Isoperimetric Inequalities for Non-Local Dirichlet Forms

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Abstract

Let $(E, \mathscr{F}, \mu)$ be a σ-finite measure space. For a non-negative symmetric measure $J(\mathrm{d}x, \mathrm{d}y) := J(x, y)\, \mu(\mathrm{d}x)\, \mu(\mathrm{d}y)$ on $E \times E$, consider the quadratic form

$$\mathscr{E}(f, f) := \frac{1}{2} \int_{E \times E} (f(x) - f(y))^2\, J(\mathrm{d}x, \mathrm{d}y)$$

in $L^2(\mu)$. We characterize the relationship between the isoperimetric inequality and the super Poincaré inequality associated with $\mathscr{E}$. In particular, sharp Orlicz-Sobolev type and Poincaré type isoperimetric inequalities are derived for stable-like Dirichlet forms on $\mathbb{R}^n$, which include the existing fractional isoperimetric inequality as a special example.

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1 Introduction

For local (i.e. differential) quadratic forms, the isoperimetric inequality is a geometric inequality using the surface area of a set to bound its volume, see, for instance [14, 22, 28] and references therein, for the study of isoperimetric inequalities and applications to symmetric diffusion processes. In this case, the surface area refers to the possibility for the associated diffusion process to exit the set.

In the non-local case, the associated process is a jump process which exits a set without hitting the boundary, so it is reasonable to replace the surface area of a set A by the jump rate from A to its complementary A^c. In this spirit, the famous Cheeger inequality [9] for the first eigenvalue was extended in [21, 12] to jump processes (see also [34] for finite Markov chains). See [10, 11, 40, 42, 33, 39, 25, 26] for the study of more general functional inequalities of symmetric jump processes using isoperimetric constants. These references only consider large jumps (i.e. the total jump rate is finite). In this paper, we aim to investigate isoperimetric inequalities for non-local forms with infinite jump rates, for which small jumps will paly a key role.

To explain our motivation more clearly, let us start from the following classical isoperimetric inequality on $\mathbb{R}^n$:

$$(1.1) \qquad \mu_\partial(\partial A) \geq n\mu(A)^{\frac{n-1}{n}}\omega_n^{\frac{1}{n}},$$

where A is a measurable subset of $\mathbb{R}^n$ with finite volume, ∂A is its boundary, ω_n is the volume of the n-dimensional unit ball, μ is the Lebesgue measure and μ_∂ is the area measure induced by μ:

$$\mu_\partial(\partial A) := \limsup_{\varepsilon \downarrow 0} \frac{\mu(\{\mathrm{dist}(\cdot, A) \leq \varepsilon\}) - \mu(A)}{\varepsilon}.$$

In particular, the equality in (1.1) holds for A being a ball. By the co-area formula, (1.1) is equivalent to the sharp L^1-Sobolev inequality (i.e. the energy form is of L^1 type)

$$(1.2) \qquad \|f\|_{\frac{n}{n-1}} \leq \frac{1}{n\omega_n^{1/n}} \int_{\mathbb{R}^n} |\nabla f(x)|\, \mathrm{d}x, \quad f \in W^{1,1}(\mathbb{R}^n),$$

where for any $p \geq 1$, $\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p\, \mathrm{d}x \right)^{1/p}$ and $W^{1,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space of differentiability 1 and integrability p. For $n > 2$, applying (1.2) to $f = |g|^{\frac{2(n-1)}{n-2}}$ and using the Cauchy-Schwarz inequality, we obtain the sharp Sobolev inequality:

$$\|g\|_{\frac{2n}{n-2}} \leq \frac{2(n-1)}{n(n-2)\omega_n^{1/n}} \left(\int_{\mathbb{R}^n} |\nabla g(x)|^2\, \mathrm{d}x \right)^{1/2}, \quad g \in W^{1,2}(\mathbb{R}^n).$$

These inequalities are also available for the α-stable Dirichlet form. For any $\alpha \in (0, 2 \wedge n)$, there exists a universal constant $C > 0$ such that the fractional Sobolev inequality

$$\|f\|_{\frac{2n}{n-\alpha}} \leq C \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}}\, \mathrm{d}x\, \mathrm{d}y \right)^{1/2}, \quad f \in C_c^\infty(\mathbb{R}^n)$$

holds. By an approximation argument, this inequality can be extended to $f \in W^{\alpha/2,2}(\mathbb{R}^d)$. Here and in what follows, for $p \geq 1$, $W^{\alpha/2,p}(\mathbb{R}^d)$ is denoted by the fractional homogeneous Sobolev space, which is the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to

$$\left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+p\alpha/2}}\, \mathrm{d}x\, \mathrm{d}y \right)^{1/p}.$$

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Correspondingly to (1.2), [23, Theorem 1.1] (or [19, Theorem 4.1] with sharp constant) gives the following L^1-Sobolev inequality

$$(1.3) \qquad \|f\|_{\frac{n}{n-\alpha/2}} \le C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha/2}} \, \mathrm{d}x \, \mathrm{d}y, \quad f \in W^{\alpha/2,1}(\mathbb{R}^n)$$

for some constant $C > 0$. The proof of (1.3) addressed in [23, 19] relies on the Hardy inequality for fractional Sobolev spaces. We note that the Sobolev embedding theorems involving the spaces $W_{\alpha/2,p}$ also can be obtained by interpolation techniques and by passing through Besov spaces, see for example [6, 7]. For the treatment of fractional Sobolev-type inequalities we can refer to [1, 4, 37, 31] and the references therein.

According to Theorem 2.1(1) below, (1.3) holds if and only if

$$(1.4) \qquad \kappa := \inf_{\mu(A) \in (0,\infty)} \frac{1}{\mu(A)^{\frac{n-\alpha/2}{n}}} \int_{A \times A^c} \frac{\mathrm{d}x \, \mathrm{d}y}{|x - y|^{n+\alpha/2}} > 0,$$

and furthermore, $\kappa \in [\frac{1}{2C}, \frac{n}{2C(n-\alpha/2)}]$, where C is the sharp constant in (1.3). Due to this fact, we also call (1.3) a Sobolev type isoperimetric inequality.

In this paper, we aim to establish isoperimetric inequalities for the following non-local form on a σ-finite measure space $(E, \mathscr{F}, \mu)$:

$$(1.5) \qquad \mathscr{E}(f, f) := \frac{1}{2} \int_{E \times E} (f(x) - f(y))^2 \, J(\mathrm{d}x, \mathrm{d}y),$$

where $J(\mathrm{d}x, \mathrm{d}y)$ is a non-negative symmetric measure on $E \times E$.

Instead of the fractional Hardy inequality used in [23, 19] and the Besov or interpolation spaces used in [6, 7], in our paper we will apply the super Poincaré inequality of $\mathscr{E}$, which was introduced by the first author in [40]. This inequality can be regarded as a deformation of the Nash-type inequality, but is easier to verify in applications. The proof here is self-contained.

We also mentioned that isoperimetric inequalities for symmetric diffusions have already been studied in the literature, see [8, 24, 40, 2, 3] and the references therein. In particular, Ledoux's approach of Buser's inequality was used in [40, 2] to illustrate the relation of the super-Poincaré inequalities with isoperimetry. A notion of Orlicz hypercontractive semi-groups was introduced in [2], and their relations with various functional inequalities were studied. A measure-Capacity sufficient condition, in the spirit of Maz'ja [22], was established for super-Poincaré inequality inequality in [3]. In the present setting, we are concerned with non-local forms. We will directly derive the equivalence of L^1 Orlicz-Sobolev inequality (involving the L^1-norm of the jumping kernel for non-local forms) and L^1-Poincaré type inequality, and also characterize the relationship between the isoperimetric inequality and the super Poincaré inequality. In particular, one of our general results (see Theorem 2.2 below) implies the following Orlicz-Sobolev type isoperimetric inequality (1.7) on $\mathbb{R}^n$.

Following [32, Section 1.3], a function $N : [0, \infty) \to [0, \infty]$ is called a Young function if it is convex and increasing with $N(0) = 0$ and $N(\infty) := \lim_{s \to \infty} N(s) = \infty$. We consider the

following Orlicz norm induced by N (see [32, Section 3.2]):

$$\|f\|_N := \inf\left\{r > 0 : \int_{\mathbb{R}^n} N\left(\frac{|f(x)|}{r}\right) dx \le 1\right\},$$

where $\inf \emptyset = \infty$ by convention. Let $L_N(\mathbb{R}^n) = \{f \in \mathscr{B}(\mathbb{R}^n) : \|f\|_N < \infty\}$. It is easy to see from the convexity and $N(0) = 0$ that $cN(s) \le N(cs)$ for $c \ge 1$. So, $N(\infty) > 0$ is equivalent to $N(\infty) = \infty$, and

$$(1.6) \qquad\qquad c^{-1}\|f\|_{cN} \le \|f\|_N \le \|f\|_{cN}, \quad c \ge 1.$$

For two Young functions N_1 and N_2, we say that N_1 is not dominated by N_2 if $\sup_{s>0} \frac{N_1(s)}{N_2(s)} = \infty$, where we set $\frac{0}{0} = 1$, $\frac{\infty}{\infty} = 1$, $\frac{r}{0} = \infty$ and $\frac{r}{\infty} = 0$ for $r > 0$. In this case, we write $N_1 \not\preceq N_2$.

For $\alpha \in (0, 2)$, let $\mathscr{H}_\alpha$ be the class of functions $h : (0, \infty) \to (0, \infty)$ satisfying

(i) $h(s)$ and $sh(s)^{-1}$ are increasing in s.

(ii) For any $s > 0$,

$$\Phi_h(s) := \int_0^s dt \int_0^{t^{-1/n}} \frac{r^{\alpha-1}}{h(r)} dr < \infty.$$

It is easy to see that Φ_h is continuous, strictly increasing and concave with $\Phi_h(0) = 0$. Thus, $N_h := \Phi_h^{-1}$ is a Young function.

Theorem 1.1. *For any $\alpha \in (0, 2)$ and $h \in \mathscr{H}_\alpha$, there exists a constant $C > 0$ such that*

$$(1.7) \qquad \|f\|_{N_h} \le C \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| \frac{h(|x - y|)}{|x - y|^{n+\alpha}} dx\, dy, \quad f \in L_{N_h}(\mathbb{R}^n),$$

which implies

$$(1.8) \qquad \inf_{\mu(A)\in(0,\infty)} \left(N_h^{-1}(\mu(A)^{-1}) \int_{A \times A^c} \frac{h(|x - y|)}{|x - y|^{n+\alpha}} dx\, dy\right) > 0.$$

Consequently:

(1) *For any $\alpha_1, \alpha_2 \in (0, 2)$, let*

$$N^\wedge_{\alpha_1,\alpha_2}(s) := s^{\frac{n}{n-\alpha_1/2}} \wedge s^{\frac{n}{n-\alpha_2/2}}, \quad N^\vee_{\alpha_1,\alpha_2}(s) := s^{\frac{n}{n-\alpha_1/2}} \vee s^{\frac{n}{n-\alpha_2/2}}, \quad s \ge 0.$$

Then there exists a constant $C > 0$ such that

$$(1.9) \quad \|f\|_{N^\wedge_{\alpha_1,\alpha_2}} \le C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha_1/2} \vee |x - y|^{n+\alpha_2/2}} dx\, dy, \quad f \in L_{N^\wedge_{\alpha_1,\alpha_2}}(\mathbb{R}^n),$$

$$(1.10) \quad \|f\|_{N^\vee_{\alpha_1,\alpha_2}} \le C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha_1/2} \wedge |x - y|^{n+\alpha_2/2}} dx\, dy, \quad f \in L_{N^\vee_{\alpha_1,\alpha_2}}(\mathbb{R}^n).$$

These inequalities are sharp in the sense that (1.9) (resp. (1.10)) fails if $N^\wedge_{\alpha_1,\alpha_2}$ (resp. $N^\vee_{\alpha_1,\alpha_2}$) is replaced by a Young function $N \not\preceq N^\wedge_{\alpha_1,\alpha_2}$(resp. $N^\vee_{\alpha_1,\alpha_2}$).

