Two Boundedness Criterions for Some Operators on Musielak-Orlicz Hardy Spaces and Applications

Qiu Xiaoli, Li Baode, Liu Xiong and Li Bo *

Abstract. Let $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ satisfy that $\varphi(x, \cdot)$, for any given $x \in \mathbb{R}^n$, is an Orlicz function and $\varphi(\cdot, t)$ is a Muckenhoupt A_∞ weight uniformly in $t \in (0, \infty)$. The (weak) Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ ($WH^\varphi(\mathbb{R}^n)$) generalizes both of the weighted (weak) Hardy space and the (weak) Orlicz Hardy space and hence has a wide generality. In this paper, two boundedness criterions for both of linear operator and positive sublinear operator from $H^\varphi(\mathbb{R}^n)$ to $H^\varphi(\mathbb{R}^n)$ or from $H^\varphi(\mathbb{R}^n)$ to $WH^\varphi(\mathbb{R}^n)$ are obtained. As applications, we establish the boundedness of Bochner-Riesz means from $H^\varphi(\mathbb{R}^n)$ to $H^\varphi(\mathbb{R}^n)$, or from $H^\varphi(\mathbb{R}^n)$ to $WH^\varphi(\mathbb{R}^n)$ in the critical case. These results are also new even when $\varphi(x, t) := \Phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, where Φ is an Orlicz function.

1 Introduction

The real-variable theory of Hardy space on the n-dimensional Euclidean space $\mathbb{R}^n$, initiated by Stein and Weiss [35], plays an important role in the harmonic analysis and partial differential equations. It is well known that Hardy space $H^p(\mathbb{R}^n)$ is a good substitute of Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$; for example, when $p \in (0, 1]$, the Riesz transforms are not bounded on $L^p(\mathbb{R}^n)$, however, they are bounded on $H^p(\mathbb{R}^n)$. Moreover, when studying the boundedness of operator in the critical case, the weak Hardy space $WH^p(\mathbb{R}^n)$ naturally appear and prove to be a good substitute of Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$. For example, if $\delta \in (0, 1]$, T is a δ-Calderón-Zygmund operator and $T^*(1) = 0$, where T^* denotes the adjoint operator of T, it is known that T is bounded on $H^p(\mathbb{R}^n)$ for all $p \in (\frac{n}{n+\delta}, 1]$ (see [1]), but T may be not bounded on $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$; however, Liu [19] proved that T is bounded from $H^{\frac{n}{n+\delta}}(\mathbb{R}^n)$ to $WH^{\frac{n}{n+\delta}}(\mathbb{R}^n)$.

Recently, Ky [15] introduced a new Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which generalizes both of the classical Hardy space [7], the weighted Hardy space [33], the Orlicz Hardy space [10, 12, 13, 14] and the weighted Orlicz Hardy space, and hence has a wide generality. Later, Liang et al. [26] further introduced a weak Musielak-Orlicz Hardy s-pace $WH^\varphi(\mathbb{R}^n)$, which covers both of the weak Hardy space [6], the weighted weak Hardy space [29], the weak Orlicz Hardy space and the weighted weak Orlicz Hardy space, as special cases. Apart from interesting theoretical considerations, the motivation to study

2010 Mathematics Subject Classification. Primary 42B30; Secondary 42B15, 46E30.

Key words and phrases. Hardy space, Musielak-Orlicz function, Muckenhoupt weight, Bochner-Riesz means.

This work is supported by National Natural Science Foundation of China (Grant Nos. 11861062, 11461065, 11661075, 11671039 & 11771043).

* Corresponding author

Musielak-Orlicz-type space comes from applications to elasticity, fluid dynamics, image processing, nonlinear PDEs and the calculus of variation (see, for example, [3, 4]). More Musielak-Orlicz-type spaces are referred to [23, 25, 24, 38, 21, 5, 39, 37]. We refer the reader to [37] for a complete survey of the real-variable theory of Musielak-Orlicz Hardy space.

On the other hand, observe that a distribution in Hardy space can be represented as a (finite or infinite) linear combination of atoms (see [16, 8]). Then, the boundedness of linear operator on Hardy space can be deduced from their behavior on atoms. More precisely, as is well known, a linear operator T (which is originally defined on smooth functions with compact support) can extend to a bounded operator from $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ to some quasi-Banach space $\mathcal{B}$ if T is bounded on $L^2(\mathbb{R}^n)$ and it maps all $(p, 2)$-atoms into uniformly bounded elements of $\mathcal{B}$; see, for example, [18, 20, 28, 31].

