})$. To prove $(RH_p) \Longrightarrow (GLY_p)$ for $p \in (2,\infty)$, we follow some ideas from [74, 76]. In particular, starting from (RH_p), we first establish a quantitative regularity estimate for solutions to the Poisson equation; see Theorem 3.6 below. As we already said, harmonic functions are *not* necessarily locally Lipschitz in a non-smooth setting. Therefore, to establish Theorem 3.6, we can neither assume nor use any Lipschitz regularity of harmonic functions. In the classical setting, the fact that quantitative regularity for harmonic functions implies quantitative regularity for solutions to the Poisson equation is easy to prove and there is even an analog for certain non-linear equations, see [81].

To overcome the difficulties attached to the non-smooth setting, we use the pointwise approach to Sobolev spaces on metric measure spaces by Hajłasz [60]; see [64, 101] and Section 2.1 below for more details. Then by using (RH_p) in the full strength, a stopping-time argument and a bootstrap argument, we obtain pointwise control on Hajłasz gradients of solutions to the Poisson equation in terms of potentials; see (3.6) below. We expect that such estimates are of independent interest.

Then, by viewing the heat kernel h_t as a solution to the Poisson equation $\mathcal{L}h_t = -\frac{\partial h_t}{\partial t}$, where a suitable estimate for $\frac{\partial h_t}{\partial t}$ can be obtained from h_t by using Cauchy transforms (cf. Sturm [107, Theorem 2.6]), we obtain (GLY_p).

To prove $(G_p) \Longrightarrow (RH_p)$ for $p \in (2,\infty]$, we first establish a reproducing formula for harmonic functions by using the finite propagation speed property; see Lemma 4.6 below. Then, by using this reproducing formula, we follow recent developments on the boundedness of spectral multipliers from [14, 20] to show that $(G_p) \Longrightarrow (RH_p)$ for all $p \in (2,\infty]$.

1.2 Applications to Riesz transforms

Let us apply the previous results to L^p-boundedness of the Riesz transform $|\nabla\mathcal{L}^{-1/2}|$. We say that (R_p) holds if this operator is continuous from $L^p(X,\mu)$ to itself. One easily checks that (R_2) follows from the definitions and spectral theory.

For $p \in (1, 2)$, it was proved by Coulhon and Duong in [32] that (R_p) holds as soon as (D) and (UE) hold (however, this condition is not necessary, see [28]). In particular, (D) and (P_2) are sufficient conditions for (R_p) to be valid in this range.

For $p > 2$, Auscher, Coulhon, Duong and Hofmann established in [7] a characterisation of the boundedness of the Riesz transform on manifolds via boundedness of the gradient of the heat semigroup. Although the characterisation in [7] was stated on manifolds, its proof indeed works on metric measure spaces, as indicated in [14, p.6]. For further information we refer to [6, 15] and references therein.

Using [7, Theorem 1.3], Theorem 1.6 above, and the open-ended character of condition (RH_p) (Lemma 5.2 below), we obtain the following result.

Theorem 1.9. *Let $(X, d, \mu, \mathscr{E})$ be a non-compact doubling Dirichlet metric measure space endowed with a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ satisfies (UE). Let $p \in (2, \infty)$. If (P_p) holds, then (RH_p), (G_p) and (R_p) are equivalent.*

Let us compare Theorems 1.6 and 1.9 with [6, Theorem 2.1]. The latter result states that on a Riemannian manifold M satisfying (D) and (P_2), there exists $p_M \in (2, \infty]$ such that for all $p_0 \in (2, p_M)$, (RH_p) for all $p \in (2, p_0)$ is equivalent to the validity of (R_p) for all $p \in (2, p_0)$. Now (R_p) easily implies (G_p) and conversely, according to [7, Theorem 2.1], under the same assumptions the validity of (G_p) for all $p \in (2, p_0)$ implies the validity of (R_p) for all $p \in (2, p_0)$. Theorems 1.6 and 1.9 contain three improvements with respect to [6, Theorem 2.1]. First, the proof of [6, Theorem 2.1] makes an essential use of 1-forms on manifolds, and we do not know how to extend the arguments from [6] to our general setting. Second, Theorems 1.6 and 1.9 state a point-to-point equivalence among (RH_p), (G_p) and (R_p), as opposed to a mere equivalence between (RH_p) for $p \in (2, p_0)$ and (G_p) for $p \in (2, p_0)$. Finally, we obtain that $p_M = +\infty$.

According to Gehring's Lemma (cf. [53, 70, 115]), our reverse Hölder inequality (RH_p) is an open-ended condition; see Lemma 5.2 below. We then have the following corollary to Theorem 1.9, which generalises the main result of [6] and a recent result [15, Theorem 1.2].

Corollary 1.10. *Let $(X, d, \mu, \mathscr{E})$ be a non-compact doubling Dirichlet metric measure space endowed with a "carré du champ". Assume that $(X, d, \mu, \mathscr{E})$ satisfies (UE).*

(i) If (P_2) holds, then the set of p's such that (R_p) holds is an interval $(1, p_0)$, with $p_0 \in (2, \infty]$.

(ii) Let $p \in (2, \infty)$. If (P_p), and one of the mutually equivalent conditions (RH_p), (G_p), (R_p), hold, then there exists $\varepsilon > 0$ such that all the mutually equivalent conditions $(RH_{p+\varepsilon})$, $(G_{p+\varepsilon})$, $(R_{p+\varepsilon})$ hold.

Remark 1.11. Even though we only assume (P_p) in Theorem 1.9 and (ii) of Corollary 1.10, recent results from [14, Theorem 6.3] and [15, Corollary 3.8] show that (P_p) together with (RH_p) or (G_p) implies (P_2).

1.3 Sobolev inequalities and isoperimetric inequality

Let $1 \le p \le q \le +\infty$. In the above setting, we say that the Sobolev inequality $(S_{p,q})$ holds if for every ball B, $B = B(x, r)$ and every Lipschitz function f, compactly supported in B, there exists C

such that

$$(S_{q,p}) \qquad \left(\fint_B |f|^q\,d\mu\right)^{1/q} \le Cr\left(\fint_B |\nabla f|^p\,d\mu\right)^{1/p}.$$

Applying the methods from [75, 76], we show in Theorem 6.1 below that on a non-compact metric measure space $(X, d, \mu, \mathscr{E})$ endowed with a "*carré du champ*" and satisfying (D_Q) and (UE), if additionally for some $p_0 \in (2, \infty)$, $(P_{p_0,\mathrm{loc}})$ and one of the conditions (RH_{p_0}), (R_{p_0}), (G_{p_0}) holds, then the Sobolev inequality $(S_{q,p_0'})$, where $p_0' < 2$ is the Hölder conjugate of p_0, $q \ge p_0'$ satisfying $1/p_0' - 1/q < 1/Q$, is valid. An analogue for the isoperimetric inequality ($p_0 = \infty$) will also be established in Theorem 6.3.

1.4 Plan of the paper

The paper is organized as follows. In Section 2, we recall and provide some basic notions and tools, which include Sobolev spaces, harmonic functions, Poisson equations and some functional calculi.

In Section 3, we provide a quantitative gradient estimate for solutions to Poisson equations, assuming (RH_p).

In Section 4, we give the proofs of Theorems 1.2 and 1.6, and their corollaries.

In Section 5, we prove Theorem 1.9, and in Section 6, we study Sobolev inequalities and the isoperimetric inequality.

In Section 7, we exhibit several examples that our results can be applied to.

In Appendix A, we provide additional details for the techniques that are used in the proofs.

Throughout the work, we always assume that our space $(X, d, \mu, \mathscr{E})$ is a non-compact doubling Dirichlet metric measure space. However, we wish to point out that our results and techniques allow a localisation for local or compact settings. In order to keep the length of this paper reasonable, we will present the localization in a forthcoming paper.

We denote by C, c positive constants which are independent of the main parameters, but which may vary from line to line. We use $\sim$ to mean that two quantities are comparable.

2 Preliminaries and auxiliary tools

2.1 Harmonic functions and Poisson equations

In this subsection, we recall some basic properties of harmonic functions and of solutions to the Poisson equation. Most of these properties have been deduced via de Giorgi-Moser-Nash theory, requiring only doubling property and Sobolev inequality.

Before we start our discussion, let us recall the notion of the reverse doubling, which for Riemannian manifolds originates in [55, Theorem 1.1]. Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D). If in addition X is connected then it is known that the so-called reverse doubling estimate is valid, see e.g. [57, Proposition 5.2]. The reverse doubling estimate ensures that, as (X, d) is non-compact, there exist $0 < Q' < Q$ and $c > 0$ such that, for all $r \geq s > 0$ and $x, y \in M$ such that $d(x, y) < r + s$,

$$c\left(\frac{r}{s}\right)^{Q'} \leq \frac{V(y,r)}{V(x,s)}, \tag{RD}$$

Notice that (UE) is equivalent to the local Sobolev inequality (LS_q), for any $q \in (2, \infty]$ satisfying $\frac{q-2}{q} < \frac{2}{Q}$, see [20, Theorem 1.2.1]. It follows from [20, Section 3.4] that under (RD) the local Sobolev estimate (LS_q) can be strengthened to the Sobolev inequality $(S_{q,2})$.

We continue with the Harnack inequality; see for instance [17, 18, 72].

Proposition 2.1. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (UE). Then there exists C only depending on C_D and C_S such that if $\mathcal{L}u = 0$ in $B(x_0, r)$, then*

$$\|u\|_{L^\infty(B(x_0,r/2))} \leq C \fint_{B(x_0,r)} |u| \, d\mu.$$

Proposition 2.2. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies $(P_{2,\mathrm{loc}})$. For each $r_0 > 0$, there exists $C = C(C_D, C_P(r_0))$ such that if u is a positive harmonic function on $B(x_0, r)$, $r < r_0$, then*

$$\sup_{y \in B(x_0,r/2)} u(y) \leq C \inf_{y \in B(x_0,r/2)} u(y).$$

Further if (P_2) holds, then the above constant C may be chosen independent of r_0.

Using the Harnack inequality, we obtain the following relation between Yau's gradient estimate and our condition (RH_∞). Since Lipschitz regularity of harmonic functions is the best one can hope for in non-smooth settings (cf. [72, 117]), we have to use essential supremum instead of pointwise supremum in (Y_∞).

Lemma 2.3. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (P_2). Then (RH_∞) holds if and only if (Y_∞) holds with $K = 0$.*

Proof. $(RH_\infty) \Longrightarrow (Y_\infty)$ with $K = 0$: Suppose that u is positive harmonic function on $2B$, $B = B(x_0, r)$. By Propositions 2.1, 2.2 and a simple covering argument, we see that

$$|\nabla u(x)| \leq \||\nabla u|\|_{L^\infty(B)} \leq \frac{C}{r} \fint_{\frac{3}{2}B} |u| \, d\mu \leq \frac{C}{r} \sup_{y \in \frac{3}{2}B} u(y) \leq \frac{C}{r} \inf_{y \in \frac{3}{2}B} u(y) \leq \frac{C}{r} u(x)$$

for a.e. $x \in B$, i.e., (Y_∞) holds with $K = 0$.

(Y_∞) with $K = 0 \Longrightarrow (RH_\infty)$: Suppose that u is a harmonic function in $2B$. Let $\delta = \|u\|_{L^\infty(\frac{3}{2}B)}$, then the strong maximum principle (cf. [18]) implies that either $u + \delta \equiv 0$ in $\frac{3}{2}B$ or $u + \delta > 0$ there. In the first case, (RH_∞) holds obviously since $|\nabla u| \equiv 0$ in B. In the second case, by Proposition 2.1 and the same covering argument, we obtain

$$\delta = \|u\|_{L^\infty(\frac{3}{2}B)} \leq C \fint_{2B} |u| \, d\mu,$$

and hence by (Y_∞) with $K = 0$ and Proposition 2.2,

$$|\nabla(u(x)+\delta)| \le \frac{C}{r}[u(x)+\delta] \le \frac{C}{r}\inf_{y\in B}[u(y)+\delta] \le \frac{C}{r}\fint_{2B}|u+\delta|\,d\mu \le \frac{C}{r}\fint_{2B}|u|\,d\mu$$

for a.e. $x \in B$. That is, (RH_∞) holds, which completes the proof. □

In what follows we will need the following Caccioppoli inequality; see [18, 72].

Lemma 2.4. *Let $(X, d, \mu, \mathscr{E})$ be a doubling Dirichlet metric measure space. Then if $\mathcal{L}f = g$ in $B := B(x_0, R)$, $g \in L^2(B)$, we have that for any $0 < r < R$*

$$\int_{B(x_0,r)} |\nabla f|^2\,d\mu \le \frac{C}{(R-r)^2}\int_{B(x_0,R)}|f|^2\,d\mu + C(R-r)^2\int_{B(x_0,R)}|g|^2\,d\mu,$$

where C only depends on C_D.

The following result was proved in [18] by using Sobolev inequalities.

Lemma 2.5. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_Q), $Q \ge 2$, and that (UE) holds. Let $p \in (\max\{Q/2, 2\}, \infty]$. Then for each $g \in L^p(B(x_0, r))$, there is a unique solution $f \in W_0^{1,2}(B(x_0, r))$ to $\mathcal{L}f = g$ in $B(x_0, r)$. Moreover*

$$\|f\|_{L^\infty(B(x_0,r))} \le Cr^2 V(x_0, r)^{-1/p}\|g\|_{L^p(B(x_0,r))},$$

where $C = C(C_D, C_S)$.

Proof. See [18, Theorem 4.1] for the existence and the given estimate; the uniqueness follows since the difference of any two solutions is harmonic, with boundary value zero in the Sobolev sense. □

Lemma 2.6. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_Q), $Q \ge 2$, and that (UE) holds. Let $q \in \left(\frac{2Q}{Q+2}, \infty\right]$. For each $B = B(x_0, r)$ and $g \in L^q(B)$, there exists $f \in W_0^{1,2}(B)$ that satisfies $\mathcal{L}f = g$ in B. Moreover, there exists a constant C such that*

$$\fint_B |f|\,d\mu \le Cr\left(\fint_B |\nabla f|^2\,d\mu\right)^{1/2} \le Cr^2\left(\fint_B |g|^q\,d\mu\right)^{1/q}.$$

Proof. Let us first prove the existence of f. Let q' be the Hölder conjugate of q. Notice that $\frac{q'-2}{q'} < \frac{2}{Q}$ and (UE) implies that $(S_{q',2})$ holds. For each $k \in \mathbb{N}$, let $g_k := \chi_{\{|g|\le k\}}\, g$. By Lemma 2.5, there exists a solution $f_k \in W_0^{1,2}(B)$ to $\mathcal{L}f_k = g_k$ in B. For all $k, j \in \mathbb{N}$, $(S_{q',2})$ yields

$$\begin{aligned}\int_B |\nabla(f_k - f_j)|^2\,d\mu &= \int_B [g_k - g_j][f_k - f_j]\,d\mu\\ &\le \|g_k - g_j\|_{L^q(B)}\|f_k - f_j\|_{L^{q'}(B)}\end{aligned}$$

$$\leq C\|g_k - g_j\|_{L^q(B)} \frac{r}{\mu(B)^{1/2-1/q'}} \||\nabla(f_k - f_j)|\|_{L^2(B)},$$

and similarly

$$\int_B |\nabla f_k|^2 \, d\mu \leq C\|g_k\|_{L^q(B)} \frac{r}{\mu(B)^{1/2-1/q'}} \||\nabla f_k|\|_{L^2(B)}. \tag{2.1}$$

Therefore, $(f_k)_k$ is a Cauchy sequence in $W_0^{1,2}(B)$, and there exists a limit $f \in W_0^{1,2}(B)$. By this and $(S_{q',2})$, we see that for each $\varphi \in W_0^{1,2}(B)$,

$$\int_B \langle \nabla f, \nabla \varphi \rangle \, d\mu = \lim_{k\to\infty} \int_B \langle \nabla f_k, \nabla \varphi \rangle \, d\mu = \lim_{k\to\infty} \int_B g_k \varphi \, d\mu = \int_B g\varphi \, d\mu,$$

where the last equality follows from the convergence $g_k \to g$ in $L^q(B)$ together with $\varphi \in W_0^{1,2}(B) \subset L^{q'}(B)$. This implies that f is a solution to $\mathcal{L}f = g$ in B.

Notice that by (2.1),

$$\fint_B |f_k|^2 \, d\mu \leq Cr^2 \fint_B |\nabla f_k|^2 \, d\mu \leq C\|g_k\|^2_{L^q(B)} \frac{r^4}{\mu(B)^{2/q}} \leq C\|g\|^2_{L^q(B)} \frac{r^4}{\mu(B)^{2/q}}.$$

By this, letting $k \to \infty$, we conclude that

$$\fint_B |f| \, d\mu \leq Cr \left(\fint_B |\nabla f|^2 \, d\mu \right)^{1/2} \leq Cr^2 \left(\fint_B |g|^q \, d\mu \right)^{1/q},$$

as desired. □

In our discussion we will also need the following result, see [18, Theorem 5.13].

Lemma 2.7. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_Q), $Q \geq 2$, and $(P_{2,\mathrm{loc}})$. Suppose that $f \in W^{1,2}(B)$, $B = B(x_0, r)$, $g \in L^p(B)$ and $\mathcal{L}f = g$ in B, where $p \in (\frac{Q}{2}, \infty] \cap (2, \infty]$. Then f is locally Hölder continuous on B.*

2.2 Functional calculus

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Let L be a non-negative, self-adjoint operator on $L^2(X, \mu)$, and let us denote its spectral decomposition by $E_L(\lambda)$. Then, for every bounded measurable function $F : [0, \infty) \to \mathbb{C}$, one defines the operator $F(L) : L^2(X, \mu) \to L^2(X, \mu)$ by the formula

$$F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda). \tag{2.2}$$

In the case of $F_z(\lambda) := e^{-z\lambda}$ for $z \in \mathbb{C}_+$, one sets $e^{-zL} := F_z(L)$ as given by (2.2), which gives a definition of the heat semigroup for complex time. By spectral theory, the family $\{e^{-zL}\}_{z\in\mathbb{C}_+}$ satisfies

$$\|e^{-zL}\|_{2\to 2} \leq 1$$

for all $z \in \mathbb{C}_+$; cf. [37, Chapter 2].