(2) *For any $\alpha \in (0,2)$ and $q, p \in \mathbb{R}$, let $\lambda \geq 2$ large enough such that both $N_\alpha^{log,q,+}(s) := \left\{ s \log^q (\lambda + s) \right\}^{\frac{n}{n-\alpha/2}}$ and $N_\alpha^{log,p,-}(s) := \left\{ s \log^p (\lambda + s^{-1}) \right\}^{\frac{n}{n-\alpha/2}}$ are Young functions. Then there exists a constant $C > 0$ such that for all $f \in L_{N_\alpha^{log,q,+}}(\mathbb{R}^n)$,*

(1.11) $$\|f\|_{N_\alpha^{log,q,+}} \leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| \frac{\{\log(2 + |x-y|^{-1})\}^q}{|x-y|^{n+\alpha/2}} \, dx \, dy,$$

and for all $f \in L_{N_\alpha^{log,p,-}}(\mathbb{R}^n)$,

(1.12) $$\|f\|_{N_\alpha^{log,p,-}} \leq C \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x) - f(y)| \frac{\{\log(2 + |x-y|)\}^p}{|x-y|^{n+\alpha/2}} \, dx \, dy.$$

These inequalities are sharp in the sense that (1.11) (resp. (1.12)) fails if $N_\alpha^{log,q,+}$ (resp. $N_\alpha^{log,p,-}$) is replaced by a Young function $N \not\preceq N_\alpha^{log,q,+}$(resp. $N_\alpha^{log,p,-}$).

According to Theorem 2.1(1) below, (1.7) and (1.8) are equivalent in more general case, so an L^1 Orlicz-Sobolev inequality of type (1.7) is also called an Orlicz-Sobolev type isoperimetric inequality. It is easy to see that when $\alpha_1 = \alpha_2 = \alpha$ and $q = p = 0$, the inequalities (1.9), (1.10), (1.11) and (1.12) coincide with (1.3). The Orlicz-Sobolev type isoperimetric inequalities (1.9)–(1.12) are equivalent to the corresponding Poincaré type ones, see Corollary 2.7 for details.

In the remainder of the paper, we will work with the form (1.5) under a general framework. In Section 2, we characterize the link between the super poincaré and isoperimteric inequalities. In Section 3, we first apply the main result derived in Section 2 to prove Theorem 1.1, then make extensions to the truncated and discrete α-stable Dirichlet forms. Finally, by using a perturbation argument, we derive isoperimetric inequalities in Section 4 for α-stable-like Dirichlet forms with finite reference measures.

2 Super Poincaré and isoperimetric inequalities: general results

Let $(E, \mathscr{F}, \mu)$ be a σ-finite measure space, and let $J(dx, dy)$ be a non-negative and symmetric measure on $E \times E$. In this section, we investigate the link between the isoperimetric inequality and the super Poincaré inequality for the following symmetric quadratic form

(2.1)
$$\mathscr{E}(f, g) := \frac{1}{2} \int_{E \times E} (f(x) - f(y))(g(x) - g(y)) J(dx, dy),$$
$$f, g \in \mathscr{D}(\mathscr{E}) := \left\{ f \in L^2(\mu) : \mathscr{E}(f, f) < \infty \right\}.$$

To ensure that $\mathscr{E}(f, f)$ does not depend on the choice of μ-versions of f, we assume that $J(dx, dy) = J(x, y) \, \mu(dx) \, \mu(dy)$ for some symmetric density $J : E \times E \to [0, \infty)$. Moreover, we assume that $\mathscr{D}(\mathscr{E})$ is dense in $L^2(\mu)$ so that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a symmetric Dirichlet form.

According to [40], we say that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ satisfies the super Poincaré inequality with rate function $\beta : (0, \infty) \to (0, \infty)$, if

$$(2.2) \qquad \|f\|_2^2 \le r\mathscr{E}(f, f) + \beta(r)\|f\|_1^2, \quad f \in \mathscr{D}(\mathscr{E}), r > 0.$$

Here and in what follows, for any $p \in [1, \infty]$, $\|\cdot\|_p$ denotes the L^p-norm with respect to μ. Since $\mathscr{E}(f, f) \ge 0$, we may and do assume that β is decreasing on $(0, \infty)$. See [40, 41, 43] and references within for the super Poincaré inequality and applications.

For a Young function N, let $\|\cdot\|_N$ be the Orlicz norm induced by N and the measure μ, and let $L_N(\mu) = \{f \in \mathscr{B}(E) : \|f\|_N < \infty\}$, where $\mathscr{B}(E)$ is the class of measurable functions on E. The left derivative of Young function N, denoted by N_-', always exists and is non-decreasing left continuous on $(0, \infty)$, see e.g. [32, Section 1.3]. For any non-negative decreasing function f on $[0, \infty)$, let

$$f^{-1}(r) := \inf\{s > 0 : f(s) \le r\}, \quad r \ge 0,$$

where $\inf \emptyset := \infty$. Similarly, for any non-negative increasing function f on $[0, \infty)$, let

$$f^{-1}(r) := \inf\{s > 0 : f(s) \ge r\}, \quad r \ge 0.$$

In the following four subsections, we first observe the equivalence of an L^1 functional inequality and the corresponding isoperimetric inequality, then investigate the link between the super Poincaré and isoperimetric inequalities, and finally extend the main results to the case with killing.

2.1 L^1 functional and isoperimetric inequalities

Consider the L^1 Orlicz-Sobolev inequality

$$(2.3) \qquad \|f\|_N \le C \int_{E \times E} |f(x) - f(y)|\, J(\mathrm{d}x, \mathrm{d}y), \quad f \in L_N(\mu),$$

and the L^1 Poincaré type inequality

$$(2.4) \qquad \|f\|_2^2 \le C_1 \int_{E \times E} |f^2(x) - f^2(y)|\, J(\mathrm{d}x, \mathrm{d}y) + C_2\|f\|_1^2, \quad f \in L^2(\mu),$$

where N is a Young function and $C, C_1, C_2 > 0$ are constants. The following result provides their equivalent isoperimetric inequalities.

Theorem 2.1. (1) *The inequality* (2.3) *implies*

$$(2.5) \qquad \inf_{\mu(A) \in (0, \infty)} \left\{N^{-1}(\mu(A)^{-1}) J(A \times A^c)\right\} \ge \kappa$$

holds for $\kappa = \frac{1}{2C}$. *On the other hand, if* $N_-'(s) > 0$ *for* $s > 0$ *such that*

$$c_N := \inf_{s>0} \frac{N(s)}{s N_-'(s)} > 0,$$

then (2.5) *implies* (2.3) *for* $C = \frac{1}{2c_N \kappa}$.

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(2) *The inequality* (2.4) *implies*

(2.6) $$\mu(A) \le 2C_1 J(A \times A^c) + \tilde{C}_2 \mu(A)^2, \quad \mu(A) \in (0, \infty)$$

for $\tilde{C}_2 = C_2$. *On the other hand,* (2.6) *implies* (2.4) *for* $C_2 = 2\tilde{C}_2$.

Proof. (1) For $A \subset E$ with $\mu(A) \in (0, \infty)$, let $f = \mathbf{1}_A$. Then

(2.7) $$\int_{E \times E} |f(x) - f(y)| \, J(\mathrm{d}x, \mathrm{d}y) = 2J(A \times A^c).$$

Moreover, for any $r > 0$, by the definition of N^{-1} we see that

$$\int_E N\big(|f(x)|/r\big) \, \mu(\mathrm{d}x) = N(r^{-1})\mu(A) \le 1$$

implies $r \ge N^{-1}(\mu(A)^{-1})$. Therefore, $\|f\|_N \ge N^{-1}(\mu(A)^{-1})$. Combining this with (2.3) and (2.7), we prove (2.5) for $\kappa = \frac{1}{2C}$.

On the other hand, let $c_N > 0$ and (2.5) hold. It suffices to prove (2.3) for $C = \frac{1}{2c_N \kappa}$ and any $f \ge 0$ with $\|f\|_N = 1$. By Fubini's theorem and (2.5), we have

(2.8)
$$
\begin{aligned}
\int_{E \times E} |f(x) - f(y)| \, J(\mathrm{d}x, \mathrm{d}y) &= 2 \int_{\{f(x) > f(y)\}} \left(\int_{f(y)}^{f(x)} \mathrm{d}r \right) J(\mathrm{d}x, \mathrm{d}y) \\
&= 2 \int_0^\infty J(\{(x, y) : f(x) > r \ge f(y)\}) \, \mathrm{d}r \ge 2\kappa \int_0^\infty \frac{\mathrm{d}r}{N^{-1}(\mu(f > r)^{-1})} \\
&= 2\kappa \int_0^\infty \frac{\mu(N(f) > N(r))}{\mu(N(f) > N(r)) N^{-1}(\mu(N(f) > N(r))^{-1})} \, \mathrm{d}r \\
&= 2\kappa \int_0^\infty \frac{\mu(N(f) > s)}{N'_-(N^{-1}(s)) \mu(N(f) > s) N^{-1}(\mu(N(f) > s)^{-1})} \, \mathrm{d}s.
\end{aligned}
$$

Since $f \ge 0$ and $\|f\|_N = 1$, we have

$$\mu(N(f) > s) \le \frac{\mu(N(f))}{s} = \frac{1}{s}, \quad s > 0.$$

Noting that $N'_- \circ N^{-1}$ is increasing, by letting $t = N^{-1}(\mu(N(f) > s)^{-1})$ we obtain

$$
\begin{aligned}
&N'_-(N^{-1}(s)) \mu(N(f) > s) N^{-1}(\mu(N(f) > s)^{-1}) \\
&\le N'_-(N^{-1}(\mu(N(f) > s)^{-1})) \mu(N(f) > s) N^{-1}(\mu(N(f) > s)^{-1}) \\
&= \frac{N'_-(t) t}{N(t)} \le \frac{1}{c_N}.
\end{aligned}
$$

Substituting into (2.8) and noting that $\|f\|_N = 1$, we arrive at

$$\int_{E \times E} |f(x) - f(y)| \, J(\mathrm{d}x, \mathrm{d}y) \ge 2\kappa c_N \mu(N(f)) = 2\kappa c_N = 2\kappa c_N \|f\|_N.$$

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Thus, (2.3) holds for $C = \frac{1}{2\kappa c_N}$.

(2) As in (1), by applying (2.4) to $f = \mathbf{1}_A$ we prove (2.6) for $\tilde{C}_2 = C_2$. On the other hand, let $f \in L^2(\mu)$ with $\|f\|_1 = 1$. Then $\mu(f^2 > s) \le s^{-1/2}$, so that, as in (2.8), (2.6) yields

$$
\begin{aligned}
C_1 \int_{E \times E} |f^2(x) - f^2(y)| \, J(\mathrm{d}x, \mathrm{d}y) &= 2C_1 \int_0^\infty J(\{f^2 > s\} \times \{f^2 \le s\}) \, \mathrm{d}s \\
&\ge \int_0^\infty \{\mu(f^2 > s) - \tilde{C}_2\mu(f^2 > s)^2\} \, \mathrm{d}s \ge \mu(f^2) - \tilde{C}_2 \int_0^\infty \frac{1}{\sqrt{s}}\mu(|f| > \sqrt{s}) \, \mathrm{d}s \\
&= \mu(f^2) - 2\tilde{C}_2 \int_0^\infty \mu(|f| > \sqrt{s}) \, \mathrm{d}\sqrt{s} = \mu(f^2) - 2\tilde{C}_2\mu(|f|) = \mu(f^2) - 2\tilde{C}_2\mu(|f|)^2.
\end{aligned}
$$

Therefore, (2.6) implies (2.4) for $C_2 = 2\tilde{C}_2$. $\qquad\square$

2.2 From super Poincaré to isoperimetric

Let P_t be the (sub-) Markov semigroup associated with the symmetric Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

Theorem 2.2. *Assume that (2.2) holds with $\beta(\infty) := \lim_{r \to \infty} \beta(r) = 0$. Let $\gamma : E \times E \to [0, \infty)$ with $\gamma(x, y) > 0$ and $\gamma(x, y) = \gamma(y, x)$ for $x \ne y$, and define*

$$
\theta_\gamma(t) := \sup_{\|g\|_\infty \le 1} \mathrm{ess}_{\mu \times \mu} \sup_{x \ne y} \frac{|P_t g(x) - P_t g(y)|}{\gamma(x, y)}, \quad t > 0.
$$

If

$$
\Phi_\gamma(s) := \int_0^s \mathrm{d}r \int_0^{\beta^{-1}(r)} \theta_\gamma(t) \, \mathrm{d}t < \infty, \quad s > 0,
$$

then $N_\gamma := \Phi_\gamma^{-1}$ is a Young function, and there exists a constant $C > 0$ such that

(2.9)
$$
\|f\|_{N_\gamma} \le C \int_{E \times E} |f(x) - f(y)|\gamma(x, y) \, J(\mathrm{d}x, \mathrm{d}y), \quad f \in L_{N_\gamma}(\mu).
$$

To prove this result, we consider the symmetric measure

$$
J_\gamma(\mathrm{d}x, \mathrm{d}y) := \gamma(x, y) \, J(\mathrm{d}x, \mathrm{d}y)
$$

on $E \times E$, and introduce the isoperimetric constants

(2.10)
$$
\kappa_\gamma(s) := \inf \left\{ \frac{J_\gamma(A \times A^c)}{\mu(A)} : \mu(A) \in (0, s) \right\}, \quad s > 0,
$$

where $\inf \emptyset := \infty$. We have the following result.

