Blow-up and Nonlinear Instability for the Magnetic Zakharov System *

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Abstract: This study deals with the generalized Zakharov system with magnetic field. First of all, we construct a kind of blow-up solutions and establish the existence of blow-up solutions to the system through considering an elliptic system. Next, we show the nonlinear instability for a kind of periodic solutions. In addition, we consider the concentration properties of blow-up solutions for the system under study. At the end of this paper, we establish the global existence of weak solutions to the Cauchy problem of the system under consideration.

Key words: Generalized Zakharov System, Blow-up solutions, Nonlinear instability, Concentration properties, Magnetic field.

MSC(2000): 35A20; 35Q55

1 Introduction

In this paper, we study the Cauchy problem of a generalized Zakharov system with magnetic field:

$$\begin{cases} i\mathbf{E}_t + \Delta\mathbf{E} - n\mathbf{E} + i(\mathbf{E}\wedge\mathbf{B}) = 0, \\ \frac{1}{c_0^2}n_{tt} - \triangle n = \triangle|\mathbf{E}|^2, \\ \triangle\mathbf{B} - i\eta\bigtriangledown\times(\bigtriangledown\times(\mathbf{E}\wedge\bar{\mathbf{E}})) + \beta\mathbf{B} = 0, \end{cases} \tag{1.1}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), n(0,x) = n_0(x), n_t(0,x) = n_1(x), \tag{1.2}$$

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where $\mathbf{E}(t,x)$ is a vector valued function from $\mathbb{R}^+\times\mathbb{R}^2$ into $\mathbb{C}^3$ and denotes the slowly varying complex amplitude of the high-frequency electric field, $n(t,x)$ is a function from $\mathbb{R}^+\times\mathbb{R}^2$ into $\mathbb{R}$ and represents the fluctuation of the electron density from its equilibrium, the self-generated magnetic field $\mathbf{B}$ is a vector-valued function from $\mathbb{R}^+\times\mathbb{R}^2$ into $\mathbb{R}^3$, $i^2=-1$, constants $\eta>0$, $\beta\leq 0$, $\bar{\mathbf{E}}$ is the complex conjugate of $\mathbf{E}$, and $\wedge$ means the exterior product of vector-valued functions. System (1.1) describes the spontaneous generation of a magnetic field in a cold plasma (see Ref. [8] for the physical derivation).

If we neglect the magnetic field, system (1.1) reduces the following classical Zakharov system:

$$\begin{cases} i\mathbf{E}_t+\triangle\mathbf{E}-n\mathbf{E}=0, \\ \frac{1}{c_0^2}n_{tt}-\triangle n=\triangle|\mathbf{E}|^2, \end{cases} \tag{ZS}$$

which describes the propagation of Langmuir waves (cf. [17]). There are many papers concerning the well-posedness of the Zakharov system (ZS) (see e.g., [1, 3, 4, 6, 12, 13, 14] and references therein). On this topic, for (1.1) there are also some works (cf. [2, 5, 7, 10, 18]).

Let $\mathbf{E}=(E_1,E_2,0)$, $\mathbf{B}=-i\eta\mathcal{F}^{-1}\left(\frac{|\xi|^2}{|\xi|^2-\beta}\mathcal{F}\left(\mathbf{E}\wedge\bar{\mathbf{E}}\right)\right)$, $E_1(t,x)$, $E_2(t,x)\in\mathbb{C}$, $x\in\mathbb{R}^2$. For $n_1\in H^{-1}$, there exist $\omega_0\in L^2(\mathbb{R}^2)$ and $\mathbf{v}_0\in L^2(\mathbb{R}^2)$ such that $n_t(0,x)=n_1=-\mathrm{div}\mathbf{v}_0+w_0$. In this case, (1.1)-(1.2) can be rewritten as follows:

$$\begin{cases} i\mathbf{E}_t+\triangle\mathbf{E}-n\mathbf{E}+i(\mathbf{E}\wedge\mathbf{B}(\mathbf{E}))=0, \\ n_t=-\mathrm{div}\mathbf{v}+w_0, \\ \frac{1}{c_0^2}\mathbf{v}_t=-\nabla(n+|E|^2), \\ \mathbf{E}(0,x)=\mathbf{E}_0(x), n(0,x)=n_0(x), \mathbf{v}(0,x)=\mathbf{v}_0(x). \end{cases} \tag{1.3}$$

In the present paper, we first study the existence of blow-up solutions for the Cauchy problem (1.1)-(1.2). We construct a kind of blow-up solutions to (1.1)-(1.2) on $[0,T)$, which has the form:

$$\mathbf{E}=(E_1,-iE_1,0),\quad n(t,x)=\frac{\omega^2}{(T-t)^2}\tilde{N}\left(\frac{x\omega}{T-t}\right), \tag{1.4}$$

where

$$E_1=\frac{\omega}{T-t}e^{i\left(\theta+\frac{|x|^2}{4(-T+t)}-\frac{\omega^2}{-T+t}\right)}\frac{\tilde{P}\left(\frac{x\omega}{T-t}\right)}{\sqrt{2}},$$

$\tilde{P}(x)=\tilde{P}(|x|)$ and $\tilde{N}(x)=\tilde{N}(|x|)$ are real valued functions on $\mathbb{R}^2$, and $\theta\in\mathbb{R}$ and $\omega>0$. In addition, let

$$\mathbf{B}=\left(0,0,\frac{\omega^2}{(T-t)^2}\tilde{B}\left(\frac{x\omega}{T-t}\right)\right), \tag{1.5}$$

where $\tilde{B}(x) = \tilde{B}(|x|)$ is a real-valued function on $\mathbb{R}^2$ and $(\tilde{P}, \tilde{N}, \tilde{B})$ solves the following system:

$$\begin{cases} \triangle \tilde{P} - \tilde{P} + \tilde{P}\tilde{B} = \tilde{N}\tilde{P}, \\ \lambda^2 \left(r^2 \tilde{N}_{rr} + 6r\tilde{N}_r + 6\tilde{N} \right) - \triangle \tilde{N} = \triangle |\tilde{P}|^2, \\ \triangle \tilde{B} + \beta c_0^2 (T-t)^2 \lambda^2 \tilde{B} = \eta \triangle |\tilde{P}|^2. \end{cases}$$

Here, $r = |x|$, $\triangle = \partial_{rr} + \frac{\partial_r}{r}$, and $\lambda = \frac{1}{\omega c_0}$. Let

$$(\tilde{P}, \tilde{N}) = \left(\frac{P}{(\eta+1)^{1/2}}, \frac{N}{\eta+1} \right)$$

and

$$\tilde{B} = \eta \mathcal{F}^{-1} \left(\frac{|\xi|^2}{|\xi|^2 - \beta c_0^2 (T-t)^2 \lambda^2} \mathcal{F}(\tilde{P}^2) \right),$$

we then obtain

$$\begin{cases} \triangle P - P + \frac{\eta}{\eta+1} P \mathcal{F}^{-1} \left(\frac{|\xi|^2}{|\xi|^2 - \beta c_0^2 (T-t)^2 \lambda^2} \mathcal{F}(P^2) \right) = \frac{1}{\eta+1} NP, \\ \lambda^2 \left(r^2 N_{rr} + 6rN_r + 6N \right) - \triangle N = \triangle |P|^2. \end{cases} \tag{1.6}$$

We shall consider the existence of solutions for (1.6) in $H_r^1 \times L_r^2$ for $\forall T > 0$, $0 \le t < T$ fixed, where $H_r^1 := \{u;\ u \in H^1(\mathbb{R}^2)$ and u is radially symmetric$\}$, $L_r^2 := \{u;\ u \in L^2(\mathbb{R}^2)$ and u is radially symmetric$\}$. If $(P_{\lambda,T-t}, N_{\lambda,T-t}) \in H_r^1 \times L_r^2$ is a solution to (1.6), then $(\mathbf{E}, n)$ defined in (1.4) is a blow-up solution to the Cauchy problem (1.1)-(1.2), which will be shown in Theorem 1.1. When $\beta = 0$, (1.6) becomes the following form

$$\begin{cases} \triangle P - P + \frac{\eta}{\eta+1} P^3 = \frac{1}{\eta+1} NP, \\ \lambda^2 \left(r^2 N_{rr} + 6rN_r + 6N \right) - \triangle N = \triangle |P|^2. \end{cases} \tag{1.7}$$

If $(P_\lambda, N_\lambda) \in H_r^1 \times L_r^2$ is a solution to (1.7), then $(\mathbf{E}, n)$ defined in (1.4) is a self-similar blow-up solution to (1.1)-(1.2) with $\beta = 0$.

The main results of this paper states as follows. At first, we have

Theorem 1.1 (Existence of blow-up solutions to (1.1)-(1.2))

For $\forall T > 0$, $0 \le t < T$, there exist λ_T with $0 < \lambda < \lambda_T$, and a solution $(P_{\lambda,T-t}, N_{\lambda,T-t})$ to (1.6) such that for $\forall\ \theta \in \mathbb{R}$,

$$\mathbf{E} = (E_1, -iE_1, 0),\ n = \frac{\omega^2 N_{\lambda,T-t}\left(\frac{x\omega}{T-t}\right)}{(T-t)^2(\eta+1)},$$

is a blow-up solution to (1.1)-(1.2) and

$$\|\mathbf{E}\|_{H^1} + \|n\|_{L^2} + \|n_t\|_{\hat{H}^{-1}} \to +\infty \text{ as } t \to T,$$

$$\mathbf{B} = \left(0, 0, \frac{\eta\omega^2}{(\eta+1)(T-t)^2}\mathcal{F}^{-1}\left(\frac{|\xi|^2}{|\xi|^2 - \beta c_0^2 (T-t)^2\lambda^2}\mathcal{F}(P^2)\right)\right).$$

Here, $E_1 = \frac{\omega}{T-t} e^{i\left(\theta + \frac{|x|^2}{4(-T+t)} - \frac{\omega^2}{-T+t}\right)} \frac{P_{\lambda, T-t}\left(\frac{x\omega}{T-t}\right)}{\sqrt{2}(\eta+1)^{1/2}}$, *and*

$$\hat{H}^{-1} := \left\{ u : \exists w \in L^2(\mathbb{R}^2) \text{ such that } u = -\nabla \cdot w \text{ and } \|u\|_{\hat{H}^{-1}} = \|w\|_{L^2} \right\}.$$

□

Next, the following theorem concerns the nonlinear instability of minimal periodic solutions to the Cauchy problem (1.1)-(1.2) with $\beta = 0$, which will be checked in Section 3.

Theorem 1.2 (Instability of minimal periodic solution to (1.1)-(1.2) with $\beta = 0$)

Let $(\mathbf{E}(t), n(t))$ be a minimal periodic solution to (1.1)-(1.2) with $\beta = 0$, where

$$\mathbf{E}(t) = \left(\frac{\omega^{\frac{1}{2}} e^{i(\theta+\omega t)} Q(\omega^{\frac{1}{2}}(x-x_0))}{\sqrt{2}(\eta+1)^{1/2}}, -i\frac{\omega^{\frac{1}{2}} e^{i(\theta+\omega t)} Q(\omega^{\frac{1}{2}}(x-x_0))}{\sqrt{2}(\eta+1)^{1/2}}, 0\right),$$

$$n = -\frac{\omega Q^2(\omega^{1/2}(x-x_0))}{\eta+1},$$

Q is the unique positive radial solution of the equation

$$\triangle V - V + V^3 = 0$$

in $\mathbb{R}^2$, $\omega > 0$, $\theta \in \mathbb{R}$, $x_0 \in \mathbb{R}^2$. Then there exists $\{(\mathbf{E}_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})\}$ such that as $\varepsilon \to 0$, $(\mathbf{E}_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (E(0), n(0), 0)$ in H_k, $k \geq 1$, and $(\mathbf{E}_\varepsilon, n_\varepsilon)$ blows up in finite time for some $T_\varepsilon > 0$ in H_1, where $(\mathbf{E}_\varepsilon, n_\varepsilon)$ is a solution to (1.1)-(1.2) for $\beta = 0$ with the initial data $(\mathbf{E}_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon})$, and $H_k = H^k(\mathbb{R}^2) \times H^{k-1}(\mathbb{R}^2) \times H^{k-2}(\mathbb{R}^2)$. That is, $(\mathbf{E}(0), n(0))$ is orbitally unstable in H_k for all $k \geq 1$ and $(\mathbf{E}(t), n(t))$ is strongly unstable in the sense of instability induced by blow-up. □

In addition, some concentration properties of blow-up solutions to the Cauchy problem (1.3) holds.