Definition 2.8 (Davies-Gaffney estimate)**.** *We say that the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the Davies-Gaffney estimate if for all open sets E and F in X, $t \in (0,\infty)$ and $f \in L^2(E)$ with $\operatorname{supp} f \subset E$, it holds that*

$$\|e^{-tL} f\|_{L^2(F)} \leq \exp\left\{-\frac{\operatorname{dist}(E,F)^2}{4t}\right\}\|f\|_{L^2(E)}, \tag{2.3}$$

where and in what follows, $\operatorname{dist}(E,F) := \inf_{x\in E,\, y\in F} d(x,y)$.

Definition 2.9 (Finite propagation speed property)**.** *We say that L satisfies the finite propagation speed property if for all $0 < t < d(E,F)$ and $E, F \subset X$, $f_1 \in L^2(E)$ and $f_2 \in L^2(F)$,*

$$\int_X \langle \cos(t\sqrt{L}) f_1, f_2\rangle \, d\mu = 0. \tag{2.4}$$

The following result was obtained by Sikora in [103]. The statement can also be found in [66, Proposition 3.4] and [33, Theorem 3.4].

Proposition 2.10. *The operator L satisfies the finite propagation speed property* (2.4) *if and only if the semigroup $\{e^{-tL}\}_{t>0}$ satisfies the Davies-Gaffney estimate* (2.3).

By the Fourier inversion formula, whenever F is an even bounded Borel-function with $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$ as

$$F(\sqrt{L}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(t)\cos(t\sqrt{L})\, dt. \tag{2.5}$$

The following result follows from [107, Theorem 0.1] (see also [65]) and [33, Theorem 3.4].

Lemma 2.11. *Let $(X, d, \mu, \mathscr{E})$ be a Dirichlet metric measure space endowed with a "carré du champ". Then the associated heat semigroup $e^{-t\mathcal{L}}$ satisfies the Davies-Gaffney estimate.*

In what follows, $\mathcal{L}$ is as above. Let $\mathscr{S}(\mathbb{R})$ denote the collection of all Schwartz functions on $\mathbb{R}$. We need the following L^2-boundedness of spectral multipliers.

Lemma 2.12. *Let $\Phi \in \mathscr{S}(\mathbb{R})$ be an even function with $\Phi(0) = 1$. Then there exists $C > 0$ such that*

$$\sup_{r>0} \|(r^2\mathcal{L})^{-1}(1-\Phi(r\sqrt{\mathcal{L}}))\|_{2\to 2} \leq C,$$

and, for each $k = 0, 1, 2, \ldots$, there exists C such that

$$\sup_{r>0} \|(r^2\mathcal{L})^k \Phi(r\sqrt{\mathcal{L}})\|_{2\to 2} \leq C.$$

Proof. We only give the proof of the first inequality; the second one follows similarly. Since $\Phi'(0) = 0$, spectral theory (cf. [37, Chapter 2]) gives

$$\left\|(r^2\mathcal{L})^{-1}(1-\Phi(r\sqrt{\mathcal{L}}))\right\|_{2\to 2} \leq \sup_{\lambda}\left|\frac{1-\Phi(r\lambda)}{r^2\lambda^2}\right| = \sup_{\lambda}\left|\frac{1-\Phi(\lambda)}{\lambda^2}\right| < \infty.$$

The proof is complete. □

Lemma 2.13. *Let $\Phi \in \mathscr{S}(\mathbb{R})$ be an even function whose Fourier transform $\hat{\Phi}$ satisfies* $\operatorname{supp}\hat{\Phi} \subset [-1, 1]$. *Then for every $\kappa \in \mathbb{Z}_+$ and $t > 0$, the operator $(t^2\mathcal{L})^\kappa \Phi(t\sqrt{\mathcal{L}})$ satisfies*

$$\int_X \langle (t^2\mathcal{L})^\kappa \Phi(t\sqrt{\mathcal{L}}) f_1, f_2 \rangle \, d\mu = 0 \tag{2.6}$$

for all $0 < t < d(E, F)$ and $E, F \subset X$, $f_1 \in L^2(E)$ and $f_2 \in L^2(F)$.

Proof. Let $\Phi_\kappa(s) := s^{2\kappa}\Phi(s)$. By noticing that $\widehat{\Phi_\kappa}(\lambda) = (-1)^\kappa \frac{d^{2\kappa}}{d\lambda^{2\kappa}} \widehat{\Phi}(\lambda)$, the conclusion follows from Lemma 2.11, Proposition 2.10 and (2.5). □

Lemma 2.14. *Let $\Phi \in \mathscr{S}(\mathbb{R})$ be an even function with $\Phi(0) = 1$. Then, for each $f \in L^2(X, \mu)$, it holds that*

$$\lim_{t\to 0^+} \left\| f - \Phi(t\sqrt{\mathcal{L}}) f \right\|_2 = 0.$$

Proof. The domain $\mathscr{D}(\mathcal{L})$ is dense in $L^2(X, \mu)$ and hence it is enough to prove Lemma 2.14 for $f \in \mathscr{D}(\mathcal{L})$. Then

$$\left\| f - \Phi(t\sqrt{\mathcal{L}}) f \right\|_2 \le Ct^2 \|\mathcal{L}f\|_2 \left\| (t^2\mathcal{L})^{-1}(1 - \Phi(t\sqrt{\mathcal{L}})) \right\|_{2\to 2}$$

and the lemma follows from Lemma 2.12. □

3 Regularity of solutions to the Poisson equation

In this section, we show that suitable regularity of harmonic functions implies a gradient estimate for solutions to the Poisson equation $\mathcal{L}f = g$.

The following result was established in [76, Proposition 3.1] under the stronger assumption of both (D_Q) and (P_2); we adapt the proof below to our our setting. Given $a > 1$ and $r > 0$, let $[\log_a r]$ be the largest integer smaller than $\log_a r$.

Proposition 3.1. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_Q), $Q \ge 2$, and that (UE) holds. Suppose that $\mathcal{L}f = g$ in $2B$, $B = B(x_0, r)$, with $g \in L^\infty(2B)$. Then, for every $q > \frac{2Q}{Q+2}$, there exists $C > 0$ such that for almost every $x \in B$,*

$$|f(x)| \le C \left\{ \fint_{2B} |f| \, d\mu + G_1(x) \right\},$$

where

$$G_1(x) := \sum_{j \le [\log_2 r]} 2^{2j} \left(\fint_{B(x, 2^j)} |g|^q \, d\mu \right)^{1/q}. \tag{3.1}$$

Proof. Let $k_0 = [\log_2 r]$ and $x \in B$. By Lemma 2.6, for each $j \le k_0$, there exists $f_j \in W_0^{1,2}(B(x, 2^j))$ such that $\mathcal{L} f_j = g$ in $B(x, 2^j)$, and

$$\fint_{B(x,2^{j-2})} |f_j(y)|\, d\mu(y) \le C \fint_{B(x,2^j)} |f_j(y)|\, d\mu(y) \le C 2^{2j} \left(\fint_{B(x,2^j)} |g|^q\, d\mu \right)^{1/q}.$$

Moreover, for each $k \le k_0 - 1$, as $\mathcal{L}(f_{k+1} - f_k) = 0$ in $B(x, 2^k)$ (notice that $B(x, 2^k) \subset 2B$), by Proposition 2.1, we have

$$\|f_{k+1} - f_k\|_{L^\infty(B(x,2^{k-2}))} \le C \fint_{B(x,2^{k-1})} |f_{k+1} - f_k|\, d\mu \le C \fint_{B(x,2^k)} |f_{k+1} - f_k|\, d\mu.$$

Thus, from the above two inequalities, for almost every $x \in B$, we deduce that

$$\begin{aligned}
|f(x)| &= \lim_{j\to-\infty} \fint_{B(x,2^{j-2})} |f(y)|\, d\mu(y) \\
&\le \limsup_{j\to-\infty} \left\{ \fint_{B(x,2^{j-2})} |f_j(y)|\, d\mu(y) + \sum_{k=j}^{k_0-1} \fint_{B(x,2^{j-2})} |f_{k+1}(y) - f_k(y)|\, d\mu(y) \right\} \\
&\quad + \limsup_{j\to-\infty} \fint_{B(x,2^{j-2})} |f_{k_0}(y) - f(y)|\, d\mu(y) \\
&\le \limsup_{j\to-\infty} \left\{ C 2^{2j} \left(\fint_{B(x,2^j)} |g|^q\, d\mu \right)^{1/q} + \sum_{k=j}^{k_0-1} \|f_{k+1} - f_k\|_{L^\infty(B(x,2^{j-2}))} + \|f_{k_0} - f\|_{L^\infty(B(x,2^{j-2}))} \right\} \\
&\le \lim_{j\to-\infty} C \sum_{k=j}^{k_0-1} \fint_{B(x,2^k)} |f_{k+1}(y) - f_k(y)|\, d\mu(y) + C \fint_{B(x,2^{k_0})} |f_{k_0} - f|\, d\mu \\
&\le C \sum_{k=-\infty}^{k_0-1} \fint_{B(x,2^k)} |f_{k+1}(y)|\, d\mu(y) + C \fint_{B(x,2^{k_0})} |f_{k_0}|\, d\mu + C \fint_{B(x,2^{k_0})} |f|\, d\mu \\
&\le C \sum_{k=-\infty}^{k_0} 2^{2k} \left(\fint_{B(x,2^k)} |g|^q\, d\mu \right)^{1/q} + C \fint_{2B} |f|\, d\mu.
\end{aligned}$$

Above, in the third inequality, we used the fact that

$$\limsup_{j\to-\infty} 2^{2j} \left(\fint_{B(x,2^j)} |g|^q\, d\mu \right)^{1/q} \le \limsup_{j\to-\infty} 2^{2j} \|g\|_{L^\infty(2B)} = 0,$$

and in the last inequality, we used the doubling condition to conclude that

$$\fint_{B(x,2^{k_0})} |f|\, d\mu \le C \fint_{2B} |f|\, d\mu.$$

The proof is complete. ◻

3.1 Harmonic functions satisfying condition (RH_∞)

The next statement deals with the case when harmonic functions satisfy condition (RH_∞). The proof of the following theorem is similar to that of [76, Theorem 3.1].

Theorem 3.2. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_Q), $Q \geq 2$, and that (UE) and $(P_{\infty,\mathrm{loc}})$ hold. Assume that (RH_∞) holds. Then if $\mathcal{L}f = g$ in $2B$, $B = B(x_0, r)$, $g \in L^\infty(2B)$, and $q > \frac{2Q}{Q+2}$, there exists $C = C(C_D, C_{LS}, C_P(1), q) > 0$ such that, for almost every $x \in B$,*

$$|\nabla f(x)| \leq C \left\{ \frac{1}{r} \fint_{2B} |f| \, d\mu + G_2(x) \right\},$$

where

$$G_2(x) := \sum_{j \leq [\log_2 r]} 2^j \left(\fint_{B(x,2^j)} |g|^q \, d\mu \right)^{1/q}. \tag{3.2}$$

In order to prove Theorem 3.2, we need the following Lipschitz estimate, which follows from (D), $(P_{\infty,\mathrm{loc}})$ and (RH_∞). Its proof, which uses a telescopic estimate, will be omitted; see for instance [101].

Lemma 3.3. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (UE) and $(P_{\infty,\mathrm{loc}})$. Assume that (RH_∞) holds. If $\mathcal{L}f = 0$ in $2B$, $B = B(x_0, r)$, then for almost all $x, y \in B(x_0, r)$ with $d(x, y) < 1/2$, it holds that*

$$|f(x) - f(y)| \leq C \frac{d(x, y)}{r} \fint_{2B} |f| \, d\mu,$$

where $C = C(C_D, C_P(1))$.

Proof of Theorem 3.2. Set $k_0 := [\log_2 r]$. Let $x, y \in B$ be Lebesgue points of f. Note that $G_1(x) \leq CrG_2(x)$ for all $x \in B$. Hence if $d(x, y) \geq r/16$, then by Proposition 3.1, we have

$$|f(x) - f(y)| \leq C \fint_{2B} |f| \, d\mu + CG_1(x) + CG_1(y) \leq Cd(x, y) \left\{ \frac{1}{r} \fint_{2B} |f| \, d\mu + G_2(x) + G_2(y) \right\}. \tag{3.3}$$

Now assume that $d(x, y) < r/16$ and $d(x, y) < \frac{1}{2}$. Choose $k_1 \in \mathbb{Z}$ such that $2^{k_1-2} \leq d(x, y) < 2^{k_1-1}$. As in the proof of Proposition 3.1, for each $j \in \mathbb{Z}$ and $j \leq k_0$, pick $f_j \in W_0^{1,2}(B(x, 2^j))$ with $\mathcal{L}f_j = g$ in $B(x, 2^j)$. By the choice of k_1, we see that for each $z \in B(y, 2^{k_1-1})$,

$$d(x, z) \leq d(x, y) + d(y, z) < 2^{k_1-1} + 2^{k_1-1} \leq 2^{k_1},$$

which further implies that $B(y, 2^{k_1-1}) \subset B(x, 2^k)$ for each $k \geq k_1$. Hence, for each $k \geq k_1$, the value $f_k(y)$ is well defined, and we have

$$|f(x) - f(y)| \leq |f(x) - f_{k_0}(x) - [f(y) - f_{k_0}(y)]|$$

$$+\sum_{j=k_1}^{k_0-1}|[f_j(x)-f_{j+1}(x)]-[f_j(y)-f_{j+1}(y)]|+|f_{k_1}(x)-f_{k_1}(y)|$$
$$=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3.$$

Let us estimate the term I_1. According to the choice of f_{k_0}, $f-f_{k_0}$ is harmonic in $B(x,2^{k_0})\subset 2B$. By using the fact that $y\in B(x,2^{k_1-1})$ together with Lemma 3.3, we conclude that

$$\begin{aligned}\mathrm{I}_1 &\le C\frac{d(x,y)}{2^{k_0}}\fint_{B(x,2^{k_0})}|f-f_{k_0}|\,d\mu\\ &\le Cd(x,y)\left\{\frac{1}{r}\fint_{2B}|f|\,d\mu+2^{k_0}\left(\fint_{B(x,2^{k_0})}|g|^q\,d\mu\right)^{1/q}\right\},\end{aligned}$$

where we used (D), estimate $\frac{1}{\mu(B(x,2^{k_0}))}\le C\frac{1}{\mu(2B)}$, and Lemma 2.6 to estimate $\fint_{B(x,2^{k_0})}|f_{k_0}|\,d\mu$.

The term I_2 can be estimated similarly as the first term. For each $k_1\le j\le k_0-1$, f_j-f_{j+1} is harmonic in $B(x,2^j)$. As $y\in B(x,2^{k_1-1})\subset\frac{1}{2}B(x,2^j)$, by using Lemma 3.3 and Lemma 2.6, we deduce that

$$\begin{aligned}\mathrm{I}_2 &= \sum_{j=k_1}^{k_0-1}|[f_j(x)-f_{j+1}(x)]-[f_j(y)-f_{j+1}(y)]|\le Cd(x,y)\left\{\sum_{j=k_1}^{k_0-1}\frac{1}{2^j}\fint_{B(x,2^j)}|f_j-f_{j+1}|\,d\mu\right\}\\ &\le Cd(x,y)\left\{\sum_{j=k_1}^{k_0}\frac{1}{2^j}\fint_{B(x,2^j)}|f_j|\,d\mu\right\}\le Cd(x,y)\left\{\sum_{j=k_1}^{k_0}2^j\left(\fint_{B(x,2^j)}|g|^q\,d\mu\right)^{1/q}\right\}.\end{aligned}$$

By Proposition 3.1 and Lemma 2.6, we see that for almost every $z\in B(x,2^{k_1-1})$,

$$\begin{aligned}|f_{k_1}(z)| &\le C\sum_{k=-\infty}^{k_1}2^{2k}\left(\fint_{B(z,2^k)}|g|^q\,d\mu\right)^{1/q}+C\fint_{B(x,2^{k_1})}|f_{k_1}|\,d\mu\\ &\le C\sum_{k=-\infty}^{k_1}2^{2k}\left(\fint_{B(z,2^k)}|g|^q\,d\mu\right)^{1/q}+C2^{2k_1}\left(\fint_{B(x,2^{k_1})}|g|^q\,d\mu\right)^{1/q},\end{aligned}$$

which together with the fact that $y\in B(x,2^{k_1+1})$ implies that

$$\mathrm{I}_3\le C2^{k_1}\left\{\sum_{k=-\infty}^{k_1}2^k\left(\fint_{B(x,2^k)}|g|^q\,d\mu\right)^{1/q}+\sum_{k=-\infty}^{k_1}2^k\left(\fint_{B(y,2^k)}|g|^q\,d\mu\right)^{1/q}\right\}.$$

Combining the estimates for the terms I_1, I_2 and I_3, and (3.3), we see that for almost all $x,y\in B$ with $d(x,y)<1/2$,

$$|f(x)-f(y)|\le Cd(x,y)\left\{\frac{1}{r}\fint_{2B}|f|\,d\mu+G_2(x)+G_2(y)\right\}.$$

Clearly, for $g \in L^\infty(2B)$, $G_2 \in L^\infty(2B)$, and hence up to a modification on a set with measure zero, f is a Lipschitz function on B.

For a locally Lipschitz function ϕ, denote by $\operatorname{Lip}\phi$ its pointwise Lipschitz constant as

$$\operatorname{Lip}\phi(x) := \limsup_{d(x,y)\to 0} \frac{|\phi(x)-\phi(y)|}{d(x,y)}. \tag{3.4}$$

By [80, Theorem 2.1] (also see [59]), we see that for almost every $x \in B$,

$$|\nabla f(x)| \le \operatorname{Lip} f(x) \le C\left\{\frac{1}{r}\fint_{B(x_0,2r)} |f|\,d\mu + \sum_{j=-\infty}^{k_0} 2^j \left(\fint_{B(x,2^j)} |g|^q\,d\mu\right)^{1/q}\right\},$$

proving the claim. ◻

We need the following potential estimate from Hajłasz and Koskela [61, Theorem 5.3]. Again, g refers both to a function defined on $2B$ and to its zero extension to the exterior of $2B$.