Motivated by all of the above mentioned facts, it is a natural and interesting problem to ask if T is a linear or a positive sublinear operator, what kind of additional conditions on T can deduce the boundedness of T from $H^\varphi(\mathbb{R}^n)$ to $H^\varphi(\mathbb{R}^n)$ or from $H^\varphi(\mathbb{R}^n)$ to $WH^\varphi(\mathbb{R}^n)$? In this paper, we shall answer these problems affirmatively. As applications, we establish the boundedness of Bochner-Riesz means from $H^\varphi(\mathbb{R}^n)$ to $H^\varphi(\mathbb{R}^n)$, or from $H^\varphi(\mathbb{R}^n)$ to $WH^\varphi(\mathbb{R}^n)$ at critical index. These results are also new even when $\varphi(x, t) := \Phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, where Φ is an Orlicz function.

An outline of this paper is as follows.

In Section 2, we recall some notions concerning Muckenhoupt weight, growth function and Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$. Then we present the completeness of weak Musielak-Orlicz Hardy space (see Theorem 2.11 below) and two boundedness criterions for T from $H^\varphi(\mathbb{R}^n)$ to $WH^\varphi(\mathbb{R}^n)$ or from $H^\varphi(\mathbb{R}^n)$ to itself (see Theorems 2.12 and 2.13 below).

Section 3 is devoted to the proofs of Theorems 2.11, 2.12 and 2.13. Here, we point out that, the uniformly σ-quasi-subadditive property of φ can be used to prove the completeness of $H^\varphi(\mathbb{R}^n)$ (see [15, Proposition 5.2] for more details). However, we don't know how to use this method to obtain the completeness of $WH^\varphi(\mathbb{R}^n)$ because of the particular form of the norm of $WH^\varphi(\mathbb{R}^n)$. Fortunately, we overcome this difficulty by borrowing some ideas from the proof of [27, Proposition 2.8] and using Aoki-Rolewicz's theorem (see Lemma 3.2 below). In the process of the proof of Theorem 2.13, the molecular characterization of $H^\varphi(\mathbb{R}^n)$ plays a key role in obtaining the boundedness criterion for T from $H^\varphi(\mathbb{R}^n)$ to itself. It is worth pointing out that the Musielak-Orlicz Hardy space has several different kinds of molecular characterization. Here, we use the molecular characterization of Li et al. [21] rather than that of Hou et al. [9].

In Section 4, we first recall the definition of Bochner-Riesz means T_R^δ. Then, as applications of Theorems 2.12 and 2.13, the boundedness of Bochner-Riesz means from $H^\varphi(\mathbb{R}^n)$ to $WH^\varphi(\mathbb{R}^n)$ (see Theorem 4.1 below) or from $H^\varphi(\mathbb{R}^n)$ to itself (see Theorem 4.2 below) is obtained. It is worth pointing out that this method is different from that used by Lu [18, Chapter 3, §5], in which the kernel of T_R^δ belongs to Campanato space was proved. However, in present setting, the corresponding conclusion that the kernel of T_R^δ belongs to Musielak-Orlicz Campanato space is still unknown due to the complex structure of Musielak-Orlicz-type space.

Finally, we make some conventions on notation. Let $\mathbb{Z}_+ := \{1, 2, \ldots\}$ and $\mathbb{N} := \{0\} \cup \mathbb{Z}_+$. For any $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$ and $\partial^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Throughout this paper, the letter C will denote a *positive constant* that may vary from line to line but will remain independent of the main variables. The *symbol* $P \lesssim Q$ stands for the inequality $P \le CQ$. If $P \lesssim Q \lesssim P$, we then write $P \sim Q$. For any sets $E,\, F \subset \mathbb{R}^n$, we use $E^\complement$ to denote the set $\mathbb{R}^n \setminus E$, $|E|$ its *n-dimensional Lebesgue measure*, χ_E its *characteristic function* and $E+F$ the *algebraic sum* $\{x+y:\ x \in E,\, y \in F\}$. For any $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the unique integer such that $s-1 < \lfloor s \rfloor \le s$. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any index $q \in [1, \infty]$, q' denotes the *conjugate index* of q, namely, $1/q + 1/q' = 1$. For any set E of $\mathbb{R}^n$, $t \in [0, \infty)$ and measurable function f, let $\varphi(E,\, t) := \int_E \varphi(x,\, t)\, dx$ and $\{|f| > t\} := \{x \in \mathbb{R}^n:\ |f(x)| > t\}$. As usual, for any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\alpha \in (0, \infty)$, let $B(x,\, r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\alpha B(x,\, r) := B(x,\, \alpha r)$.

2 Notions and main results

In this section, we first recall the notion concerning the Musielak-Orlicz Hardy space via the non-tangential grand maximal function, and then present the completeness of weak Musielak-Orlicz Hardy space and two boundedness criterions for some operators on Musielak-Orlicz Hardy space.