Theorem 1.3 (Concentration properties of blow-up solutions)

If $\|\mathbf{E}\|_{H^1} + \|n\|_{L^2} + \|\mathbf{v}\|_{L^2} \to +\infty$ as $t \to T$, where $(\mathbf{E}, n, \mathbf{v})$ is a blow-up solution to (1.3) in $H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ on $[0, T)$, then the following properties hold:

(1) If $n_t(0) \in \hat{H}^{-1}$ *and* $\mathbf{E}, n$ *are radial functions of* $|x|$*, then one has*

$$\forall R > 0, \ \liminf_{t \to T} \|\mathbf{E}(t,x)\|_{L^2(B(0,R))} \geq \|Q\|_{L^2}.$$

In addition, provided that

$$\frac{\|Q\|_{L^2}^2}{\eta + 1} < \|\mathbf{E}_0\|_{L^2}^2 < \frac{\delta \|Q\|_{L^2}^2}{\eta}, \ \textit{where} \ \frac{\eta}{\eta+1} < \delta < 1, \tag{1.8}$$

then there exists $m_n(\|\mathbf{E}_0\|_{L^2}^2) > 0$ *such that*

$$\forall R > 0, \ \liminf_{t \to T} \|n(t,x)\|_{L^1(B(0,R))} \geq m_n.$$

(2) If $n_t(0) \in \hat{H}^{-1}$ *and* $\mathbf{E}, n$ *are non-radial functions of* $|x|$*, there is then a function* $t \to x(t) \in \mathbb{R}^2$ *such that*

$$\forall R > 0, \ \liminf_{t \to T} \|\mathbf{E}(t,x)\|_{L^2(B(x(t),R))} \geq \|Q\|_{L^2}^2.$$

Moreover, under the assumption (1.8), there exist $m_n(\|\mathbf{E}_0\|_{L^2}^2) > 0$ *and a function* $t \to x(t) \in \mathbb{R}^2$ *such that*

$$\liminf_{t \to T} \|n(t,x)\|_{L^1(B(x(t),R))} \geq m_n.$$

(3) If $n_t(0) \in H^{-1}$*,* $n_t(0) \notin \hat{H}^{-1}$ *and* $\mathbf{E}, n$ *are radial functions of* $|x|$*, there is then a sequence* $t_k \to T$ *as* $k \to +\infty$ *such that*

$$\forall R > 0, \ \liminf_{k \to \infty} \|\mathbf{E}(t_k,x)\|_{L^2(B(0,R))} \geq \|Q\|_{L^2}.$$

In addition, under the assumption (1.8), there exists $t_k \to T$ *as* $k \to +\infty$ *such that*

$$\liminf_{k \to \infty} \|n(t_k,x)\|_{L^1(B(0,R))} \geq m_n.$$

(4) If $n_t(0) \in H^{-1}$*,* $n_t(0) \notin \hat{H}^{-1}$ *and* $\mathbf{E}, n$ *are non-radial functions of* $|x|$*, there then exist* $t_k \to T$ *as* $k \to +\infty$ *and* x_k *such that*

$$\forall R > 0, \ \liminf_{k \to +\infty} \|\mathbf{E}(t_k,x)\|_{L^2(B(x_k,R))} \geq \|Q\|_{L^2}.$$

Furthermore, under the assumption (1.8), there exist $t_k \to T$ *as* $k \to +\infty$ *and* x_k *such that*

$$\liminf_{k \to +\infty} \|n(t_k,x)\|_{L^1(B(x_k,R))} \geq m_n. \qquad \square$$

At last, the following global existence result for the Cauchy problem (1.1)-(1.2) is valid.

Theorem 1.4 (Global existence for the case $\|\mathbf{E}_0\|_{L^2}^2 \le \frac{\|Q\|_{L^2}^2}{\eta+1}$)

If $\mathbf{E}_0 \in H^1(\mathbb{R}^2)$, $n_0 \in L^2(\mathbb{R}^2)$, $n_1 \in H^{-1}(\mathbb{R}^2)$ and $\|\mathbf{E}_0\|_{L^2}^2 \le \frac{\|Q\|_{L^2}^2}{\eta+1}$, then there exists a global weak solution $(\mathbf{E}, n)$ to the Cauchy problem (1.1)-(1.2) such that

$$\mathbf{E} \in L^\infty\left(\mathbb{R}^+; H^1(\mathbb{R}^2)\right), \quad n \in L^\infty\left(\mathbb{R}^+; L^2(\mathbb{R}^2)\right). \qquad \square$$

2 Existence of blow-up solutions to (1.1)-(1.2)

In this section, we will prove Theorem 1.1.

2.1 Some properties of solutions to (1.6)

In this subsection, we give several lemmas and propositions concerning the properties of solutions to (1.6). Since $T-t$ is fixed, for convenience, we denote $(P_{\lambda,T-t}, N_{\lambda,T-t})$ by (P_λ, N_λ).

Lemma 2.1 Assume that $(\mathbf{E}, n, \mathbf{v})$ is a regular solution to (1.3). Then $(\mathbf{E}, n, \mathbf{v})$ satisfies

1) $\forall t \in (0,T)$, $\|\mathbf{E}(t)\|_{L^2}^2 = \|\mathbf{E}_0\|_{L^2}^2$;

2) $\frac{dI(t)}{dt} = \int_{\mathbb{R}^2} w_0(n + |\mathbf{E}|^2)$, where

$$\begin{aligned} I(t) &= I(\mathbf{E}(t), n(t), \mathbf{v}(t)) \\ &= \int_{\mathbb{R}^2} |\nabla \mathbf{E}|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |n|^2 + \frac{1}{2c_0^2}\int_{\mathbb{R}^2} |\mathbf{v}|^2 + \int_{\mathbb{R}^2} n|\mathbf{E}|^2 \\ &\quad -\frac{\eta}{2}\int_{\mathbb{R}^2} \frac{|\xi|^2}{|\xi|^2 - \beta} |\mathcal{F}(\mathbf{E} \wedge \bar{\mathbf{E}})|^2 d\xi. \end{aligned}$$

Proof. Multiplying the first equation of (1.3) by $\bar{\mathbf{E}}$, we obtain 1). Multiplying the first equation of (1.3) by $\bar{\mathbf{E}}_t$, the second equation of (1.3) by n and the third equation of (1.3) by $\mathbf{v}$, we derive 2). $\square$

By a direct computation, we obtain

Proposition 2.2 If $\{(P_{\lambda,T-t}, N_{\lambda,T-t})\} \subset H_r^1 \times L_r^2$ is a sequence of nontrivial solutions to (1.6) in the sense of distribution and $\inf_{0\le t<T}(\|P_{\lambda,T-t}\|_{H^1} + \|N_{\lambda,T-t}\|_{H^1}) \ge c > 0$, then $(\mathbf{E}, n)$ defined in (1.4) is a solution to (1.1)-(1.2), and $(\mathbf{E}, n, \mathbf{v})$ is a solution to (1.3), where $\mathbf{v}(x,t) = \frac{x}{r}\frac{\omega^2}{-(T-t)^3} r N_\lambda\left(\frac{r\omega}{T-t}\right)$, $n_t = \nabla \cdot \mathbf{v}$, and

$$\left(\mathbf{E}(t), n(t), \frac{\partial n}{\partial t}\right) \in H^1 \times L^2 \times \hat{H}^{-1},$$

$$\|\mathbf{E}(t)\|_{H^1} + \|n(t)\|_{L^2} + \left\|\frac{\partial n}{\partial t}\right\|_{\hat{H}^{-1}} \to +\infty \text{ as } t \to T, \tag{2.1}$$

$$\|\mathbf{E}(t)\|_{L^2} = \|P_\lambda\|_{L^2}, \tag{2.2}$$

$$I(t) = \frac{\omega^2}{(T-t)^2}\left[\int_{\mathbb{R}^2}(|\nabla P_\lambda(x)|^2 + \frac{N_\lambda P_\lambda}{\eta+1}) + \frac{1}{2(\eta+1)}\int_{\mathbb{R}^2}(\lambda^2|x|^2+1)N_\lambda^2 \right.$$
$$\left. -\frac{\eta}{2(\eta+1)}\int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}|\mathcal{F}(P_\lambda^2)|^2\right] + \frac{1}{4\omega^2}\int_{\mathbb{R}^2}|x|^2P_\lambda^2,$$

which implies by Lemma 2.1 that

$$\int_{\mathbb{R}^2}\left(|\nabla P_\lambda(x)|^2 + \frac{N_\lambda P_\lambda}{\eta+1}\right) + \frac{1}{2(\eta+1)}\int_{\mathbb{R}^2}\left(\lambda^2|x|^2+1\right)N_\lambda^2$$
$$-\frac{\eta}{2(\eta+1)}\int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}|\mathcal{F}(P_\lambda^2)|^2 = 0. \tag{2.3}$$

□

Lemma 2.3(Weinstein [15]) If $u \in H^1(\mathbb{R}^2)$, then

$$\frac{1}{2}\|u\|_{L^4(\mathbb{R}^2)}^4 \le \frac{\|u\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}\|\nabla u\|_{L^2(\mathbb{R}^2)}^2. \tag{2.4}$$

□

Proposition 2.4 If $(P_\lambda, N_\lambda) \in H_r^1 \times L_r^2$ is a nontrivial solution to (1.6) in the sense of distributions, then we have

1) $\displaystyle\int_{\mathbb{R}^2}\left(|\nabla P_\lambda|^2 + |P_\lambda|^2\right) = \frac{1}{\eta+1}\left(\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}|\mathcal{F}(P_\lambda^2)|^2 - \int_{\mathbb{R}^2}N_\lambda|P_\lambda|^2\right),$

2) $\displaystyle\int_{\mathbb{R}^2}|P_\lambda|^2 = \frac{1}{2(\eta+1)}\left(\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}|\mathcal{F}(P_\lambda^2)|^2 + \int_{\mathbb{R}^2}(\lambda^2|x|^2+1)|N_\lambda|^2\right),$

3) $\displaystyle\int_{\mathbb{R}^2}|P_\lambda|^2 > \int_{\mathbb{R}^2}|Q|^2.$

Proof. **Step 1** Multiplying the first equation of (1.6) by P_λ and then integrating in $\mathbb{R}^2$, we obtain 1).

Step 2 By 1) and (2.3), we drive 2).

Step 3 Using (2.3), we get

$$(\eta+1)\int_{\mathbb{R}^2}|\nabla P_\lambda|^2 = -\frac{1}{2}\int_{\mathbb{R}^2}(P_\lambda^2+N_\lambda)^2 + \frac{\eta+1}{2}\int_{\mathbb{R}^2}P_\lambda^4 - \frac{1}{2}\int_{\mathbb{R}^2}\lambda^2|x|^2N_\lambda^2$$
$$-\frac{\eta}{2}\left(\int_{\mathbb{R}^2}P_\lambda^4 - \int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}|\mathcal{F}(P_\lambda^2)|^2\right).$$

From the above equality, it follows that

$$(\eta+1)\left(\int_{\mathbb{R}^2}|\nabla P_\lambda|^2 - \frac{1}{2}\int_{\mathbb{R}^2}P_\lambda^4\right) + \frac{1}{2}\int_{\mathbb{R}^2}\left(P_\lambda^2+N_\lambda\right)^2 + \frac{1}{2}\int_{\mathbb{R}^2}\lambda^2|x|^2N_\lambda^2$$
$$+\frac{\eta}{2}\left(\int_{\mathbb{R}^2}P_\lambda^4 - \int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}|\mathcal{F}(P_\lambda^2)|^2\right) = 0,$$

which yields that

$$\int_{\mathbb{R}^2}|\nabla P_\lambda|^2 - \frac{1}{2}\int_{\mathbb{R}^2}P_\lambda^4 < 0. \tag{2.5}$$

By (2.5) and Lemma 2.3, we conclude 3). □

Lemma 2.5

1) (**Regularity of (1.6)).**

If $(P_\lambda, N_\lambda) \in H^1 \times L^2$ is a radially symmetric solution to (1.6) in the sense of distribution, then $(P_\lambda, N_\lambda) \in C^\infty \times C^\infty$ and is a classical solution to (1.6).