Theorem 3.4. *Let $(X,d,\mu,\mathscr{E})$ be a Dirichlet metric measure space satisfying (D_Q), $Q \ge 2$. Let $B = B(x_0,r) \subset X$, $g \in L^q(2B)$, and G_2 be defined via* (3.2). *Then*

(i) for $q \in \left(\frac{2Q}{Q+2}, Q\right)$ and $p < q^* = \frac{Qq}{Q-q}$,

$$\|G_2\|_{L^p(B)} \le Cr\mu(B)^{1/p-1/q}\|g\|_{L^q(2B)};$$

(ii) for $q \in \left(\frac{2Q}{Q+2}, Q\right)$, $q < p < Q$ and $p^* = \frac{Qq}{Q-q}$,

$$\|G_2\|_{L^{p^*}(B)} \le Cr\mu(B)^{-1/Q}\|g\|_{L^p(2B)};$$

(iii) for $q > Q$

$$\|G_2\|_{L^\infty(B)} \le Cr\mu(B)^{-1/q}\|g\|_{L^q(2B)}.$$

Theorem 3.4 allows us to obtain the following corollary to Theorem 3.2.

Corollary 3.5. *Let $(X,d,\mu,\mathscr{E})$ be a Dirichlet metric measure space satisfying (D_Q), $Q \ge 2$, and assume that (UE) and $(P_{\infty,\mathrm{loc}})$ hold. Assume that (RH_∞) holds. Then for every $f \in W^{1,2}(2B)$, $B = B(x_0,r)$, satisfying $\mathcal{L}f = g$ with $g \in L^q(2B)$, $q > Q$, we have*

$$\||\nabla f|\|_{L^\infty(B)} \le C\left\{\frac{1}{r}\fint_{2B} |f|\,d\mu + r\left(\fint_{2B} |g|^q\,d\mu\right)^{1/q}\right\}.$$

where $C = C(Q, C_Q, C_P) > 0$.

Proof. If $q = \infty$, then the conclusion follows from Theorem 3.2.

Suppose now that $q \in (Q,\infty)$. Let $f_0 \in W^{1,2}_0(2B)$ be the solution to $\mathcal{L}f_0 = g$ in $2B$. Then $f - f_0 \in W^{1,2}(2B)$ and $\mathcal{L}(f - f_0) = 0$. Moreover, for each $k \in \mathbb{N}$, set again $g_k := \chi_{\{|g|\le k\}}\, g$, and let

$f_k \in W_0^{1,2}(2B)$ be the solution to $\mathcal{L} f_k = g_k$ in $2B$. By using Lemma 2.5, Theorems 3.2 and 3.4, we conclude that

$$\||\nabla f_k|\|_{L^\infty(B)} \le C\left\{\frac{1}{r}\fint_{2B}|f_k|\,d\mu + r\mu(B)^{-1/q}\|g_k\|_{L^q(2B)}\right\} \le \frac{Cr}{\mu(B)^{1/q}}\|g_k\|_{L^q(2B)} \le \frac{Cr}{\mu(B)^{1/q}}\|g\|_{L^q(2B)}.$$

On the other hand, notice that

$$\begin{aligned}\int_{2B}|\nabla(f_0 - f_k)|^2\,d\mu &= \int_{2B}(f_0 - f_k)(g - g_k)\,d\mu\\ &\le (\|f_0\|_{L^\infty(2B)} + \|f_k\|_{L^\infty(2B)})\,\|g - g_k\|_{L^1(2B)} \to 0,\end{aligned}$$

as $k \to \infty$, which, together with the preceding inequality, implies that

$$\||\nabla f_0|\|_{L^\infty(B)} \le Cr\left(\fint_{B(x_0,2r)}|g|^q\,d\mu\right)^{1/q}.$$

Combining this with (RH_∞) for $f - f_0$ yields that

$$\||\nabla f|\|_{L^\infty(B)} \le C\left\{\frac{1}{r}\fint_{2B}|f|\,d\mu + r\left(\fint_{B(x_0,2r)}|g|^q\,d\mu\right)^{1/q}\right\},$$

as desired. □

3.2 Harmonic functions satisfying condition (RH_p) for $p \in (2,\infty)$

Let us now turn to the case when only a reverse Hölder inequality (RH_p), $p \in (2,\infty)$, holds for gradients of harmonic functions. In this case, we do not know how to get pointwise estimates for the gradients of solutions to Poisson equations. As a substitute for this, we provide a quantitative L^p-estimate as follows.

Theorem 3.6. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (D_Q), $Q \ge 2$, and that $(P_{2,\mathrm{loc}})$ and (UE) hold. Assume that (RH_p) holds for some $p \in (2,\infty)$. Let $q \in \left(\frac{pQ}{Q+p}, p\right]$ with $\frac{1}{q} - \frac{1}{p} < \frac{1}{Q}$. Then for every $f \in W^{1,2}(2B)$, $B = B(x_0, r)$, satisfying $\mathcal{L} f = g$ with $g \in L^q(2B)$, it holds that*

$$\left(\fint_{B(x_0,r)}|\nabla f|^p\,d\mu\right)^{1/p} \le \frac{C}{r}\left(\fint_{B(x_0,2r)}|f|\,d\mu + r^2\left(\fint_{B(x_0,2r)}|g|^q\,d\mu\right)^{1/q}\right),$$

where $C = C(p, q, C_Q, C_P(1), C_{LS})$.

Remark 3.7. We note that Theorem 3.6 also applies to the case $p = \infty$, which yields a weaker version than Theorem 3.2 and Corollary 3.5, as a stronger version of Poincaré inequality is assumed. In Theorem 3.6 we assumed a Poincaré inequality $(P_{2,\mathrm{loc}})$, which is stronger than $(P_{\infty,\mathrm{loc}})$ assumed in Theorem 3.2 and Corollary 3.5. The reason is that, for the case (RH_∞), harmonic functions are Lipschitz continuous and we can use the pointwise Lipschitz constant (3.4), while for the case (RH_p), $p < \infty$, harmonic functions are not necessarily Lipschitz continuous and we need instead to use the Hajłasz gradient (see Step 2 of proof of Theorem 3.6 below) which requires a stronger Poincaré inequality.

We need the following well-known Christ's dyadic cube decomposition for metric measure spaces (X, d, μ) from [31]; also see [69, Theorem 1.2].

Lemma 3.8 (Christ's dyadic cubes)**.** *There exists a collection of open subsets* $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$*, where* I_k *denotes a certain (possibly finite) index set depending on* k*, and constants* $\delta \in (0,1)$*,* $a_0 \in (0,1)$ *and* $a_1 > a_0$ *with* $a_1 \in (0, \infty)$ *such that*

(i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$ *for all* $k \in \mathbb{Z}$*;*

(ii) if $i > k$*, then either* $Q_\alpha^i \subset Q_\beta^k$ *or* $Q_\alpha^i \cap Q_\beta^k = \emptyset$*;*

(iii) for each k *and all* $\alpha \neq \beta \in I_k$*,* $Q_\alpha^k \cap Q_\beta^k = \emptyset$*;*

(iv) for each (k, α) *and each* $i < k$*, there exists a unique* β *such that* $Q_\alpha^k \subset Q_\beta^i$*;*

(v) $\operatorname{diam}(Q_\alpha^k) \leq a_1 \delta^k$*;*

(vi) each Q_α^k *contains a ball* $B(z_\alpha^k, a_0 \delta^k)$*.*

Remark 3.9. (i) In the above lemma, we can require δ and a_1 to be as small as we wish. This can been done by removing some generations, for instance, $2k+1$-generations, from the set; also see [69, Theorem 1.2].

(ii) Under the doubling condition (D), it is easy to see via conditions (iii) and (v) above that $X = \cup_\alpha \bar{Q}_\alpha^k$ for each k.

The doubling condition allows us to conclude the following bounded overlap property for the balls $B(z_\alpha^k, a_1 \delta^k)$ that contain $B(z_\alpha^k, a_0 \delta^k)$ from Lemma 3.8.

Proposition 3.10. *Let* (X, d, μ) *be a Dirichlet metric measure space satisfying* (D_Q) *for some* $Q \geq 2$*. For each* α *and* k*, let* $B_\alpha^k = B(z_\alpha^k, a_1 \delta^k)$*. Then for each dilation* $t > 1$*, there exists a constant* $C(t, a_0, a_1, C_Q, Q) > 0$ *such that for each* k*,*

$$\sum_\alpha \chi_{tB_\alpha^k}(x) \leq C(t, a_1, C_Q, Q).$$

Proof. For each $x \in X$, let

$$C(x, k) = \sum_\alpha \chi_{tB_\alpha^k}(x).$$

Fix a point x_0 and $k \in \mathbb{Z}$, and consider the ball $B(x_0, 2ta_1\delta^k)$. Then there exist $C(x_0, k)$ distinct balls, say $\{B_{\alpha_j}^k\}_{j \leq C(x_0,k)}$, that are inside $B(x_0, 2ta_1\delta^k)$. Using the doubling condition and the properties (iii) and (vi) of the dyadic cubes from Lemma 3.8, we see that

$$\begin{aligned} V(x_0, 2ta_1\delta^k) &\geq \sum_{j=1}^{C(x_0,k)} V(z_{\alpha_j}^k, a_0\delta^k) \geq \sum_{j=1}^{C(x_0,k)} \frac{a_0^Q}{C_Q(4ta_1)^Q} \mu(4tB_{\alpha_j}^k) \\ &\geq \frac{C(x_0,k) a_0^Q}{C_Q(4ta_1)^Q} V(x_0, 2ta_1\delta^k). \end{aligned}$$

Therefore, we conclude that

$$C(x_0, k) \leq C_Q(4ta_1/a_0)^Q,$$

which completes the proof. □

We need the following geometric consequence of doubling; see [64] for instance.

Lemma 3.11. *Let (X, d, μ) be a doubling Dirichlet metric measure space. Then there exists a constant N_μ such that for each $k \geq 1$, every $2^{-k}r$-separated set in any ball $B(x, r)$ in X has at most N_μ^k elements.*

We shall make use of the Hardy-Littlewood maximal functions.

Definition 3.12 (Hardy-Littlewood maximal function)**.** *For any locally integrable function f on X, its Hardy-Littlewood maximal function is defined as*

$$\mathcal{M}f(x) := \sup_{B:\, x\in B} \fint_B |f|\, d\mu,$$

where B is any ball. For $p > 1$, we define the p-Hardy-Littlewood maximal function as

$$\mathcal{M}_p f(x) := \sup_{B:\, x\in B} \left(\fint_B |f|^p\, d\mu \right)^{1/p}.$$

Using the Poincaré inequality, it readily follows that $\mathcal{M}_2(|\nabla f|)$ generates a Hajłasz gradient in the following sense; see Appendix A.2. The proof uses a telescopic argument, which is by now classical; see for instance [61] and the monographs [19, 64].

Lemma 3.13. *Assume that $(X, d, \mu, \mathscr{E})$ satisfies (D) and $(P_{2,\mathrm{loc}})$. Then for each $\beta \in (0, 1)$ and $r_0 > 0$, there exists $C = C(C_D, \beta, C_P(r_0)) > 0$ such that, for all $f \in W^{1,2}(B)$, $B = B(x_0, r)$, it holds for almost all $x, y \in \beta B$, that*

$$|f(x) - f(y)| \leq C d(x, y) \left(\mathcal{M}_2(|\nabla f|\chi_B)(x) + \mathcal{M}_2(|\nabla f|\chi_B)(y) \right).$$

Moreover, if f is continuous on B, then the above inequality holds for all $x, y \in \beta B$.

Let us now turn to the proof of the main gradient estimate. Recall that, under (D_Q) together with $(P_{2,\mathrm{loc}})$, every solution f to the equation $\mathcal{L}f = g$ with $g \in L^\infty$ is locally Hölder continuous according to Lemma 2.7: there exists a modification $\tilde{f}$ such that $\tilde{f} = f$ a.e., and every point is a Lebesgue point of $\tilde{f}$. Thus, in what follows, we may assume that every point is a Lebesgue point of our solution.

Proof of Theorem 3.6. For simplicity, we assume that $a_1 = 1$ and $\delta = \frac{1}{4}$ in Lemma 3.8.

We divide the proof into five steps. In first four steps we prove that the statement is valid under the additional assumption that $g \in L^\infty(2B)$. Then in the last step we use truncations to remove this additional assumption and conclude the proof.

Step 1. *Construction of a chain of balls.*

Let $k_0 = [-\log_4 r]$ be the largest integer smaller than $-\log_4 r$, and set $B^{k_0} = B(x_0, 6r)$. Fix a dyadic decomposition as in Lemma 3.8. For each $k > k_0$, let

$$I_{B,k} := \{\alpha : \ Q_\alpha^{k+2} \cap B(x_0, r) \neq \emptyset\},$$

and

$$\mathcal{F}_k = \left\{ B_\alpha^k := B(z_\alpha^{k+2}, 2^{-2k}) : \alpha \in I_{B,k} \right\}.$$

Then, by Proposition 3.10, we see that for each $k > k_0$, it holds that

$$\sum_{\alpha \in I_{B,k}:\, B_\alpha^k \in \mathcal{F}_k} \chi_{B_\alpha^k}(x) \le C_Q (64/a_0)^Q.$$

From the properties of our dyadic cubes (Lemma 3.8), we see that:

(i) $B(x_0, r) \subset \cup_{\alpha \in I_{B,k}} B_\alpha^k$ for all $k > k_0$;

(ii) for each $B_\alpha^k \in \mathcal{F}_k$, there exist balls $B_\alpha^j \in \mathcal{F}_j$, $k_0 < j < k$, such that for all $k_0 < j < k$, $B_\alpha^{j+1} \subset \frac{1}{3} B_\alpha^j$, and $B_\alpha^{k_0+1} \subset \frac{1}{3} B^{k_0} = 2B$.

We call the collection $B_\alpha^{k_0+1}, \ldots, B_\alpha^{k-1}$ a chain associated to B_α^k (and hence to Q_α^k).

Proof of (ii): If $B_\alpha^k \in \mathcal{F}_k$, then $Q_\alpha^{k+2} \cap B(x_0, r) \neq \emptyset$. Therefore, there exists Q_α^{k+1} that contains Q_α^{k+2} and hence, $Q_\alpha^{k+1} \cap B(x_0, r) \neq \emptyset$ and $d(z_\alpha^{k+2}, z_\alpha^{k+1}) \le 2^{-2k-2}$ (by Lemma 3.8 (v)).

For each $x \in B_\alpha^k$,

$$d(x, z_\alpha^{k+1}) < 2^{-2k-2} + 2^{-2k} = \frac{5}{4} 2^{-2k} < \frac{5}{16} 2^{-2k+2}.$$

From this, we conclude that $\frac{1}{3} B_\alpha^{k-1} \supset B_\alpha^k$.

In what follows, for each $B_\alpha^k \in \mathcal{F}_k$, we fix a chain from (ii).

Step 2. *Construction of a Hajłasz gradient via the chain.*

We first assume that $\mathcal{L} f = g$ in $6B = B^{k_0}$, $B = B(x_0, r)$, and $g \in L^\infty(6B)$. In the last step of the proof, we will complete the proof by using $2B$ instead of $6B$.

Let $f_{k_0} \in W_0^{1,2}(B(x_0, 6r))$ be the solution to

$$\mathcal{L} f_{k_0} = g$$

in $B(x_0, 6r)$; the existence of a unique solution is guaranteed by Lemma 2.5. For each $k > k_0$ and $B_\alpha^k \in \mathcal{F}_k$, $a \in I_{B,k}$, we let $f_{\alpha,k} \in W_0^{1,2}(B_\alpha^k)$ be the solution to the Poisson equation

$$\mathcal{L} f_{\alpha,k} = g$$

in B_α^k. Then by Lemma 2.6, we see that for each $k \ge k_0$ and every $a \in I_{B,k}$,

$$\fint_{B_\alpha^k} |f_{\alpha,k}| \, d\mu \le C 2^{-4k} \left(\fint_{B_\alpha^k} |g|^q \right)^{1/q}. \tag{3.5}$$

In what follows, for consistency, we will write f_{k_0} as f_{α,k_0}, $\alpha \in I_{B,k_0}$, although there is only one element in I_{B,k_0}.

Define a function w_{k_0} on $6B = B(x_0, 6r)$ by setting

$$w_{k_0}(x) = \mathcal{M}_2(|\nabla(f - f_{\alpha,k_0})| \chi_{3B})(x).$$

For each $k > k_0$ and every $\alpha \in I_{B,k}$, let $\alpha' \in I_{B,k-1}$ the unique one such that $Q_\alpha^{k+2} \subset Q_{\alpha'}^{k+1}$. Define

$$w_k(x) := \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \mathcal{M}_2(|\nabla(f_{\alpha,k} - f_{\alpha',k-1})|\chi_{\frac{1}{2}B_\alpha^k})(x)\chi_{B_\alpha^k}(x)$$

on $6B$.

We also set

$$G_3(x) := \sum_{j=2k_0-4}^{\infty} 2^{-j} \left(\fint_{B(x,2^{-j})} |h|^q \, d\mu \right)^{1/q},$$

where h is the zero extension of g to $X \setminus 6B$.

Claim: There exists $C = C(C_D, C_{LS}, C_P(1)) > 0$ such that for all $x, y \in B(x_0, r)$ with $d(x, y) < 1/2$, it holds that

$$|f(x) - f(y)| \le Cd(x, y) \left\{ G_3(x) + G_3(y) + \sum_{k \ge k_0} w_k(x) + \sum_{k \ge k_0} w_k(y) \right\}. \tag{3.6}$$

Proof of the claim: Let $x, y \in B$ such that $d(x, y) < 1/2$. If $d(x, y) \ge \frac{r}{64}$, then

$$|f(x) - f(y)| \le |(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| + |f_{\alpha,k_0}(x) - f_{\alpha,k_0}(y)|,$$

where by Lemma 3.13 with $(P_{2,\mathrm{loc}})$ for balls with radii at most one and $\beta = \frac{1}{2}$, we have

$$\begin{aligned} &|(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| \\ &\quad \le Cd(x, y) \left[\mathcal{M}_2 \left(|\nabla(f - f_{\alpha,k_0})|\chi_{3B} \right)(x) + \mathcal{M}_2 \left(|\nabla(f - f_{\alpha,k_0})|\chi_{3B} \right)(y) \right], \end{aligned}$$

and by Proposition 3.1 and (3.5) we have

$$|f_{\alpha,k_0}(x) - f_{\alpha,k_0}(y)| \le Cd(x, y)[G_3(x) + G_3(y)].$$

The above two estimates complete the case $d(x, y) \ge \frac{r}{64}$.