Recall that a function $\Phi : [0, \infty) \to [0, \infty)$ is called an *Orlicz function*, if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t\to\infty} \Phi(t) = \infty$.

Given a function $\varphi : \mathbb{R}^n \times [0, \infty) \to [0, \infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x, \cdot)$ is an Orlicz function, φ is said to be of *uniformly lower* (resp. *upper*) *type p* with $p \in (0, \infty)$, if there exists a positive constant $C := C_\varphi$ such that, for any $x \in \mathbb{R}^n$, $t \in [0, \infty)$ and $s \in (0, 1]$ (resp. $s \in [1, \infty)$),

$$\varphi(x,\, st) \le C s^p \varphi(x,\, t).$$

The *critical uniformly lower type index* and the *critical uniformly upper type index* of φ are, respectively, defined by

$$i(\varphi) := \sup\{p \in (0, \infty) : \varphi \text{ is of uniformly lower type } p\}, \tag{2.1}$$

and

$$I(\varphi) := \inf\{p \in (0, \infty) : \varphi \text{ is of uniformly upper type } p\}. \tag{2.2}$$

Observe that $i(\varphi)$ or $I(\varphi)$ may not be attainable, namely, φ may not be of uniformly lower type $i(\varphi)$ or of uniformly upper type $I(\varphi)$; see below for some examples.

Definition 2.1. Let $q \in [1, \infty)$. A function $\varphi(\cdot\,, t) : \mathbb{R}^n \to [0, \infty)$ is said to satisfy the *uniform Muckenhoupt condition*, denoted by $\varphi \in \mathbb{A}_q$, if there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$ and $t \in (0, \infty)$, when $q = 1$,

$$\frac{1}{|B|} \int_B \varphi(x,\, t)\, dx \left\{ \operatorname*{ess\,sup}_{x \in B} [\varphi(x,\, t)]^{-1} \right\} \le C$$

and, when $q \in (1, \infty)$,

$$\frac{1}{|B|} \int_B \varphi(x,\, t)\, dx \left\{ \frac{1}{|B|} \int_B [\varphi(x,\, t)]^{-\frac{1}{q-1}}\, dx \right\}^{q-1} \le C.$$

Let $\mathbb{A}_\infty := \bigcup_{q \in [1,\, \infty)} \mathbb{A}_q$. The *critical weight index* of $\varphi \in \mathbb{A}_\infty$ is defined as follows:

$$q(\varphi) := \inf\{q \in [1,\, \infty) :\ \varphi \in \mathbb{A}_q\}. \tag{2.3}$$

Observe that, if $q(\varphi) \in (1,\, \infty)$, then $\varphi \notin \mathbb{A}_{q(\varphi)}$, and there exists $\varphi \notin \mathbb{A}_1$ such that $q(\varphi) = 1$ (see, for example, [11]).

Definition 2.2. ([15, Definition 2.1]) A function $\varphi : \mathbb{R}^n \times [0,\, \infty) \to [0,\, \infty)$ is called a *growth function* if the following conditions are satisfied:

(i) φ is a *Musielak-Orlicz function*, namely,

(a) the function $\varphi(x,\, \cdot) : [0,\, \infty) \to [0,\, \infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$,

(b) the function $\varphi(\cdot\,, t)$ is a Lebesgue measurable function on $\mathbb{R}^n$ for all $t \in [0,\, \infty)$;

(ii) $\varphi \in \mathbb{A}_\infty$;

(iii) φ is of uniformly lower type p for some $p \in (0,\, 1]$ and of uniformly upper type 1.

Clearly, $\varphi(x,\, t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty$ and Φ is an Orlicz function of lower type p for some $p \in (0,\, 1]$ and of upper type 1. It is well known that, for $p \in (0,\, 1]$, if $\Phi(t) := t^p$ for all $t \in [0,\, \infty)$, then Φ is an Orlicz function of lower type p and of upper p; for $p \in [1/2,\, 1]$, if $\Phi(t) := t^p/\ln(e+t)$ for all $t \in [0,\, \infty)$, then Φ is an Orlicz function of lower type q for $q \in (0,\, p)$ and of upper type p; for $p \in (0,\, 1/2]$, if $\Phi(t) := t^p \ln(e+t)$ for all $t \in [0,\, \infty)$, then Φ is an Orlicz function of lower type p and of upper type q for $q \in (p,\, 1]$. Recall that if an Orlicz function is of upper type $p \in (0,\, 1)$, then it is also of upper type 1. Another typical and useful growth function is

$$\varphi(x,\, t) := \frac{t^\alpha}{[\ln(e+|x|)]^\beta + [\ln(e+t)]^\gamma}$$

for all $(x,\, t) \in \mathbb{R}^n \times [0,\, \infty)$, with any $\alpha \in (0,\, 1]$, $\beta \in [0,\, \infty)$ and $\gamma \in [0,\, 2\alpha(1+\ln 2)]$; more precisely, $\varphi \in \mathbb{A}_1$, φ is of uniformly upper type α and $i(\varphi) = \alpha$ which is not attainable (see [15]).