2) (**An equivalent system of (1.6)).**

Let $(P_\lambda, N_\lambda) \in H^1 \times L^2 \cap C^\infty \times C^\infty$ be radially symmetric. Then system (1.6) is equivalent to the following system:

$$\begin{cases} \triangle P - P + \frac{\eta}{\eta+1}P\mathcal{F}^{-1}\left(\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda^2}\mathcal{F}(P^2)\right) = \frac{1}{\eta+1}NP, \\ N(r) = \frac{1}{(\lambda^2r^2-1)^{3/2}}\int_{\frac{1}{\lambda}}^{r} 2P(s)P'(s)(\lambda^2s^2-1)^{1/2}ds. \end{cases} \tag{2.6}$$

3) (**Decay solution of (1.6) at infinity).**

If $(P_\lambda, N_\lambda) \in H^1 \times L^2$ is a solution of (1.6) in the sense of distribution, then there exists constants $\delta > 0$ and $C_k > 0$ for $k \geq 0$ such that

$$\forall k \geq 0,\ \forall x,\ |P_\lambda^{(k)}(x)| \leq C_k e^{-\delta|x|},\ |N_\lambda^{(k)}(x)| \leq \frac{C_k}{1+|x|^{k+3}}.$$ □

Remark 2.1. The proof of Lemma 2.5 is similar to that of the same result as the following elliptic system

$$\begin{cases} \triangle P - P = NP, \\ \lambda^2(r^2N_{rr} + 6rN_r + 6N) - \triangle N = \triangle|P|^2, \end{cases}$$

which was given in [3]. $\square$

Proposition 2.6 (Asymptotics behavior of solution (P_λ, N_λ) as $\lambda \to 0$)

If $(P_{\lambda_n}, N_{\lambda_n}) \in H^1 \times L^2$ is a nontrivial radially symmetric solution to (1.6) in the sense of distributions, $\lambda_n \to 0$ as $n \to +\infty$, and there exists $C > 0$ such that $\|P_{\lambda_n}\|_{L^2} \le C$, then there is a subsequence $\{(P_{\lambda_n}, N_{\lambda_n})\}$ and a radially symmetric solution V to

$$\triangle V - V + V^3 = 0 \text{ in } \mathbb{R}^2, \tag{2.7}$$

such that

$$(P_{\lambda_n}, N_{\lambda_n}) \to (V, -V^2) \text{ in } H^1 \times L^2 \text{as } \lambda_n \to 0.$$

Moreover, if $P_{\lambda_n}(r) \ge 0$ for $\forall r \ge 0$, then $V = Q$.

Proof. From 2) of Proposition 2.4, we obtain

$$\int_{\mathbb{R}^2} |N_{\lambda_n}|^2 \le c, \text{ and } \frac{\eta}{\eta+1} \int_{\mathbb{R}^2} \frac{|\xi|^2}{|\xi|^2 - \beta c_0^2 (T-t)^2 \lambda_n^2} |\mathcal{F}(P_{\lambda_n}^2)|^2 \le c.$$

Using Hölder's inequality and Lemma 2.3, we derive from 1) and 2) in Proposition 2.4 as well as the above two inequalities that

$$\int_{\mathbb{R}^2} \left(|\nabla P_{\lambda_n}|^2 + |P_{\lambda_n}|^2\right) \le c + c\left(\int_{\mathbb{R}^2} |N_{\lambda_n}|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |P_{\lambda_n}|^4\right)^{\frac{1}{2}}$$

$$\le c + c\left(\int_{\mathbb{R}^2} |N_{\lambda_n}|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |P_{\lambda_n}|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} (|\nabla P_{\lambda_n}|^2 + |P_{\lambda_n}|^2)\right)^{\frac{1}{2}}$$

$$\le c + c\left(\int_{\mathbb{R}^2} (|\nabla P_{\lambda_n}|^2 + |P_{\lambda_n}|^2)\right)^{\frac{1}{2}},$$

which concludes that

$$\int_{\mathbb{R}^2} (|\nabla P_{\lambda_n}|^2 + |P_{\lambda_n}|^2) \le c.$$

Since H_r^1 and L_r^2 are both reflexive Banach spaces, there exist $P \in H_r^1$ and $N \in L_r^2$ such that

$$P_{\lambda_n} \rightharpoonup P \text{ in } H_r^1, \text{ and } N_{\lambda_n} \rightharpoonup N \text{ in } L_r^2 \text{ as } n \to +\infty.$$

Since the imbedding $H_r^1 \hookrightarrow L_r^p$, $2 < p < +\infty$, is compact, $|P_{\lambda_n}|^2 P_{\lambda_n} \to |P|^2 P$ in L_r^2, and

$$\triangle |P_{\lambda_n}|^2 \to \triangle |P|^2, \; N_{\lambda_n} P_{\lambda_n} \to NP$$

in the sense of distribution. From

$$\frac{\eta}{\eta+1}P_{\lambda_n}\mathcal{F}^{-1}\left(\frac{\beta c_0^2(T-t)^2\lambda_n^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda_n^2}\mathcal{F}(|P_{\lambda_n}|^2)\right)\to 0 \text{ in } L_r^2 \text{ as } n\to+\infty, \tag{2.8}$$

it follows that

$$\frac{\eta}{\eta+1}P_{\lambda_n}\mathcal{F}^{-1}\left(\frac{|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda_n^2}\mathcal{F}(|P_{\lambda_n}|^2)\right)\to \frac{\eta}{\eta+1}P|P|^2 \text{ in } L_r^2.$$

Therefore, (P,N) is a solution to the system

$$\begin{cases} \triangle P - P + \frac{\eta}{\eta+1}|P|^2P = \frac{1}{\eta+1}NP, \\ -\triangle N = \triangle |P|^2, \end{cases}$$

in the sense of distribution. Hence, there exists V (a radially symmetric solution to (2.7)) such that

$$P = V, \quad N = -V^2.$$

Since $P_{\lambda_n}\to V$ in L_r^4, one has $|P_{\lambda_n}|^2\to |V|^2$ in L_r^2, and $N_{\lambda_n}\rightharpoonup -V^2$ in L_r^2 as $n\to+\infty$. Thus, using (2.8), we have

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\mathbb{R}^2}(|\nabla P_{\lambda_n}|^2+|P_{\lambda_n}|^2)\\
&=\lim_{n\to+\infty}\frac{1}{\eta+1}\left(\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda_n^2}|\mathcal{F}(P_{\lambda_n}^2)|^2-\int_{\mathbb{R}^2}N_{\lambda_n}|P_{\lambda_n}|^2\right)\\
&=\frac{\eta}{\eta+1}\int_{\mathbb{R}^2}|V|^4+\frac{1}{\eta+1}\int_{\mathbb{R}^2}|V|^4=\int_{\mathbb{R}^2}|V|^4=\int_{\mathbb{R}^2}(|\nabla V|^2+|V|^2),
\end{aligned}$$

where we apply the identity $\int_{\mathbb{R}^2}|V|^4=\int_{\mathbb{R}^2}(|\nabla V|^2+|V|^2)$ with equation (2.7). Therefore, one has

$$P_{\lambda_n}\to V \text{ in } H_r^1 \text{ as } n\to+\infty.$$

Since $N_{\lambda_n}\rightharpoonup -V^2$ in L_r^2 as $n\to+\infty$, by the weakly lower semi-continuity of norm, we get

$$\int_{\mathbb{R}^2}|V|^4\le\liminf_{n\to+\infty}\int_{\mathbb{R}^2}|N_{\lambda_n}|^2. \tag{2.9}$$

On the other hand, by 2) of Proposition 2.4, we have

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^2}|N_{\lambda_n}|^2$$

$$\le \limsup_{n\to+\infty}\left(2(\eta+1)\int_{\mathbb{R}^2}|P_{\lambda_n}|^2-\int_{\mathbb{R}^2}\frac{\eta|\xi|^2}{|\xi|^2-\beta c_0^2(T-t)^2\lambda_n^2}|\mathcal{F}(P_{\lambda_n}^2)|^2\right)$$
$$=2(\eta+1)\int_{\mathbb{R}^2}|V|^2-\eta\int_{\mathbb{R}^2}|V|^4=\int_{\mathbb{R}^2}|V|^4, \tag{2.10}$$

where we use $P_{\lambda_n}\to V$ in H_r^1 as $n\to+\infty$, (2.8) and the Pohozaev identity $\int_{\mathbb{R}^2}|V|^4=2\int_{\mathbb{R}^2}|V|^2$ with equation (2.7). By $N_{\lambda_n}\rightharpoonup -V^2$ in L_r^2 as $n\to+\infty$, we derive from (2.9) and (2.10) that

$$N_{\lambda_n}\to -V^2 \text{ in } L_r^2 \text{ as } n\to+\infty.$$

In view of $P_{\lambda_n}\ge 0$, and $P_{\lambda_n}\to V$ in H_r^1 as $n\to+\infty$, by 3) of Proposition 2.4, we get $V\ge 0$ and $V\ne 0$. Applying the uniqueness theorem of positive radial solutions to (2.7), which was proved in [9], we know that $V=Q$. □

Proposition 2.7 (Asymptotics behavior of solution $(P_{\lambda,T-t},N_{\lambda,T-t})$ **as** $t\to T$**)**

Let $\lambda>0$ and $T>0$ be fixed. If $(P_{\lambda,T-t_n},N_{\lambda,T-t_n})\in H_r^1\times L_r^2$ is a nontrivial radially symmetric solution to (1.6) in the sense of distribution, $t_n\to T$ as $n\to+\infty$, and there exists $C>0$ such that $\|P_{\lambda,T-t_n}\|_{L^2}\le C$, then there is a subsequence $\{(P_{\lambda,T-t_n},N_{\lambda,T-t_n})\}$ such that

$$(P_{\lambda,T-t_n},N_{\lambda,T-t_n})\to(P_\lambda,N_\lambda) \text{ in } H^1\times L^2 \text{ as } \ t_n\to T,$$

where $(P_\lambda,N_\lambda)\in H^1\times L^2$ is a nontrivial radially symmetric solution to (1.7) in the sense of distribution.

Proof. As is shown in the proof of Proposition 2.6, it follows from $\|P_{\lambda,T-t_n}\|_{L^2}\le C$ that $\|P_{\lambda,T-t_n}\|_{H^1}\le c$ and $\|N_{\lambda,T-t_n}\|_{L^2}\le c$ for some positive constant c. Thus, there exist a subsequence denoted again by $(P_{\lambda,T-t_n},N_{\lambda,T-t_n})$ and $(P_\lambda,N_\lambda)\in H_r^1\times L_r^2$ such that

$$(P_{\lambda,T-t_n},N_{\lambda,T-t_n})\rightharpoonup(P_\lambda,N_\lambda) \text{ in } H^1\times L^2 \text{as } t_n\to T.$$

Then it follows from $B_{\lambda,T-t_n}\to\eta P_\lambda^2\in L^2$as $t_n\to T$ that (P_λ,N_λ) is a radially symmetric solution to (1.6) in the sense of distribution. Similar to the proof of Proposition 2.6, we obtain that

$$(P_{\lambda,T-t_n},N_{\lambda,T-t_n})\to(P_\lambda,N_\lambda) \ \text{ in } H^1\times L^2 \text{ as } t_n\to T. \qquad \square$$

2.2 Existence of solutions to (1.6)

In this subsection, we prove the existence of solutions to (1.6) and establish some properties for them.

Theorem 2.8 (Existence of solutions (P_λ, N_λ) to (1.6))

For $\forall T > 0$, $0 \le t < T$, there exists a solution (P_λ, N_λ) to (1.6) for some λ_T with $0 < \lambda < \lambda_T$. Moreover, $(P_\lambda, N_\lambda) \to (Q, -Q^2)$ in $H^1 \times L^2$as $\lambda \to 0$.$\square$

We shall prove this theorem by using Banach fixed point theorem and the maximum principle at the end of this section.