Suppose now $d(x, y) < \frac{r}{64}$ and $d(x, y) < 1/2$. Then there exists $k > k_0$ such that $1/2^{2k+6} \le d(x, y) < 1/2^{2k+4}$. From the properties of dyadic cubes, Lemma 3.8, we see that there exists a cube Q_α^{k+2} such that $x \in \bar{Q}_\alpha^{k+2}$. Noticing that $B_\alpha^k = B(z_\alpha^{k+2}, 2^{-2k})$, we see that

$$d(y, z_\alpha^{k+2}) \le d(x, y) + d(x, z_\alpha^{k+2}) < \frac{1}{2^{2k+4}} + \frac{1}{2^{2k+4}} = \frac{1}{2^{2k+3}},$$

and hence, $x, y \in \frac{1}{3}B_\alpha^k$.

Let $\{B_\alpha^j \in \mathcal{F}_j\}_{k_0 \le j < k}$ be the chain of balls such that $\frac{1}{3}B_\alpha^j \supset B_\alpha^{j+1}$, whose existence is guaranteed by **Step 1** (ii). Applying a telescopic argument, we obtain

$$|f(x) - f(y)| \le |(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| + \sum_{j=k_0}^{k-1} |(f_{\alpha,j} - f_{\alpha,j+1})(x) - (f_{\alpha,j} - f_{\alpha,j+1})(y)|$$

$$+|f_{\alpha,k}(x)-f_{\alpha,k}(y)|.$$

By using Lemma 3.13 with $(P_{2,\mathrm{loc}})$ for balls with radii at most one, $\beta=\frac{2}{3}$ for $k>k_0$ and $\beta=\frac{1}{2}$ for k_0, we conclude that

$$\begin{aligned}
&|(f-f_{\alpha,k_0})(x)-(f-f_{\alpha,k_0})(y)|+\sum_{j=k_0}^{k-1}|(f_{\alpha,j}-f_{\alpha,j+1})(x)-(f_{\alpha,j}-f_{\alpha,j+1})(y)|\\
&\le Cd(x,y)\left\{w_{k_0}(x)+w_{k_0}(y)+\sum_{k_0<j\le k}w_j(x)+\sum_{k_0<j\le k}w_j(y)\right\}\\
&\le Cd(x,y)\left\{\sum_{k\ge k_0}w_k(x)+\sum_{k\ge k_0}w_k(y)\right\}.
\end{aligned}$$

On the other hand, by using (3.5), Proposition 3.1 and that $d(x,y)\approx 2^{-2k}$, we see that

$$\begin{aligned}
|f_{\alpha,k}(x)-f_{\alpha,k}(y)| &\le |f_{\alpha,k}(x)|+|f_{\alpha,k}(y)|\\
&\le Cd(x,y)\left\{\sum_{j=2k}^{\infty}2^{-j}\left(\fint_{B(x,2^{-j})}|g|^q\,d\mu\right)^{1/q}+\sum_{j=2k}^{\infty}2^{-j}\left(\fint_{B(y,2^{-j})}|g|^q\,d\mu\right)^{1/q}\right\}\\
&\le Cd(x,y)\left\{G_3(x)+G_3(y)\right\}.
\end{aligned}$$

The above two estimates imply the claim.

Step 3. *Claim:* For $q\le p$ with $1/q-1/p<1/Q$, we have the estimate

$$\left\|G_3+\sum_{k\ge k_0}w_k\right\|_{L^p(B(x_0,r))}\le\frac{C(q)[V(x_0,6r)]^{1/p}}{r}\left(\fint_{B(x_0,6r)}|f|\,d\mu+r^2\left(\fint_{B(x_0,6r)}|g|^q\,d\mu\right)^{1/q}\right).$$

We begin by estimating the L^p-norm of the second term on the left-hand side. For a ball $B^k_\alpha\in\mathcal{F}_k$, let $B^{k-1}_{\alpha'}=B^{k-1}_{\alpha'(\alpha)}\in\mathcal{F}_{k-1}$ be the ball from the definition of w_k; then it satisfies $\frac{1}{3}B^{k-1}_{\alpha'}\supset B^k_\alpha\in\mathcal{F}_k$. Notice that $f_{k,\alpha}-f_{\alpha,k-1}$ is harmonic on B^k_α. Hence (RH_p), (3.5) and the boundedness of the usual Hardy-Littlewood maximal operator on $L^{p/2}$ with $p>2$ gives, for all $k>k_0$ and $\alpha\in I_{B,k}$, that

$$\begin{aligned}
\int_X\left[\mathcal{M}_2(|\nabla(f_{\alpha,k}-f_{\alpha,k-1})|\chi_{\frac{1}{2}B^k_\alpha})(x)\chi_{B^k_\alpha}(x)\right]^p d\mu(x) &\le \int_{\frac{1}{2}B^k_\alpha}|\nabla(f_{\alpha,k}-f_{\alpha,k-1})|^p\,d\mu(x)\\
&\le C\mu(B^k_\alpha)\left(\frac{2^{2k}}{\mu(B^k_\alpha)}\int_{B^k_\alpha}|(f_{\alpha,k}-f_{\alpha,k-1})|\,d\mu\right)^p\\
&\le C\mu(B^{k-1}_{\alpha'})2^{-2pk}\left(\frac{1}{\mu(B^{k-1}_{\alpha'})}\int_{B^{k-1}_{\alpha'}}|g|^q\,d\mu\right)^{p/q}.
\end{aligned}$$

Combining this with the fact that the sets $\{B_\alpha^k\}_{\alpha\in I_{B,k}}$ have uniformly bounded overlaps for each k, we have that for each $k > k_0$,

$$\begin{aligned}
\|w_k\|^p_{L^p(B(x_0,6r))} &\le C(\mu)\int_{B(x_0,6r)}\left(\sum_{\alpha\in I_{B,k}:B_\alpha^k\in\mathcal{F}_k}\mathcal{M}_2(|\nabla(f_{\alpha,k}-f_{\alpha,k-1})|\chi_{\frac12 B_\alpha^k})(x)\chi_{B_\alpha^k}(x)\right)^p d\mu(x)\\
&\le C(\mu)\sum_{\alpha\in I_{B,k}:B_\alpha^k\in\mathcal{F}_k}\int_{B_\alpha^k}\left(\mathcal{M}_2(|\nabla(f_{\alpha,k}-f_{\alpha,k-1})|\chi_{\frac12 B_\alpha^k})(x)\right)^p d\mu(x)\\
&\le C(\mu)\sum_{\alpha\in I_{B,k}:B_\alpha^k\in\mathcal{F}_k}\mu(B^{k-1}_{\alpha'(\alpha)})2^{-2pk}\left(\frac{1}{\mu(B^{k-1}_{\alpha'(\alpha)})}\int_{B^{k-1}_{\alpha'(\alpha)}}|g|^q\,d\mu\right)^{p/q}\\
&\overset{\text{doubling}}{\le} C(\mu)\sum_{\alpha\in I_{B,k}:B_\alpha^k\in\mathcal{F}_k}V(x_0,6r)^{1-\frac{p}{q}}\frac{2^{2kQ(\frac{p}{q}-1)}}{r^{Q(\frac{p}{q}-1)}}2^{-2pk}\left(\int_{B^{k-1}_{\alpha'(\alpha)}}|g|^q\,d\mu\right)^{p/q}\\
&\le C(\mu)V(x_0,6r)^{1-\frac{p}{q}}\frac{2^{2kQ(\frac{p}{q}-1)}}{r^{Q(\frac{p}{q}-1)}}2^{-2pk}\left(\int_{B(x_0,6r)}|g|^q\,d\mu\right)^{p/q}. \qquad (3.7)
\end{aligned}$$

Above the last inequality relies on $q \le p$ and the fact that

$$\sum_{\alpha\in I_{B,k}:B_\alpha^k\in\mathcal{F}_k}\chi_{B^{k-1}_{\alpha'(\alpha)}}(x)\le C(\mu,a_0).$$

Indeed, since $B_\alpha^k\subset\frac13 B^{k-1}_{\alpha'(\alpha)}$, we have $Q_\alpha^{k+2}\subset\frac13 B^{k-1}_{\alpha'}$. For each $\alpha'\in I_{B,k-1}$, let

$$I_{\alpha',k} := \left\{\alpha : \alpha\in I_{B,k}, B_\alpha^k\subset\frac13 B^{k-1}_{\alpha'}\right\}.$$

By using dyadic cubes again, we see that

$$\mu(B^{k-1}_{\alpha'})\ge\sum_{\alpha\in I_{\alpha',k}}V(z_\alpha^{k+2},a_0 2^{-2k-4})\ge\sum_{\alpha\in I_{\alpha',k}}\frac{C(\mu)a_0^Q}{(2^8)^Q}V(z_\alpha^{k+2},2^{-2k+4})\ge\sum_{\alpha\in I_{\alpha',k}}\frac{C(\mu)a_0^Q}{(2^8)^Q}\mu(B^{k-1}_{\alpha'}),$$

which implies that $\#(I_{\alpha',k})\le\frac{2^{8Q}}{C(\mu)(a_0)^Q}$. Therefore, we conclude that

$$\sum_{\alpha\in I_{B,k}:B_\alpha^k\in\mathcal{F}_k}\chi_{B^{k-1}_{\alpha'(\alpha)}}(x)\le\sum_{\alpha'\in I_{B,k-1}:B^{k-1}_{\alpha'}\in\mathcal{F}_{k-1}}\chi_{B^{k-1}_{\alpha'}}(x)\cdot\#(I_{\alpha',k})\le C(\mu,a_0).$$

By (3.7)

$$\|w_k\|_{L^p(B(x_0,2r))}\le C(\mu)V(x_0,6r)^{\frac1p-\frac1q}\frac{2^{2kQ(\frac1q-\frac1p)}}{r^{Q(\frac1q-\frac1p)}}2^{-2k}\left(\int_{B(x_0,6r)}|g|^q\,d\mu\right)^{1/q}.$$

Therefore, by the Minkowski inequality,

$$\left\|\sum_{k>k_0} w_k\right\|_{L^p(B(x_0,r))} \le C(\mu)V(x_0,6r)^{\frac{1}{p}-\frac{1}{q}} r\left(\int_{B(x_0,6r)} |g|^q\,d\mu\right)^{1/q}$$

provided $q \le p$ and $\frac{1}{q}-\frac{1}{p} < \frac{1}{Q}$.

By applying Lemma 2.6 and (RH_p), we conclude that

$$\begin{aligned}\|w_{k_0}\|_{L^p(B(x_0,r))} &\le V(x_0,2r)^{1/p}\frac{C}{r}\fint_{B(x_0,2r)} |f-f_{\alpha,k_0}|\,d\mu \\ &\le \frac{CV(x_0,2r)^{1/p}}{r}\left(\fint_{B(x_0,6r)} |f|\,d\mu + r^2\left(\fint_{B(x_0,6r)} |g|^q\,d\mu\right)^{1/q}\right), \qquad (3.8)\end{aligned}$$

which completes the estimate for the L^p-integral of the second term on the left-hand side.

Regarding the first term, an estimate similar to the one in Theorem 3.4 (see also [61, Theorem 5.3]) yields

$$\|G_3\|_{L^p(B(x_0,r))} \le C(\mu)rV(x_0,6r)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B(x_0,6r)} |g|^q\,d\mu\right)^{1/q}.$$

The claim then follows by combining the last inequality with (3.7) and (3.8).

Step 4. *Completion of the L^∞ case.*

For each $y_0 \in B(x_0, r/2)$, let $0 \le \psi_r \le 1$ be a one-parameter family of Lipschitz cut-off functions such that

$$\begin{aligned}&\psi_r(x) = 1 \text{ whenever } x \in B(y_0,\min\{r/4,1/8\}),\\ &\psi_r(x) = 0 \text{ whenever } x \in X\setminus B(y_0,\min\{r/2,1/4\}), \text{ and } |\nabla\psi_r(x)| \le \frac{C}{\min\{r,1\}}. \qquad (3.9)\end{aligned}$$

Then by **Step 2**, we see that, for all $x,\ y \in X$,

$$\begin{aligned}&|(f\psi_r)(x)-(f\psi_r)(y)|\\ &\le Cd(x,y)\left\{\left(\frac{|f(x)|}{\min\{r,1\}} + G_3(x) + \sum_{k\ge k_0} w_k(x)\right)\chi_{2B}(x) + \left(\frac{|f(y)|}{\min\{r,1\}} + G_3(y) + \sum_{k\ge k_0} w_k(y)\right)\chi_{2B}(y)\right\}.\end{aligned}$$

Recall our assumption that $g \in L^\infty(6B)$. Therefore, by applying Lemma 2.5 to f_{α,k_0} and Proposition 2.1 to $f - f_{\alpha,k_0}$, we see that $f \in L^\infty(2B)$. This, together with **Step 3**, implies that $f\psi_r \in W_0^{1,p}(B(y_0,\min\{r/2,1/4\}))$; see Appendix A.2.

By (3.6) and the pointwise estimate of the gradient of a Sobolev function (see Appendix A.2) for $f\psi_r$, we conclude that

$$|\nabla f(x)| = |\nabla(f\psi_r)(x)| \le CG_3(x) + C\sum_{k\ge k_0} w_k(x)$$

for a.e. $x \in B(y_0, \min\{r/4, 1/8\})$. By the arbitrariness of y_0, we see this estimate holds for a.e. $x \in B(x_0, r/2)$. This together with the estimate from **Step 3** yields

$$\||\nabla f|\|_{L^p(B(x_0,r/2))} \le \frac{CV(x_0,6r)^{1/p}}{r}\left(\fint_{B(x_0,6r)}|f|\,d\mu + r^2\left(\fint_{B(x_0,6r)}|g|^q\,d\mu\right)^{1/q}\right). \tag{3.10}$$

Let us now replace $B(x_0, 6r)$ on the R.H.S. by $2B$, $B = B(x_0, r)$. By using Lemma 3.11, we see that $B(x_0, r)$ contains at most N_μ^5 separate balls with radii $r/32$. Fix such a maximal collection, which we for simplicity assume to have exactly N_μ^5 elements. Denote these balls by $\{B(x_i, r/32)\}_{1\le i\le N_\mu^5}$. Then

$$B(x_0, r) \subset \bigcup_{1\le i\le N_\mu^5} B(x_i, r/16).$$

By applying the estimate (3.10) to each $B(x_i, \frac{12r}{16})$ yields

$$\begin{aligned}\||\nabla f|\|_{L^p(B(x_i,r/16))} &\le \frac{CV(x_i,\frac{12r}{16})^{1/p}}{r}\left(\fint_{B(x_i,\frac{12r}{16})}|f|\,d\mu + r^2\left(\fint_{B(x_i,\frac{12r}{16})}|g|^q\,d\mu\right)^{1/q}\right)\\ &\le \frac{CV(x_0,2r)^{1/p}}{r}\left(\fint_{B(x_0,2r)}|f|\,d\mu + r^2\left(\fint_{B(x_0,2r)}|g|^q\,d\mu\right)^{1/q}\right).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\||\nabla f|\|_{L^p(B(x_0,r))} &\le \sum_{i=1}^{N_\mu^5}\||\nabla f|\|_{L^p(B(x_i,r/16))}\\ &\le \frac{CV(x_0,2r)^{1/p}}{r}\left(\fint_{B(x_0,2r)}|f|\,d\mu + r^2\left(\fint_{B(x_0,2r)}|g|^q\,d\mu\right)^{1/q}\right),\end{aligned}$$

which completes the proof in the case $g \in L^\infty(2B)$.

Step 5. *Truncation argument.*

Once again, for each $k \in \mathbb{N}$, let $g_k := \chi_{\{|g|\le k\}}\, g$, and let $f_k \in W_0^{1,2}(2B)$ be the solution to $\mathcal{L}f_k = g_k$ in $2B$.

By Lemma 2.6, we see that there exists a solution $f_0 \in W_0^{1,2}(2B)$ to $\mathcal{L}f_0 = g$ in $2B$, with

$$\fint_{2B}|f_0|\,d\mu \le Cr\left(\fint_{2B}|\nabla f_0|^2\,d\mu\right)^{1/2} \le Cr^2\left(\fint_{2B}|g|^q\,d\mu\right)^{1/q}. \tag{3.11}$$

For each $k \in \mathbb{N}$, by using Lemma 2.6 again, we obtain

$$\fint_{2B}|f_0 - f_k|\,d\mu \le Cr\left(\fint_{2B}|\nabla(f_0 - f_k)|^2\,d\mu\right)^{1/2} \le Cr^2\left(\fint_{2B}|g - g_k|^q\,d\mu\right)^{1/q},$$

since $f_0 - f_k \in W_0^{1,2}(2B)$. Consequently $f_k \to f_0$ in $W_0^{1,2}(2B)$.

By Lemma 2.6 and Theorem 3.6, we have for each $k \in \mathbb{N}$ that

$$\left(\fint_{B(x_0,r)} |\nabla f_k|^p \, d\mu\right)^{1/p} \le \frac{C}{r}\left(\fint_{B(x_0,2r)} |f_k| \, d\mu + r^2 \left(\fint_{B(x_0,2r)} |g_k|^q \, d\mu\right)^{1/q}\right)$$
$$\le Cr\left(\fint_{B(x_0,2r)} |g_k|^q \, d\mu\right)^{1/q} \le Cr\left(\fint_{B(x_0,2r)} |g|^q \, d\mu\right)^{1/q}.$$

Letting $k \to \infty$, we conclude that

$$\left(\fint_{B(x_0,r)} |\nabla f_0|^p \, d\mu\right)^{1/p} \le Cr\left(\fint_{B(x_0,2r)} |g|^q \, d\mu\right)^{1/q}.$$

By applying this together with (RH_p) for the harmonic function $f - f_0$ on $B(x_0, r)$ and (3.11) we obtain

$$\left(\fint_{B(x_0,r)} |\nabla f|^p \, d\mu\right)^{1/p} \le \left(\fint_{B(x_0,r)} |\nabla(f - f_0)|^p \, d\mu\right)^{1/p} + \left(\fint_{B(x_0,r)} |\nabla f_0|^p \, d\mu\right)^{1/p}$$
$$\le \frac{C}{r} \fint_{B(x_0,2r)} |f - f_0| \, d\mu + Cr\left(\fint_{B(x_0,2r)} |g|^q \, d\mu\right)^{1/q}$$
$$\le \frac{C}{r} \fint_{B(x_0,2r)} |f| \, d\mu + Cr\left(\fint_{B(x_0,2r)} |g|^q \, d\mu\right)^{1/q},$$

as desired.

□

Corollary 3.5 and Theorems 3.6 yield the following quantitative Hölder regularity of solutions to the Poisson equation.