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Lemma 2.3. *For any increasing function* $G : [0, \infty) \to [0, \infty)$ *with* $G(0) = 0$ *and* $G(s) > 0$ *for* $s > 0$, *it holds that*

(2.11) $$\int_E \mathrm{d}\mu \int_0^{|f|} \kappa_\gamma \left(G(s)^{-1}\right) \mathrm{d}s \leq \frac{1}{2} \int_{E \times E} |f(x) - f(y)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y), \quad \mu(G(|f|)) = 1.$$

Consequently:

(1) *If* $\kappa_\gamma(s) > 0$ *for some* $s > 0$, *then*

(2.12) $$\|f\|_2^2 \leq \frac{1}{2\kappa_\gamma(s)} \int_{E \times E} |f^2(x) - f^2(y)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y) + \frac{2}{s} \|f\|_1^2, \quad f \in L^2(\mu).$$

(2) *If* $\kappa_\gamma(s) > 0$ *for all* $s > 0$ *such that*

$$\Phi(t) := \int_0^t \frac{\mathrm{d}r}{\kappa_\gamma(r^{-1})} < \infty, \quad t > 0,$$

then $N := \Phi^{-1}$ *is a Young function, and*

(2.13) $$\|f\|_N \leq \frac{1}{2} \int_{E \times E} |f(x) - f(y)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y), \quad f \in L_N(\mu).$$

Proof. For any $f \in \mathscr{B}(E)$ with $\mu(G(|f|)) = 1$, we have

$$\mu(|f| > s) \leq G(s)^{-1}, \quad s > 0.$$

As in (2.8), this and the definition of $\kappa_\gamma(s)$ imply that

$$\frac{1}{2} \int_{E \times E} |f(x) - f(y)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y) \geq \frac{1}{2} \int_{E \times E} \big||f|(x) - |f|(y)\big| \, J_\gamma(\mathrm{d}x, \mathrm{d}y)$$

$$= \int_0^\infty J_\gamma(\{|f| > u\} \times \{|f| \leq u\}) \, \mathrm{d}u \geq \int_0^\infty \kappa_\gamma \left(G(u)^{-1}\right) \mu(|f| > u) \, \mathrm{d}u$$

$$= \int_E \mathrm{d}\mu \int_0^{|f|} \kappa_\gamma \left(G(u)^{-1}\right) \mathrm{d}u.$$

We have proved (2.11). Below we prove assertions (1) and (2) respectively.

Assertion (1). For any $f \in \mathscr{B}(E)$ with $\|f\|_1 = 1$, we have

(2.14) $$\mu(|f| > s) \leq \frac{1}{s}, \quad s > 0.$$

Since $\kappa_\gamma(s)$ is decreasing in s, applying (2.11) to f^2 with $G(s) = s^{1/2}$, we derive

(2.15) $$\frac{1}{2} \int_{E \times E} |f^2(x) - f^2(y)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y) \geq \int_E \mathrm{d}\mu \int_0^{f^2} \kappa_\gamma \left(u^{-1/2}\right) \mathrm{d}u$$

$$= \int_0^\infty \kappa_\gamma \left(u^{-1/2}\right) \mu(f^2 > u) \, \mathrm{d}u \geq \kappa_\gamma(s) \int_{s^{-2}}^\infty \mu(f^2 > u) \, \mathrm{d}u, \quad s > 0.$$

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On the other hand, by (2.14) we have

$$\int_0^{s^{-2}} \mu(f^2 > u)\, \mathrm{d}u \le \int_0^{s^{-2}} \frac{1}{\sqrt{u}}\, \mathrm{d}u = \frac{2}{s}, \quad s > 0.$$

Combining this with (2.15) and

$$\|f\|_2^2 = \int_0^\infty \mu(f^2 > u)\, \mathrm{d}u,$$

we prove (2.12).

Assertion (2). Let $N = \Phi^{-1}$. Then N satisfies $N(0) = 0$, and solves the equation

$$(2.16) \qquad \frac{\mathrm{d}N(s)}{\mathrm{d}s} = \kappa_\gamma\big(N(s)^{-1}\big) \quad \text{a.e. } s > 0,$$

where $\frac{\mathrm{d}N(s)}{\mathrm{d}s}$ denotes the Radon-Nikodym derivative of N with respect to the Lebesgue measure. Since $\kappa_\gamma(s)$ is strictly positive and decreasing in s, and since $\Phi(t) < \infty$ for $t > 0$, it is easy to deduce from (2.16) that N is a Young function, and

$$\int_0^{|f|} \kappa_\gamma\big(N(s)^{-1}\big)\, \mathrm{d}s = \int_0^{|f|} \frac{\mathrm{d}N(s)}{\mathrm{d}s}\, \mathrm{d}s = N(|f|).$$

Combining this with (2.11) leads to

$$1 \le \frac{1}{2} \int_{E \times E} |f(x) - f(y)|\, J_\gamma(\mathrm{d}x, \mathrm{d}y), \quad \mu(N(|f|)) = 1,$$

which in turn implies (2.13). $\qquad\square$

According to Lemma 2.3, for the proof of Theorem 2.2 we only need to estimate the isoperimetric constants $\kappa_\gamma(s)$ using (2.2). The following result can be regarded as an extension of a result of [8] (see also [24]) to non-local forms.

Lemma 2.4. *Let γ and $\theta_\gamma(t)$ be in Theorem 2.2. If*

$$\Theta_\gamma(t) := \int_0^t \theta_\gamma(s)\, \mathrm{d}s < \infty, \quad t \ge 0,$$

then the super Poincaré inequality (2.2) implies

$$(2.17) \qquad \kappa_\gamma(s) \ge \frac{1 - \mathrm{e}^{-1}}{2\Theta_\gamma(\beta^{-1}(1/(2s)))}, \quad s > 0.$$

Proof. For any $f, g \in \mathscr{D}(\mathscr{E})$, we have

$$\mu(g(f - P_t f)) = \mu(f(g - P_t g)) = \int_0^t \mathscr{E}(f, P_s g)\, \mathrm{d}s$$

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$$= \frac{1}{2} \int_0^t \mathrm{d}s \int_{E \times E} (P_s g(x) - P_s g(y))(f(x) - f(y)) \, J(\mathrm{d}x, \mathrm{d}y)$$

$$\leq \frac{\|g\|_\infty}{2} \left(\int_0^t \theta_\gamma(s) \, \mathrm{d}s \right) \int_{E \times E} |f(y) - f(x)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y)$$

$$= \frac{\Theta_\gamma(t) \|g\|_\infty}{2} \int_{E \times E} |f(y) - f(x)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y).$$

Thus,

(2.18) $$\mu(|f - P_t f|) \leq \frac{\Theta_\gamma(t)}{2} \int_{E \times E} |f(y) - f(x)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y).$$

Next, by [40, (3.4)], the super Poincaré inequality (2.2) is equivalent to

$$\|P_t f\|_2^2 \leq \|f\|_2^2 \exp(-2t/r) + \beta(r) \|f\|_1^2 (1 - \exp(-2t/r)), \quad t, r > 0.$$

In particular, for any $A \subset E$ with $\mu(A) < \infty$, we have

(2.19) $$\|P_{t/2} \mathbf{1}_A\|_2^2 \leq \mu(A) \exp(-t/r) + \mu(A)^2 \beta(r) (1 - \exp(-t/r)), \quad t, r > 0.$$

Now, for $s > 0$ and $A \subset E$ with $0 < \mu(A) < s$, (2.18) gives

(2.20) $$\mu(|\mathbf{1}_A - P_t \mathbf{1}_A|) \leq \frac{\Theta_\gamma(t)}{2} \int_{E \times E} |\mathbf{1}_A(y) - \mathbf{1}_A(x)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y) = \Theta_\gamma(t) J_\gamma(A \times A^c).$$

On the other hand, we have

$$\mu(|\mathbf{1}_A - P_t \mathbf{1}_A|) = \int_A (1 - P_t \mathbf{1}_A) \, \mathrm{d}\mu + \int_{A^c} P_t \mathbf{1}_A \, \mathrm{d}\mu$$

$$\geq \int_A (1 - P_t \mathbf{1}_A) \, \mathrm{d}\mu = \mu(A) - \int_A P_t \mathbf{1}_A \, \mathrm{d}\mu = \mu(A) - \|P_{t/2} \mathbf{1}_A\|_2^2.$$

This together with (2.19) yields that for any $t, r > 0$,

$$\mu(|\mathbf{1}_A - P_t \mathbf{1}_A|) \geq \mu(A) \big(1 - \mu(A)\beta(r)\big) \big(1 - \exp(-t/r)\big).$$

Taking $r = t = \beta^{-1}(1/(2\mu(A)))$ in the inequality above, we get that

$$\mu(|\mathbf{1}_A - P_t \mathbf{1}_A|) \geq \frac{1 - \mathrm{e}^{-1}}{2} \mu(A).$$

Combining this with (2.20) we arrive at

$$\frac{J_\gamma(A \times A^c)}{\mu(A)} \geq \frac{1 - \mathrm{e}^{-1}}{2\Theta_\gamma(\beta^{-1}(1/(2\mu(A))))} \geq \frac{1 - \mathrm{e}^{-1}}{2\Theta_\gamma(\beta^{-1}(1/(2s)))},$$

where in the last inequality we have used the facts that $0 < \mu(A) < s$, β is decreasing and Θ_γ is increasing. Therefore, (2.17) holds. $\qquad\square$

Proof of Theorem 2.2. Let Φ_γ be in Theorem 2.2. Since $\beta(s)$ is strictly positive and deceasing on $(0,\infty)$, it is easy to see that $N_\gamma := \Phi_\gamma^{-1}$ is a Young function. Since $\beta(\infty) = 0$, by Lemma 2.4 we have $\kappa_\gamma(s) > 0$ for all $s > 0$, and

$$\Phi(t) := \int_0^t \frac{\mathrm{d}u}{\kappa_\gamma(u^{-1})} \leq \frac{2}{1-\mathrm{e}^{-1}} \int_0^t \Theta_\gamma\big(\beta^{-1}(u/2)\big)\,\mathrm{d}u$$
$$\leq \frac{4}{1-\mathrm{e}^{-1}} \int_0^t \Theta_\gamma\big(\beta^{-1}(r)\big)\,\mathrm{d}r = \frac{4}{1-\mathrm{e}^{-1}}\Phi_\gamma(t), \quad t \geq 0.$$

Thus, $N_\gamma(s) := \Phi_\gamma^{-1}(s) \leq \Phi^{-1}(4s/(1-\mathrm{e}^{-1})) := N(4s/(1-\mathrm{e}^{-1}))$. Combining this with (2.13) and (1.6), we prove (2.9). $\qquad\square$

2.3 From isoperimetric to super Poincaré

Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be given by (2.1). For a non-negative symmetric function γ on $E \times E$, let $J_\gamma(\mathrm{d}x, \mathrm{d}y) := \gamma(x,y)\,J(\mathrm{d}x,\mathrm{d}y)$, and $\kappa_\gamma(s)$ be the isoperimetric constant defined by (2.10). For a Young function N, we aim to deduce the super Poincaré inequality (2.2) from the Orlicz-Sobolev type isoperimetric inequality

$$(2.21) \qquad \|f\|_N \leq C \int_{E \times E} |f(x) - f(y)|\, J_\gamma(\mathrm{d}x, \mathrm{d}y), \quad f \in L_N(\mu).$$

To this end, we also consider the Poincaré type isoperimetric inequality

$$(2.22) \qquad \|f\|_2^2 \leq r \int_{E \times E} |f^2(x) - f^2(y)|\, J_\gamma(\mathrm{d}x, \mathrm{d}y) + \beta_1(r)\|f\|_1^2, \quad r > 0, f \in L^2(\mu)$$

for some decreasing function $\beta_1 : (0, \infty) \to (0, \infty)$.