In fact, if (P_λ, N_λ) is a solution to (1.6), where

$$\begin{aligned} &P_\lambda = Q + h_\lambda, \quad N_\lambda = F_\lambda((Q+h_\lambda)^2), \\ &F_\lambda(u) = \frac{1}{(\lambda^2 r^2 - 1)^{3/2}} \int_{\frac{1}{\lambda}}^{r} (u(s))' (\lambda^2 r^2 - 1)^{1/2} ds, \end{aligned} \tag{2.11}$$

then

$$\begin{aligned} &\triangle(Q+h_\lambda) - (Q+h_\lambda) \\ &\quad + \frac{\eta}{\eta+1}(Q+h_\lambda)\mathcal{F}^{-1}\left(\frac{|\xi|^2}{|\xi|^2 - \beta c_0^2 (T-t)^2\lambda^2}\mathcal{F}((Q+h_\lambda)^2)\right) \\ &\quad = \frac{F_\lambda((Q+h_\lambda)^2)(Q+h_\lambda)}{\eta+1}, \end{aligned}$$

that is,

$$\begin{aligned} &\triangle h_\lambda - h_\lambda + 3Q^2 h_\lambda \\ &= \frac{Q^3 + 3Q^2 h_\lambda - \eta h_\lambda^3 - 3\eta h_\lambda^2 Q + F_\lambda((Q+h_\lambda)^2)(Q+h_\lambda)}{\eta+1} + G_\lambda(Q, h_\lambda), \end{aligned}$$

where

$$G_\lambda(Q, h_\lambda) = -\frac{\eta}{\eta+1}(Q+h_\lambda)\mathcal{F}^{-1}\left(\frac{\beta c_0^2 (T-t)^2 \lambda^2}{|\xi|^2 - \beta c_0^2 (T-t)^2\lambda^2}\mathcal{F}((Q+h_\lambda)^2)\right).$$

By the definition of F_λ, we have

$$\begin{aligned} &Q^3 + 3Q^2 h_\lambda - \eta h_\lambda^3 - 3\eta h_\lambda^2 Q + F_\lambda((Q+h_\lambda)^2)(Q+h_\lambda) \\ &\quad = Z_\lambda(h_\lambda) + l_\lambda(h_\lambda) + q_\lambda(h_\lambda) + C_\lambda(h_\lambda), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} Z_\lambda(h_\lambda) &= (F_\lambda(Q^2) + Q^2)Q, \\ l_\lambda(h_\lambda) &= (F_\lambda(Q^2) + Q^2)h_\lambda + 2(F_\lambda(Qh_\lambda) + Qh_\lambda)Q, \\ q_\lambda(h_\lambda) &= -3\eta h_\lambda^2 Q + F_\lambda(h_\lambda^2)Q + 2F_\lambda(Qh_\lambda)h_\lambda, \\ C_\lambda(h_\lambda) &= -\eta h_\lambda^3 + F_\lambda(h_\lambda^2)h_\lambda. \end{aligned}$$

Since $L = (\triangle - Id + 3Q^2)^{-1}$ is a bounded operator in H_r^1 and there exists $C > 0$ such that $\|L(u)\|_{H^2} \leq C\|u\|_{L^2}$ for $u \in H_r^1$, which was proved in [3], we know that (P_λ, N_λ) is a solution to (1.6), where $P_\lambda = Q + h_\lambda$, $N_\lambda = F_\lambda((Q + h_\lambda)^2)$, if and only if h_λ is a fixed point of the operator

$$T_\lambda(h_\lambda) = L\left(\frac{Z_\lambda(h_\lambda) + l_\lambda(h_\lambda) + q_\lambda(h_\lambda) + C_\lambda(h_\lambda)}{\eta + 1} + G_\lambda(Q, h_\lambda)\right). \quad (2.13)$$

We will show that T_λ is a contraction mapping in the set $B_{\delta_0} = \{u \in H_r^2, \|u\|_{H^2} \leq \delta_0\}$. Now, we give two key lemmas.

Lemma 2.9 ([3]) There exists λ_0 such that for $0 < \lambda < \lambda_0$, $u, v, w \in H_r^2$,

$$\|L(F_\lambda(uv)w)\|_{H^2} \leq c_{\lambda_0}\|F_\lambda(uv)\|_{L^\infty}\|w\|_{L^2} \leq c_{\lambda_0}\|u\|_{H^2}\|v\|_{H^2}\|w\|_{H^2}, \quad (2.14)$$

$$\|L((F_\lambda(Qu) + Qu)v)\|_{H^2} \leq c_{\lambda_0}\lambda^2\|u\|_{H^2}\|v\|_{H^2}. \quad (2.15)$$

□

Lemma 2.10 For $\forall\varepsilon > 0$, $T > 0$, there exists $\lambda_{\varepsilon,T} > 0$ such that for $0 < \lambda < \lambda_{\varepsilon,T}$,

$$\|G_\lambda(Q, h_\lambda)\|_{L^2} \leq \varepsilon, \quad (2.16)$$

where $\|h_\lambda\|_{H^1} \leq c$.

Proof. By the properties of Fourier transform, we have

$$\|G_\lambda(Q, h_\lambda)\|_{L^2} = \left\|\frac{\eta}{\eta + 1}(Q + h_\lambda)\mathcal{F}^{-1}\left(\frac{\beta c_0^2(T - t)^2\lambda^2}{|\xi|^2 - \beta c_0^2(T - t)^2\lambda^2}\mathcal{F}((Q + h_\lambda)^2)\right)\right\|_{L^2}$$

$$= \sup_{\|v\|_{L^2}=1} \frac{\eta}{\eta + 1}\int_{\mathbb{R}^2} v(Q + h_\lambda)\mathcal{F}^{-1}\left(\frac{\beta c_0^2(T - t)^2\lambda^2}{|\xi|^2 - \beta c_0^2(T - t)^2\lambda^2}\mathcal{F}((Q + h_\lambda)^2)\right)$$

$$= \sup_{\|v\|_{L^2}=1} \frac{\eta}{\eta + 1}\int_{\mathbb{R}^2} \frac{\beta c_0^2(T - t)^2\lambda^2}{|\xi|^2 - \beta c_0^2(T - t)^2\lambda^2}\mathcal{F}(v(Q + h_\lambda))\mathcal{F}((Q + h_\lambda)^2)$$

$$= \sup_{\|v\|_{L^2}=1} \frac{\eta}{\eta + 1}\int_{\Omega_1+\Omega_2+\Omega_3} \frac{\beta c_0^2(T - t)^2\lambda^2}{|\xi|^2 - \beta c_0^2(T - t)^2\lambda^2}\mathcal{F}(v(Q + h_\lambda))\mathcal{F}((Q + h_\lambda)^2).$$

Here,

$$\Omega_1 = \left\{\xi \in \mathbb{R}^2 :\ |\xi|^2 \leq -\beta c_0^2(T - t)^2\lambda^2\right\},$$

$$\Omega_2 = \left\{\xi \in \mathbb{R}^2 :\ -\beta c_0^2(T - t)^2\lambda^2 < |\xi|^2 < -N\beta c_0^2(T - t)^2\lambda^2\right\},$$

and

$$\Omega_3 = \left\{\xi \in \mathbb{R}^2 :\ |\xi|^2 \geq -N\beta c_0^2(T - t)^2\lambda^2\right\}.$$

Since $v(Q+h_\lambda),\ (Q+h_\lambda)^2 \in L^1(\mathbb{R}^2)$ implies that $\mathcal{F}(v(Q+h_\lambda)),\ \mathcal{F}((Q+h_\lambda)^2) \in L^\infty(\mathbb{R}^2)$, one has that there exists $c>0$ such that

$$\sup_{\|v\|_{L^2}=1} \frac{\eta}{\eta+1} \int_{\Omega_1+\Omega_2} \frac{\beta c_0^2 (T-t)^2 \lambda^2}{|\xi|^2 - \beta c_0^2 (T-t)^2\lambda^2} \mathcal{F}(v(Q+h_\lambda))\mathcal{F}((Q+h_\lambda)^2)$$
$$\leq c\left(|\beta| c_0^2 (T-t)^2\lambda^2 + |\beta| c_0^2 (T-t)^2 \lambda^2\right).$$

By the Hölder inequality and the Plancherel Theorem, we have

$$\begin{aligned}
&\frac{\eta}{\eta+1} \int_{\Omega_3} \frac{\beta c_0^2 (T-t)^2 \lambda^2}{|\xi|^2 - \beta c_0^2 (T-t)^2\lambda^2} \mathcal{F}(v(Q+h_\lambda))\mathcal{F}((Q+h_\lambda)^2) \\
&\leq \frac{1}{N}\frac{\eta}{\eta+1}\left(\int_{\mathbb{R}^2} |\mathcal{F}(v(Q+h_\lambda))|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\mathcal{F}((Q+h_\lambda)^2)|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{N}\frac{\eta}{\eta+1}\left(\int_{\mathbb{R}^2} |v(Q+h_\lambda)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |Q+h_\lambda|^4\right)^{\frac{1}{2}} \\
&\leq \frac{1}{N}\frac{\eta}{\eta+1}\|v\|_{L^2}\|Q+h_\lambda\|_{H^2}^3.
\end{aligned}$$

Thus for $\|h_\lambda\|_{H^1} \leq c$, there exists $c>0$ such that

$$\|G_\lambda(Q,h_\lambda)\|_{L^2} \leq c\lambda^2 + cN\lambda^2 + \frac{c}{N}.$$

Therefore, for given $\varepsilon>0$, there exist N_ε large enough and λ_ε small enough such that for $N \geq N_\varepsilon$, $0<\lambda\leq\lambda_\varepsilon$,

$$\|G_\lambda(Q,h_\lambda)\|_{L^2} \leq \varepsilon.$$

The proof of Lemma 2.10 is completed. □

Now, we prove Theorem 2.8.

Proof of Theorem 2.8.

a) **Existence of fixed points.** We prove the existence of solutions to (1.6) by Banach fixed pointed theorem. For any $\delta>0$, we define

$$\Sigma_\delta = \{h \in H_r^2 :\ \|h\|_{H_r^2} \leq \delta\}.$$

It is sufficient to show that there exist $\delta_0>0$ and $\lambda_T>0$ such that for all $0<\lambda<\lambda_T$, T_λ is a contraction mapping of the set Σ_{δ_0}.