Suppose that φ is a Musielak-Orlicz function. Recall that the *Musielak-Orlicz space* L^φ is defined to be the set of all measurable functions f such that, for some $\lambda \in (0,\, \infty)$,

$$\int_{\mathbb{R}^n} \varphi\left(x,\, \frac{|f(x)|}{\lambda}\right) dx < \infty$$

equipped with the Luxembourg-Nakano (quasi-)norm

$$\|f\|_{L^\varphi} := \inf\left\{\lambda \in (0,\, \infty) :\ \int_{\mathbb{R}^n} \varphi\left(x,\, \frac{|f(x)|}{\lambda}\right) dx \le 1\right\}.$$

Similarly, the *weak Musielak-Orlicz space* WL^φ is defined to be the set of all measurable functions f such that, for some $\lambda \in (0, \infty)$,

$$\sup_{t\in(0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) < \infty$$

equipped with the quasi-norm

$$\|f\|_{WL^\varphi} := \inf\left\{\lambda \in (0, \infty) : \sup_{t\in(0,\infty)} \varphi\left(\{|f| > t\}, \frac{t}{\lambda}\right) \le 1\right\}.$$

Remark 2.3. Let ω be a classical Muckenhoupt weight and Φ an Orlicz function.

(i) If $\varphi(x, t) := \omega(x)t^p$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ with $p \in (0, \infty)$, then L^φ (resp. WL^φ) is reduced to weighted Lebesgue space L^p_ω (resp. weighted weak Lebesgue space WL^p_ω), and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

(ii) If $\varphi(x, t) := \omega(x)\Phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, then L^φ (resp. WL^φ) is reduced to weighted Orlicz space L^Φ_ω (resp. weighted weak Orlicz space WL^Φ_ω), and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

In what follows, we denote by $\mathcal{S}$ the *space of all Schwartz functions* and by $\mathcal{S}'$ its *dual space* (namely, the *space of all tempered distributions*). For any $m \in \mathbb{N}$, let

$$\mathcal{S}_m := \left\{\psi \in \mathcal{S} : \sup_{\alpha\in\mathbb{N}^n,\,|\alpha|\le m+1} \sup_{x\in\mathbb{R}^n} (1 + |x|)^{(m+2)(n+1)}|\partial^\alpha\psi(x)| \le 1\right\}.$$

Then, for any $m \in \mathbb{N}$ and $f \in \mathcal{S}'$, the *non-tangential grand maximal function* f^*_m of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f^*_m(x) := \sup_{\psi\in\mathcal{S}_m} \sup_{|y-x|<t,\,t\in(0,\infty)} |f * \psi_t(y)|,$$

where, for any $t \in (0, \infty)$, $\psi_t(\cdot) := t^{-n}\psi(\frac{\cdot}{t})$. When

$$m = m(\varphi) := \left\lfloor n\left(\frac{q(\varphi)}{i(\varphi)} - 1\right)\right\rfloor, \tag{2.4}$$

we denote f^*_m simply by f^*, where $q(\varphi)$ and $i(\varphi)$ are as in (2.3) and (2.1), respectively.

Definition 2.4. ([15, Definition 2.2]) Let φ be a growth function as in Definition 2.2 and $m \in [m(\varphi), \infty) \cap \mathbb{N}$, where $m(\varphi)$ is as in (2.4). The *Musielak-Orlicz Hardy space* H^φ_m is defined as the set of all $f \in \mathcal{S}'$ such that $f^*_m \in L^\varphi$ equipped with the (quasi-)norm

$$\|f\|_{H^\varphi_m} := \|f^*_m\|_{L^\varphi}.$$

Definition 2.5. ([26, Definition 2.3]) Let φ be a growth function as in Definition 2.2 and $m \in [m(\varphi), \infty) \cap \mathbb{N}$, where $m(\varphi)$ is as in (2.4). The *weak Musielak-Orlicz Hardy space* WH_m^φ is defined as the set of all $f \in \mathcal{S}'$ such that $f_m^* \in WL^\varphi$ equipped with the quasi-norm

$$\|f\|_{WH_m^\varphi} := \|f_m^*\|_{WL^\varphi}.$$

Remark 2.6. By [22, Lemma 2.13], we know that, if $m \in [m(\varphi), \infty) \cap \mathbb{N}$, then the definition of H_m^φ is independent of m. Analogously, by [26, Throrem 3.5] and the same argument as in the proof of [22, Lemma 2.13], we know that, if $m \in [m(\varphi), \infty) \cap \mathbb{N}$, then the definition of WH_m^φ is also independent of m. Therefore, from now on, we denote H_m^φ and WH_m^φ with $m \in [m(\varphi), \infty) \cap \mathbb{N}$ simply by H^φ and WH^φ, respectively.