Theorem 2.5. *Assume* (2.21) *for some Young function N such that $s \mapsto s^{-1}N(s)$ is increasing on $(0, \infty)$. Then:*

(1) *For any $s > 0$,*
$$\kappa_\gamma(s) \geq \frac{1}{2CsN^{-1}(s^{-1})}.$$

(2) (2.22) *holds with*
$$\beta_1(r) := 2\inf\big\{s > 0 : \ Cs^{-1}N^{-1}(s) \leq r\big\}, \quad r > 0.$$

(3) *If the density $J(x,y) := \frac{J(\mathrm{d}x,\mathrm{d}y)}{\mu(\mathrm{d}x)\mu(\mathrm{d}y)}$ and γ satisfy*

$$(2.23) \qquad c_\gamma := \mathrm{ess}_\mu \sup_x \int_E \gamma(x,y)^2 J(x,y)\,\mu(\mathrm{d}y) < \infty,$$

then (2.2) *holds with*
$$\beta(r) := 4\inf\left\{s > 0 : \ s^{-1}N^{-1}(s) \leq \frac{\sqrt{r}}{2C\sqrt{2c_\gamma}}\right\}, \quad r > 0.$$

Proof. For any $s > 0$ and $A \subset E$ with $\mu(A) \in (0, s)$, take $f = N^{-1}(\mu(A)^{-1})\mathbf{1}_A$. Then $\|f\|_N = 1$ and due to (2.21),

$$1 \le 2C J_\gamma(A \times A^c)N^{-1}(\mu(A)^{-1}).$$

Therefore,

$$\kappa_\gamma(s) \ge \frac{1}{2C} \inf_{r \in (0,s)} \frac{1}{rN^{-1}(r^{-1})}.$$

Since $sN^{-1}(s^{-1})$ is increasing in $s > 0$, this implies (1).

It is easy to see that (2) follows from (1) and Lemma 2.3(1). It remains to prove (3). By (1), Lemma 2.3(1), and the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
\|f\|_2^2 &\le CsN^{-1}(s^{-1}) \int_{E \times E} |f^2(x) - f^2(y)| \, J_\gamma(\mathrm{d}x, \mathrm{d}y) + \frac{2}{s}\|f\|_1^2 \\
&\le CsN^{-1}(s^{-1}) \left(\int_{E \times E} (f(x) - f(y))^2 \, J(\mathrm{d}x, \mathrm{d}y) \right)^{1/2} \\
&\qquad \times \left(\int_{E \times E} (f(x) + f(y))^2 \gamma(x, y)^2 \, J(\mathrm{d}x, \mathrm{d}y) \right)^{1/2} + \frac{2}{s}\|f\|_1^2 \\
&\le 2C\sqrt{2c_\gamma}sN^{-1}(s^{-1})\sqrt{\mathscr{E}(f, f)\|f\|_2^2} + \frac{2}{s}\|f\|_1^2 \\
&\le \frac{1}{2}\|f\|_2^2 + 4C^2 c_\gamma (sN^{-1}(s^{-1}))^2 \mathscr{E}(f, f) + \frac{2}{s}\|f\|_1^2, \quad s > 0,
\end{aligned}
$$

where in the third inequality we have used (2.23). This implies (2.2) for the desired β. $\quad\square$

Similarly, we have the following result.

Theorem 2.6. *Assume that (2.22) holds with $\beta_1(\infty) := \lim_{r\to\infty} \beta_1(r) = 0$. Then:*

(1) *For any $s > 0$,*

$$\kappa_\gamma(s) \ge \frac{1}{4\beta_1^{-1}(1/(2s))}.$$

(2) *If*

$$\Phi(t) := 4 \int_0^t \beta_1^{-1}(r/2) \, \mathrm{d}r < \infty, \quad t > 0,$$

then (2.21) holds with $N := \Phi^{-1}$.

(3) *(2.23) implies (2.2) with*

$$\beta(r) := 2\beta_1 \left(\sqrt{r}/(2\sqrt{2c_\gamma}) \right), \quad r > 0.$$

Proof. By Theorem 2.1(2), (2.22) implies

$$\mu(A) \le 2r J_\gamma(A \times A^c) + \beta_1(r)\mu(A)^2, \quad r > 0.$$

Thus,

$$1 \le 2r \frac{J_\gamma(A \times A^c)}{\mu(A)} + \beta_1(r)\mu(A) \le 2r \frac{J_\gamma(A \times A^c)}{\mu(A)} + \beta_1(r)s, \quad r > 0.$$

Taking $r = \beta_1^{-1}((2s)^{-1})$ in the inequality above, we get that

$$\frac{J_\gamma(A \times A^c)}{\mu(A)} \ge \frac{1}{4\beta_1^{-1}(1/(2s))}.$$

This implies (1).

(2) immediately follows from (1) and Lemma 2.3(2), and (3) can be proved by the argument for Theorem 2.5(3). $\square$

As a consequence of Theorem 2.5(2) and Theorem 2.6(2), we have the following correspondence of (2.21) and (2.22).

Corollary 2.7. *Let $p_1, p_2, p > 1$ and $q \in \mathbb{R}$ be constants. Then,*

(1) *(2.21) holds with $N(s) = s^{p_1} \wedge s^{p_2}$ if and only if (2.22) holds with*

$$\beta_1(r) := c(r^{-\frac{p_1}{p_1-1}} \vee r^{-\frac{p_2}{p_2-1}}), \quad r > 0$$

for some constant $c > 0$.

(2) *(2.21) holds with $N(s) = s^{p_1} \vee s^{p_2}$ if and only if (2.22) holds with*

$$\beta_1(r) := c(r^{-\frac{p_1}{p_1-1}} \wedge r^{-\frac{p_2}{p_2-1}}), \quad r > 0$$

for some constant $c > 0$.

(3) *Let $\lambda \ge 2$ such that $N(s) := s^p \{\log(\lambda + s^{-1})\}^q$ is Young function and $s \mapsto s^{-1}N(s)$ is increasing on $(0, \infty)$. Then, (2.21) holds with $N(s)$ if and only if (2.22) holds with*

$$\beta_1(r) := cr^{-\frac{p}{p-1}} \{\log(2 + r)\}^{-\frac{q}{p-1}}, \quad r > 0$$

for some constant $c > 0$.

(4) *Let $\lambda \ge 2$ such that $N(s) := s^p \{\log(\lambda + s)\}^q$ is Young function and $s \mapsto s^{-1}N(s)$ is increasing on $(0, \infty)$. Then, (2.21) holds with $N(s)$ if and only if (2.22) holds with*

$$\beta_1(r) := cr^{-\frac{p}{p-1}} \{\log(2 + r^{-1})\}^{-\frac{q}{p-1}}, \quad r > 0$$

for some constant $c > 0$.

2.4 Extension to the case with killing

We will add a potential term to the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ given in (2.1). Let V be a non-negative measurable function on $(E, \mathscr{F})$ such that the class

$$\mathscr{D}(\mathscr{E}_V) := \left\{ f \in L^2(\mu) : \mathscr{E}_V(f, f) := \mathscr{E}(f, f) + \int_E f^2 V(\mathrm{d}x) < \infty \right\}$$

is dense in $L^2(\mu)$, where $V(\mathrm{d}x) := V(x)\,\mu(\mathrm{d}x)$. Then $(\mathscr{E}_V, \mathscr{D}(\mathscr{E}_V))$ is a Shrödinger type symmetric energy form in $L^2(\mu)$, where

$$(2.24) \qquad \mathscr{E}_V(f, g) := \mathscr{E}(f, g) + \int_E f(x)g(x)\,V(\mathrm{d}x), \quad f, g \in \mathscr{D}(\mathscr{E}_V).$$

It is standard that by enlarging the state space we are able to reduce to present setting to the case without killing, see [21, 12]. More precisely, let $\bar{E} := E \cup \{\Delta\}$ for an additional state Δ, and define

$$\bar{\mu}(\mathrm{d}x) = \mathbf{1}_E(x)\,\mu(\mathrm{d}x) + \delta_\Delta(\mathrm{d}x),$$
$$\bar{J}(\mathrm{d}x, \mathrm{d}y) = \mathbf{1}_{E \times E} J(\mathrm{d}x, \mathrm{d}y) + \mathbf{1}_{\{\Delta\} \times E}(x, y)\,\delta_\Delta(\mathrm{d}x)\,V(\mathrm{d}y) + \mathbf{1}_{E \times \{\Delta\}}(x, y)\,V(\mathrm{d}x)\,\delta_\Delta(\mathrm{d}y),$$

where δ_Δ is the Dirac measure at point Δ. Since $J(\mathrm{d}x, \mathrm{d}y) = J(x, y)\,\mu(\mathrm{d}x)\,\mu(\mathrm{d}y)$, we have

$$\bar{J}(\mathrm{d}x, \mathrm{d}y) = \bar{J}(x, y)\,\bar{\mu}(\mathrm{d}x)\,\bar{\mu}(\mathrm{d}y),$$

where

$$\bar{J}(x, y) := \mathbf{1}_{E \times E}(x, y)J(x, y) + \mathbf{1}_{\{\Delta\} \times E}(x, y)V(y) + \mathbf{1}_{E \times \{\Delta\}}(x, y)V(x).$$

Next, for a non-negative symmetric function γ on $E \times E$ and a non-negative function ξ on E, let

$$\bar{\gamma}(x, y) = \mathbf{1}_{E \times E}(x, y)\gamma(x, y) + \mathbf{1}_{\{\Delta\} \times E}(x, y)\xi(y) + \mathbf{1}_{E \times \{\Delta\}}(x, y)\xi(x).$$

Then for any $x \in E$,

$$\int_{\bar{E}} \bar{\gamma}(x, y)^2 \bar{J}(x, y)\,\bar{\mu}(\mathrm{d}y) = \int_E \gamma(x, y)^2 J(x, y)\,\mu(\mathrm{d}y) + \xi(x)^2 V(x).$$

Finally, for any measurable function f on E, we extend it into $\bar{f}$ defined on $\bar{E}$ and by letting $\bar{f}(\Delta) = 0$. Then

$$(2.25) \quad \mathscr{E}_V(f, g) = \bar{\mathscr{E}}(\bar{f}, \bar{g}) := \frac{1}{2} \int_{\bar{E} \times \bar{E}} (\bar{f}(x) - \bar{f}(y))(\bar{g}(x) - \bar{g}(y))\,\bar{J}(\mathrm{d}x, \mathrm{d}y), \quad f, g \in \mathscr{D}(\mathscr{E}_V).$$

Let P_t^V be the (sub)-Markov semigroup on $L^2(\mu)$ associated to $(\mathscr{E}_V, \mathscr{D}(\mathscr{E}_V))$, while $\bar{P}_t$ is the corresponding semigroup on $L^2(\bar{\mu})$. We have

$$\bar{\theta}(t) := \sup_{\|\bar{g}\|_\infty \leq 1} \mathrm{ess}_{\bar{\mu} \times \bar{\mu}} \sup_{x \neq y} \frac{|\bar{P}_t \bar{g}(x) - \bar{P}_t \bar{g}(y)|}{\bar{\gamma}(x, y)}$$

$$= \sup_{\|g\|_\infty \le 1} \mathrm{ess}_{\mu \times \mu} \sup_{x \ne y} \max \left\{ \frac{|P_t g(x) - P_t g(y)|}{\gamma(x, y)}, \ \frac{|P_t g(x)|}{\xi(x)} \right\}, \quad t > 0.$$

With the aid of all the notations above, by applying Theorem 2.2 and Theorem 2.5 to $\bar{\mathscr{E}}$ and $\bar{\mu}$ we obtain the following result.

Theorem 2.8. *Suppose that the super Poincaré inequality (2.2) holds for $(\mathscr{E}_V, \mathscr{D}(\mathscr{E}_V))$ replacing $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ with some decreasing function $\beta : (0, \infty) \to (0, \infty)$ satisfying that $\beta(\infty) = 0$. If*

$$\bar{\Phi}_\gamma(s) := \int_0^s \mathrm{d}r \int_0^{\beta^{-1}(r)} \bar{\theta}(t) \, \mathrm{d}t < \infty, \quad s > 0,$$

then $\bar{N}_\gamma := \bar{\Phi}_\gamma^{-1}$ is a Young function, and there exists a constant $C > 0$ such that

$$(2.26) \qquad \|f\|_{\bar{N}_\gamma} \le C \left(\int_{E \times E} |f(x) - f(y)| \gamma(x, y) \, J(\mathrm{d}x, \mathrm{d}y) + \int_E |f(x)| \xi(x) \, V(\mathrm{d}x) \right)$$

holds for all $f \in L_{\bar{N}_\gamma}(\mu)$.
  On the other hand, suppose that

$$\sup_{x \in E} \left(\int_E \gamma(x, y)^2 J(x, y) \, \mu(\mathrm{d}y) + \xi(x)^2 V(x) \right) < \infty.$$

If (2.26) holds for some Young function N replacing $\bar{N}_\gamma$ and satisfying that $s \mapsto s^{-1} N(s)$ is increasing on $(0, \infty)$, then there exist constants $c_1, c_2 > 0$ such that (2.2) holds for $(\mathscr{E}_V, \mathscr{D}(\mathscr{E}_V))$ replacing $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ with

$$\beta(r) := c_1 \inf \left\{ s > 0 : \ s^{-1} N^{-1}(s) \le c_2 \sqrt{r} \right\}, \quad r > 0.$$

3 Proof of Theorem 1.1 and extensions

3.1 Proof of Theorem 1.1

By Theorem 2.1(1), it suffices to prove (1.7) and assertions (1) and (2). Let $E = \mathbb{R}^n$ and $\mu(\mathrm{d}x)$ be the Lebesgue measure. Consider the symmetric α-stable process on $\mathbb{R}^n$ with jumping kernel

$$J(x, y) := \mathbf{1}_{\{x \ne y\}} |x - y|^{-(n+\alpha)}, \quad x, y \in \mathbb{R}^n.$$

Let P_t be the Markov semigroup generated by the Dirichlet form

$$\mathscr{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} (f(x) - f(y))(g(x) - g(y)) J(x, y) \, \mathrm{d}x \, \mathrm{d}y, \quad f, g \in \mathscr{D}(\mathscr{E}).$$

It is well known that for some constant $c_1 \ge 1$, we have the heat kernel upper bound (see for instance [17, Theorem 3.2])

$$(3.1) \qquad \|P_t\|_{L^1(\mu) \to L^\infty(\mu)} \le \frac{c_1}{t^{n/\alpha}}, \quad t > 0,$$

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as well as the gradient estimate (see for instance [36, Theorem 1.3 and Example 1.4])

$$(3.2) \qquad \|\nabla P_t f\|_\infty := \sup_{x \in \mathbb{R}^n} \limsup_{y \to x} \frac{|P_t f(y) - P_t f(x)|}{|y - x|} \leq \frac{c_1}{t^{1/\alpha}} \|f\|_\infty, \quad f \in L^\infty(\mu), t > 0.$$

The heat kernel upper bound (3.1) is equivalent to the Sobolev/Nash inequality with dimension $2n/\alpha$ (see [15] or [17, Section 3]), or the super Poincaré inequality (2.2) with

$$(3.3) \qquad \beta(r) = c_2 r^{-n/\alpha}, \quad r > 0$$

for some constant $c_2 \geq 1$, see [40, 41] or [43].