From (2.15) in Lemma 2.9 and $h,\ h_1,\ h_2 \in \Sigma_{\delta_0}$, we obtain

$$\left\|\frac{1}{\eta+1}L(Z_\lambda(h))\right\|_{H_r^2} \leq C\lambda^2,$$

$$\left\| \frac{1}{\eta+1} L(l_\lambda(h)) \right\|_{H_r^2} \leq C\lambda^2 \|h\|_{H_r^2},$$

and

$$\left\| \frac{1}{\eta+1} L(l_\lambda(h_1) - l_\lambda(h_2)) \right\|_{H_r^2} \leq C\lambda^2 \|h_1 - h_2\|_{H_r^2}.$$

Applying (2.14) in Lemma 2.9 and $h \in \Sigma_{\delta_0}$, we have

$$\left\| \frac{1}{\eta+1} L(q_\lambda(h)) \right\|_{H_r^2} \leq C\|h\|_{H_r^2}^2,$$

$$\left\| \frac{1}{\eta+1} L(C_\lambda(h)) \right\|_{H_r^2} \leq C\|h\|_{H_r^2}^3,$$

$$\left\| \frac{1}{\eta+1} L(q_\lambda(h_1) - q_\lambda(h_2)) \right\|_{H_r^2} \leq C\left(\|h_1\|_{H_r^2} + \|h_2\|_{H_r^2}\right) \|h_1 - h_2\|_{H_r^2},$$

and

$$\left\| \frac{1}{\eta+1} L(C_\lambda(h_1) - C_\lambda(h_2)) \right\|_{H_r^2} \leq C(\|h_1\|_{H_r^2}^2 + \|h_2\|_{H_r^2}^2)\|h_1 - h_2\|_{H_r^2}.$$

Therefore,

$$\|T_\lambda(h)\|_{H_r^2} \leq C\left(\lambda^2 + \lambda^2\|h\|_{H_r^2} + \|h\|_{H_r^2}^2 + \|h\|_{H_r^2}^3 + \|G_\lambda(Q,h)\|_{L^2}\right),$$

and

$$\|T_\lambda(h_1) - T_\lambda(h_2)\|_{H_r^2} \leq \|G_\lambda(Q,h_1) - G_\lambda(Q,h_1)\|_{L^2})$$
$$+C\|h_1 - h_2\|_{H_r^2}\left(\lambda^2 + \|h_1\|_{H_r^2} + \|h_2\|_{H_r^2} + \|h_1\|_{H_r^2}^2 + \|h_2\|_{H_r^2}^2\right).$$

Thus, from Lemma 2.10, we know that there exist $\delta_0 > 0$ and $\lambda_T > 0$ such that for all $0 < \lambda < \lambda_T$,

$$T_\lambda(h) \in \Sigma_{\delta_0} \text{ for } h \in \Sigma_{\delta_0},$$

and for all h_1, $h_2 \in \Sigma_{\delta_0}$,

$$\|T_\lambda(h_1) - T_\lambda(h_2)\|_{H_r^2} \leq \frac{1}{2}\|h_1 - h_2\|_{H_r^2}.$$

Thus, for all $0 < \lambda < \lambda_T$, T_λ is a contraction mapping of the set Σ_{δ_0}. By Banach fixed point Theorem, we know that there exists a unique fixed point of the mapping T_λ in the set Σ_{δ_0}, i.e., there exists a solution (P_λ, N_λ) to

(1.6).

b) **Continuity of solutions** (P_λ, N_λ) **with respect to** λ **in** $H^1 \times L^2$. Applying Lemma 2.9 and Lemma 2.10, with the dominated convergence theorem, we obtain the uniform continuity of the function $T_\lambda(h) : \mathbb{R}^+ \times H_r^2 \to H_r^2$. Thus, we get the continuity of h_λ in H_r^2 with respect to λ, i.e., the continuity of $P_\lambda = Q + h_\lambda$ in H_r^2 with respect to λ. Thus, we prove that $N_\lambda = F_\lambda((P_\lambda)^2)$ is continuous in L_r^2 with λ. □

Proof of Theorem 1.1. Using Theorem 2.8, Proposition 2.8 and Proposition 2.2, we obtain the results in Theorem 1.1. □

3 Instability of minimal periodic solutions to (1.1)-(1.2) with $\beta = 0$

In this section, we prove Theorem 1.2 by applying Theorem 1.1. We first consider a kind of minimal periodic solutions to (1.1)-(1.2), which has the form:

$$(\mathbf{E}(t), n(t)) = (e^{i\omega t}\mathbf{V}(x), |\mathbf{V}(x)|^2),$$

where

$$\mathbf{V}(x) = \left(\frac{V_1(x)}{\sqrt{2(\eta+1)}}, -i\frac{V_1(x)}{\sqrt{2(\eta+1)}}, 0 \right),$$

$\triangle V_1 - \omega V_1 + |V_1|^2|V_1| = 0$, $\omega > 0$ and $\|V_1\|_{L^2} = \|Q\|_{L^2}$. Applying the uniqueness of positive radial solutions to $\triangle V - V + V^3 = 0$ in $\mathbb{R}^2$, we obtain that there exist $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^2$ such that

$$V_1(x) = \omega^{\frac{1}{2}} e^{i\theta} Q(\omega^{\frac{1}{2}}(x - x_0)).$$

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Let

$$\tilde{c}_0 = c_0 \omega^{-\frac{1}{2}}.$$

Applying Theorem 1.1, we conclude that there exists a solution $(P_\varepsilon, N_\varepsilon)$ to (1.7) for some ε_0 with $0 < \lambda = \varepsilon < \varepsilon_0$, which is is a blow-up solution to (1.1)-(1.2) with $c_0 = \tilde{c}_0$ and

$$\|\tilde{\mathbf{E}}_\varepsilon\|_{H^1} + \|\tilde{n}_\varepsilon\|_{L^2} + \|\frac{\partial \tilde{n}_\varepsilon}{\partial t}\|_{\hat{H}^{-1}} \to +\infty \text{ as } t \to T_\varepsilon,$$

where for $\forall\ \theta_\varepsilon \in \mathbb{R}$,

$$\tilde{\mathbf{E}}_\varepsilon = (\tilde{E}_{1\varepsilon}, -i\tilde{E}_{1\varepsilon}, 0),\ \tilde{n}_\varepsilon = \frac{\omega_\varepsilon^2 N_\varepsilon(\frac{x\omega_\varepsilon}{T_\varepsilon - t})}{(T_\varepsilon - t)^2(\eta+1)},$$

and

$$\tilde{E}_{1\varepsilon} = \frac{\omega_\varepsilon}{T_\varepsilon - t} e^{i\left(\theta_\varepsilon + \frac{|x|^2}{4(-T_\varepsilon+t)} - \frac{\omega_\varepsilon^2}{-T_\varepsilon+t}\right)} \frac{P_\varepsilon(\frac{x\omega_\varepsilon}{T_\varepsilon - t})}{\sqrt{2}(\eta+1)^{1/2}}.$$

Moreover, according to Theorem 2.8, we get

$$(P_\varepsilon, N_\varepsilon) \to (Q, -Q^2) \text{ in } H^1 \times L^2 \text{ as } \varepsilon \to 0.$$

Choosing

$$\omega_\varepsilon = \frac{1}{\tilde{c}_0\varepsilon}, \quad T_\varepsilon = \frac{1}{\tilde{c}_0\varepsilon}, \quad \theta_\varepsilon = \frac{-1}{\tilde{c}_0\varepsilon},$$

we obtain that $(\tilde{\mathbf{E}}_\varepsilon, \tilde{n}_\varepsilon)$ is a blow-up solution to (1.1)-(1.2) with $c_0 = \tilde{c}_0$ and the initial data $\tilde{\mathbf{E}}_\varepsilon(0) = \tilde{\mathbf{E}}_{0\varepsilon}$, $\tilde{n}_\varepsilon(0) = \tilde{n}_{0\varepsilon}$, $\frac{\partial \tilde{n}_\varepsilon}{\partial t}(0) = \tilde{n}_{1\varepsilon}$, where

$$\tilde{\mathbf{E}}_{0\varepsilon} = \left(e^{i\tilde{c}_0\varepsilon\frac{|x|^2}{4}} \frac{P_\varepsilon(x)}{\sqrt{2}(\eta+1)^{1/2}}, -ie^{i\tilde{c}_0\varepsilon\frac{|x|^2}{4}} \frac{P_\varepsilon(x)}{\sqrt{2}(\eta+1)^{1/2}}, 0 \right),$$

$$\tilde{n}_{0\varepsilon} = \frac{N_\varepsilon(x)}{(\eta+1)}, \quad \tilde{n}_{1\varepsilon} = \tilde{c}_0\varepsilon(|x|N_\varepsilon'(x) + 2N_\varepsilon(x)),$$

$$(\tilde{\mathbf{E}}_{0\varepsilon}, \tilde{n}_{0\varepsilon}, \tilde{n}_{1\varepsilon}) = \left(\tilde{\mathbf{E}}_Q, -\frac{Q^2}{\eta+1}, 0 \right) \text{ in } H^1 \times L^2 \times H^{-1} \text{ as } \varepsilon \to 0,$$

and

$$\tilde{\mathbf{E}}_Q = \left(\frac{Q}{\sqrt{2}(\eta+1)^{1/2}}, -i\frac{Q}{\sqrt{2}(\eta+1)^{1/2}}, 0 \right).$$

Let

$$\mathbf{E}_\varepsilon(t,x) = e^{i\theta}\omega^{\frac{1}{2}}\tilde{\mathbf{E}}_\varepsilon\left(\omega t, \omega^{\frac{1}{2}}(x-x_0)\right),$$
$$n_\varepsilon(t,x) = \omega\tilde{n}_\varepsilon\left(\omega t, \omega^{\frac{1}{2}}(x-x_0)\right).$$

We obtain that $(\mathbf{E}_\varepsilon(t,x), n_\varepsilon(t,x))$ is a blow-up solution to (1.1)-(1.2) with the initial data

$$\mathbf{E}_\varepsilon(0,x) = \mathbf{E}_{0\varepsilon}(x) = e^{i\theta}\omega^{\frac{1}{2}}\tilde{\mathbf{E}}_{0\varepsilon}\left(\omega^{\frac{1}{2}}(x-x_0)\right),$$

$$n_\varepsilon(0,x) = n_{0\varepsilon}(x) = \omega\tilde{n}_{0\varepsilon}\left(\omega^{\frac{1}{2}}(x-x_0)\right),$$

$$n_{t\varepsilon}(0,x) = n_{1\varepsilon}(x) = \omega^2\tilde{n}_{1\varepsilon}\left(\omega^{\frac{1}{2}}(x-x_0)\right).$$

Furthermore, for all $k \geq 1$, we also have

$$(\mathbf{E}_{0\varepsilon}, n_{0\varepsilon}, n_{1\varepsilon}) \to (E(0), n(0), 0) \; in \; H_k \; as \; \varepsilon \to 0. \qquad \square$$

4 Concentration properties of blow-up solutions to (1.3)

In this section, we first give some lemmas and propositions which are key to the proof of Theorem 1.3.

Lemma 4.1 (Merle [4]) Assume that there exists a sequence $(\mathbf{v}_k, N_k) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that as $k \to +\infty$,

$$\int_{\mathbb{R}^2} |\mathbf{v}_k|^2 \to C_1 > 0, \quad \int_{\mathbb{R}^2} N_k |\mathbf{v}_k|^2 \to -C_3 < 0,$$

$$\int_{\mathbb{R}^2} |\nabla \mathbf{v}_k|^2 + \frac{1}{2} \int_{\mathbb{R}^2} |N_k|^2 \to C_2 > 0.$$

Then there exist a constant $C_4 = C_4(C_1, C_2, C_3) > 0$ and a sequence x_k such that

$$\int_{|x-x_k|<1} |N_k| > C_4. \qquad \square$$

Lemma 4.2 Assume that $\{v_m\}$ is bounded in $H^1(\mathbb{R}^2)$ and

$$\sup_{y\in\mathbb{R}^2} \int_{B(y,R)} |v_m|^2 dx \to 0 \quad \text{for some} \quad R > 0.$$

Then $v_m \to 0$ in $L^4(\mathbb{R}^2)$.