Corollary 3.14. *Let $(X, d, \mu, \mathscr{E})$ be a Dirichlet metric measure space satisfying (D_Q) with $Q \ge 2$, and suppose that (UE) holds. Assume that (RH_p) and $(P_{2,\mathrm{loc}})$ hold for some $p \in (Q, \infty]$. Let $q > \max\{Q/2, 1\}$ and $\alpha := \alpha(p, q) = \min\{1 - Q/p, 2 - Q/q\}$. If $\mathcal{L}f = g$ in $B(x_0, r)$ with $g \in L^q(B)$, then f belongs to $C^{\alpha}_{\mathrm{loc}}(B)$.*

4 Elliptic equations vs parabolic equations

4.1 From elliptic equations to parabolic equations

In this section, we give quantitative gradient estimates for the heat kernel by using the regularity of solutions to the Poisson equation.

To begin with, let us recall that, under (D) and (UE), we have the estimate

$$\left|\frac{\partial h_t}{\partial t}(x, y)\right| \le \frac{C}{t\, V(y, \sqrt{t})} \exp\left\{-\frac{d^2(x, y)}{ct}\right\}, \tag{4.1}$$

for the time derivative of the heat kernel for all $t > 0$; see [20, 99, 107, 108].

A version of the following result, requiring the slightly stronger condition (P_2), has been established in [74]. The proof below follows the ideas of the proof of [74, Theorem 3.2].

Proposition 4.1. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (UE) and $(P_{\infty,\mathrm{loc}})$. Then (RH_∞) implies (GLY_∞).*

Proof. By using Theorem 3.2 and following the proof of [74, Theorem 3.2], we conclude the claim. □

Our next result follows via the argument in [7, p 941].

Proposition 4.2. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (UE). Then (GLY_∞) implies (G_p) for all $p \in [1, \infty]$.*

Proof. By decomposing X into the union of $B(x, \sqrt{t})$ and the sets $B(x, 2^k\sqrt{t}) \setminus B(x, 2^{k-1}\sqrt{t})$ for $k \geq 1$, one sees via (D) that

$$\int_X \frac{1}{V(x, \sqrt{t})} \exp\left\{-\frac{d^2(x,y)}{ct}\right\} d\mu(y) \leq C(C_D). \tag{4.2}$$

The conclusion then follows from this and (GLY_∞). □

We will also need the following observation.

Proposition 4.3. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (UE). Suppose that $(P_{2,\mathrm{loc}})$ and (RH_p) hold for some $p \in (2, \infty)$. Then (GLY_p) holds.*

Proof. Decompose the space X into $B = B(y, 2\sqrt{t})$ and the sets $B(y, 2^{k+1}\sqrt{t}) \setminus B(y, 2^k\sqrt{t})$, $k \geq 1$. Denote $B(y, 2^{k+1}\sqrt{t}) \setminus B(y, 2^k\sqrt{t})$ by $U_k(B)$. By Theorem 3.6, (UE) and (4.1), we see that

$$\begin{aligned}
&\||\nabla_x h_t(\cdot, y)|\|_{L^p(B)} \\
&\quad\leq \frac{CV(y, 4\sqrt{t})^{1/p}}{\sqrt{t}} \left(\fint_{B(y,4\sqrt{t})} |h_t(x,y)|\, d\mu(x) + t \left(\fint_{B(y,4\sqrt{t})} \left|\frac{\partial}{\partial t} h_t(x,y)\right|^p d\mu(x) \right)^{1/p} \right) \\
&\quad\leq \frac{C}{\sqrt{t}} V(y, \sqrt{t})^{1/p-1}.
\end{aligned}$$

Let $\{B_{k,j} = B(x_{k,j}, \sqrt{t}/2)\}_j$ be a maximal set of pairwise disjoint balls with radius $2^{-1}\sqrt{t}$ in $B(y, 2^{k+1}\sqrt{t})$. Then it is easy to see that

$$B(y, 2^{k+1}\sqrt{t}) \subset \cup_j B(x_{k,j}, \sqrt{t})$$

and

$$\sum_j \chi_{4B_{k,j}}(x) \leq C(C_D).$$

Therefore, by applying Theorem 3.6, (D), (UE), and (4.1), we conclude that

$$\begin{aligned}
&\int_{U_k(B)} |\nabla_x h_t(x,y)|^p \,d\mu(x) \\
&\le \sum_{j:\,2B_{k,j}\cap U_k(B)\neq\emptyset} \int_{2B_{k,j}} |\nabla_x h_t(x,y)|^p \,d\mu(x) \\
&\le \sum_{j:\,2B_{k,j}\cap U_k(B)\neq\emptyset} \frac{C\mu(4B_{k,j})}{t^{p/2}} \left(\fint_{4B_{k,j}} |h_t(x,y)|\,d\mu(x) + t\left(\fint_{4B_{k,j}} \left|\frac{\partial}{\partial t} h_t(x,y)\right|^p \,d\mu(x)\right)^{1/p}\right)^p \\
&\le \sum_{j:\,2B_{k,j}\cap U_k(B)\neq\emptyset} \frac{C\mu(4B_{k,j})}{t^{p/2}} V(y,\sqrt{t})^{-p} \exp\left\{\frac{-c2^{2k}t}{t}\right\} \\
&\le CV(y,2^{k+2}\sqrt{t}) \frac{\exp\left\{-c2^{2k}\right\}}{t^{p/2}V(y,\sqrt{t})^p} \le CV(y,\sqrt{t})2^{kQ} \frac{\exp\left\{-c2^{2k}\right\}}{t^{p/2}V(y,\sqrt{t})^p} \\
&\le C\frac{\exp\left\{-c2^{2k}\right\}}{t^{p/2}V(y,\sqrt{t})^{p-1}}.
\end{aligned}$$

This together with the estimate on $\|\nabla_x h_t(\cdot,y)\|_{L^p(B)}$ from the beginning of the proof allow us to deduce that there exists $\gamma>0$ such that

$$\int_X |\nabla_x h_t(x,y)|^p \exp\left\{\gamma d^2(x,y)/t\right\} d\mu(x) \le \frac{C}{t^{p/2}V(y,\sqrt{t})^{p-1}},$$

which completes the proof. □

The following conclusion follows via the argument in [7, p. 944] applied to our setting.

Proposition 4.4. *Assume that the doubling Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (UE). Then (GLY_p) for some $p\in(2,\infty)$ implies (G_p).*

Remark 4.5. If (P_p) holds, then one can also use the open-ended property of the reverse Hölder inequality (RH_p) (Lemma 5.2 below), Theorem 3.6 and the Hardy-Littlewood maximal operator to prove the fact that (RH_p) $(p\in(2,\infty))$ yields (G_p). We will not go through this argument and leave the details to interested readers.

4.2 From parabolic equations to elliptic equations

In this section, we show that (G_p) implies (RH_p). We begin with an abstract reproducing formula for harmonic functions.

Lemma 4.6 (Reproducing formula)**.** *Let $(X,d,\mu,\mathscr{E})$ be a doubling Dirichlet metric measure space. Assume that (UE) holds, Let $\Phi\in\mathscr{S}(\mathbb{R})$ be an even function whose Fourier transform $\hat{\Phi}$ satisfies $\operatorname{supp}\hat{\Phi}\subset[-1,1]$ and $\Phi(0)=1$. Then if $u\in W^{1,2}(3B)$ is harmonic on $3B$, $B=B(x_0,r)$, for each $0<t\le r$, $u=\Phi(t\sqrt{\mathcal{L}})u$ as functions in $W^{1,2}(B)$.*

Proof. Since $\Phi'(0) = 0$, the function $\tilde{\Phi}(s) := s^{-1}\Phi'(s) \in \mathscr{S}(\mathbb{R})$ extends to an analytic function which satisfies a Paley-Wiener estimate with the same exponent as Φ; see [96] or Appendix A.4. By applying Lemma 2.13 to the functions $t^{2\kappa}\Phi(t)$, $\kappa \in \mathbb{Z}_+$, and $\tilde{\Phi}$, we conclude that the operators $(t^2\mathcal{L})^{\kappa}\Phi(t\sqrt{\mathcal{L}})$ and $(t^2\mathcal{L})^{-1/2}\Phi'(t\sqrt{\mathcal{L}})$ satisfy

$$\int_X \langle (t^2\mathcal{L})^{\kappa}\Phi(t\sqrt{\mathcal{L}})f_1, f_2\rangle \, d\mu = 0, \tag{4.3}$$

and

$$\int_X \langle (t^2\mathcal{L})^{-1/2}\Phi'(t\sqrt{\mathcal{L}})f_1, f_2\rangle \, d\mu = 0, \tag{4.4}$$

for all $0 < t < d(E, F)$ with $E, F \subset X$, $f_1 \in L^2(E)$, and $f_2 \in L^2(F)$.

Let ψ be a Lipschitz cut-off function such that $\psi = 1$ on $\frac{8}{3}B$, $\psi = 0$ outside $3B$. Let $\varepsilon \in (0, r/4)$. For each $g \in L^2(\frac{3}{2}B)$ with support in $\overline{\frac{3}{2}B}$, by (4.3) and Lemma 2.12 we have

$$\Phi(\varepsilon\sqrt{\mathcal{L}})g \in \mathscr{D}(\mathcal{L})$$

with support in $\overline{\frac{7}{4}B}$. Since $\Phi(0) = 1$, we have

$$1 - \Phi(t\sqrt{\mathcal{L}}) = -\int_0^t \sqrt{\mathcal{L}}\Phi'(s\sqrt{\mathcal{L}})\, ds,$$

which together with (4.4) implies that

$$\int_X \langle (t^2\mathcal{L})^{-1}(1 - \Phi(t\sqrt{\mathcal{L}}))f_1, f_2\rangle \, d\mu = \int_0^t \int_X \langle (t^2\mathcal{L})^{-1}\sqrt{\mathcal{L}}\Phi'(s\sqrt{\mathcal{L}})f_1, f_2\rangle \, d\mu\, ds = 0, \tag{4.5}$$

for all $0 < t < d(E, F)$ with $E, F \subset X$, $f_1 \in L^2(E)$, and $f_2 \in L^2(F)$. This together with Lemma 2.12 implies that for each $t \le r$

$$(t^2\mathcal{L})^{-1}(1 - \Phi(t\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g \in \mathscr{D}(\mathcal{L}), \tag{4.6}$$

with support in $\overline{\frac{11}{4}B}$. By this, the self-adjointness of $\mathcal{L}$ and the fact that u is harmonic on $3B$, we obtain that

$$\begin{aligned}
\int_X \langle (1 - \Phi(tr\sqrt{\mathcal{L}}))u, \Phi(\varepsilon\sqrt{\mathcal{L}})g\rangle \, d\mu &= \int_X \langle u, (1 - \Phi(tr\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g\rangle \, d\mu \\
&= \int_X \langle u\psi, (1 - \Phi(tr\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g\rangle \, d\mu \\
&= r^2 \int_{3B} \langle \nabla u, \nabla (r^2\mathcal{L})^{-1}(1 - \Phi(tr\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g\rangle \, d\mu \\
&= 0.
\end{aligned}$$

Since g is arbitrary, and by Lemma 2.14 $\Phi(\varepsilon\sqrt{\mathcal{L}})g \to g$ in $L^2(X, \mu)$ as $\varepsilon \to 0$, we find that $(1 - \Phi(tr\sqrt{\mathcal{L}}))u = 0$ in $L^2(B)$. Hence $u(x) = \Phi(tr\sqrt{\mathcal{L}})u(x)$ for a.e. $x \in \frac{3}{2}B$. Therefore, $u = \Phi(tr\sqrt{\mathcal{L}})u$ in $W^{1,2}(B)$ for each $t \le 1$. The proof is complete. □

Remark 4.7. Notice that, for each $f \in L^2(X,\mu)$, $\Phi(r\sqrt{\mathcal{L}})f \in W^{1,2}(X)$ and

$$\||\nabla\Phi(r\sqrt{\mathcal{L}})f|\|_2 = \|\sqrt{\mathcal{L}}\Phi(r\sqrt{\mathcal{L}})f\|_2 \le \frac{C}{r}\|f\|_2,$$

see Lemma 2.12.

Corollary 4.8. *Assume that the doubling Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (UE). Let $\Phi \in \mathscr{S}(\mathbb{R})$ be an even function whose Fourier transform $\hat{\Phi}$ satisfies $\operatorname{supp}\hat{\Phi} \subset [-1,1]$ and $\Phi(0) = 1$. Then if $u \in W^{1,2}(X)$ is harmonic on $3B$, $B = B(x_0,r)$, for each $0 < t \le 1$, u equals $\Phi(tr\sqrt{\mathcal{L}})u$ as functions in $W^{1,2}(B)$.*

Proof. Notice that by Lemma 4.6, for each $0 < t \le 1$, $u(x) = \Phi(tr\sqrt{\mathcal{L}})(u\chi_{3B})(x)$, a.e. $x \in B$. On the other hand, by (4.3), we see that

$$\Phi(tr\sqrt{\mathcal{L}})(u\chi_{X\setminus 3B})(x) = 0$$

on B, which allows us to conclude that for each $0 < t \le 1$, $u = \Phi(tr\sqrt{\mathcal{L}})u$ in $W^{1,2}(B)$. □

The main result of this section reads as follows.

Theorem 4.9. *Assume that the doubling Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies (UE). If (G_{p_0}) holds for some $p_0 \in (2,\infty]$, then (RH_{p_0}) holds.*

Proof. Let $\Phi \in \mathscr{S}(\mathbb{R})$ be an even function whose Fourier transform $\hat{\Phi}$ satisfies $\operatorname{supp}\hat{\Phi} \subset [-1/2,1/2]$ and $\Phi(0) = 1$. Then it follows that $\Phi^2 \in \mathscr{S}(\mathbb{R})$ and $\operatorname{supp}\hat{\Phi^2} \subset [-1,1]$. In the proof, for simplicity we denote $V(x,r)$ by $V_r(x)$.

Step 1. *Boundedness of the spectral multipliers.*

Claim 1. We first claim that, for each $p \in [1,2]$, there exists $C > 0$ such that

$$\sup_{r>0}\|V_r^{1/p-1/2}\Phi(r\sqrt{\mathcal{L}})\|_{p\to2} \le C.$$

By [20, Proposition 4.1.1] and the fact that $\sup_{t>0}|\Phi(t)(1+t^2)^N| < \infty$, one has

$$\begin{aligned}
\|V_r^{1/p-1/2}\Phi(r\sqrt{\mathcal{L}})\|_{p\to2} &= \|V_r^{1/p-1/2}\Phi(r\sqrt{\mathcal{L}})(1+r^2\mathcal{L})^N V_r^{1/2-1/p}V_r^{1/p-1/2}(1+r^2\mathcal{L})^{-N}\|_{p\to2}\\
&\le C\|\Phi(r\sqrt{\mathcal{L}})(1+r^2\mathcal{L})^N\|_{2\to2}\|V_r^{1/p-1/2}(1+r^2\mathcal{L})^{-N}\|_{p\to2}\\
&\le C\|V_r^{1/p-1/2}(1+r^2\mathcal{L})^{-N}\|_{p\to2},
\end{aligned}$$

where we choose $N > Q$ with Q the number from (D_Q). Notice that for any $f \in L^p(X,\mu)$ one has

$$\begin{aligned}
\|V_r^{1/p-1/2}(1+r^2\mathcal{L})^{-N}f\|_2 &\le C\int_0^\infty\left(\int_X\left|e^{-s}s^{N-1}V_r(x)^{1/p-1/2}e^{-sr^2\mathcal{L}}f(x)\right|^2 d\mu(x)\right)^{1/2} ds\\
&\le C\int_0^\infty e^{-s}s^{N-1}\|V_r^{1/p}e^{-sr^2\mathcal{L}}f\|_\infty^{1-p/2}\left(\int_X\left|e^{-sr^2\mathcal{L}}f(x)\right|^p d\mu(x)\right)^{1/2} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^\infty e^{-s}s^{N-1}\left\|\frac{V_r}{V_{\sqrt{s}r}}\right\|_\infty^{1/p-1/2}\|f\|_p^{1-p/2}\|f\|_p^{p/2}\,ds\\
&\leq C\int_0^\infty e^{-s}s^{N-1}\frac{1}{(s\wedge 1)^{Q/2(1/p-1/2)}}\|f\|_p\,ds\\
&\leq C\|f\|_p.
\end{aligned}
$$

Above in the third inequality, we used the fact that

$$
\begin{aligned}
|V_r^{1/p}(x)e^{-sr^2\mathcal{L}}f(x)| &\leq \frac{CV_r(x)^{1/p}}{V_{\sqrt{sr^2}}(x)^{1/p}}\int_X \frac{V_{\sqrt{sr^2}}(x)^{1/p}}{V_{\sqrt{sr^2}}(x)}e^{-\frac{d(x,y)^2}{c\sqrt{sr^2}}}|f(y)|\,d\mu(y)\\
&\leq \frac{CV_r(x)^{1/p}}{V_{\sqrt{sr^2}}(x)^{1/p}}\|f\|_p\left(\int_X \frac{1}{V_{\sqrt{sr^2}}(x)}e^{-\frac{d(x,y)^2}{c\sqrt{sr^2}}}\,d\mu(y)\right)^{(p-1)/p}\\
&\leq \frac{CV_r(x)^{1/p}}{V_{\sqrt{sr^2}}(x)^{1/p}}\|f\|_p.
\end{aligned}
$$

The claim is proved.

Claim 2. For each $p\in(2,\infty]$, if (G_p) holds, then there exists $C>0$ such that

$$
\sup_{r>0}\|rV_r^{1-1/p}|\nabla\Phi(r\sqrt{\mathcal{L}})^2|\|_{1\to p}\leq C.
$$

By Claim 1 and [20, Proposition 4.1.1] again, we have

$$
\begin{aligned}
\|rV_r^{1-1/p}|\nabla\Phi(r\sqrt{\mathcal{L}})^2|\|_{1\to p} &= \|rV_r^{1-1/p}|\nabla\Phi(r\sqrt{\mathcal{L}})|V_r^{-1/2}V_r^{1/2}\Phi(r\sqrt{\mathcal{L}})\|_{1\to p}\\
&\leq C\|rV_r^{1-1/p}|\nabla\Phi(r\sqrt{\mathcal{L}})|V_r^{-1/2}\|_{2\to p}\|V_r^{1/2}\Phi(r\sqrt{\mathcal{L}})\|_{1\to 2}\\
&\leq C\|rV_r^{1-1/p}|\nabla\Phi(r\sqrt{\mathcal{L}})|V_r^{-1/2}\|_{2\to p}\\
&\leq Cr\||\nabla\Phi(r\sqrt{\mathcal{L}})|V_r^{1/2-1/p}\|_{2\to p}\\
&\leq Cr\||\nabla(1+r^2\mathcal{L})^{-1}|\|_{p\to p}\|(1+r^2\mathcal{L})\Phi(r\sqrt{\mathcal{L}})V_r^{1/2-1/p}|\|_{2\to p}.
\end{aligned}
$$

Claim 1 together with a duality argument easily implies

$$
\sup_{r>0}\|(1+r^2\mathcal{L})\Phi(r\sqrt{\mathcal{L}})V_r^{1/2-1/p}|\|_{2\to p}<\infty,
$$

while (G_p) implies that

$$
\||\nabla(1+r^2\mathcal{L})^{-1}|\|_{p\to p}\leq C\int_0^\infty\left\||\nabla e^{-t(1+r^2)\mathcal{L}}|\right\|_{p\to p}\,dt\leq\frac{C}{r}.
$$

Combining these two estimate proves the second claim.