Now, for any h satisfying conditions (i) and (ii), the gradient estimate (3.2) yields

$$(3.4) \qquad \begin{aligned} \theta(t) &:= \sup_{\|g\|_\infty \leq 1} \text{ess}_{\mu \times \mu} \sup_{x \neq y} \frac{|P_t g(x) - P_t g(y)|}{h(|x - y|)} \leq \sup_{s > 0} \frac{2c_1}{h(s)} \left(1 \wedge \frac{s}{t^{1/\alpha}}\right) \\ &= 2c_1 \sup_{s > 0} \left(\frac{1}{h(s)} \wedge \frac{s}{h(s)t^{1/\alpha}}\right) = \frac{2c_1}{h(t^{1/\alpha})}, \quad t > 0, \end{aligned}$$

where the last step follows from the fact that $h(s)^{-1}$ is decreasing while $sh(s)^{-1}$ is increasing so that the sup is reached at $s = t^{1/\alpha}$ which solves $\frac{1}{h(s)} = \frac{s}{h(s)t^{1/\alpha}}$.

Finally, let $\gamma(x, y) = h(|x - y|)$. By (3.3) and (3.4), we have

$$\Phi_\gamma(s) := \int_0^s \mathrm{d}r \int_0^{\beta^{-1}(r)} \theta(t) \, \mathrm{d}t \leq 2c_1 \int_0^s \mathrm{d}r \int_0^{(r/c_2)^{-\alpha/n}} \frac{\mathrm{d}t}{h(t^{1/\alpha})} \leq c_3 \Phi_h(c_4 s), \quad s \geq 0$$

for some constants $c_3, c_4 \geq 1$. Therefore, by Theorem 2.2 and the property (1.6), we prove (1.7) for some constant $C > 0$.

Below we verify (1.9)–(1.12) and their sharpness respectively.

(a) **For** (1.9). Let $\alpha = 1$ and $h(s) = s^{1-\alpha_1/2} \wedge s^{1-\alpha_2/2}$ for $s \geq 0$. Then

$$\Phi_h(s) := \int_0^s \mathrm{d}t \int_0^{t^{-1/n}} \frac{\mathrm{d}r}{h(r)} \leq c_5 \left(s^{\frac{n-\alpha_1/2}{n}} \vee s^{\frac{n-\alpha_2/2}{n}}\right), \quad s \geq 0$$

holds for some constant $c_5 > 0$. So,

$$N_h := \Phi_h^{-1} \geq c_6 N_{\alpha_1, \alpha_2}^{\wedge}$$

holds for some constant $c_6 > 0$. Therefore, (1.9) follows from (1.7).

To verify the sharpness of (1.9), let N be a Young function such that $N \npreceq N_{\alpha_1, \alpha_2}^{\wedge}$. We have

$$(3.5) \qquad \left(\limsup_{s \downarrow 0} + \limsup_{s \uparrow \infty}\right) \frac{N(s)}{N_{\alpha_1, \alpha_2}^{\wedge}(s)} = \infty.$$

Let

$$f_s(x) := (s - |x|)^+, \quad s > 0, x \in \mathbb{R}^n.$$

Then
$$|f_s(x) - f_s(y)| \le (s \wedge |x - y|)\{\mathbf{1}_{B(0,s)}(x) + \mathbf{1}_{B(0,s)}(y)\}.$$

Thus, there exist constants $c_7, c_8 > 0$ such that

(3.6)
$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f_s(x) - f_s(y)|}{|x - y|^{n+\alpha_1/2} \vee |x - y|^{n+\alpha_2/2}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\le 2 \int_{B(0,s)} \mathrm{d}x \int_{\mathbb{R}^n} \frac{s \wedge |x - y|}{|x - y|^{n+\alpha_1/2} \vee |x - y|^{n+\alpha_2/2}} \, \mathrm{d}y$$
$$\le c_7 \int_{B(0,s)} (s^{1-\alpha_1/2} \wedge s^{1-\alpha_2/2}) \, \mathrm{d}x \le c_8(s^{n+1-\alpha_1/2} \wedge s^{n+1-\alpha_2/2}), \quad s \ge 0.$$

If (1.9) holds for N replacing $N^\wedge_{\alpha_1,\alpha_2}$, then

$$\|f_s\|_N \le c_0(s^{n+1-\alpha_1/2} \wedge s^{n+1-\alpha_2/2}), \quad s > 0$$

holds for some constant $c_0 > 0$. Therefore, there exist constants $c_9, c_{10} > 0$ such that

(3.7)
$$1 \ge \int_{\mathbb{R}^n} N\Big(\big\{c_0(s^{n+1-\alpha_1/2} \wedge s^{n+1-\alpha_2/2})\big\}^{-1} f_s(x)\Big) \, \mathrm{d}x$$
$$\ge \int_{\{s/4 \le |x| \le 3s/4\}} N\big(c_9(s^{\alpha_1/2-n} \vee s^{\alpha_2/2-n})\big) \, \mathrm{d}x$$
$$\ge c_{10} s^n N\big(c_9(s^{\alpha_1/2-n} \vee s^{\alpha_2/2-n})\big), \quad s > 0.$$

Noting that

$$\inf_{s>0} s^n N^\wedge_{\alpha_1,\alpha_2}\big(c_9(s^{\alpha_1/2-n} \vee s^{\alpha_2/2-n})\big) > 0,$$

from (3.5) and (3.7) we conclude that

$$1 \ge \Big(\limsup_{s \downarrow 0} + \limsup_{s \uparrow \infty}\Big) c_{10} s^n N\big(c_9(s^{\alpha_1/2-n} \vee s^{\alpha_2/2-n})\big) = \infty,$$

which is impossible.

(b) **For (1.10).** Let $\alpha = 1$ and $h(s) = s^{1-\alpha_1/2} \vee s^{1-\alpha_2/2}$ for $s \ge 0$. Then

$$\Phi_h(s) := \int_0^s \mathrm{d}t \int_0^{t^{-1/n}} \frac{\mathrm{d}r}{h(r)} \le c_5'\big(s^{\frac{n-\alpha_1/2}{n}} \wedge s^{\frac{n-\alpha_2/2}{n}}\big), \quad s \ge 0$$

holds for some constant $c_5' > 0$. So,

$$N_h := \Phi_h^{-1} \ge c_6' N^\vee_{\alpha_1,\alpha_2}$$

holds for some constant $c_6' > 0$. Therefore, (1.10) follows from (1.7).

As in (a), we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f_s(x) - f_s(y)|}{|x - y|^{n+\alpha_1/2} \wedge |x - y|^{n+\alpha_2/2}} \mathrm{d}x\mathrm{d}y \le c_7'(s^{n+1-\alpha_1/2} \vee s^{n+1-\alpha_2/2}), \quad s > 0.$$

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for some constant $c_7' > 0$. Moreover, for any Young function N with $N \not\preceq N^\wedge_{\alpha_1,\alpha_2}$, and for any constant $c_0 > 0$, we have

$$\left(\limsup_{s \downarrow 0} + \limsup_{s \uparrow \infty} \right) \int_{\mathbb{R}^n} N \left(\{ c_0 (s^{n+1-\alpha_1/2} \vee s^{n+1-\alpha_2/2}) \}^{-1} f_s(x) \right) dx = \infty,$$

so that (1.10) does not hold for N replacing $N^\vee_{\alpha_1,\alpha_2}$.

(c) **For** (1.11). Let $\lambda_0 \geq 2$ such that

$$h(s) := s^{\alpha/2} \{ \log(\lambda_0 + s^{-1}) \}^q, \quad s \geq 0$$

satisfies condition (i). Then there exists a constant $c_{11} > 0$ such that

$$\Phi_h(s) := \int_0^s dt \int_0^{t^{-\frac{1}{n}}} \frac{r^{(\alpha/2)-1}}{\{ \log(\lambda_0 + r^{-1}) \}^q} dr \leq \frac{c_{11} s^{\frac{n-\alpha/2}{n}}}{\{ \log(2 + s) \}^q}, \quad s \geq 0.$$

Thus,

$$N_h := \Phi_h^{-1} \geq c_{12} N_\alpha^{log,q,+}$$

holds for some constant $c_{12} > 0$. Therefore, (1.11) follows from (1.7). The sharpness can be verified with reference functions f_s as above.

(d) **For** (1.12). We take

$$h(s) := s^{\alpha/2} \{ \log(\lambda_0 + s) \}^p, \quad s \geq 0$$

for some $\lambda_0 \geq 2$ large enough such that condition (i) is satisfied. Then there is a constant $c_{11}' > 0$ such that

$$\Phi_h(s) := \int_0^s dt \int_0^{t^{-\frac{1}{n}}} \frac{r^{(\alpha/2)-1}}{\{ \log(\lambda_0 + r) \}^p} dr \leq \frac{c_{11}' s^{\frac{n-\alpha/2}{n}}}{\{ \log(2 + s^{-1}) \}^p}, \quad s \geq 0.$$

Hence,

$$N_h := \Phi_h^{-1} \geq c_{12}' N_\alpha^{log,p,-}$$

holds for some constant $c_{12}' > 0$. Therefore, from (1.7) we can get (1.12). Similar to (c), one can verify the sharpness of (1.12) by using reference functions f_s as above.

3.2 Extension to the truncated α-stable form

Theorem 3.1. *Let $n \geq 2$, $\alpha \in (0,2)$, and let $h : (0,\infty) \to (0,\infty)$ satisfy condition* (i) *in Theorem 1.1 and*

(ii')

$$\tilde{\Phi}_h(s) := \int_0^s dt \int_0^{t^{-\frac{\alpha}{n}} \vee t^{-\frac{2}{n}}} \frac{dr}{h(r^{\frac{1}{\alpha}} \wedge r^{\frac{1}{2}})} < \infty, \quad s \geq 0.$$

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Let $\tilde{N}_h = \tilde{\Phi}_h^{-1}$. Then there exists a constant $C > 0$ such that

$$(3.8) \qquad \|f\|_{\tilde{N}_h} \leq C \int_{\{|x-y|\leq 1\}} |f(x) - f(y)| \frac{h(|x-y|)}{|x-y|^{n+\alpha}} \, dx \, dy, \quad f \in L_{\tilde{N}_h}(\mathbb{R}^n).$$

Consequently, for $\tilde{N}_\alpha(s) := s^{\frac{n}{n-\alpha/2}} \wedge s^{\frac{n}{n-1}}$, there exists a constant $C > 0$ such that

$$(3.9) \qquad \|f\|_{\tilde{N}_\alpha} \leq C \int_{\{|x-y|\leq 1\}} \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha/2}} \, dx \, dy, \quad f \in L_{\tilde{N}_\alpha}(\mathbb{R}^n).$$

This inequality fails if $\tilde{N}_\alpha$ is replaced by a Young function $N \not\gtrsim \tilde{N}_\alpha$.