Proof. By interpolation inequalities, for $v \in H^1(\mathbb{R}^2)$ we have

$$\|v\|^4_{L^4(B(y,R))} \leq c \|v\|^2_{L^2(B(y,R))} \|v\|^2_{H^1(B(y,R))},$$

where c is a positive constant. Let $B_1 = B(0, R)$, $B_2 = B(y_2, R)$, where $y_2 \in \partial B(0, R)$, $B_3 = B(y_3, R)$, $B_4 = B(y_4, R)$, $\{y_3, y_4\} = \partial B_1 \cap \partial B_2$,..., we can cover $\mathbb{R}^2$ by the above balls of radius R such that each point of $\mathbb{R}^2$ is contained in at most 3 balls. Therefore, by the above inequality,

$$\|v_m\|^4_{L^4(\mathbb{R}^2)} \leq c \sup_{y\in\mathbb{R}^2} \int_{B(y,R)} |v_m|^2 dx \|v_m\|^2_{H^1(\mathbb{R}^2)}, \tag{4.1}$$

By the assumptions of the lemma, $v_m \to 0$ in $L^4(\mathbb{R}^2)$. $\square$

Proposition 4.3 Assume that $\mathbf{E}_k \in H^1(\mathbb{R}^2)$, $\|\mathbf{E}_k\|^2_{L^2} = \|\mathbf{E}_0\|^2_{L^2} > 0$, $n_k \in L^2(\mathbb{R}^2)$, $\mathbf{v}_k \in L^2(\mathbb{R}^2)$, and there exist $R_0 > 0$ and $\delta_0 > 0$ such that

$$\sup_{y\in\mathbb{R}^2} \int_{|y-x|<R_0} |\mathbf{E}_k|^2 \leq \|Q\|^2_{L^2} - \delta_0, \tag{4.2}$$

or for $\frac{\|Q\|_{L^2}^2}{1+\eta} < \|\mathbf{E}_0\|_{L^2}^2 < \frac{\delta\|Q\|_{L^2}^2}{\eta}$ with $\frac{\eta}{\eta+1} < \delta < 1$, there is a constant $m_n(\|\mathbf{E}_0\|_{L^2}^2) > 0$ such that

$$\sup_{y\in\mathbb{R}^2}\int_{|y-x|<R_0}|n_k(x)| \leq m_n(\|\mathbf{E}_0\|_{L^2}^2) - \delta_0. \tag{4.3}$$

Then there are $C_1 > 0$ and $C_2 > 0$ such that

$$-C_1 + C_2\int_{\mathbb{R}^2}(|\nabla\mathbf{E}_k|^2 + |n_k|^2 + |\mathbf{v}_k|^2) \leq I(\mathbf{E}_k, n_k, \mathbf{v}_k).$$

In order to prove Proposition 4.3, we first define some functionals:

$$M(\mathbf{E}, n) = \int_{\mathbb{R}^2}|\nabla\mathbf{E}|^2 + \frac{1}{2}\int_{\mathbb{R}^2}|n|^2 + \int_{\mathbb{R}^2}n|\mathbf{E}|^2 - \frac{\eta}{2}\int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\mathbf{E}\wedge\bar{\mathbf{E}})|^2,$$

$$G(\mathbf{E}) = \int_{\mathbb{R}^2}|\nabla\mathbf{E}|^2 - \frac{1}{2}\int_{\mathbb{R}^2}|\mathbf{E}|^4 - \frac{\eta}{2}\int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\mathbf{E}\wedge\bar{\mathbf{E}})|^2,$$

$$G^*(\mathbf{E}) = \int_{\mathbb{R}^2}|\nabla\mathbf{E}|^2 - \frac{\eta+1}{2}\int_{\mathbb{R}^2}|\mathbf{E}|^4.$$

It is clear from $\beta \leq 0$ that

$$M(\mathbf{E}, n) \geq G(\mathbf{E}) \geq G^*(\mathbf{E}).$$

Now we begin to prove Proposition 4.3 by contradiction.

Proof of Proposition 4.3. By the definition of $M(\mathbf{E}, n)$ and $I(\mathbf{E}, n, \mathbf{v})$, we only need to prove that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$-C_1 + C_2\int_{\mathbb{R}^2}(|\nabla\mathbf{E}_k|^2 + |n_k|^2) \leq M(\mathbf{E}_k, n_k). \tag{4.4}$$

Assume that there would be no positive constants $C_1 > 0$ and $C_2 > 0$ satisfying (4.4). Then

$$\lambda_k^2 := \int_{\mathbb{R}^2}|\nabla\mathbf{E}_k|^2 + \frac{1}{2}\int_{\mathbb{R}^2}|n_k|^2 \to +\infty \text{ as } k \to +\infty, \tag{4.5}$$

and

$$\limsup_{k\to\infty}\frac{M(\mathbf{E}_k, n_k)}{\lambda_k^2} \leq 0. \tag{4.6}$$

Indeed, if $\lambda_k \leq C$, then we have $M(\mathbf{E}_k, n_k) \leq C$ by using $\|\mathbf{E}_k\|_{L^2}^2 = \|\mathbf{E}_0\|_{L^2}^2$, which implies (4.4). If $\lim_{k\to+\infty}\dfrac{M(\mathbf{E}_K, n_k)}{\lambda_k^2} = C > 0$, then there exists $k_0 > 0$, for all $k \geq k_0$, $\frac{M(\mathbf{E}_k,n_k)}{\lambda_k^2} \geq \frac{C}{2}$, which also concludes (4.4).

Let

$$\tilde{\mathbf{E}}_k(x) = \frac{1}{\lambda_k}\mathbf{E}_k\left(\frac{x}{\lambda_k}\right), \text{ and } \tilde{n}_k(x) = \frac{1}{\lambda_k}n_k\left(\frac{x}{\lambda_k}\right).$$

Using the assumptions of Proposition 4.3 and (4.5), we obtain

$$\int_{\mathbb{R}^2}|\tilde{\mathbf{E}}_k(x)|^2 = \int_{\mathbb{R}^2}|\tilde{\mathbf{E}}_0|^2, \quad \int_{\mathbb{R}^2}\left(|\nabla\tilde{\mathbf{E}}_k(x)|^2 + \frac{1}{2}|\tilde{n}_k(x)|^2\right) = 1. \tag{4.7}$$

1) We shall prove (4.4) under the assumption (4.2). At first, combining (4.2) with (4.5), one has, for $\forall R > 0$, that

$$\liminf_{k\to+\infty}\sup_y\int_{|y-x|<R}|\tilde{\mathbf{E}}_k(x)|^2 \le \|Q\|_{L^2}^2 - \delta_0. \tag{4.8}$$

By (4.7) and the Sobolev inequality, there exist positive constants C_1 and C_2 such that

$$C_1 \le \int_{\mathbb{R}^2}|\tilde{\mathbf{E}}_k(x)|^4 \le C_2 \text{ and } C_1 \le \int_{\mathbb{R}^2}\left(|\nabla\tilde{\mathbf{E}}_k(x)|^2 + |\tilde{\mathbf{E}}_k(x)|^2\right) \le C_2. \tag{4.9}$$

By Lemma 4.2, we derive from (4.9) that there exists a positive constants δ_1 (depending only on $\|\mathbf{E}_0\|_{L^2}^2$) and a sequence x_k^1 such that

$$\int_{|x-x_k^1|<1}|\tilde{\mathbf{E}}_k(x)|^2 \ge \delta_1.$$

By the techniques of Concentration-Compactness Principle (see [11]) for the case of dichotomy, we obtain that there exist $\tilde{E}_k^1$ and $\tilde{\mathbf{E}}_k^{1,R}(x)$ (going if necessary to a subsequence) such that

$$\tilde{\mathbf{E}}_k(x) = \tilde{\mathbf{E}}_k^1(x) + \tilde{\mathbf{E}}_k^{1,R}(x),$$

where

$$\tilde{\mathbf{E}}_k^1(x + x_k^1) \rightharpoonup \psi_1 \text{ in } H^1, \tag{4.10}$$

$$\int_{|x|<1}|\tilde{\mathbf{E}}_k^1(x + x_k^1)|^2 \ge \delta_1, \qquad \|\tilde{\mathbf{E}}_k^1\|_{L^2}^2 + \|\tilde{\mathbf{E}}_k^{1,R}(x)\|_{L^2}^2 \to \|\mathbf{E}_0\|_{L^2}^2,$$

$$\delta_1 \le \lim_{k\to\infty}\|\tilde{\mathbf{E}}_k^1(x)\|_{L^2}^2 \le \|Q\|_{L^2}^2 - \delta_0,$$

and

$$\limsup_{k\to+\infty}G(\tilde{\mathbf{E}}_k^1) + \limsup_{k\to+\infty}G(\tilde{\mathbf{E}}_k^{1,R}(x)) \le \limsup_{k\to+\infty}G(\tilde{\mathbf{E}}_k) \le 0. \tag{4.11}$$

By the weakly lower semi-continuity of norm, we derive from (4.10) and (4.11) that

$$G(\psi_1) + \limsup_{k\to+\infty} G(\tilde{\mathbf{E}}_k^{1,R}(x)) \le 0, \text{ and } \delta_1 \le \|\psi_1\|_{L^2}^2 \le \|Q\|_{L^2}^2 - \delta_0,$$

which implies that there exists $k_0 > 0$ such that $\forall k \ge k_0$,

$$G(\tilde{\mathbf{E}}_k^{1,R}(x)) \le \frac{G(\psi_1)}{2} < 0. \tag{4.12}$$

If $\|\tilde{\mathbf{E}}_k^{1,R}(x)\|_{L^2}^2 \le \|Q\|_{L^2}^2$, we then get by Lemma 2.3 that $G(\tilde{\mathbf{E}}_k^{1,R}(x)) \ge 0$, which is contradictory to (4.12).

If $\|\tilde{\mathbf{E}}_k^{1,R}(x)\|_{L^2}^2 > \|Q\|_{L^2}^2$, then we derive from (4.12) that there exists a positive constant C depending only on $\|\mathbf{E}_0\|_{L^2}^2$ such that $\int_{\mathbb{R}^2} |\tilde{\mathbf{E}}_k^{1,R}(x)|^4 > C$.

Similarly, by Lemma 4.2, there exist $\delta_1 > 0$ and x_k^2 such that

$$\int_{|x-x_k^2|<1} |\tilde{\mathbf{E}}_k^{1,R}(x)|^2 \ge \delta_1.$$

Using the same procedure as above, we obtain that there exist $\tilde{\mathbf{E}}_k^2$ and $\tilde{\mathbf{E}}_k^{2,R}(x)$ such that

$$\tilde{\mathbf{E}}_k^{1,R}(x) = \tilde{\mathbf{E}}_k^2 + \tilde{\mathbf{E}}_k^{2,R},$$

where $\tilde{\mathbf{E}}_k^2$ has the same properties as $\tilde{\mathbf{E}}_k^1$ and $\tilde{\mathbf{E}}_k^{2,R}(x)$ as $\tilde{\mathbf{E}}_k^{1,R}(x)$.

Applying the above procedure p times such that

$$\|\tilde{\mathbf{E}}_k^{p,R}\|_{L^2}^2 \le \|Q\|_{L^2}^2, \tag{4.13}$$

we have

$$G\left(\tilde{\mathbf{E}}_k^{p,R}\right) \le \frac{G(\psi_1)}{2} < 0, \text{ for p large enough,}$$

which is contradictory to (4.13). The proof of (4.4) under the assumption (4.2) is completed.

2) In the following, we shall prove (4.4) under the assumption (4.3).

Since $\|\tilde{\mathbf{E}}_k\|_{L^2}^2 = \|\mathbf{E}_0\|_{L^2}^2 < \frac{\delta}{\eta}\|Q\|_{L^2}^2$, by Lemma 2.3, we have

$$\delta\|\nabla\tilde{\mathbf{E}}_k\|_{L^2}^2 \ge \frac{\frac{\delta}{2}\|\tilde{\mathbf{E}}_k\|_{L^4}^4\|Q\|_{L^2}^2}{\|\tilde{\mathbf{E}}_k\|_{L^2}^2} > \frac{\eta}{2}\|\tilde{\mathbf{E}}_k\|_{L^4}^4 \ge \frac{\eta}{2}\int_{\mathbb{R}^2} |\mathcal{F}(\tilde{\mathbf{E}}_k \wedge \bar{\tilde{\mathbf{E}}}_k)|^2. \tag{4.14}$$

On the other hand, we derive from (4.6) and (4.14) that

$$\limsup_{k\to+\infty}((1-\delta)\left(\int_{\mathbb{R}^2} |\nabla\tilde{\mathbf{E}}|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |\tilde{n}_k|^2 + \int_{\mathbb{R}^2} \tilde{n}_k|\tilde{\mathbf{E}}_k|^2\right)$$

$$
\begin{aligned}
&\leq \limsup_{k\to+\infty}\left((1-\delta+\delta)\left(\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{E}}|^2+\frac{1}{2}\int_{\mathbb{R}^2}|\tilde{n}_k|^2\right)\right.\\
&\qquad\left.-\frac{\eta}{2}\int_{\mathbb{R}^2}|\mathcal{F}(\tilde{\mathbf{E}}_k\wedge\bar{\tilde{\mathbf{E}}}_k)|^2+\int_{\mathbb{R}^2}\tilde{n}_k|\tilde{\mathbf{E}}_k|^2\right)\\
&\leq \limsup_{k\to+\infty} M(\tilde{\mathbf{E}}_k,\tilde{n}_k)\\
&\leq \limsup_{k\to+\infty}\frac{M(\mathbf{E}_k,n_k)}{\lambda_k^2}\\
&\leq 0,
\end{aligned}
$$

which implies that

$$\int_{\mathbb{R}^2}\tilde{n}_k|\tilde{\mathbf{E}}_k|^2\to -C\leq -(1-\delta)$$

as $k\to\infty$ (going if necessary to a subsequence), where we have used the Sobolev inequality.