Remark 2.7. Let ω be a classical Muckenhoupt weight and Φ an Orlicz function.

(i) If $\varphi(x, t) := \omega(x)t^p$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$ with $p \in (0, 1]$, then H^φ (resp. WH^φ) is reduced to weighted Hardy space H_ω^p (resp. weighted weak Hardy space WH_ω^p), and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

(ii) If $\varphi(x, t) := \omega(x)\Phi(t)$ for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$, then H^φ (resp. WH^φ) is reduced to weighted Orlicz Hardy space H_ω^Φ (resp. weighted weak Orlicz Hardy space WH_ω^Φ), and particularly, when $\omega \equiv 1$, the corresponding unweighted spaces are also obtained.

Definition 2.8. ([15, Definition 2.4]) Let φ be a growth function as in Definition 2.2.

(i) A triplet (φ, q, N) is said to be *admissible*, if $q \in (q(\varphi), \infty]$ and $N \in [m(\varphi), \infty) \cap \mathbb{N}$, where $q(\varphi)$ and $m(\varphi)$ are as in (2.3) and (2.4), respectively.

(ii) For an admissible triplet (φ, q, N), a measurable function a is called a (φ, q, N)-*atom* associated with some ball $B \subset \mathbb{R}^n$ if it satisfies the following three conditions:

(a) a is supported in B;

(b) $\|a\|_{L_\varphi^q(B)} \le \|\chi_B\|_{L^\varphi}^{-1}$, where

$$\|a\|_{L_\varphi^q(B)} := \begin{cases} \sup_{t \in (0, \infty)} \left[\frac{1}{\varphi(B, t)} \int_B |a(x)|^q \varphi(x, t)\, dx\right]^{1/q}, & q \in [1, \infty), \\ \|a\|_{L^\infty}, & q = \infty; \end{cases}$$

(c) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$.

Definition 2.9. ([21, Definition 2.6]) Let φ be a growth function as in Definition 2.2.

(i) A quadruple $(\varphi, q, N, \varepsilon)$ is said to be *admissible*, if $q \in (q(\varphi), \infty]$, $N \in [m(\varphi), \infty) \cap \mathbb{N}$ and $\varepsilon \in (0, \infty)$ satisfying $\varepsilon > \max\{q(\varphi)/i(\varphi), N/n + 1\}$, where $q(\varphi)$, $m(\varphi)$ and $i(\varphi)$ are as in (2.3), (2.4) and (2.1), respectively.

(ii) For an admissible quadruple $(\varphi,\, q,\, N,\, \varepsilon)$, a measurable function M is called a $(\varphi,\, q,\, N,\, \varepsilon)$-*molecule* associated with some ball $B \subset \mathbb{R}^n$ if it satisfies the following three conditions:

(a) $\|M\|_{L^q_\varphi(B)} \le \|\chi_B\|^{-1}_{L^\varphi}$;

(b) for any $j \in \mathbb{N}$ and $y \in (2^{j+1}B) \setminus (2^j B)$,

$$|M(y)| \le 2^{-nj\varepsilon}\|\chi_B\|^{-1}_{L^\varphi};$$

(c) $\int_{\mathbb{R}^n} M(x)x^\alpha\, dx = 0$ for any $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$.

Definition 2.10. Let X and Y be two function spaces. An operator T: $D \subset X \to Y$ is called a positive sublinear operator if, for any $x \in \mathbb{R}^n$, the following conditions are satisfied:

(i) $T(f)(x) \ge 0$;

(ii) $T(\alpha f)(x) \le |\alpha| T(f)(x)$, where $\alpha \in \mathbb{C}$;

(iii) $T(f+g)(x) \le T(f)(x) + T(g)(x)$.

The main results of this paper are as follows, the proofs of which are given in next section.

Theorem 2.11. *Let φ be a growth function as in Definition 2.2. The weak Musielak-Orlicz Hardy space WH^φ is complete.*

Theorem 2.12. *Let φ be a growth function as in Definition 2.2 satisfying $I(\varphi) \in (0,\, 1)$, and $m \in [m(\varphi),\, \infty) \cap \mathbb{N}$, where $I(\varphi)$ and $m(\varphi)$ are as in* (2.2) *and* (2.4)*, respectively. Suppose that a linear or a positive sublinear operator T is bounded on L^2. If there exists a positive constant C such that, for any $\lambda \in (0,\, \infty)$ and multiple of a $(\varphi,\, q,\, N)$-atom $b(\cdot)$ associated with some ball $B \subset \mathbb{R}^n$,*

$$\sup_{\alpha\in(0,\,\infty)} \varphi\left(\left\{(T(b))^*_m > \alpha\right\},\, \frac{\alpha}{\lambda}\right) \le C\varphi\left(B,\, \frac{\|b\|_{L^q_\varphi(B)}}{\lambda}\right), \tag{2.5}$$

then T extends uniquely to a bounded operator from H^φ to WH^φ.