Proof. Consider the following truncated α-stable Dirichlet form

$$\mathscr{E}(f,g) := \frac{1}{2} \int_{\{|x-y|\leq 1\}} (f(x) - f(y))(g(x) - g(y)) \frac{1}{|x-y|^{d+\alpha}} \, dx \, dy, \quad f, g \in \mathscr{D}(\mathscr{E}).$$

Let P_t be the associated Markov semigroup. Then, by [16, Proposition 2.2] and [36, Theorem 1.3 and Example 1.5], we have

$$(3.10) \qquad \|P_t\|_{L^1(\mu) \to L^\infty(\mu)} \leq \frac{c_1}{t^{n/\alpha} \wedge t^{n/2}}, \quad t > 0,$$

and

$$(3.11) \qquad \|\nabla P_t f\|_\infty \leq \frac{c_1}{t^{1/\alpha} \wedge t^{1/2}} \|f\|_\infty, \quad f \in L^\infty(\mu), t > 0$$

for some constant $c_1 \geq 1$. By (3.10) and [41, Theorem 4.5], the super Poincaré inequality (2.2) holds with

$$(3.12) \qquad \beta(r) = c_2(r^{-n/\alpha} \vee r^{-n/2}), \quad r > 0$$

for some constant $c_2 \geq 1$. On the other hand, by (3.11) and the argument of (3.4), for any h satisfying condition (i),

$$(3.13) \qquad \begin{aligned} \theta(t) :&= \sup_{\|g\|_\infty \leq 1} \mathrm{ess}_{\mu \times \mu} \sup_{x \neq y} \frac{|P_t g(x) - P_t g(y)|}{h(|x-y|)} \\ &\leq \sup_{s>0} \frac{2c_1}{h(s)} \left(1 \wedge \frac{s}{t^{1/\alpha} \wedge t^{1/2}} \right) = \frac{2c_1}{h(t^{1/\alpha} \wedge t^{1/2})}, \quad t > 0. \end{aligned}$$

Thus, let $\gamma(x,y) = h(|x-y|)$. By (3.12) and (3.13), for $s > 0$,

$$\Phi_\gamma(s) := \int_0^s dr \int_0^{\beta^{-1}(r)} \theta(t) \, dt \leq 2c_1 \int_0^s dr \int_0^{(r/c_2)^{-\alpha/n} \vee (r/c_2)^{-2/n}} \frac{dt}{h(t^{1/\alpha} \wedge t^{1/2})} \leq c_3 \tilde{\Phi}_h(c_4 s)$$

holds with some constants $c_3, c_4 \geq 1$. Therefore, by (ii') and Theorem 2.2 we prove (3.8) for some constant $C > 0$.

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Below we verify (3.9) and its sharpness. Let $h(s) = s^{\alpha/2} \vee s$ for $s \geq 0$. Then

$$\tilde{\Phi}_h(s) := \int_0^s \mathrm{d}r \int_0^{r^{-\alpha/n} \vee r^{-2/n}} \frac{\mathrm{d}t}{h(t^{1/\alpha} \wedge t^{1/2})} \leq c_5\big(s^{\frac{n-\alpha/2}{n}} \vee s^{\frac{n-1}{n}}\big), \quad s \geq 0$$

holds for some constant $c_5 > 0$, where in the inequality we have used the fact that $n \geq 2$. So,

$$\tilde{N}_h := \tilde{\Phi}_h^{-1} \geq c_6 \tilde{N}_\alpha$$

holds for some constant $c_6 > 0$. Therefore, (3.9) follows from (1.7).

Let f_s be the function in the argument of Theorem 1.1. Then there exists a constant $c_7 > 0$ such that

$$\int_{\{|x-y| \leq 1\}} \frac{|f_s(x) - f_s(y)|}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \leq 2 \int_{B(0,s)} \mathrm{d}x \int_{\{|y-x| \leq 1\}} \frac{s \wedge |x-y|}{|x-y|^{n+\alpha/2}} \, \mathrm{d}y$$
$$\leq c_7(s^{n+1-\alpha/2} \wedge s^n), \quad s \geq 0.$$

Let N be a Young function such that $N \not\preceq \tilde{N}_\alpha$. We have

$$\Big(\limsup_{s \downarrow 0} + \limsup_{s \uparrow \infty} \Big) \frac{N(s)}{\tilde{N}_\alpha(s)} = \infty.$$

Suppose that (3.9) holds for N. Then

$$\|f_s\|_N \leq c_0(s^{n+1-\alpha/2} \wedge s^n), \quad s > 0$$

holds for some constant $c_0 > 0$, so that

$$1 \geq \int_{\mathbb{R}^n} N\Big(\big\{ c_0(s^{n+1-\alpha/2} \wedge s^n) \big\}^{-1} f_s(x) \Big) \, \mathrm{d}x$$
$$\geq \int_{\{s/4 \leq |x| \leq 3s/4\}} N\big(c_8(s^{\alpha/2-n} \vee s^{1-n})\big) \, \mathrm{d}x \geq c_9 s^n N\big(c_8(s^{\alpha/2-n} \vee s^{1-n})\big), \quad s > 0.$$

Combining this with all the estimates above, we obtain that

$$1 \geq \Big(\limsup_{s \downarrow 0} + \limsup_{s \uparrow \infty} \Big) c_9 s^n N\big(c_8(s^{\alpha/2-n} \vee s^{1-n})\big) = \infty,$$

which is impossible. Therefore, (3.9) does not hold for N, and so we verify the sharpness of (3.9). $\qquad \square$

3.3 Extension to discrete α-stable Dirichlet form

In this subsection, let $E = \mathbb{Z}^n$ and μ be the counting measure. Under this setting, the Orlicz norm $\| \cdot \|_N$ for a Young function N is essentially determined by $N(s)$ for small $s > 0$. In particular, for any two Young functions N_1 and N_2, $\| \cdot \|_{N_1} \leq c \| \cdot \|_{N_2}$ for some constant $c > 0$

if and only if there is a constant $c' > 0$ such that $N_1(s) \leq c' N_2(s)$ holds for all $s \in [0, 1]$. Moreover, since $|x - y| \geq 1$ for $x \neq y$, we have $2 \leq 2 + |x - y|^{-1} \leq 3$, and for $\alpha_1 \leq \alpha_2$,

$$|x - y|^{n+\alpha_1/2} \vee |x - y|^{n+\alpha_2/2} = |x - y|^{n+\alpha_2/2}, \quad |x - y|^{n+\alpha_1/2} \wedge |x - y|^{n+\alpha_2/2} = |x - y|^{n+\alpha_1/2}.$$

Therefore, in assertion (1) of Theorem 1.1 we will take $\alpha_1 = \alpha_2 = \alpha$, and in assertion (2) we only consider $N_\alpha^{\log,p,-} := \left\{ s \log^p (\lambda + s^{-1}) \right\}^{\frac{n}{n-\alpha/2}}$ with some constant $\lambda \geq 2$.

Theorem 3.2. *Assertions in Theorem 1.1 hold for the counting measure μ on $\mathbb{Z}^n$ replacing the Lebesgue measure on $\mathbb{R}^n$. More precisely, for any $\alpha \in (0, 2)$ and $h \in \mathscr{H}_\alpha$, there exists a constant $C > 0$ such that*

$$\|f\|_{N_h} \leq C \sum_{x,y \in \mathbb{Z}^n, x \neq y} |f(x) - f(y)| \frac{h(|x - y|)}{|x - y|^{n+\alpha}}, \quad f \in L_{N_h}(\mathbb{Z}^n).$$

Consequently:

(1) *There exists a constant $C > 0$ such that*

$$\|f\|_{\frac{n}{n-\alpha/2}} \leq C \sum_{x,y \in \mathbb{Z}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{n+\alpha/2}}, \quad f \in L^{\frac{n}{n-\alpha/2}}(\mathbb{Z}^n).$$

This inequality fails if $\| \cdot \|_{\frac{n}{n-\alpha/2}}$ is replaced by $\| \cdot \|_{L_N}$ for a Young function N with

$$\limsup_{s \to 0} s^{\frac{n-\alpha/2}{n}} N(s) = \infty.$$

(2) *For any $\alpha \in (0, 2)$ and $p \in \mathbb{R}$, there exists a constant $C > 0$ such that for all $f \in L_{N_\alpha^{\log,p,-}}(\mathbb{Z}^n)$,*

$$\|f\|_{N_\alpha^{\log,p,-}} \leq C \sum_{x,y \in \mathbb{Z}^n, x \neq y} |f(x) - f(y)| \frac{\{\log(2 + |x - y|)\}^p}{|x - y|^{n+\alpha/2}}.$$

This inequality fails if $N_\alpha^{\log,p,-}$ is replaced by a Young function N with

$$\limsup_{s \to 0} \frac{N(s)}{N_\alpha^{\log,p,-}(s)} = \infty.$$

Proof. According to the proof of Theorem 1.1, it suffices to construct a symmetric sub-Markov semigroup P_t on $L^2(\mu)$ such that the associated Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is comparable with

$$\mathscr{E}_\alpha(f, f) := \frac{1}{2} \sum_{x,y \in \mathbb{Z}^n, x \neq y} \frac{(f(x) - f(y))^2}{|x - y|^{n+\alpha}}, \quad \mathscr{D}(\mathscr{E}_\alpha) := \{f \in L^2(\mu), \mathscr{E}_\alpha(f, f) < \infty\},$$

i.e., $\mathscr{D}(\mathscr{E}) = \mathscr{D}(\mathscr{E}_\alpha)$ and there exists a constant $C \geq 1$ such that

$$(3.14) \qquad C^{-1}\mathscr{E}(f,f) \leq \mathscr{E}_\alpha(f,f) \leq C\mathscr{E}(f,f), \quad f \in \mathscr{D}(\mathscr{E}) = \mathscr{D}(\mathscr{E}_\alpha);$$

and moreover, both (3.1) and (3.2) are satisfied for P_t, where in (3.2)

$$\|\nabla P_t f\|_\infty := \sup_{x,y \in \mathbb{Z}^n, |x-y|=1} |P_t f(x) - P_t f(y)|.$$

Condition (3.1) can be easily deduced from the Nash inequality for $(\mathscr{E}_\alpha, \mathscr{D}(\mathscr{E}_\alpha))$ (see for example [30, Proposition 2.1]), but to prove explicit gradient estimate (3.2) we need additional arguments. Below we first construct the required semigroup P_t then verify these two estimates.

(1) **Construction of P_t.** Let $q_k(x,y)$ and Q^k be the transition function and the semigroup for discrete time simple random walk $Y' = (Y_k')_{k \geq 0}$ on $\mathbb{Z}^n$, respectively. It is known (see [38] or [20, Theorem 5.1]) that there are constants $c_i > 0$ $(i = 1, \cdots 5)$ so that

$$(3.15) \qquad q_k(x,y) \leq \frac{c_1}{k^{n/2}} \exp\left(\frac{-c_2|x-y|^2}{k}\right), \quad x,y \in \mathbb{Z}^n, k \geq 1,$$

$$(3.16) \qquad q_k(x,y) + q_{k+1}(x,y) \geq \frac{c_3}{k^{n/2}} \exp\left(\frac{-c_4|x-y|^2}{k}\right), \quad x,y \in \mathbb{Z}^n, k \geq |x-y|$$

and

$$(3.17) \qquad \begin{aligned} \|\nabla Q^k f\|_\infty &:= \sup_{x \in \mathbb{Z}^n} \sup_{y \in \mathbb{Z}^n, |y-x|=1} |Q^k f(y) - Q^k f(x)| \\ &\leq c_5 k^{-1/2} \|f\|_\infty, \quad k \geq 1, f \in L^\infty(\mathbb{Z}^n). \end{aligned}$$

Consider the discrete subordination of Y' by the Bernstein function $\psi(r) = r^{\alpha/2}$ with $\alpha \in (0,2)$, see [5]. Denote by $X' = (X_k')_{k \geq 0}$ the corresponding discrete time subordinated Markov chain on $\mathbb{Z}^n$, and by $p_k(x,y)$ the transition function of X'. Then, according to [5, Proposition 2.3 and Example 2.1],

$$(3.18) \qquad p_1(x,y) = \sum_{k=1}^\infty c(\psi,k) q_k(x,y), \quad k \geq 1, x,y \in \mathbb{Z}^n,$$

where

$$c(\psi,k) = \frac{\alpha}{2\Gamma(1-\alpha/2)} \frac{\Gamma(k-\alpha/2)}{\Gamma(k+1)}.$$

We claim that

$$(3.19) \qquad \frac{c_0^{-1}}{|x-y|^{n+\alpha}} \leq p_1(x,y) \leq \frac{c_0}{|x-y|^{n+\alpha}}, \quad x,y \in \mathbb{Z}^n, x \neq y$$

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holds for some constant $c_0 \geq 1$. Indeed, by [18],

$$(3.20) \qquad c_6^{-1} k^{-(\alpha/2+1)} \leq c(\psi, k) \leq c_6 k^{-(\alpha/2+1)}, \quad k \geq 1$$

holds for some constant $c_7 \geq 1$. Then, by (3.16), (3.18) and (3.20), we have

$$p_1(x, y) \geq \frac{1}{2} \left(\sum_{k=1}^{\infty} c(\psi, k) q_k(x, y) + \sum_{k=1}^{\infty} c(\psi, k+1) q_{k+1}(x, y) \right)$$

$$\geq c_7 \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha/2}} (q_k(x, y) + q_{k+1}(x, y)) \geq c_8 \sum_{k=|x-y|^2}^{\infty} \frac{1}{k^{1+(\alpha+n)/2}} \geq \frac{c_9}{|x-y|^{n+\alpha}}.$$

On the other hand, according to (3.15), (3.18) and (3.20),

$$p_1(x, y) \leq c_7 \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha/2}} q_k(x, y)$$

$$\leq c_9 \left(\sum_{k=|x-y|^2}^{\infty} \frac{1}{k^{1+(\alpha+n)/2}} + \sum_{k=1}^{|x-y|^2} \frac{1}{k^{1+(\alpha+n)/2}} \exp\left(\frac{-c_2|x-y|^2}{k} \right) \right)$$

$$\leq \frac{c_{10}}{|x-y|^{n+\alpha}}.$$

Thus, (3.19) is proved.