Step 2. *Completion of the proof.*

Suppose first that $u \in W^{1,2}(3B)$, $B = B(x_0, r)$, satisfies $\mathcal{L}u = 0$ in $3B$. By Claim 2 and the validity of (G_{p_0}), we then have

$$\left\| rV_r^{1-1/p_0} \left|\nabla\Phi(r\sqrt{\mathcal{L}})^2(u\chi_{3B})(\cdot)\right|\right\|_{p_0} \le C\|u\|_{L^1(3B)}.$$

The doubling condition together with Lemma 4.6 implies that

$$\||\nabla u|\|_{L^{p_0}(B)} \le \frac{1}{rV_r(x_0)^{1-1/p_0}}\|rV_r^{1-1/p_0}|\nabla\Phi(r\sqrt{\mathcal{L}})^2(u\chi_{3B})(\cdot)|\|_{p_0} \le C\frac{1}{rV_r(x_0)^{1-1/p_0}}\|u\|_{L^1(3B)},$$

i.e.,

$$\left(\fint_B |\nabla u|^{p_0}\,d\mu\right)^{1/p_0} \le \frac{C}{r}\fint_{3B}|u|\,d\mu.$$

Finally following the same argument as in **Step 4** of proof of Theorem 3.6, we see that (RH_{p_0}) holds, which completes the proof. □

Remark 4.10. Using Claim 1 from Step 1 and [20, Proposition 4.1.1] one can see that for each $r > 0$

$$\begin{aligned}
\|V_r\Phi(r\sqrt{\mathcal{L}})^2\|_{1\to\infty} &= \|V_r\Phi(r\sqrt{\mathcal{L}})V_r^{-1/2}V_r^{1/2}\Phi(r\sqrt{\mathcal{L}})\|_{1\to\infty}\\
&\le \|V_r\Phi(r\sqrt{\mathcal{L}})V_r^{-1/2}\|_{2\to\infty}\|V_r^{1/2}\Phi(r\sqrt{\mathcal{L}})\|_{1\to 2}\\
&\le \|\Phi(r\sqrt{\mathcal{L}})V_r^{1/2}\|_{2\to\infty}\|V_r^{1/2}\Phi(r\sqrt{\mathcal{L}})\|_{1\to 2}\\
&\le C.
\end{aligned}$$

This together with Lemma 4.6 then gives a simple proof of Proposition 2.1.

We can now finish the proofs of Theorem 1.2 and Theorem 1.6, and their corollaries.

Proof of Theorem 1.2. $(RH_\infty) \implies (GLY_\infty)$ is contained in Proposition 4.1, $(GLY_\infty) \implies (G_\infty)$ is straightforward and is contained in Proposition 4.2 (see [7, p.919]), and $(G_\infty) \implies (RH_\infty)$ is contained in Theorem 4.9.

$(GBE) \implies (GLY_\infty)$ follows from [7, Lemma 3.3] whose proof only requires (D) and (UE). Notice that (GLY_∞) together with (UE) implies (LY), and therefore (P_2); see [14, Theorem 3.4]. Using (D) and (P_2), $(GLY_\infty) \implies (GBE)$ then also follows from the same proof of [7, Lemma 3.3]. □

Proof of Corollary 1.3. Note that (P_2) implies (P_∞) and (LY) (cf. [99, 108]), in particular $(P_{\infty,\mathrm{loc}})$ and (UE). □

Proof of Corollary 1.4. If (X, d, μ) is a Riemannian manifold, then for any locally smooth function v with bounded gradient ∇v on a ball B, $B = B(x_0, r)$, it holds that

$$\fint_B |v - v_B|\,d\mu \le \fint_B\fint_B |v(x) - v(y)|\,d\mu(x)\,d\mu(y) \le Cr\||\nabla v|\|_{L^\infty(B)}.$$

Since harmonic functions are locally smooth on a Riemannian manifold, this together with the assumption (RH_∞) implies that the conclusion of Lemma 3.3 holds under the current assumptions. Therefore, (D) and (UE) are enough to guarantee $(RH_\infty) \Longrightarrow (GLY_\infty)$ if (X, d, μ) is a Riemannian manifold, by the proof of Theorem 1.2.

The implications $(GLY_\infty) \Longrightarrow (G_\infty)$ and $(G_\infty) \Longrightarrow (RH_\infty)$ are contained in Proposition 4.2 and Theorem 4.9, respectively, requiring only (D) and (UE).

$(GBE) \Longrightarrow (GLY_\infty)$ is straightforward; see [7, Lemma 3.3]. On the other hand, since under (D) and (GLY_∞), (P_2) holds by [34, Corollary 2.2] (see also [14, Theorem 3.4]), one can apply [7, Lemma 3.3] to see that $(GLY_\infty) \Longrightarrow (GBE)$. □

Proof of Theorem 1.6. $(RH_p) \Longrightarrow (GLY_p)$ is contained in Proposition 4.3, $(GLY_p) \Longrightarrow (G_p)$ is explained in Proposition 4.4, and $(G_p) \Longrightarrow (RH_p)$ is contained in Theorem 4.9. □

Proof of Corollary 1.7. The conclusion holds, since (P_2) implies $(P_{2,\mathrm{loc}})$ and (UE) (cf. [20, 99, 107]). □

5 Riesz transforms

In this section we apply our results to the Riesz transform. The following result was essentially proved by Auscher, Coulhon, Duong and Hofmann [7]; see [14]. As we already said, (D) together with (P_2) guarantees (R_p) for all $p \in (1, 2]$, see [32].

Theorem 5.1. *Assume that the doubling Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies (P_2). Let $p_0 \in (2, \infty)$. Then the following statements are equivalent:*

(i) (R_p) holds for all $p \in (2, p_0)$.

(ii) (G_p) holds for all $p \in (2, p_0)$.

First we record the open-ended character of condition (RH_p).

Lemma 5.2. *Let $(X, d, \mu, \mathscr{E})$ be a doubling Dirichlet metric measure space.*

(i) If (P_2) holds, then there exists $\varepsilon > 0$, such that (RH_p) holds for each $p \in (2, 2 + \varepsilon)$.

(ii) If there exists $p_0 \in (2, \infty)$ such that (P_{p_0}) and (RH_{p_0}) holds, then there exists $\varepsilon_1 > 0$ such that (RH_p) holds for each $p \in (2, p_0 + \varepsilon_1)$.

Proof. (i) By the self-improving property of (P_2) from [78] (see Appendix A.3), we have that there exists $0 < \tilde{\varepsilon} < 1$ such that for each ball $B = B(x, r)$ and every $v \in W^{1,2}(B)$

$$\fint_B |v - v_B|\, d\mu \le Cr \left(\fint_B |\nabla v|^{2-\tilde{\varepsilon}}\, d\mu \right)^{1/(2-\tilde{\varepsilon})},$$

where C is independent of B and v. Therefore, by Lemma 2.4, for each $u \in W^{1,2}(2B)$ satisfying $\mathcal{L}u = 0$ in $2B$, $B = B(x_0, r)$, it holds that

$$\left(\fint_B |\nabla u|^2\, d\mu \right)^{1/2} = \left(\fint_B |\nabla (u - u_{2B})|^2\, d\mu \right)^{1/2} \le \frac{C}{r} \fint_{2B} |u - u_{2B}|\, d\mu \le C \left(\fint_{2B} |\nabla u|^{2-\tilde{\varepsilon}}\, d\mu \right)^{1/(2-\tilde{\varepsilon})}.$$

By applying the Gehring Lemma (cf. [115] or [19, Chapter 3]), we see that there exists $\varepsilon > 0$ such that, for each $p \in (2, 2+\varepsilon)$,

$$\left(\fint_{B(x_0,r/2)} |\nabla u|^p \, d\mu\right)^{1/p} \le C\left(\fint_{B(x_0,r)} |\nabla u|^{2-\tilde{\varepsilon}} \, d\mu\right)^{1/(2-\tilde{\varepsilon})} \le C\left(\fint_{B(x_0,r)} |\nabla u|^2 \, d\mu\right)^{1/2} \le \frac{C}{r}\fint_{2B} |u| \, d\mu.$$

Applying the geometric doubling lemma, Lemma 3.11, as in Step 4 of the proof of Theorem 3.6, we conclude that (RH_p) holds for each $p \in (2, 2+\varepsilon)$.

(ii) The second statement follows by noticing that (P_{p_0}) implies $(P_{p_0-\hat{\varepsilon}})$ for some $\hat{\varepsilon} > 0$ (cf. [78] or Appendix A.3). This and (RH_{p_0}) imply

$$\left(\fint_B |\nabla u|^{p_0} \, d\mu\right)^{1/p_0} = \left(\fint_B |\nabla(u-u_{2B})|^{p_0} \, d\mu\right)^{1/p_0} \le \frac{C}{r}\fint_{2B} |u-u_{2B}| \, d\mu \le \left(\fint_{2B} |\nabla u|^{p_0-\hat{\varepsilon}} \, d\mu\right)^{1/(p_0-\hat{\varepsilon})},$$

if $u \in W^{1,2}(2B)$ satisfies $\mathcal{L}u = 0$ in $2B$, $B = B(x_0, r)$.

Using the Gehring Lemma once more gives the existence of $\varepsilon_1 > 0$ such that (RH_p) holds for each $p \in (2, p_0 + \varepsilon_1)$. □

We can now prove Theorem 1.9 by using Theorem 1.6 and the lemma above.

Proof of Theorem 1.9. Notice that under the assumption of (D), (UE) and (P_p), (RH_p) or (G_p) implies (P_2); see [15, Corollary 2.8] and [14, Theorem 6.3]. The equivalence of (RH_p) and (G_p) follows from Corollary 1.7, and we only need to prove that $(G_p) \Longleftrightarrow (R_p)$.

Step 1. $(R_p) \Longrightarrow (G_p)$.

This is well known (cf. [7]), but we recall the argument for the sake of completeness. Assume (R_p). By analyticity of the heat semigroup on $L^p(X,\mu)$ (cf. [104])

$$\|\mathcal{L}^{1/2}e^{-t\mathcal{L}}\|_{p\to p} \le \frac{C}{\sqrt{t}}.$$

Therefore, we conclude via (R_p) that

$$\||\nabla H_t|\|_{p\to p} = \||\nabla \mathcal{L}^{-1/2}\mathcal{L}^{1/2}H_t|\|_{p\to p} = \||\nabla \mathcal{L}^{-1/2}\mathcal{L}^{1/2}e^{-t\mathcal{L}}|\|_{p\to p} \le \frac{C}{\sqrt{t}},$$

i.e., (G_p) holds.

Step 2. $(G_p) \Longrightarrow (R_p)$.

Suppose that (G_p) holds. According to Corollary 1.7, we know that (RH_p) holds. By Lemma 5.2, there exists $\varepsilon_1 > 0$ such that (RH_q) holds for each $q \in (2, p+\varepsilon_1)$. This, together with Theorem 1.6 and Theorem 5.1 above, yields that (R_q) holds for each $q \in (2, p+\varepsilon_1)$, and in particular, (R_p) holds, as desired. □

Corollary 1.10 now easily follows from Lemma 5.2 and Theorem 1.9.

Proof of Corollary 1.10. This corollary follows by combining Theorem 1.9 and Lemma 5.2. □

Remark 5.3. One can also find a characterization of boundedness of local Riesz transforms via boundedness of the gradient heat semigroup for small time, $(G_p^{\rm loc})$, in [7]. We expect that the ideas of this paper can be employed to show that L^p-boundedness of the local Riesz transform is point-to-point equivalent to $(G_p^{\rm loc})$ for each $p \in (2,\infty)$.

6 Sobolev inequalities and isoperimetric inequality

In this section, following the central idea of [75, 76] and using Theorem 3.2, we show that (RH_p) for $p > 2$ yields a Sobolev inequality or an isoperimetric inequality. Combining this and Theorem 1.9, we find a new necessary condition for quantitative regularity of harmonic functions and heat kernels, and for boundedness of Riesz transforms.

6.1 Sobolev inequalities

Recall the definition of the Sobolev inequality $(S_{q,p})$ given in Section 1.3. In our setting, under (D) and (UE), $(S_{q,2})$ holds for some $q > 2$ (see Section 2.1) and hence by Hölder $(S_{q,p})$ holds for every $p \geq 2$. Here we are interested in the non-trivial range $p \in [1,2)$.

Theorem 6.1. *Let $(X,d,\mu,\mathscr{E})$ be a Dirichlet metric measure space. Assume that $(X,d,\mu,\mathscr{E})$ satisfies (D_Q), $Q > 2$, and that (UE) and $(P_{2,\rm loc})$ hold. Let $p_0 \in (2,\infty)$. Suppose that one of the mutually equivalent conditions (RH_{p_0}), (GLY_{p_0}), (G_{p_0}), holds. Then the Sobolev inequality $(S_{q,p})$ holds for all $p \in [\frac{p_0}{p_0-1}, 2]$ and $q \in [1, \frac{pQ}{Q-p})$.*

Proof. Let $p_0' = \frac{p_0}{p_0-1}$ and $q \in [1, \frac{p_0'Q}{Q-p_0'})$. Then the conjugate exponent q' of q satisfies $q' > \frac{Qp_0}{Q+p_0}$. For any $B = B(x_0,r)$ and $g \in L^{q'}(B)$, let $f \in W_0^{1,2}(B)$ be the solution to $\mathcal{L}f = g$ in $B(x_0,2r)$, see Lemma 2.5. For a compactly supported Lipschitz function h on B, we have

$$\begin{aligned}
\left|\int_B h(x)g(x)\,d\mu(x)\right| &= \left|\int_B h\mathcal{L}f\,d\mu\right| \\
&= \left|\int_B \langle\nabla h,\nabla f\rangle\,d\mu\right| \\
&\leq C\||\nabla h|\|_{L^{p_0'}(B)}\||\nabla f|\|_{L^{p_0}(B)} \\
&= C\||\nabla h|\|_{L^{p_0'}(B)}[V(x_0,r)]^{1/p_0}\left(\fint_B |\nabla f|^{p_0}\right)^{1/p_0}.
\end{aligned}$$

Thus, by Theorem 3.6,

$$\left|\int_B h(x)g(x)\,d\mu(x)\right| \leq C\||\nabla h|\|_{L^{p_0'}(B)}\frac{[V(x_0,r)]^{1/p_0}}{r}\left(\fint_{B(x_0,2r)}|f|\,d\mu + r^2\left(\fint_{B(x_0,2r)}|g|^{q'}\,d\mu\right)^{1/q'}\right).$$

Since $p_0 > 2$, $\frac{1}{q'} < \frac{1}{p_0} + \frac{1}{Q} < \frac{1}{2} + \frac{1}{Q}$, and therefore we may apply Lemma 2.6, which yields

$$\fint_{B(x_0,2r)} |f|\, d\mu \le Cr^2 \left(\fint_{B(x_0,2r)} |g|^{q'}\, d\mu\right)^{1/q'},$$

and hence

$$\left|\int_B h(x)g(x)\, d\mu(x)\right| \le C\||\nabla h|\|_{L^{p_0'}(B)} r\, [V(x_0,r)]^{1/p_0} \left(\fint_{B(x_0,2r)} |g|^{q'}\, d\mu\right)^{1/q'}.$$

Taking the supremum over all g with $\|g\|_{L^{q'}(B)} \le 1$ yields

$$\left(\fint_B |h|^q\, d\mu\right)^{1/q} \le Cr \left(\fint_B |\nabla h|^{p_0'}\, d\mu\right)^{1/p_0'},$$

i.e. $(S_{q,p_0'})$. Finally $(S_{q,p})$ follows by the Hölder inequality for every $p \in [p_0', 2]$ and $q \in [1, \frac{Qp}{Q-p})$, as desired. □

6.2 Isoperimetric inequality

In this section, we give an application of Theorem 1.2 to isoperimetric inequalities. The following definition of perimeter can be found in [2, 89] (see Appendix A.2).

For an open set $\Omega \subset X$, denote by $\mathrm{Lip}(\Omega)$ ($\mathrm{Lip}_{\mathrm{loc}}(\Omega)$) the space of all (locally) Lipschitz functions on Ω, and by $\mathrm{Lip}_0(\Omega)$ the space of all Lipschitz functions with compact support in Ω. Denote by $\mathscr{B}(X)$ the collection of all Borel sets in X.

Definition 6.2. *Let $E \in \mathscr{B}(X)$ and $\Omega \subset X$ open. The perimeter of E in Ω, denoted by $P(E,\Omega)$, is defined by*

$$P(E,\Omega) = \inf\left\{\liminf_{h\to\infty} \int_\Omega |\nabla v_h|\, d\mu :\ \{v_h\}_h \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega), v_h \to \chi_E \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\}. \tag{6.1}$$

E is a set of finite perimeter in X if $P(E,X) < \infty$.

The following proof is adapted from [76]. We include it for completeness.

Theorem 6.3. *Let $(X,d,\mu,\mathscr{E})$ be a Dirichlet metric measure space. Assume that $(X,d,\mu,\mathscr{E})$ satisfies (D_Q), $Q \ge 2$, and that $(P_{\infty,\mathrm{loc}})$ and (UE) hold. Suppose that one of the mutually equivalent conditions (RH_∞), (GLY_∞), (G_∞), (GBE), holds. Then, for every bounded Borel set E and every $x \in E$,*

$$\mu(E)^{1-\frac{1}{Q}} \le C\frac{r}{[V(x,r)]^{1/Q}} P(E,X).$$

where we choose $r >$ diam(E) such that $E \subset B(x,r)$.