Using Lemma 4.1, we obtain that there exist a constant $C>0$ and a sequence x_k such that

$$\int_{|x-x_k|<1}|\tilde{n}_k|>C>0. \tag{4.15}$$

On the other hand, by the assumption (4.2) and the definition of $\tilde{n}_k$, using the dominated convergence theorem, we have

$$\liminf_{k\to+\infty}\left(\sup_y\int_{|x-y|<R}|\tilde{n}_k|\right)\to 0 \ as\ R\to 0,$$

which is contradictory to (4.15). This completes the proof of Proposition 4.3. □

Now we begin to prove Theorem 1.3.

Proof of Theorem 1.3.

(1) We shall prove the first part of Theorem 1.3 by contradiction for the case: $n_t(0)\in\hat{H}^{-1}$ and $(\mathbf{E},n)$ is radial. Assume that there exist $\delta_0>0$, $R_0>0$ and a sequence $t_k\to T$ as $k\to\infty$ such that

$$\int_{|x|<R_0}|\mathbf{E}(t_k,x)|^2\leq\|Q\|_{L^2}^2-\delta_0, \tag{4.16}$$

or for $\frac{\|Q\|_{L^2}^2}{1+\eta}\leq\|\mathbf{E}_0\|_{L^2}^2\leq\frac{\delta\|Q\|_{L^2}^2}{\eta}$ with $\frac{\eta}{\eta+1}<\delta<1$,

$$\liminf_{k\to+\infty}\left(\int_{|x|<R}|n(t_k,x)|\right)\to 0,\ \text{as } R\to 0. \tag{4.17}$$

Let

$$\mathbf{E}_k(x) = \frac{1}{\lambda_k}\mathbf{E}\left(t_k, \frac{x}{\lambda_k}\right), \text{ and } n_k(x) = \frac{1}{\lambda_k^2} n\left(t_k, \frac{x}{\lambda_k}\right),$$

where $\lambda_k^2 = \|\nabla \mathbf{E}(t_k, x)\|_{L^2}^2 \to \infty$ as $k \to +\infty$.

Indeed, assume that $\|\nabla \mathbf{E}(t)\| \leq C$ for $t \in [0, T)$. From $\|\mathbf{E}(t)\|_{L^2}^2 = \|\mathbf{E}_0\|_{L^2}^2$, one has $\|\mathbf{E}(t)\|_{H^1}^2 \leq C$ and

$$G(\mathbf{E}(t)) = \int_{\mathbb{R}^2} |\nabla \mathbf{E}|^2 - \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{E}|^4 - \frac{\eta}{2}\int_{\mathbb{R}^2} \frac{|\xi|^2}{|\xi|^2 - \beta} |\mathcal{F}(\mathbf{E} \wedge \bar{\mathbf{E}})|^2 \leq C.$$

Lemma 2.1 then implies that

$$\begin{aligned} \frac{dI(t)}{dt} &\leq 2\int_{\mathbb{R}^2} \omega_0^2 + \int_{\mathbb{R}^2} (n + |\mathbf{E}|^2)^2 \\ &\leq C + G(\mathbf{E}) + \frac{1}{2}\int_{\mathbb{R}^2} \left(n + |\mathbf{E}|^2\right)^2 \\ &\leq C + I(t). \end{aligned}$$

Thus, by the Gronwall Lemma, we have $I(t) \leq C$, which contradicts $\|\mathbf{E}(t)\|_{H^1} + \|n(t)\|_{L^2} + \|\mathbf{v}(t)\|_{L^2} \to +\infty$ as $t \to T$.

According to the definitions of $\mathbf{E}_k$, n_k, G^* and M, we have

$$\int_{\mathbb{R}^2} |\nabla \mathbf{E}_k|^2 = 1, \quad \int_{\mathbb{R}^2} |\mathbf{E}_k|^2 = \int_{\mathbb{R}^2} |\mathbf{E}_0|^2, \tag{4.18}$$

$$G^*(\mathbf{E}_k) = \frac{1}{\lambda_k^2} G^*(\mathbf{E}(t_k, x)) = \frac{1}{\lambda_k^2} G^*(\mathbf{E}(t_k)),$$

and

$$M(\mathbf{E}_k, n_k) = \frac{1}{\lambda_k^2} M(\mathbf{E}(t_k), n(t_k)).$$

Since $n_t(0) \in \hat{H}^{-1}$, which implies that $\omega_0 = 0$, Lemma 2.1 yields that for $0 \leq t < T$,

$$I(\mathbf{E}(t), n(t), \mathbf{v}(t)) = I(\mathbf{E}_0, n_0, \mathbf{v}_0) = I_0.$$

From $M(\mathbf{E}, n) \leq I(\mathbf{E}, n, \mathbf{v})$, it follows that

$$G^*(\mathbf{E}(t_k)) \leq M(\mathbf{E}(t_k), n_k(t_k)) \leq I(\mathbf{E}(t_k), n(t_k), \mathbf{v}(t_k)) \leq I_0,$$

and

$$G^*(\mathbf{E}_k) \leq M(\mathbf{E}_k, n_k) = \frac{1}{\lambda_k^2} M(\mathbf{E}(t_k), n_k(t_k)) \leq \frac{I_0}{\lambda_k^2} \to 0 \text{ as } k \to \infty. \tag{4.19}$$

Hence, one obtains that

$$\limsup_{k\to\infty} G^*(\mathbf{E}_k) \le 0$$

and

$$\limsup_{k\to\infty} M(\mathbf{E}_k, n_k) \le 0.$$

On the other hand, one has

$$\liminf_{k\to\infty} \int_{\mathbb{R}^2} |\mathbf{E}_k|^4 \ge \frac{2}{\eta+1} \liminf_{k\to\infty} (\int_{\mathbb{R}^2} |\nabla \mathbf{E}_k|^2 - G^*(\mathbf{E}_k)) \ge \frac{2}{\eta+1} > 0, \quad (4.20)$$

$$\limsup_{k\to\infty} \frac{1}{2} \int_{\mathbb{R}^2} n_k^2 \le \frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{E}_k|^4 + \frac{\eta}{2} \int \frac{|\xi|^2}{|\xi|^2 - \beta} |\mathcal{F}(\mathbf{E}_k \wedge \bar{\mathbf{E}}_k)|^2 \le C, \quad (4.21)$$

which are derived from (4.18), $\limsup_{k\to\infty} M(\mathbf{E}_k, n_k) \le 0$, and

$$\limsup_{k\to\infty} \frac{1}{2} \int_{\mathbb{R}^2} (n_k + |\mathbf{E}_k|^2)^2 - \frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{E}_k|^4 - \frac{\eta}{2} \int \frac{|\xi|^2}{|\xi|^2 - \beta} |\mathcal{F}(\mathbf{E}_k \wedge \bar{\mathbf{E}}_k)|^2 \le 0.$$

According to (4.18) and (4.20), there exist $(\tilde{\mathbf{E}}, \tilde{n}) \in H_r^1 \times L_r^2$ and a subsequence of $\{(\mathbf{E}_k, n_k)\}$, denoted again by $\{(\mathbf{E}_k, n_k)\}$, such that

$$\mathbf{E}_k \rightharpoonup \tilde{\mathbf{E}} \text{ in } H_r^1 \text{ and } n_k \rightharpoonup \tilde{n} \text{ in } L_r^2 \text{ as } k \to +\infty.$$

Since the embedding $H_r^2 \hookrightarrow L_r^p (2 < p < +\infty)$ is compact, one has $\mathbf{E}_k \to \tilde{\mathbf{E}}$ in L_r^P, Therefore, from (4.20), it follows that

$$\int_{\mathbb{R}^2} |\tilde{\mathbf{E}}|^4 \ge \frac{2}{\eta+1}, \text{ and } \tilde{\mathbf{E}} \ne 0. \quad (4.22)$$

Moreover, we derive from (4.16) that

$$\int_{\mathbb{R}^2} |\tilde{\mathbf{E}}|^2 \le \|Q\|_{L^2}^2 - \delta_0, \quad (4.23)$$

and from (4.17) that

$$\tilde{n} = 0. \quad (4.24)$$

Thus $\mathbf{E}_k \to \tilde{\mathbf{E}}$ in L_r^4 and $n_k \rightharpoonup \tilde{n}$ in L_r^2 imply that

$$\int_{\mathbb{R}^2} n_k |\mathbf{E}_k|^2 \to \int_{\mathbb{R}^2} \tilde{n} |\tilde{\mathbf{E}}|^2.$$

By (4.19), we have $M(\tilde{\mathbf{E}}, \tilde{n}) \le \liminf_{k\to\infty} M(\mathbf{E}_k, n_k) \le 0$, that is,

$$\int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{E}}|^2 - \frac{\eta+1}{2} \int_{\mathbb{R}^2} |\tilde{\mathbf{E}}|^4 + \frac{1}{2} \int_{\mathbb{R}^2} \left(\tilde{n} + |\tilde{\mathbf{E}}|^2\right)^2$$

$$+\frac{\eta}{2}\int_{\mathbb{R}^2}\left(|\tilde{\mathbf{E}}|^4-\frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\tilde{\mathbf{E}}\wedge\bar{\tilde{\mathbf{E}}})|^2\right)\leq 0,$$

which yields that

$$\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{E}}|^2-\frac{\eta+1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{E}}|^4\leq 0. \tag{4.25}$$

However, by Lemma 2.3 and (4.23), we have

$$\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{E}}|^2-\frac{\eta+1}{2}\int_{\mathbb{R}^2}|\tilde{\mathbf{E}}|^4>0,$$

which contradicts (4.25).

On the other hand, under the assumption (1.8), we have

$$\delta\int_{\mathbb{R}^2}|\nabla\mathbf{E}_k|^2\geq\frac{\eta}{2}\int\frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\mathbf{E}_k\wedge\bar{\mathbf{E}}_k)|^2.$$

Then from the above inequality and (4.19), it follows that

$$(1-\delta)\int_{\mathbb{R}^2}|\nabla\mathbf{E}_k|^2+\int_{\mathbb{R}^2}n_k|\mathbf{E}_k|^2+\int_{\mathbb{R}^2}n_k^2\leq 0.$$

Since $n_k\rightharpoonup\tilde{n}=0$ in L_r^2 and $\int_{\mathbb{R}^2}n_k|\mathbf{E}_k|^2\to\int_{\mathbb{R}^2}\tilde{n}|\tilde{\mathbf{E}}|^2$ as $k\to+\infty$, we have

$$(1-\delta)\int_{\mathbb{R}^2}|\nabla\tilde{\mathbf{E}}|^2\leq 0,$$

which is contradictory to

$$\int_{\mathbb{R}^2}|\tilde{\mathbf{E}}|^4\geq\frac{2}{1+\eta}\text{ and }\tilde{\mathbf{E}}\neq 0.$$

The proof of (1) of Theorem 1.3 is completed.

(2) Here, we show (2) for the case: $n_t(0)\in\hat{H}^{-1}$ and $(\mathbf{E},n)$ is non-radial. Let $m_n(\|\mathbf{E}_0\|_{L^2})$ be defined in Proposition 4.3. Assume that there is a subsequence $t_k\to T$ as $k\to+\infty$, $R_0>0$, $\delta_0>0$ such that

$$\liminf_{k\to+\infty}\left(\sup_y\int_{|x-y|<R_0}|\mathbf{E}(t_k,x)|^2dx\right)\leq\|Q\|_{L^2}^2-\delta_0$$

or

$$\liminf_{k\to+\infty}\left(\sup_y\int_{|x-y|<R}|n(t_k,x)|dx\right)\leq m_n(\|\mathbf{E}_0\|_{L^2})-\delta_0.$$

Applying Proposition 4.3 with $(\mathbf{E}(t_k), n(t_k), \mathbf{v}(t_k))$, we obtain

$$\int_{\mathbb{R}^2} |\nabla \mathbf{E}(t_k)|^2 + |n(t_k)|^2 + |\mathbf{v}(t_k)|^2 \leq c \text{ as } t_k \to T,$$

which is a contradiction. Thus, there exist $x(t)$ and $y(t)$ such that, for $\forall R > 0$,

$$\liminf_{t\to T} \int_{|x-x(t)|<R} |\mathbf{E}(t,x)|^2 \geq |Q|_{L^2}^2$$

and

$$\liminf_{t\to T} \int_{|x-y(t)|<R} |n(t_k,x)| \geq m_n(\|\mathbf{E}_0\|_{L^2}) > 0,$$

which concludes the proof of (2) of Theorem 1.3.