Let N_t be a Poisson process independent of X' and Y'. Set $X_t = X'_{N_t}$ and $Y_t = Y'_{N_t}$ for all $t \geq 0$. Therefore, by (3.19), $X = (X_t)_{t \geq 0}$ is a continuous time symmetric Markov chain on $\mathbb{Z}^n$ such that the associated Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is comparable with $\mathscr{E}_\alpha$, i.e., (3.14) holds for some constant $C > 1$. Let P_t be the Markov semigroup of X_t.

(2) **Proofs of** (3.1) **and** (3.2). Let Q_t be the Markov semigroup of Y_t. We have

$$Q_t = e^{-t} \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!}, \quad t > 0.$$

Then, by (3.15), for any $f \in L^\infty(\mathbb{Z}^n)$ and $t > 0$,

$$(3.21) \qquad \|Q_t f\|_\infty \leq e^{-t} \|f\|_\infty + c_1 e^{-t} \sum_{k=1}^{\infty} \frac{t^k k^{-n/2}}{k!} \|f\|_\infty$$

$$\leq e^{-t} \|f\|_\infty + c_{11} t^{-d/2} \|f\|_\infty \leq \frac{c_{12}}{t^{n/2}} \|f\|_\infty,$$

where in the second inequality we have used the expansion for inverse moments of Poisson distribution, see [45, (29) in Corollary 3]. By (3.17), we also have that for any $f \in L^\infty(\mathbb{Z}^n)$ and $t > 0$,

$$(3.22) \qquad \|\nabla Q_t f\|_\infty \leq e^{-t} \|f\|_\infty + c_6 e^{-t} \sum_{k=1}^{\infty} \frac{t^k k^{-1/2}}{k!} \|f\|_\infty \leq \frac{c_{13}}{t^{1/2}} \|f\|_\infty,$$

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where in the last inequality we have used again [45, (29) in Corollary 3].

On the other hand, let S_t be the $\alpha/2$-subordinator, which is independent of X, X', Y and Y'. According to [29, Proposition 1.2], we know that X_t and Y_{S_t} enjoy the same distribution. That is,

$$P_t f = \int_0^\infty Q_s f\, \mathbb{P}(S_t \in \mathrm{d}s), \qquad t \geq 0, f \in L^\infty(\mathbb{Z}^n).$$

This along with (3.21) and (3.22) yields

$$\frac{\|P_t f\|_\infty}{\|f\|_\infty} \leq c_{12} \int_0^\infty s^{-n/2}\, \mathbb{P}(S_t \in \mathrm{d}s) \leq c_{14} t^{-n/\alpha}, \quad t \geq 0, f \in L^\infty(\mathbb{Z}^n)$$

and

$$\frac{\|\nabla P_t f\|_\infty}{\|f\|_\infty} \leq c_{13} \int_0^\infty s^{-1/2}\, \mathbb{P}(S_t \in \mathrm{d}s) \leq c_{15} t^{-1/\alpha}, \quad t \geq 0, f \in L^\infty(\mathbb{Z}^n),$$

where we used the fact that $\mathbb{E}S_t^{-\lambda} = c_{\alpha,\lambda} t^{-2\lambda/\alpha}$ for all $t, \lambda > 0$, see [35, (25.5)]. Therefore, (3.1) and (3.2) hold. $\qquad\square$

4 Isoperimetric inequalities for α-stable-like Dirichlet forms: a perturbation argument

Let $n \geq 2$ and $\alpha \in (0,2)$. Let $W \in \mathscr{B}(\mathbb{R}^n)$ be such that $\mu_W(\mathrm{d}x) := \mathrm{e}^{-W(x)}\, \mathrm{d}x$ is a probability measure. Consider the following α-stable-like Dirichlet form $(\mathscr{E}_{\alpha,W}, \mathscr{D}(\mathscr{E}_{\alpha,W}))$:

(4.1)
$$\mathscr{E}_{\alpha,W}(f,f) := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}}\, \mathrm{d}y\, \mu_W(\mathrm{d}x),$$
$$\mathscr{D}(\mathscr{E}_{\alpha,W}) := \{f \in L^2(\mathbb{R}^n, \mu_W) : \mathscr{D}(\mathscr{E}_{\alpha,W})(f,f) < \infty\}.$$

Obviously, $C_c^\infty(\mathbb{R}^n) \subset \mathscr{D}(\mathscr{E}_{\alpha,W})$. See [44, 13] for explicit criteria of Poincaré-type (i.e., Poincaré, weak Poincaré and super Poincaré) inequalities for this Dirichlet form.

Since it is not clear how to verify the regularity property (e.g. gradient estimates) for the associated semigroup P_t, we could not apply Theorem 2.2 for non-negative symmetric function $\gamma(x,y)$ satisfying $\gamma(x,y) \to 0$ as $y \to x$. So, in this section, we will establish isoperimetric inequalities for $(\mathscr{E}_{\alpha,W}, \mathscr{D}(\mathscr{E}_{\alpha,W}))$ by using a perturbation argument. The main result of this section is the following.

Theorem 4.1. *Let $n \geq 2$ and $\alpha \in (0,2)$. Let $W \in \mathscr{B}(\mathbb{R}^n)$ be such that $\mu_W(\mathrm{d}x) = \mathrm{e}^{-W(x)}\, \mathrm{d}x$ is a probability measure. Set*

$$\Phi(l) := \inf_{|x| \geq l} \frac{\mathrm{e}^{W(x)}}{|x|^{n+\alpha/2}}, \quad l \geq 1.$$

(1) *If $\lim_{l \to \infty} \Phi(l) > 0$, then there are constants $c_1, c_2 > 0$ such that for any $f \in C_c^\infty(\mathbb{R}^n)$,*

(4.2)
$$\mu_W(f^2) \leq c_1 \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y\, \mu_W(\mathrm{d}x) + c_2 \mu_W(|f|)^2.$$

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(2) *Let W be locally Lipschitz continuous. If $\lim_{l\to\infty} \Phi(l) = \infty$, then for any $r > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$,*

$$(4.3) \qquad \mu_W(f^2) \leq r \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}} \, \mathrm{d}y \, \mu_W(\mathrm{d}x) + \beta(r)\mu_W(|f|)^2,$$

where

$$\beta(r) = \inf\left\{ 2c_1 \left(s^{-2n/\alpha} + s^{-n}\right) \left(\sup_{|z| \leq l+1} \mathrm{e}^{W(z)/2} \right) : s + \frac{1}{\Phi(l-1)} \leq c_2(r \wedge 1)\right.$$

$$\left. and \quad \sup_{|z| \leq l+2} \mathrm{e}^{2|\nabla W(z)|} \leq \frac{c_3}{s} \right\}$$

with some constants $c_1, c_2, c_3 > 0$.

To prove Theorem 4.1, we will make perturbation to the following Poincaré type isoperimetric inequality for the truncated α-stable Dirichlet from on $\mathbb{R}^n$.

Lemma 4.2. *There is a constant $c > 0$ such that for all $f \in C_c^\infty(\mathbb{R}^n)$,*

$$\int f(x)^2 \, \mathrm{d}x \leq r \int_{\{|x-y|\leq 1\}} \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}} \, \mathrm{d}x \, \mathrm{d}y + c(r^{-2n/\alpha} + r^{-n}) \left(\int |f|(x) \, \mathrm{d}x \right)^2, \quad r > 0.$$

Proof. This follows from (3.9) and Corollary 2.7(1). $\qquad\square$

For any $D \subset \mathbb{R}^n$, consider the isoperimetric constant

$$(4.4) \qquad \begin{aligned} \kappa_W(D) &:= \frac{1}{2} \inf\left\{ \frac{1}{\mu_W(A)} \int_{A \times A^c} \frac{\mathrm{e}^{W(x)} + \mathrm{e}^{W(y)}}{|x-y|^{n+\alpha/2}} \mu_W(\mathrm{d}x) \mu_W(\mathrm{d}y) : A \subset D, \mu_W(A) > 0 \right\} \\ &= \inf\left\{ \frac{1}{\mu_W(f)} \int \frac{|f(x) - f(y)|}{|x-y|^{n+\alpha/2}} \, \mathrm{d}y \, \mu_W(\mathrm{d}x) : f \geq 0, f|_{D^c} = 0, \mu_W(f) > 0 \right\}, \end{aligned}$$

where the second equality in (4.4) can be verified by the co-area formula, see [21, Theorem 3.1].

Lemma 4.3. *Let $n \geq 2$ and $\alpha \in (0,2)$. Let $B_l = \{|\cdot| < l\}$ for $l > 0$.*

(1) *If*

$$(4.5) \qquad \lim_{l\to\infty} \kappa_W(B_l^c) > 0,$$

then (4.2) holds with some constants $c_1, c_2 > 0$.

(2) *Let W be locally Lipschitz continuous. If*

$$(4.6) \qquad \lim_{l\to\infty} \kappa_W(B_l^c) = \infty,$$

then (4.3) *holds with*

$$\beta(r) = \inf\left\{2c_1\left(s^{-2n/\alpha} + s^{-n}\right)\left(\sup_{|z|\leq l+1} e^{W(z)/2}\right) : s + \frac{1}{\kappa_W(B_{l-1}^c)} \leq c_2(r \wedge 1)\right.$$

$$\left.and \sup_{|z|\leq l+2} e^{2|\nabla W(z)|} \leq \frac{c_3}{s}\right\}$$

for some constants $c_1, c_2, c_3 > 0$.

Proof. For any $l > k \geq 1$, let $\psi_{l,k} \in C_c^1(\mathbb{R}^n)$ such that $\psi_{l,k}(x) = 1$ for all $|x| \leq l$, $\psi_{l,k}(x) = 0$ for all $|x| > l + k$, and $|\nabla\psi_{l,k}| \leq 2/k$ on $\mathbb{R}^n$. Then, according to Lemma 4.2, for any $f \in C_c^\infty(\mathbb{R}^n)$ and $r > 0$,

$$\int f^2(x)e^{-W(x)}\mathbf{1}_{\{|x|\leq l\}}\,dx$$

$$\leq \int f^2(x)e^{-W(x)}\psi_{l,k}(x)\,dx$$

$$\leq r\int_{\{|x-y|\leq 1\}}\frac{|f^2(x)e^{-W(x)}\psi_{l,k}(x) - f^2(y)e^{-W(y)}\psi_{l,k}(y)|}{|x-y|^{n+\alpha/2}}\,dx\,dy$$

$$+ c_1\left(r^{-2n/\alpha} + r^{-n}\right)\left(\int |f|(x)e^{-W(x)/2}\psi_{l,k}(x)^{1/2}\,dx\right)^2$$

$$\leq r\int_{\{|x-y|\leq 1\}}\frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}}\,dy\,e^{-W(x)}\psi_{l,k}(x)\,dx$$

$$+ r\int f^2(x)e^{-W(x)}\,dx\int_{\{|x-y|\leq 1\}}\frac{|\psi_{l,k}(x) - \psi_{l,k}(y)|}{|x-y|^{n+\alpha/2}}\,dy$$

$$+ r\int f^2(x)e^{-W(x)}\,dx\int_{\{|x-y|\leq 1\}}\frac{|1 - e^{W(x)-W(y)}|}{|x-y|^{n+\alpha/2}}\psi_{l,k}(y)\,dy$$

$$+ c_1\left(r^{-2n/\alpha} + r^{-n}\right)\left(\sup_{|z|\leq l+k} e^{W(z)/2}\right)\left(\int |f|(x)e^{-W(x)}\,dx\right)^2$$

$$\leq r\int_{\{|x-y|\leq 1\}}\frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}}\,dy\,e^{-W(x)}\,dx$$

$$+ \frac{c_2 r}{k}\left(\int f^2(x)e^{-W(x)}\,dx\right) + c_3 r\left(\sup_{|z|\leq l+k+1} e^{2|\nabla W(z)|}\right)\left(\int f^2(x)e^{-W(x)}dx\right)$$

$$+ c_1\left(r^{-2n/\alpha} + r^{-n}\right)\left(\sup_{|z|\leq l+k} e^{W(z)/2}\right)\left(\int |f|(x)e^{-W(x)}\,dx\right)^2,$$

where in the last inequality we have used the facts that for all $x \in \mathbb{R}^d$,