Proof. Let E be a bounded Borel set in X. We can find a ball $B = B(x, r)$ with center in E and radius $r > \mathrm{diam}(E)$ such that $E \subset\subset B$.

Consider the Poisson equation $\mathcal{L}f = \chi_E$ in $2B$. Then there exists a solution $u \in W_0^{1,2}(2B)$ to the equation by Lemma 2.6. By using (RH_∞) and Theorem 3.2, we obtain that for each $p > \frac{2Q}{Q+2}$, there exists $C = C(C_D, C_{LS}, C_P(1), p) > 0$ such that, for almost every $y \in B$,

$$|\nabla f(y)| \le C \left\{ \frac{1}{r} \fint_{2B} |f|\, d\mu + \sum_{j \le [\log_2 r]} 2^j \left(\fint_{B(y,2^j)} |\chi_E|^p \, d\mu \right)^{1/p} \right\}.$$

By Lemma 2.6 we have

$$\frac{1}{r} \fint_{2B} |f|\, d\mu \le Cr \left(\fint_B |\chi_E|^Q \, d\mu \right)^{1/Q} \le \frac{Cr\mu(E)^{1/Q}}{\mu(B)^{1/Q}}. \tag{6.2}$$

Fix $p \in (\frac{2Q}{Q+2}, Q)$. A direct calculation (cf. [76, Proposition 4.1]) shows that for any $y \in B$

$$\sum_{j \le [\log_2 r]} 2^j \left(\fint_{B(y,2^j)} |\chi_E|^p \, d\mu \right)^{1/p} \le C \frac{r}{\mu(B)^{1/Q}} \mu(E)^{1/Q}. \tag{6.3}$$

By the definition of perimeter, we may choose a sequence of Lipschitz functions $\{v_h\}_h \subset \mathrm{Lip}_0(B)$, $v_h \to \chi_E$ in $L^1(B)$ such that

$$\lim_{h \to \infty} \int_B |\nabla v_h|\, d\mu = P(E, X).$$

As f is a solution to the Poisson equation $\mathcal{L}u = \chi_E$ in $2B$, we then have for each $h \in \mathbb{N}$,

$$\int_{2B} \nabla u \cdot \nabla v_h \, d\mu = \int_{2B} \chi_E v_h \, d\mu = \int_E v_h \, d\mu.$$

Since $\operatorname{supp} v_h \subset B$, by using the estimates (6.2) and (6.3), and passing h to infinity, we obtain

$$\begin{aligned} \mu(E) &= \lim_{h \to \infty} \|v_h\|_{L^1(B)} = \lim_{h \to \infty} \int_{2B} \nabla u \cdot \nabla v_h \, d\mu \le \lim_{h \to \infty} \|\nabla v_h\|_{L^1(B)} \||\nabla u|\|_{L^\infty(B)} \\ &\le CP(E, X) \frac{r}{\mu(B)^{1/Q}} \mu(E)^{1/Q}, \end{aligned}$$

which gives the conclusion and completes the proof. □

Remark 6.4. We remark that Theorem 6.1 and Theorem 6.3 admit localisation. Since the arguments are the same as for the global versions, we leave them to interested readers.

7 Examples

In this section, we apply our results to several concrete examples of interest. Notice that since our assumptions are quite mild ((D), (UE) and $(P_{2,\mathrm{loc}})$), our results have broad applications. Below we will mainly concentrate on three different settings, and we refer the readers to [4, 5, 7, 12, 17, 46, 47, 80, 112] for more examples.

7.1 Riemannian metric measure spaces

Let us begin with some examples arising from Riemannian geometry.

Example 1. Riemannian metric measure spaces with Ricci curvature bounded from below, i.e., $RCD^*(K,N)$ spaces, $K \in \mathbb{R}$ and $N \in [1,\infty)$; see [5, 46, 54]. Examples satisfying $RCD^*(K,N)$ include complete Riemannian manifolds with dimension not bigger than N and Ricci curvature not less than K, and complete Alexandrov spaces with dimension not bigger than N and curvature not less than K. An important fact is that the $RCD^*(K,N)$ condition is stable under Gromov-Hausdorff convergence, which means that a Gromov-Hausdorff limit, of a sequence of manifolds satisfying $RCD^*(K,N)$, satisfies also $RCD^*(K,N)$.

The $RCD^*(K,N)$ condition can be defined as follows; see [5, 46, 54]. Let $(X,d,\mu,\mathscr{E})$ be a Dirichlet metric measure space satisfying $\operatorname{supp}\mu = X$ and $V(x,r) \le Ce^{cr^2}$ for some $C,c > 0$, $x \in X$ and each $r > 0$. We call $(X,d,\mu,\mathscr{E})$ a $RCD^*(K,N)$ space, where $K \in \mathbb{R}$ and $N \in [1,\infty)$, if for all $f \in \mathscr{D}$ and each $t > 0$, it holds that

$$|\nabla H_t f(x)|^2 + \frac{4Kt^2}{N(e^{2Kt}-1)}|\mathcal{L}H_t f(x)|^2 \le e^{-2Kt}H_t(|\nabla f|^2)(x). \tag{7.1}$$

Equivalently, (X,d,μ) is a $RCD^*(K,N)$ space if the Cheeger energy is a quadratic form and $CD^*(K,N)$ condition holds; see [5, 46].

Under the $RCD^*(K,N)$ condition, the (local) doubling condition was established in [88, 110], and the (local) Poincaré inequality was established in [94]. The doubling condition and Poincaré inequality have the same behaviour as in the case of classical smooth manifolds.

Gradient estimates for harmonic functions and heat kernels on $RCD^*(K,N)$ spaces were established in [52, 73, 74, 118]. Our results recover these gradient estimates in a more obvious and simple way. By the validity of the (local) doubling condition and (local) Poincaré inequality, the definition (7.1) implies directly (RH_∞), (G_∞) and (R_p) for all $p \in (1,\infty)$ if $K \ge 0$, and their local versions if $K < 0$.

Example 2. On an n-dimensional conical manifold with compact basis N without boundary, $C(N) := \mathbb{R}^+ \times N$, let λ_1 be the smallest nonzero eigenvalue of the Laplacian on the basis (see [29, 92] for studies on the first eigenvalue). By a result of Li [82], the Riesz transform is bounded on $L^p(C(N))$ for all $p \in (1,p_0)$ and not bounded for $p \ge p_0$, where

$$p_0 := n\left(\frac{n}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_1}\right)^{-1}$$

if $\lambda_1 < n-1$ and $p_0 = \infty$ otherwise; see also [7].

Therefore by Theorem 1.9, we see that (RH_p) and (G_p) hold for all $p < p_0$. Moreover, if $\lambda_1 < n-1$, then (RH_p) and (G_p) do not hold on $C(N)$ for any $p \ge p_0$.

Example 3. By a result of Zhang [116], it is known that Yau's gradient estimate for harmonic functions is globally stable under certain perturbations of the metric in the following sense.

Let M be an n-dimensional Riemannian manifold, $n > 2$, suppose that the volume of each ball $B(x,r)$ is comparable with r^n for any $x \in M$ and $r > 0$, and assume that the L^2-Poincaré inequality holds. If

$$Ric(x) \geq -\frac{\varepsilon}{1 + d(x,x_0)^{2+\delta}}$$

for a fixed $x_0 \in M$, $\delta > 0$ and a sufficiently small $\varepsilon > 0$, then Yau's gradient estimate (Y_∞) holds with $K = 0$. This holds, in particular, if M is a small compact perturbation of a manifold of dimension at least 3 that has nonnegative Ricci curvature and maximum volume growth, i.e., $V(x,r) \sim r^n$.

By Lemma 2.3, (Y_∞) with $K = 0$ is equivalent to our (RH_∞). Therefore, by Theorem 1.2, we see that (RH_∞), (G_∞), (GLY_∞) and (GBE) hold on these spaces.

Example 4. Let M be a Riemannian manifold that is the union of a compact part, M_0, and a finite number of Euclidean ends, $\mathbb{R}^n \setminus B(0,1)$, $n \geq 3$, each of which carries the standard metric. The volume of balls in M grows as $V(x,r) \sim r^n$, in particular, volume is a doubling measure. Moreover, (UE) holds as a consequence of the Sobolev inequality (LS_q), $q > 2$. By [23], the Riesz transform is bounded on $L^p(M)$ if and only if $p \in (1,n)$. Since (R_p) implies (G_p), Theorem 1.6 implies that (RH_p) also holds if $1 < p < n$. Notice also that, while $(P_{2,\mathrm{loc}})$ holds on M, (P_p) does not hold for any $p \leq n$; see [23, 32]. Indeed, if (P_n) holds, then by [78] one has $(P_{n-\epsilon})$ for some $\epsilon > 0$. This together with $(G_{n-\epsilon})$ will imply (P_2) by [20], which however is impossible.

Actually, it is rather easy to see that (RH_p) holds on M for $p < n$. Suppose that u is a harmonic function on $2B$. There is nothing to prove if r is small, since in this case, it holds

$$\||\nabla u|\|_{L^\infty(B)} \leq \frac{C}{r} \fint_B |u|\, d\mu.$$

If $r >> 1$, then by applying the pointwise Yau's gradient estimate (Y_∞) to $u + \|u\|_{L^\infty(\frac{3}{2}B)}$, we conclude that

$$|\nabla u(x)| \leq \frac{C}{1 + \operatorname{dist}(x, M_0)} \left(u(x) + \|u\|_{L^\infty(\frac{3}{2}B)} \right)$$

for each $x \in B$, which implies, if $p < n$,

$$\left(\fint_B |\nabla u|^p\, d\mu \right)^{1/p} \leq C\|u\|_{L^\infty(\frac{3}{2}B)} \left(\fint_B \frac{1}{(1 + \operatorname{dist}(x, M_0))^p}\, d\mu \right)^{1/p} \leq \frac{C}{r}\|u\|_{L^\infty(\frac{3}{2}B)} \leq \frac{C}{r} \fint_{2B} |u|\, d\mu.$$

Notice that, however, $(\widetilde{RH}_p)$ does not hold on M for any $p > 2$. Indeed, if $(\widetilde{RH}_p)$ holds, then we have

$$\left(\int_B |\nabla u|^p\, d\mu \right)^{1/p} \leq C\mu(B)^{1/p-1/2} \left(\int_{2B} |\nabla u|^2\, d\mu \right)^{1/2},$$

if u is harmonic on $2B$. By [84, Theorem 2.1], there exists a bounded, non-constant harmonic function u with finite Dirichlet energy. Applying the above estimate to u and letting the radius of B tend to infinity, we see that $\||\nabla u|\|_p = 0$, which cannot be true.

Example 5. Consider a complete, non-compact, connected Riemannian manifold M. Suppose that a finitely generated discrete group G acts properly and freely on M by isometries, such that the orbit space $M_1 = M/G$ is a compact manifold. In other words, M is a Galois covering manifold of the compact Riemannian manifold M_1, with deck transformation group (isomorphic to) G. The most simple example is $M = \mathbb{R}^D$ endowed with a Riemannian metric which is periodic under the standard action of $G = \mathbb{Z}^D$ by translations.

Assuming that G has polynomial volume growth of some order $D \geq 1$, Dungey [40, Theorem 1.1] (see also [39]) showed that (GLY_∞) holds on M. Our Theorem 1.2 then implies that (RH_∞), (G_∞), (GLY_∞) and (GBE) hold on these spaces. Indeed, by using the group structure of M, it is relative easier to show that (RH_∞) holds on M; see Appendix A.5.

7.2 Carnot-Carathéodory spaces

A large class of examples that our results can be applied to come from Carnot-Carathéodory spaces; we refer the readers to [11, 12, 49, 58, 61, 71, 90] for background and recent developments.

Let M be a smooth, connected manifold and μ a Borel measure. Let $\{X_i\}_{i=1,\cdots,m}$ be Lipschitz vector fields on M, with real coefficients. The "carré du champ" operator Γ is given as

$$\Gamma(f) := \sum_{i=1}^{m} |X_i f|^2$$

for each $f \in C^\infty(M)$, where the corresponding Dirichlet form $\int_M \sum_i X_i f X_i g \, d\mu$ generalises a second-order diffusion operator L.

A tangent vector $v \in T_x M$ is called subunit for L at x if $v = \sum_{i=1}^m a_i X_i(x)$, with $\sum_{i=1}^m a_i^2 \leq 1$; see [48]. A Lipschitz curve $\gamma : [0, T] \mapsto M$ is called subunit for L if $\gamma'(t)$ is subunit for L at $\gamma(t)$ for a.e. $t \in [0, T]$. The subunit length of γ, $\ell(\gamma)$, is given as T. We assume that for any $x, y \in M$, there always exists a subunit curve γ joining x to y. The Carnot-Carathéodory distance then is defined as

$$d_{cc}(p, q) := \inf\{\ell(\gamma) : \ \gamma \text{ is a subunit curve joining } p \text{ to } q\}.$$

Notice that for any $x, y \in M$, the Carnot-Carathéodory distance $d_{cc}(x, y)$ is the same as $d(x, y)$ induced from the Dirichlet forms; see [12, 22].

Once again, our results can be applied to this setting as soon as a (local) doubling condition and an (local) L^2-Poincaré inequality are available. Notice that all Carnot groups equipped with the Lebesgue measure and the natural vector fields satisfy an L^2-Poincaré inequality; see [61] for instance.

For general vector fields satisfying the Hörmander condition (cf. [48, 61, 71, 90]), it is known that the doubling condition and L^2-Poincaré inequality hold locally with constants depending on the balls under consideration, which is not sufficient in order to apply our results. However, the potential estimates for the Poisson equation from Section 3, Theorem 3.5 and Theorem 3.2, still work in these settings.

As we recalled in the introduction, by Theorem 1.2, (RH_∞), (GLY_∞), (G_∞) and (GBE) hold on any Lie groups of polynomial growth (cf. [1, 97]), and more generally, on sub-Riemannian manifolds satisfying Baudoin-Garofalo's curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ (cf. [12]) with $\rho_1 \geq 0, \rho_2 > 0, \kappa \geq 0$ and $d > 0$.

Examples satisfying $CD(\rho_1, \rho_2, \kappa, d)$ include all Sasakian manifolds whose horizontal Webster-Tanaka-Ricci curvature is bounded from below, all Carnot groups with step two, and wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is bounded from below; see [12, Section 2].

7.3 Degenerate (sub-)elliptic/parabolic equations

Our results are also applicable to degenerate (sub-)elliptic/parabolic equations on Euclidean spaces. It is of course also possible to extend these degenerate equations to general metric measure spaces. For instance, one may consider a Dirichlet form given by

$$\int_X \langle \nabla f(x) \cdot \nabla g(x) \rangle w(x)\, dx,$$

where $\langle \nabla f, \nabla g \rangle$ is the natural energy density of energy on an infinitesimally Hilbertian space (X, d, μ) (cf. [4]), or ∇ is the Cheeger differential operator (cf. [24]), and w is a suitable weight.

We focus on degenerate elliptic/parabolic equations on Euclidean spaces, and we refer the reader to [17, p.133] and [50] for more examples of degenerate (sub-)elliptic equations.

Let w be a Muckenhoupt A_2-weight or a qc-weight, where by qc-weight we mean that $w = |J_f|^{1-\frac{2}{n}}$, where $|J_f|$ denotes the Jacobian of a quasiconformal mapping f on $\mathbb{R}^n$; see [17, 47]. Let $A := (A_{ij}(x))_{i,j=1}^n$ be a symmetric matrix of functions on $\mathbb{R}^n$ satisfying the *degenerate ellipticity condition*, namely, there exist constants $0 < \lambda \leq \Lambda < \infty$ such that, for all $\xi \in \mathbb{R}^n$,

$$\lambda w(x)|\xi|^2 \leq \langle A\xi,\, \xi \rangle \leq \Lambda w(x)|\xi|^2. \tag{7.2}$$

For all $f,\ g \in C_c^\infty(\mathbb{R}^n)$, consider the Dirichlet form given by

$$\int_{\mathbb{R}^n} A(x)\nabla f(x) \cdot \nabla g(x)\, dx. \tag{7.3}$$

Then the intrinsic distance d and the usual Euclidean metric d_E are comparable, that is $d \sim d_E$. On the space $(\mathbb{R}^n, d_E, w(x)\, dx)$, the doubling condition is a well-known property of a Muckenhoupt weight or follows from properties of quasiconformal mappings, and an L^2-Poincaré inequality was established in [47]. From this, one can deduce that a doubling condition and a weak L^2-Poincaré inequality, i.e.

$$\fint_B |f - f_B|\, d\mu \leq Cr \left(\fint_{cB} |\nabla f|^2\, d\mu \right)^{1/2}, \tag{7.4}$$

for all $B = B(x, r)$, for some constant $c \geq 1$, hold on $(\mathbb{R}^n, d, w(x)\, dx)$. By using the results from [61, Section 9], together with the fact that $(\mathbb{R}^n, d, w(x)\, dx)$ is geodesic, we see that $(\mathbb{R}^n, d, w(x)\, dx)$ supports a scale-invariant L^2-Poincaré inequality.

Therefore, our results are applicable to $(\mathbb{R}^n, d, w(x)\,dx)$ as well. We would like to point out that Caffarelli and Peral [21] established a $W^{1,p}$-estimate for elliptic equations in divergence form by using the technique of approximation to a reference equation. Shen [102] employed the techniques from [21] to prove the equivalence of (R_p) and (RH_p), for *uniformly elliptic* operators of divergence form on $\mathbb{R}^n$. Recently, for degenerate elliptic operators with A being complex-valued and satisfying suitable weighted condition, Cruz-Uribe et al. [36] obtained the boundedness of the Riesz transform in an open interval containing 2.

For degenerate equations satisfying condition (7.2) for some A_2-weight or qc-weight, although the heat kernel and harmonic functions are known to be Hölder continuous (cf. [17, 106, 107, 108]), harmonic functions and the heat kernel are not Lipschitz in general; see the examples from the introductions of [73, 79] for instance.