Theorem 2.13. *Let φ be a growth function as in Definition 2.2 and $a(\cdot)$ be a $(\varphi,\, q,\, N)$-atom associated with some ball $B \subset \mathbb{R}^n$. Suppose that a linear or a positive sublinear operator T is bounded on L^2. If $T(a)$ is a harmless constant multiple of a $(\varphi,\, q,\, N,\, \varepsilon)$-molecule, then T extends uniquely to a bounded operator from H^φ to H^φ.*

Remark 2.14. Let ω be a classical Muckenhoupt weight and Φ an Orlicz function. When $\varphi(x,\, t) := \omega(x)\Phi(t)$ for all $(x,\, t) \in \mathbb{R}^n \times [0,\, \infty)$, we have $H^\varphi = H^\Phi_\omega$. In this case, Theorems 2.11, 2.12 and 2.13 hold true for weighted Orlicz Hardy space. Even when $\omega \equiv 1$, the above results are also new.

3 Proofs of Theorems 2.11, 2.12 and 2.13

To prove Theorems 2.11, 2.12 and 2.13, we need some auxiliary lemmas. The proof of the following lemma is identical to that of [15, Proposition 5.1], the details being omitted.

Lemma 3.1. *Let φ be a growth function as in Definition 2.2. Then $WH^{\varphi} \subset \mathcal{S}'$ and the inclusion is continuous.*

Recall that a *quasi-normed linear space* $\mathcal{B}$ is a linear space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}} = 0$ if and only if $f = \mathbf{0}$), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a constant K no less than 1 such that, for any $f, g \in \mathcal{B}$, $\|f+g\|_{\mathcal{B}} \le K\left(\|f\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}\right)$.

Lemma 3.2. ([30, Aoki-Rolewicz's theorem]) *Let $\mathcal{B}$ be a quasi-normed linear space and K a constant associated with $\mathcal{B}$ as above. Then, for any $\{f_i\}_{i\in\mathbb{Z}_+} \subset \mathcal{B}$,*

$$\left\|\sum_{i=1}^{\infty} f_i\right\|_{\mathcal{B}}^{\gamma} \le \sum_{i=1}^{\infty} \|f_i\|_{\mathcal{B}}^{\gamma},$$

where $\gamma := [\log_2(2K)]^{-1}$.

Proof of Theorem 2.11. We show this theorem by borrowing some ideas from the proof of [27, Proposition 2.8]. To prove that WH^{φ} is complete, we divide our proof in three steps.

Firstly, without loss of generality, we take a sequence $\{f_j\}_{j\in\mathbb{Z}_+} \subset WH^{\varphi}$ such that, for any $j \in \mathbb{Z}_+$, $\|f_j\|_{WH^{\varphi}} \le 2^{-j}$.

The next thing to do in the proof is to find some f in WH^{φ}. Since $\{\sum_{j=1}^{k} f_j\}_{k\in\mathbb{Z}_+}$ is a Cauchy sequence in WH^{φ}, from Lemma 3.1, it follows that $\{\sum_{j=1}^{k} f_j\}_{k\in\mathbb{Z}_+}$ is also a Cauchy sequence in $\mathcal{S}'$, which, together with the completeness of $\mathcal{S}'$, implies that there exists some $f \in \mathcal{S}'$ such that $\sum_{j=1}^{k} f_j$ converges to f as $k \to \infty$ in $\mathcal{S}'$. Thus, for any $\psi \in \mathcal{S}$, the series $\sum_{j=1}^{k} f_j * \psi$ converges to $f*\psi$ pointwisely as $k \to \infty$. Therefore, for any $x \in \mathbb{R}^n$, we have

$$f^*(x) \le \sum_{j\in\mathbb{Z}_+} (f_j)^*(x).$$

By this and Lemma 3.2, we know that there exists some $\gamma \in (0, 1]$ associated with WL^{φ} such that

$$\|f\|_{WH^{\varphi}}^{\gamma} = \|f^*\|_{WL^{\varphi}}^{\gamma} \le \left\|\sum_{j\in\mathbb{Z}_+} (f_j)^*\right\|_{WL^{\varphi}}^{\gamma} \le \sum_{j\in\mathbb{Z}_+} \|(f_j)^*\|_{WL^{\varphi}}^{\gamma} \le \sum_{j\in\mathbb{Z}_+} 2^{-j\gamma} < \infty.$$