(4.7) $$\int_{\{|x-y|\leq 1\}}\frac{|\psi_{l,k}(x) - \psi_{l,k}(y)|}{|x-y|^{n+\alpha/2}}\,dy \leq \left(\sup_{z\in\mathbb{R}^n}|\nabla\psi_{l,k}|(z)\right)\int_{\{|x-y|\leq 1\}}\frac{1}{|x-y|^{n-1+\alpha/2}}\,dy \leq \frac{c_2}{k}$$

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and, by the elementary inequality $|e^r - 1| \leq e^{|r|}|r| \leq e^{2|r|}$ for all $r \in \mathbb{R}$,

$$\int_{\{|x-y|\leq 1\}} \frac{|1 - e^{W(x)-W(y)}|}{|x-y|^{n+\alpha/2}} \psi_{l,k}(y)\, \mathrm{d}y$$

$$\leq \sup_{|x|\leq l+k+1} \int_{\{|x-y|\leq 1, |y|\leq l+k\}} \frac{|1 - e^{W(x)-W(y)}|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y$$

$$\leq \sup_{|x|\leq l+k+1} \int_{\{|x-y|\leq 1, |y|\leq l+k\}} \frac{e^{|W(x)-W(y)|}|W(x) - W(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y$$

$$\leq \left(\sup_{|z|\leq l+k+1} e^{|\nabla W(z)|}|\nabla W(z)| \right) \int_{\{|x-y|\leq 1\}} \frac{1}{|x-y|^{n-1+\alpha/2}}\, \mathrm{d}y$$

$$\leq c_3 \left(\sup_{|z|\leq l+k+1} e^{2|\nabla W(z)|} \right).$$

On the other hand, by the definition of $\kappa_W(B_{l-k}^c)$, we have

$$\int f^2(x)e^{-W(x)} \left(1 - \mathbf{1}_{\{|x|\leq l\}} \right)\, \mathrm{d}x$$

$$\leq \int f^2(x)e^{-W(x)}(1 - \psi_{l-k,k}(x))\, \mathrm{d}x$$

$$\leq \frac{1}{\kappa_W(B_{l-k}^c)} \int \frac{|f^2(x)(1 - \psi_{l-k,k}(x)) - f^2(y)(1 - \psi_{l-k,k}(y))|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y\, e^{-W(x)}\, \mathrm{d}x$$

$$\leq \frac{1}{\kappa_W(B_{l-k}^c)} \int \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y\, e^{-W(x)}\, \mathrm{d}x$$

$$+ \frac{1}{\kappa_W(B_{l-k}^c)} \int \frac{|f^2(x)\psi_{l-k,k}(x) - f^2(y)\psi_{l-k,k}(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y\, e^{-W(x)}\, \mathrm{d}x$$

$$\leq \frac{2}{\kappa_W(B_{l-k}^c)} \int \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y\, e^{-W(x)}\, \mathrm{d}x$$

$$+ \frac{1}{\kappa_W(B_{l-k}^c)} \int f^2(x)e^{-W(x)}\, \mathrm{d}x \int \frac{|\psi_{l-k,k}(x) - \psi_{l-k,k}(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y$$

$$\leq \frac{2}{\kappa_W(B_{l-k}^c)} \int \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}}\, \mathrm{d}y\, e^{-W(x)}\, \mathrm{d}x$$

$$+ \frac{c_2}{k\, \kappa_W(B_{l-k}^c)} \int f^2(x)e^{-W(x)}\, \mathrm{d}x,$$

where the last inequality follows from (4.7) again.

Combining both inequalities above, we have

$$
\begin{aligned}
\mu_W(f^2) \leq {} & \left(r + \frac{2}{\kappa_W(B^c_{l-k})}\right) \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f^2(x) - f^2(y)|}{|x-y|^{n+\alpha/2}} \, dy \, e^{-W(x)} \, dx \\
& + c_1 \left(r^{-2n/\alpha} + r^{-n}\right) \left(\sup_{|z| \leq l+k} e^{W(z)/2}\right) \mu_W(|f|)^2 \\
& + \left(\left(\frac{c_2}{k} + c_3 \sup_{|z| \leq l+k+1} e^{2|\nabla W(z)|}\right) r + \frac{c_2}{k \, \kappa_W(B^c_{l-k})}\right) \mu_W(f^2).
\end{aligned}
$$

(4.8)

(1) Taking $l = 2k$, we have

$$
\begin{aligned}
& \left(\frac{c_2}{k} + c_3 \sup_{|z| \leq l+k+1} e^{2|\nabla W(z)|}\right) r + \frac{c_2}{k \, \kappa_W(B^c_{l-k})} \\
& = \left(\frac{c_2}{k} + c_3 \sup_{|z| \leq 3k+1} e^{2|\nabla W(z)|}\right) r + \frac{c_2}{k \, \kappa_W(B^c_k)}.
\end{aligned}
$$

Since $\kappa_W(B^c_l)$ is increasing with respect to l, under (4.5) we can choose $k \geq 1$ large enough and then take $r > 0$ small enough such that

$$
\left(\frac{c_2}{k} + c_3 \sup_{|z| \leq 3k+1} e^{2|\nabla W(z)|}\right) r + \frac{c_2}{k \, \kappa_W(B^c_k)} \leq \frac{1}{2}.
$$

This along with (4.8) yields (4.2).

(2) Taking $k = 1$ in (4.8) and using (4.6), we know that (4.3) holds with

$$
\begin{aligned}
\beta(r) = \inf \Big\{ & 2c_1 \left(s^{-2n/\alpha} + s^{-n}\right) \left(\sup_{|z| \leq l+1} e^{W(z)/2}\right) : s + \frac{2}{\kappa_W(B^c_{l-1})} \leq \frac{r}{2} \\
& \text{and } \left(c_2 + c_3 \sup_{|z| \leq l+2} e^{2|\nabla W(z)|}\right) s + \frac{c_2}{\kappa_W(B^c_{l-1})} \leq \frac{1}{2} \Big\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\beta(r) \leq \inf \Big\{ & 2c_1 \left(s^{-2n/\alpha} + s^{-n}\right) \left(\sup_{|z| \leq l+1} e^{W(z)/2}\right) : s + \frac{1}{\kappa_W(B^c_{l-1})} \leq c_4(r \wedge 1) \\
& \text{and } \sup_{|z| \leq l+2} e^{2|\nabla W(z)|} s \leq c_5 \Big\}.
\end{aligned}
$$

Then, we prove (4.3) with the desired β. $\qquad\square$

Lemma 4.4. *For any $\alpha \in (0,2)$, there is a constant $c_0 > 0$ such that*

$$
\kappa_W(B^c_l) \geq c_0 \inf_{|x| \geq l} \frac{e^{W(x)}}{|x|^{n+\alpha/2}}, \quad l \geq 1.
$$

Proof. According to the definition of $\kappa_W(B_l^c)$, we have

$$\kappa_W(B_l^c) \geq \frac{1}{2} \inf_{A \subset B_l^c} \inf_{x \in A} \int_{A^c} \frac{e^{W(x)-W(y)} + 1}{|x-y|^{n+\alpha/2}} \, dy \geq \frac{1}{2} \inf_{|x| \geq l} \int_{\{|y|<l\}} \frac{e^{W(x)-W(y)} + 1}{|x-y|^{n+\alpha/2}} \, dy$$

$$\geq c_1 \inf_{|x| \geq l} e^{W(x)} \int_{\{|y| \leq 1\}} \frac{1}{|x-y|^{n+\alpha/2}} \, dy \geq c_2 \inf_{|x| \geq l} \frac{e^{W(x)}}{|x|^{n+\alpha/2}}.$$

This proves the desired assertion. $\qquad\square$

Theorem 4.1 is a direct consequence of Lemmas 4.3 and 4.4, and so we omit the proof here.

The example below indicates that Theorem 4.1 is sharp in some sense.

Example 4.5. *Let* $W(x) = \frac{1}{2}(n + \varepsilon) \log(1 + |x|^2) + c_{n,\varepsilon}$ *for* $\varepsilon > 0$.

(1) (4.2) *holds if and only if* $\varepsilon \geq \alpha/2$.

(2) (4.3) *holds if and only if* $\varepsilon > \alpha/2$. *Furthermore, when* $\varepsilon > \alpha/2$, (4.3) *holds with*

$$\beta(r) = c_1(1 \wedge r)^{-\frac{2n}{\alpha} - \frac{n+\varepsilon}{2\varepsilon - \alpha}}.$$

Proof. The sufficiency for both conclusions is easily seen from Theorem 4.1. To verify the necessary, we will make use of the reference functions used in [44, Corollary 1.1]. For any $l \geq 1$, define $g_l \in C^\infty(\mathbb{R}^n)$ such that $|\nabla g_l| \leq 2/l$ and

$$g_l(x) \begin{cases} = 0, & \text{if } |x| \leq l, \\ \in [0,1], & \text{if } |x| \in [l, 2l], \\ = 1, & \text{if } |x| \geq 2l. \end{cases}$$

Then there exists a constant $c_0 > 0$ independent of l such that for all $x \in \mathbb{R}^n$ and $l \geq 1$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{|g_l^2(y) - g_l^2(x)|}{|x-y|^{n+\alpha/2}} \, dy &\leq 2 \int_{\mathbb{R}^n} \frac{|g_l(y) - g_l(x)|}{|x-y|^{n+\alpha/2}} \, dy \\
&\leq \frac{4}{l} \int_{\{|x-y| \leq l\}} \frac{1}{|y-x|^{n+\alpha/2-1}} \, dy + 2 \int_{\{|x-y| \geq l\}} \frac{1}{|x-y|^{n+\alpha/2}} \, dy \\
&\leq \frac{c_0}{l^{\alpha/2}}.
\end{aligned}
$$
(4.9)

Obviously,

(4.10) $$\mu_W(g_l^2) \geq \frac{c_1}{l^\varepsilon}, \quad \mu_W(g_l)^2 \leq \frac{c_2}{l^{2\varepsilon}}, \quad l \geq 1$$

hold for some constants $c_1, c_2 > 0$. Note that, since $1 - g_l \in C_c^\infty(\mathbb{R}^n)$, we can directly apply g_l into (4.2) and (4.3).

(1) Combining (4.9) with (4.10), we see that for any $c > 0$,

$$\lim_{l \to \infty} \frac{1}{\mu_W(g_l^2) - c\mu_W(g_l)^2} \int \frac{|g_l^2(y) - g_l^2(x)|}{|x - y|^{n+\alpha/2}} \, \mathrm{d}y \, \mu_W(\mathrm{d}x) \le \lim_{l \to \infty} \frac{c_0 l^{-\alpha/2}}{c_1 l^{-\varepsilon} - cc_2 l^{-2\varepsilon}} = 0$$

provided $\varepsilon \in (0, \alpha/2)$. Thus, the inequality (4.2) does not hold.

(2) We first prove that if $\varepsilon \le \alpha/2$, then for any $\beta : (0, \infty) \to (0, \infty)$ the inequality (4.3) does not hold. Indeed, if this inequality holds, then, by (4.9) and (4.10),

$$\frac{c_1}{l^\varepsilon} \le \frac{c_0 r}{l^{\alpha/2}} + \frac{c_2 \beta(r)}{l^{2\varepsilon}}, \quad r > 0, l \ge 1$$

holds for some constants $c_0, c_1, c_2 > 0$. Since $\varepsilon \in (0, \alpha/2]$, we obtain

$$c_1 \le \lim_{l \to \infty} \frac{c_0 r}{l^{(\alpha/2) - \varepsilon}} + \lim_{l \to \infty} \frac{c_2 \beta(r)}{l^\varepsilon} \le c_0 r, \quad r > 0.$$

Letting $r \to 0$ we conclude that $c_1 \le 0$, which is however impossible. Furthermore, by Theorem 4.1(2), it is easy to prove (4.3) with the desired rate function $\beta(r)$. $\square$

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