Moreover, given an explicit $p > 2$, we do not even know if the gradients of harmonic functions or heat kernels are locally L^p-integrable. Indeed, in view of Corollary 1.10 and Theorem 1.9, we see that there exists $\varepsilon > 0$ (implicit), such that (RH_p) and (G_p) hold for $p \in (2, 2+\varepsilon)$. However, for an explicitly given $p > 2$, the assumption $w \in A_2$ alone is not sufficient for quantitative L^p-regularity of harmonic functions or heat kernels, in view of Theorem 6.1. Since if (RH_p) or (G_p) holds for some $p > 2$, then one has a Sobolev inequality $(S_{p',q})$ for some $q > p'$ on $(\mathbb{R}^n, d, w(x)\,dx)$, and it is well-known that $w \in A_2$ is not sufficient to guarantee such a Sobolev inequality for (small) p'. It would be of great interest to know how to quantify the regularity of harmonic functions and heat kernels in this case.

Finally we apply our results to the simplest possible form of degenerate elliptic operators in dimension one. The correspond to the Dirichlet form

$$Q_\alpha(f, g) = \int_{\mathbb{R}} |x|^\alpha f'(x) \cdot g'(x)\,dx$$

for some $\alpha > 0$ on $L^2(\mathbb{R}, |x|^\alpha dx)$. The corresponding intrinsic distance coincides with the Euclidean distance. Note that the weight $\omega_\alpha(x) = |x|^\alpha$ belongs to Muckenhoupt class A_p only if $\alpha + 1 < p$. Observe that in the range $0 \le \alpha < 1$ any harmonic function for the operator discussed here is of the form $a\,\mathrm{sign}(x)|x|^{1-\alpha} + b$ for some constants $a, b \in \mathbb{R}$. Hence a simple calculation shows that (RH_p) holds if and only if $\alpha(1-p) > -1$, i.e. $p < (1+\alpha)/\alpha$. It follows from examples and the results obtained in [95] that the L^2-Poincaré inequality holds if and only if $\alpha < 1$ or equivalently if $\omega_\alpha \in A_2$. Now it follows from Theorem 1.9 that for $0 < \alpha < 1$, (R_p) holds also if and only if $p < (1+\alpha)/\alpha$. This range of validity of (R_p) was first obtained in [62, Theorem 5.3] (see also [62, Section 6.3]). Theorem 1.9 yields this result avoiding relatively tedious calculations. We point out that the heat kernel and harmonic functions are usually discontinuous (at the point $x = 0$) for $\alpha \ge 1$, see [45]. We refer the reader to [62] for more about the Riesz transform.

A Appendix

A.1 Sobolev spaces on domains

Let $U \subset X$ be an open set. The local Sobolev space $W^{1,2}_{\rm loc}(U)$ is defined to be the collection of all functions $f \in L^2_{\rm loc}(U)$, such that for any compact set $K \subset U$ there exists $F_K \in \mathscr{D}$ satisfying $f = F_K$ a.e. on K; see [59, Definition 2.3]. Notice that bounded closed sets are compact (cf. [59, Theorem 2.11]). So we can find a sequence of compact sets $\{U_j\}_{j=1}^\infty$ such that U_j is contained in the interior of U_{j+1} and $\cup_{j=1}^\infty U_j = U$. Write F_j for F_{U_j}.

By the locality of the Dirichlet form $\mathscr{E}$, for a given $f \in W^{1,2}_{\rm loc}(U)$, one has that for any $j \in \mathbb{N}$ and measurable set $E \subset U_j$

$$\int_{U_j} \chi_E \, d\Gamma(F_j, F_j) = \int_{U_j} \chi_E \, d\Gamma(F_{j+1}, F_{j+1}).$$

This implies that we may consistently define $|\nabla f|$ on U by setting

$$|\nabla f|^2 = \frac{d\Gamma(F_j, F_j)|_U}{d\mu}$$

on U_j.

Now for each $p \geq 2$, the Sobolev space $W^{1,p}(U)$, defined as the collection of all functions $f \in W^{1,2}_{\rm loc}(U)$ satisfying f, $|\nabla f| \in L^p(U)$, is well defined.

A.2 Equivalence of differently defined Sobolev spaces

There are several different types of Sobolev spaces on metric measure spaces: Hajłasz Sobolev spaces [60], Newtonian Sobolev spaces [101], Cheeger's Sobolev spaces [24] etc. We refer the readers to the monographs by Heinonen, Koskela, Shanmugalingam and Tyson [64] and A. Björn and J. Björn [19] for these studies.

We first recall the definition of Hajłasz Sobolev spaces.

Definition A.1. *Let $1 \leq p \leq \infty$. Given a measurable function f on X, a non-negative measurable function g on X is called a Hajłasz gradient of f if there is a set $E \subset X$ with $\mu(E) = 0$ such that for all $x, y \in X \setminus E$,*

$$|f(x) - f(y)| \leq d(x, y)[g(x) + g(y)]. \tag{A.1}$$

The Hajłasz-Sobolev space $M^{1,p}(X)$ is defined to be the set of all functions $f \in L^p(X, \mu)$ that have a Hajłasz gradient $g \in L^p(X, \mu)$. The norm on this space is given by

$$\|f\|_{M^{1,p}(X)} := \|f\|_p + \inf_g \|g\|_p,$$

where the infimum is taken over all Hajłasz gradients of f.

It is known that $M^{1,p}(X)$ embedded continuously into the Newtonian Sobolev space $N^{1,p}(X)$, which was introduced by Shanmugalingam; see [101, Theorem 4.8]. Notice that the embedding $M^{1,p}(X) \hookrightarrow N^{1,p}(X)$ actually holds on any metric measure space (X,d) equipped with a Borel regular measure μ; see [77, Theorem 1.3].

Under the requirements of doubling and Poincaré inequality (P_2), it is known that

$$W^{1,2}(X) = N^{1,2}(X) = W^{1,2}(X),$$

and that Lipschitz functions are dense in these spaces; see [101, 80]. Moreover, for any function $f \in W^{1,2}(X)$, the square root of its density energy, $|\nabla f|$, equals its approximate pointwise Lipschitz constant, $\mathrm{apLip} f$; see [80, Theorem 2.2]. Here,

$$\mathrm{apLip} f(x) := \inf_A \limsup_{y\in A:\, d(x,y)\to 0} \frac{|f(x)-f(y)|}{d(x,y)},$$

where the infimum is taken over all Borel sets $A \subset X$ with a point of density at x.

We note that, without the validity of Poincaré inequality, the above conclusions are not true in general. In particular, it may happen that, $M^{1,p}(X) \subsetneq N^{1,p}(X)$, see [64, 101] for instance.

Nevertheless, for locally Lipschitz functions ϕ, assuming only doubling but not Poincaré, one still has that $|\nabla\phi| = \mathrm{Lip}\,\phi$, a.e.; see [59, Remark 2.20] and [80, Theorem 2.1]. Therefore, our perimeter $P(E;\Omega)$ (see Definition 6.2) coincides with that from [2, 89].

A.3 Self-improving property of Poincaré inequalities

The self-improving property of Poincaré inequality was obtained by Keith and Zhong [78] on a complete doubling metric space. In our setting, together with the fact (X,d) is geodesic, their result gives: if for some $p \in (1,\infty)$ it holds for each ball $B = B(x_0, r)$ and every Lipschitz function f that

$$\fint_B |f - f_B|\, d\mu \le Cr\left(\fint_B |\mathrm{Lip} f|^p\, d\mu\right)^{1/p},$$

then there exists $q \in (1,p)$ such that the above inequality holds with p replaced by q.

From the previous subsection, we see that for each locally Lipschitz function ϕ it holds $|\nabla\phi| = \mathrm{Lip}\,\phi$. This together with a density argument implies that, if (P_p) holds for some $p \ge 2$, then there exists $1 < q < p$ such that for any ball $B = B(x_0, r)$ and any $g \in W^{1,p}(B)$ it holds that

$$\fint_B |g - g_B|\, d\mu \le Cr\left(\fint_B |\nabla g|^q\, d\mu\right)^{1/q},$$

where C is independent of B and g.

Notice that however (P_∞) does not have the self-improving property, see [41, 42].

A.4 Paley-Wiener Estimate

Suppose that $F \in \mathscr{S}(\mathbb{R})$ satisfies $\operatorname{supp}\hat{F} \subset [-1,1]$ and $F(0)=0$, where $\hat{F}$ denotes the Fourier transform of F. By the Paley-Wiener theorem (see [96]), we can extend F to analytic function, $F(z)$, on $\mathbb{C}$, and so also $G(z) = F(z)/z$. It holds obviously that for $|z| \ge 1$, $|G(z)| \le |F(z)|$. Note that Paley-Wieners estimate $|F(z)| \le C_N(1+|z|)^{-N}e^{B|\mathrm{Im}(z)|}$ for $z \in [-B,B]$ is a condition only for large $|z|$, so $|G(z)|$ satisfies the same estimates as $|F(z)|$, which implies that $\hat{G} \subset [-1,1]$.

From the above discussion, we see that if F is a Schwartz function, then $G, \hat{F}, \hat{G}$ are all Schwartz functions. It is easy to note that $d_t\hat{G}(t) = \hat{F}(t)$, so if $\hat{F} \subset [-1,1]$ then $\hat{G}(t)$ is constant on both half lines $(-\infty,-1]$ and $[1,\infty)$. But $\hat{G}(t)$ is a Schwartz function, so it has to converge to zero at the ends, which means that $\operatorname{supp}\hat{G} \subset [-1,1]$.

A.5 Gradient estimates on covering manifolds

Let us provide a proof of (RH_∞) on covering manifolds in **Example 5** from Section 7.

Theorem A.2. *Let M be a complete, non-compact, connected Riemannian manifold. Suppose that a finitely generated discrete group G acts properly and freely on M by isometries, such that the orbit space $M_1 = M/G$ is a compact manifold. Assume that G has polynomial volume growth of some order $D \ge 1$. Then (RH_∞) holds on M.*

Let us observe that, due to the group action and the polynomial volume growth of G, (D) and (P_2) hold on M; see [99]. Moreover, by Yau's gradient estimate (cf. [30, 114]), $(RH_{\infty,\mathrm{loc}})$ holds, i.e., for each $r_0 > 1$, there exists $C(r_0) > 0$ such that if u is harmonic in $2B$, $B = B(x_0, r)$, $r < r_0$, it holds that

$$\||\nabla u|\|_{L^\infty(B)} \le \frac{C(r_0)}{r} \fint_{2B} |u|\, d\mu. \qquad (RH_{\infty,\mathrm{loc}})$$

Proof. Since $(RH_{\infty,\mathrm{loc}})$ holds, we only need to prove (RH_∞) for balls of large radii.

Since (D) and (P_2) hold, by applying Proposition 2.2 there exist $C > 0$ and $\gamma \in (0,1)$, such that for each ball $B = B(x_0, r)$ and if u is harmonic on $2B$, it holds for all $x, y \in B(x_0, r)$ that

$$|u(x) - u(y)| \le C\frac{d^\gamma(x,y)}{r^\gamma} \fint_{2B} |u|\, d\mu; \qquad (A.2)$$

see for instance [18].

We may assume that $r > 1$ is large enough so that $B = B(x_0, r)$ contains a copy of the fundamental domain X. Then $g \cdot X$ are pairwise disjoint for different $g \in G$, and $M \setminus (G \cdot X)$ is of measure zero. For simplicity of notions we assume that u is harmonic on $8B$.

Claim 1: For each $x \in X$, $g \in G$ such that $g \cdot x \in B(x_0, r)$, it holds that

$$|u(x) - u(g\cdot x)| \le C\frac{\rho(g)}{r} \fint_{4B} |u|\, d\mu. \qquad (A.3)$$

Proof of Claim 1. If $d(x, g\cdot x) \ge 2^{-16}r$, then (A.3) is obvious by (A.2). Consider now the $d(x, g\cdot x) < 2^{-16}r$. Let $k \in \mathbb{N}$ such that $2^k < r\rho(g)^{-1} \le 2^{k+1}$ (remember $r >> 1$ and $\rho(g) \ge 1$). For each $j \in \{1, \cdots, k\}$, notice that $g^{2^j} \cdot x \in 2B$, since $d(g^{2^j} \cdot x, x) < r$. By using (A.2) twice, we see that for $1 \le j \le k$ it holds that

$$\left|[u(x) - u(g^{2^j} \cdot x)] - [u(g^{2^j} \cdot x) - u(g^{2^{j+1}} \cdot x)]\right| \le C\frac{d^\gamma(x, g^{2^j} \cdot x)}{r^\gamma} \fint_{3B} |u(x) - u(g^{2^j} \cdot x)|\, d\mu(x) \\ \le C\frac{2^{2j\gamma}\rho(g)^{2\gamma}}{r^{2\gamma}} \fint_{4B} |u|\, d\mu.$$

Using the identity

$$u(x) - u(g \cdot x) = \sum_{j=0}^{k} 2^{-j-1}[u(x) - 2u(g^{2^j} \cdot x) + u(g^{2^{j+1}} \cdot x)] + 2^{-k-1}[u(x) - u(g^{2^{k+1}} \cdot x)]$$

together with the above estimate and $2^k < r\rho(g)^{-1} \le 2^{k+1}$, we see that for $\gamma < \beta < 2\gamma$ and $\beta \le 1$, it holds that

$$\begin{aligned} |u(x) - u(g \cdot x)]| &\le \sum_{j=0}^{k} 2^{-j-1}|u(x) - 2u(g^{2^j} \cdot x) + u(g^{2^{j+1}} \cdot x)| + 2^{-k-1}|u(x) - u(g^{2^{k+1}} \cdot x)| \\ &\le \sum_{j=0}^{k} C2^{-j-1}\frac{2^{2j\gamma}\rho(g)^{2\gamma}}{r^{2\gamma}} \fint_{4B} |u|\, d\mu + C2^{-k-1}\frac{2^{k\gamma}\rho(g)^\gamma}{r^\gamma} \fint_{4B} |u|\, d\mu \\ &\le C \fint_{4B} |u|\, d\mu \left(\sum_{j=0}^{k} 2^{-j-1+2j\gamma-2k\gamma} + 2^{-k-1}\right) \\ &\le C \fint_{4B} |u|\, d\mu \left(\sum_{j=0}^{k} 2^{j(2\gamma-1)-k(2\gamma-\beta)}2^{-k\beta} + 2^{-k-1}\right) \\ &\le C2^{-k\beta} \fint_{4B} |u|\, d\mu \le C\frac{\rho(g)^\beta}{r^\beta} \fint_{4B} |u|\, d\mu. \end{aligned}$$

Repeating this argument sufficiently many times, we conclude that (A.3) holds.

Claim 2: There exists a finite set $J \subset G$, with $e \in J$, such that

$$\sup_{x,y \in \cup_{g\in J} g\cdot X} |u(x) - u(y)| \le C \sup_{g\in J, x\in X} |u(x) - u(g \cdot x)|.$$

Proof of Claim 2. Take $y_0 \in X$ and fix $0 < r_1$ such that $X \subset B(y_0, r_1)$. Then by Proposition 2.1, there exists $r_2 > r_1$ such that

$$\text{(A.4)} \qquad \sup_{x,y\in B(y_0,r_1)} |u(x) - u(y)| \le \frac{1}{2} \sup_{x,y\in B(y_0,r_2)} |u(x) - u(y)|.$$

Let $J \subset G$ be the collection of $g \in G$ such that $g \cdot X \cap B(y_0, r_2) \neq \emptyset$. Then $B(y_0, r_2) \subset \cup_{g\in J} g \cdot X$ and J only has finitely many elements. For $x, y \in \cup_{g\in J} g \cdot X$, take $\tilde{x}, \tilde{y} \in X$ and $g, h \in G$ such that $x = g \cdot \tilde{x}$ and $y = h \cdot \tilde{y}$. Then by (A.4) we obtain

$$|u(x) - u(y)| \le |u(g \cdot \tilde{x}) - u(\tilde{x})| + |u(h \cdot \tilde{y}) - u(\tilde{y})| + |u(\tilde{x}) - u(\tilde{y})|$$

$$
\begin{aligned}
&\le 2 \sup_{g\in J, x\in X} |u(x) - u(g\cdot x)| + \sup_{\tilde{x},\tilde{y}\in X} |u(\tilde{x}) - u(\tilde{y})| \\
&\le 2 \sup_{g\in J, x\in X} |u(x) - u(g\cdot x)| + \frac{1}{2} \sup_{\tilde{x},\tilde{y}\in B(y_0,r_2)} |u(\tilde{x}) - u(\tilde{y})|,
\end{aligned}
$$

which, together with the fact $B(y_0, r_2) \subset \cup_{g\in J} g\cdot X$, implies that

$$
\sup_{x,y\in \sup_{g\in J} g\cdot X} |u(x) - u(y)| \le C \sup_{g\in J, x\in X} |u(x) - u(g\cdot x)|.
$$

We can now complete the proof.

Recall that $y_0 \in X \subset B$, $X \subset B(x_0, r_1) \subset B(x_0, r_2) \subset \cup_{g\in J} g\cdot X$. Fix $r_3 > r_2$ such that $\cup_{g\in J} g\cdot X \subset B(x_0, r_3)$. Notice $J \subset G$ is a fixed finite set.

Since $(RH_{\infty,\mathrm{loc}})$ holds, we may assume that $r > r_1 + r_3$ is large enough, so that for each $h \in \{h \in G : h\cdot X \cap B \neq \emptyset\}$, $hg\cdot X \subset 2B$ for each $g \in J$. By $(RH_{\infty,\mathrm{loc}})$, together with the previous two claims, we obtain for each $h \in \{h \in G : h\cdot X \cap B \neq \emptyset\}$,

$$
\begin{aligned}
\||\nabla u|\|_{L^\infty(h\cdot X)} &\le \||\nabla (u - u(h\cdot y_0))|\|_{L^\infty(B(h\cdot y_0, r_1))} \le C \fint_{B(h\cdot y_0, r_2)} |u - u(h\cdot y_0)|\, d\mu \\
&\le C \fint_{\cup_{g\in J} hg\cdot X} |u - u(h\cdot y_0)|\, d\mu \le C \sup_{x,y\in \cup_{g\in J} gh\cdot X} |u(x) - u(y)| \\
&\le C \sup_{g\in J, x\in X} |u(h\cdot x) - u(hg\cdot x)| \le \frac{C}{r} \fint_{8B} |u|\, d\mu.
\end{aligned}
$$

This implies that

$$
\||\nabla u|\|_{L^\infty(B)} \le \sup_{h\in G: h\cdot X\cap B\neq\emptyset} \||\nabla u|\|_{L^\infty(h\cdot X)} \le \frac{C}{r} \fint_{8B} |u|\, d\mu.
$$

A covering argument similar to that of **Step 4** in the proof of Theorem 3.6 then gives (RH_∞). □

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