Finally, we still to show that $\sum_{j=1}^{k} f_j \to f$ as $k \to \infty$ in WH^{φ}. Applying Lemma 3.2 again, we know that there exists some $\widetilde{\gamma} \in (0, 1]$ associated with WH^{φ} such that

$$\left\|f - \sum_{j=1}^{k} f_j\right\|_{WH^{\varphi}} = \left\|\sum_{j=k+1}^{\infty} f_j\right\|_{WH^{\varphi}} \le \left(\sum_{j=k+1}^{\infty} \|f_j\|_{WH^{\varphi}}^{\widetilde{\gamma}}\right)^{1/\widetilde{\gamma}}$$

$$\le \left(\sum_{j=k+1}^{\infty} 2^{-j\widetilde{\gamma}}\right)^{1/\widetilde{\gamma}} \sim 2^{-k} \to 0 \text{ as } k \to \infty.$$

This finishes the proof of Theorem 2.11. □

Lemma 3.3. *Let X and Y be two linear spaces. Suppose $T : D \subset X \to Y$ is a positive sublinear operator as in Definition 2.10. Then, for any $f, g \in D$,*

$$|T(f) - T(g)| \le T(f - g).$$

Proof. Applying Definition 2.10(ii), we obtain that

$$T(-f) \le |-1|\, T(f) = T(f) \le |-1|\, T(-f) = T(-f),$$

therefore, $T(-f) = T(f)$. By Definition 2.10(iii), we know that

$$T(f) - T(g) = T(f - g + g) - T(g) \le T(f - g) + T(g) - T(g) = T(f - g).$$

Similarly,

$$T(g) - T(f) \le T(g - f).$$

From the above two inequalities and $T(-f) = T(f)$, we deduce that $|T(f) - T(g)| \le T(f - g)$. This finishes the proof of Lemma 3.3. □

The following lemma gives the superposition principle of weak type estimates.

Lemma 3.4. ([2, Lemma 7.13]) *Let φ be a growth function as in Definition 2.2 satisfying $I(\varphi) \in (0, 1)$, where $I(\varphi)$ is as in (2.2). Assume that $\{f_j\}_{j\in\mathbb{Z}_+}$ is a sequence of measurable functions such that, for some $\lambda \in (0, \infty)$,*

$$\sum_{j\in\mathbb{Z}_+} \sup_{\alpha\in(0,\infty)} \varphi\left(\{|f_j| > \alpha\}, \frac{\alpha}{\lambda}\right) < \infty.$$

Then there exists a positive constant C, depending only on φ, such that, for any $\eta \in (0, \infty)$,

$$\varphi\left(\left\{\sum_{j\in\mathbb{Z}_+} |f_j| > \eta\right\}, \frac{\eta}{\lambda}\right) \le C \sum_{j\in\mathbb{Z}_+} \sup_{\alpha\in(0,\infty)} \varphi\left(\{|f_j| > \alpha\}, \frac{\alpha}{\lambda}\right).$$

By an argument similar to that used in the proof of [15, Lemma 4.3], we easily obtain the following lemma, the details being omitted.

Lemma 3.5. *Let φ be a growth function as in Definition 2.2. For a given positive constant $\widetilde{C}$, there exists a positive constant C such that, for any $\lambda \in (0, \infty)$,*

$$\sup_{\alpha\in(0,\infty)} \varphi\left(\{|f| > \alpha\}, \frac{\alpha}{\lambda}\right) \le \widetilde{C} \text{ implies that } \|f\|_{WL^{\varphi}} \le C\lambda.$$

Definition 3.6. ([15, Definition 2.4]) For an admissible triplet $(\varphi,\, q,\, N)$, the *Musielak-Orlicz atomic Hardy space* $H^{\varphi,\, q,\, N}_{\mathrm{at}}$ is defined as the set of all $f \in \mathcal{S}'$ which can be represented as a linear combination of $(\varphi,\, q,\, N)$-atoms, that is, $f = \sum_j b_j$ in $\mathcal{S}'$, where b_j for each j is a multiple of some $(\varphi,\, q,\, N)$-atom supported in some ball B_j, with the property

$$\sum_j \varphi\left(B_j,\, \|b_j\|_{L^q_\varphi(B_j)}\right) < \infty.$$

Define

$$\Lambda_q\left(\{b_j\}_j\right) := \inf\left\{\lambda \in (0,\, \infty) :\ \sum_j \varphi\left(B_j,\, \frac{\|b_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \le 1\right\}$$

and

$$\|f\|_{H^{\varphi,\, q,\, N}_{\mathrm{at}}} := \inf\left\{\Lambda_q(\{b_j\}_j)\right\},$$

where the infimum is taken over all admissible decompositions of f as above.