(3) Now, we prove (3) and (4) for the case: $n_t(0) \in H^{-1}$ but $n_t(0) \notin \hat{H}^{-1}$.

Assume that there is no sequence $t_k \to T$ such that, for $\forall R > 0$,

$$\liminf_{k\to+\infty} \left(\sup_y \int_{|x-y|<R} |\mathbf{E}(t_k,x)|^2 dx \right) \geq \|Q\|_{L^2}^2,$$

or

$$\liminf_{k\to+\infty} \left(\sup_y \int_{|x-y|<R} |n(t_k,x)| dx \right) \geq m_n(\|\mathbf{E}_0\|_{L^2}).$$

Then there are $R_0, \delta_0 > 0$ such that, for $\forall t \in [0,T)$,

$$\sup_y \int_{|x-y|<R_0} |\mathbf{E}(t,x)|^2 dx \leq \|Q\|_{L^2}^2 - \delta_0$$

or

$$\sup_y \int_{|x-y|<R_0} |n(t,x)| dx \leq m_n(\|\mathbf{E}_0\|_{L^2}) - \delta_0.$$

Applying Proposition 4.3, we obtain, for $\forall t \in [0,T)$,

$$\int_{\mathbb{R}^2} |\nabla \mathbf{E}(t_k)|^2 + |n(t_k)|^2 + |\mathbf{v}(t_k)|^2 \leq C_1 I(t) + C_2, \tag{4.26}$$

In addition, from Lemmas 2.1, it follows for $\forall t \in [0,T)$ that

$$\begin{aligned} I(t) &\leq I(0) + \int_0^t I'(s) ds \\ &\leq c \left(1 + \int_0^t (\|w_0\|_{L^2}^2 + \|n(s)\|_{L^2}^2 + \|\mathbf{E}(s)\|_{L^2}^2 \right) ds) \\ &\leq c \left(1 + \int_0^t (|n(s)|_{L^2}^2 + |\nabla \mathbf{E}(s)|_{L^2}^2) ds \right) \\ &\leq c \left(1 + \int_0^t (\|\nabla \mathbf{E}(s)\|_{L^2}^2 + \|n(s)\|_{L^2}^2 + \|\mathbf{v}(s)\|_{L^2}^2) ds \right). \end{aligned} \tag{4.27}$$

Using the Gronwall lemma, we derive from (4.26) and (4.27) that

$$\forall t \in [0,T),\ \|\nabla \mathbf{E}(t)\|_{L^2}^2 + \|n(t)\|_{L^2}^2 + \|\mathbf{v}(t)\|_{L^2}^2 \leq C,$$

or equivalently,

$$\forall t \in [0,T),\ |\mathbf{E}(t), n(t), n_t(t)|_{H_1} \leq C,$$

which is a contradiction.

We remark that in the radial case, we only need to choose $x_k = 0$ in Theorem 1.3 in view of the obvious symmetry reasons and conservation of the L^2 norm.

The proof of Theorem 1.3 is completed. $\square$

5 Global existence for the case $\|\mathbf{E}_0\|_{L^2}^2 \leq \frac{\|Q\|_{L^2}^2}{\eta+1}$

In this section, we prove Theorem 1.4. On one hand, we prove the global existence of weak solutions to (1.3) for the case $\|\mathbf{E}_0\|_{L^2}^2 < \frac{\|Q\|_{L^2}^2}{\eta+1}$. On the other hand, we use Proposition 4.3 to prove the global existence for the case $\|\mathbf{E}_0\|_{L^2}^2 = \frac{\|Q\|_{L^2}^2}{\eta+1}$.

Theorem 5.1 If $\mathbf{E}_0 \in H^1(\mathbb{R}^2)$, $n_0 \in L^2(\mathbb{R}^2)$, $\mathbf{v}_0 \in L^2(\mathbb{R}^2)$ and $\|\mathbf{E}_0\|_{L^2}^2 < \frac{1}{\eta+1}\|Q\|_{L^2}^2$, then there is a global weak solution $\mathbf{E} \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^2))$, $n \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))$, $\mathbf{v} \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2))$ to (1.3), and $(\mathbf{E}, n, \mathbf{B}(\mathbf{E}))$ is a weak solution to (1.1) with initial data $\mathbf{E}_0, n_0, n_1 = -\text{div}\mathbf{v}_0 + w_0$.

Proof. Here we only give the uniform *a priori* estimates for the solutions to (1.3). For more details of the proof of Theorem 5.1, we can refer to [10]. By Lemma 2.1, we have

$$\frac{dI(t)}{dt} = \int_{\mathbb{R}^2} w_0(n + |\mathbf{E}|^2) \leq 2\int_{\mathbb{R}^2} w_0^2 + \frac{1}{2}\int_{\mathbb{R}^2}(n + |\mathbf{E}|^2)^2.$$

We note that $\int_{\mathbb{R}^2}|\nabla \mathbf{E}|^2 - \frac{\eta}{2}\int_{\mathbb{R}^2}|\mathbf{E}|^4 > 0$ for $\|\mathbf{E}\|_{L^2}^2 < \frac{\|Q\|_{L^2}^2}{1+\eta}$, which is true from Lemma 2.3, $\int_{\mathbb{R}^2}|\mathbf{E}|^4 \geq \int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\mathbf{E}\wedge\bar{\mathbf{E}})|^2 d\xi$ and the definition of I, where

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^2}|\nabla \mathbf{E}|^2 + \frac{1}{2}\int_{\mathbb{R}^2}|n|^2 + \frac{1}{2c_0^2}\int_{\mathbb{R}^2}|\mathbf{v}|^2 + \int_{\mathbb{R}^2} n|\mathbf{E}|^2 \\ &\quad -\frac{\eta}{2}\int_{\mathbb{R}^2}\frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\mathbf{E}\wedge\bar{\mathbf{E}})|^2 d\xi \end{aligned}$$

$$= \int_{\mathbb{R}^2} |\nabla \mathbf{E}|^2 - \frac{1+\eta}{2}\int_{\mathbb{R}^2} |\mathbf{E}|^4 + \frac{1}{2}\int_{\mathbb{R}^2} (n+|\mathbf{E}|^2)^2 + \frac{1}{2c_0^2}\int_{\mathbb{R}^2} |\mathbf{v}|^2$$
$$+\frac{\eta}{2}\left(\int_{\mathbb{R}^2} |\mathbf{E}|^4 - \int_{\mathbb{R}^2} \frac{|\xi|^2}{|\xi|^2-\beta}|\mathcal{F}(\mathbf{E}\wedge\bar{\mathbf{E}})|^2 d\xi\right)^2.$$

Thus, we conclude that

$$\frac{dI(t)}{dt} \leq 2\int_{\mathbb{R}^2} w_0^2 + \frac{1}{2}\int_{\mathbb{R}^2} (n+|\mathbf{E}|^2)^2 \leq 2\int_{\mathbb{R}^2} w_0^2 + I(t),$$

which together with the Gronwall Lemma implies that

$$I(t) \leq C(I(0), \|w_0\|_{L^2}). \tag{5.1}$$

On the other hand, in view of the Hölder inequality, the Young inequality and Lemma 2.3, we derive from (5.1) that

$$\begin{aligned}
\|\nabla \mathbf{E}\|_{L^2}^2 &+ \frac{1}{2}\|n\|_{L^2}^2 + \frac{1}{2c_0^2}\|\mathbf{v}\|_{L^2}^2 \\
&\leq C + \|n\|_{L^2}\|E\|_{L^4}^2 + \frac{\eta}{2}\|\mathbf{E}\|_{L^4}^4 \\
&\leq C + b^2\|n\|_{L^2} + \frac{1}{4b^2}\|\mathbf{E}\|_{L^4}^4 + \frac{\eta}{2}\|\mathbf{E}\|_{L^4}^4 \\
&\leq C + b^2\|n\|_{L^2} + \left(\frac{1}{2b^2}+\eta\right)\frac{\|\mathbf{E}\|_{L^2}^2}{\|Q\|_{L^2}^2}\|\nabla \mathbf{E}\|_{L^2}^2,
\end{aligned}$$

where $0 < b^2 \leq \frac{1}{2}$. Letting $b^2 = \frac{1}{2}$, we obtain

$$\|\nabla \mathbf{E}\|_{L^2}^2 \leq C, \text{ and } \|\mathbf{v}\|_{L^2}^2 \leq C.$$

Furthermore, letting $0 < b < \frac{1}{2}$, we have $\|n\|_{L^2}^2 \leq C$.

Proof of Theorem 1.4 for the case $\|\mathbf{E}_0\|_{L^2}^2 = \frac{\|Q\|_{L^2}^2}{\eta+1}$.

Here we shall prove the global existence of weak solutions to (1.3) for the case $\|\mathbf{E}_0\|_{L^2}^2 = \frac{\|Q\|_{L^2}^2}{\eta+1}$ by contradiction. Assume that there exists $T > 0$ such that $\|\mathbf{E}\|_{H^1} + \|n\|_{L^2} + \|\mathbf{v}\|_{L^2} \to +\infty$ as $t \to T$. Applying Lemma 2.3 and noting that $\|\mathbf{E}\|_{L^2}^2 = \|\mathbf{E}_0\|_{L^2}^2 = \frac{\|Q\|_{L^2}^2}{\eta+1}$, we get

$$\int_{\mathbb{R}^2} |\nabla \mathbf{E}|^2 - \frac{\eta}{2}\int_{\mathbb{R}^2} |\mathbf{E}|^4 \geq 0. \tag{5.2}$$

Similarly, one has

$$I(t) \leq C(\|w_0\|_{L^2}, I(0)). \tag{5.3}$$

By the definition of I, we derive from (5.2) and (5.3) that

$$G^*(\mathbf{E}(t)) \leq C, \ \|\mathbf{v}\|_{L^2} \leq C, \text{ and } \int_{\mathbb{R}^2} (n + |\mathbf{E}|^2)^2 \leq C. \tag{5.4}$$

By $|\mathbf{E}(t)|^2 = (n + |\mathbf{E}(t)|^2) - n$ and (5.4), we obtain

$$\||\mathbf{E}(t)|^2\|_{H^{-1}} \leq C. \tag{5.5}$$

Indeed, we can derive from $n_t = \nabla \cdot \mathbf{v} + w_0$ that

$$\begin{aligned}\|n(t)\|_{H^{-1}} &\leq \|n_0\|_{H^{-1}} + \int_0^t \|n_t(s)\|_{H^{-1}} ds \\ &\leq C + \int_0^t (\|\mathbf{v}(s)\|_{L^2} + \|w_0\|_{L^2}) ds \leq 0. \end{aligned} \tag{5.6}$$

Combining (5.4) with (5.6), we establish (5.5).

In the proof of (1) of Theorem 1.3, we note that if $\|\mathbf{E}\|_{H^1} + \|n\|_{L^2} + \|\mathbf{v}\|_{L^2} \to +\infty$ as $t \to T$, then $\|\nabla \mathbf{E}\|_{H^1} \to +\infty$ as $t \to T$. Thus, applying Proposition 4.3, we obtain that there is $x(t)$ such that

$$|\mathbf{E}(t, x + x(t))|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \text{ as } t \to T,$$

in the distribution sense, where $\delta_{x=0}$ is the usual Dirac function. Moreover, by (5.5), we have

$$\|Q\|_{L^2}^2 \delta_{x=0} \in H^{-1},$$

which is impossible. Therefore, the solution $(\mathbf{E}(t), n(t))$ to (1.1)-(1.2) exists globally.

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