Lemma 3.7. ([22, Lemma 2.13]) *Let $(\varphi,\, q,\, N)$ be an admissible triplet as in Definition 2.9. If $m \in [m(\varphi),\, \infty) \cap \mathbb{N}$, where $m(\varphi)$ is as in* (2.4)*, then*

$$H^\varphi_m = H^{\varphi,\, q,\, N}_{\mathrm{at}}$$

with equivalent (quasi-)norms.

Lemma 3.8. ([37, Remark 4.1.4]) *Let φ be a growth function as in Definition 2.2. Then $H^\varphi \cap L^2$ is dense in H^φ.*

Lemma 3.9. *Let $\mathcal{B}$ be a quasi-normed linear space equipped with the quasi-norm $\|\cdot\|_{\mathcal{B}}$. For any $\{f_k\}_{k\in\mathbb{Z}_+} \subset \mathcal{B}$ and $f \in \mathcal{B}$, if $\lim_{k\to\infty} \|f_k - f\|_{\mathcal{B}} = 0$, then*

$$\lim_{k\to\infty} \|f_k\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}\,.$$

Proof. By Lemma 3.2, we obtain that, for any $k \in \mathbb{Z}_+$,

$$\|f_k\|^\gamma_{\mathcal{B}} - \|f\|^\gamma_{\mathcal{B}} = \|f_k - f + f\|^\gamma_{\mathcal{B}} - \|f\|^\gamma_{\mathcal{B}} \le \|f_k - f\|^\gamma_{\mathcal{B}}\,,$$

where γ is a harmless constant as in Lemma 3.2. Similarly, we have

$$\|f\|^\gamma_{\mathcal{B}} - \|f_k\|^\gamma_{\mathcal{B}} \le \|f - f_k\|^\gamma_{\mathcal{B}}\,,$$

which, together with the above inequality, implies that

$$\left|\|f_k\|^\gamma_{\mathcal{B}} - \|f\|^\gamma_{\mathcal{B}}\right| \le \|f_k - f\|^\gamma_{\mathcal{B}} \to 0 \text{ as } k \to \infty.$$

This finishes the proof of Lemma 3.9. $\square$

Proof of Theorem 2.12. We first assume that $f \in H^\varphi \cap L^2$. By the well known Calderón reproducing formula (see also [22, Theorem 2.14]), we know that there exist complex numbers $\{\lambda_j\}_{j\in\mathbb{Z}_+}$ and $(\varphi,\, q,\, N)$-atoms $\{a_j\}_{j\in\mathbb{Z}_+}$ associated with balls $\{B_j\}_{j\in\mathbb{Z}_+}$ such that

$$f = \lim_{k\to\infty} \sum_{j=1}^{k} \lambda_j a_j =: \lim_{k\to\infty} f_k \text{ in } \mathcal{S}' \text{ and also in } L^2. \tag{3.1}$$

From Lemma 3.3, the assumption that the linear or positive sublinear operator T is bounded on L^2, and (3.1), it follows that

$$\lim_{k\to\infty} \|T(f) - T(f_k)\|_{L^2} \le \lim_{k\to\infty} \|T(f-f_k)\|_{L^2} \lesssim \lim_{k\to\infty} \|f - f_k\|_{L^2} = 0,$$

which implies that

$$T(f) = \lim_{k\to\infty} T(f_k) \le \lim_{k\to\infty} \sum_{j=1}^{k} T(\lambda_j a_j) = \sum_{j=1}^{\infty} T(\lambda_j a_j) \text{ almost everywhere.} \tag{3.2}$$

By this, Lemma 3.4 and (2.5) with taking $\lambda = \Lambda_q(\{\lambda_j a_j\})$, we obtain that, for any $m \in [m(\varphi),\, \infty) \cap \mathbb{N}$ and $\alpha \in (0,\, \infty)$,

$$\begin{aligned}
\varphi\left(\{(T(f))^*_m > \alpha\},\, \frac{\alpha}{\Lambda_q(\{\lambda_j a_j\}_j)}\right) &\le \varphi\left(\left\{\sum_{j=1}^{\infty} (T(\lambda_j a_j))^*_m > \alpha\right\},\, \frac{\alpha}{\Lambda_q(\{\lambda_j a_j\}_j)}\right) \\
&\lesssim \sum_{j=1}^{\infty} \sup_{\alpha\in(0,\,\infty)} \varphi\left(\left\{(T(\lambda_j a_j))^*_m > \alpha\right\},\, \frac{\alpha}{\Lambda_q(\{\lambda_j a_j\}_j)}\right) \\
&\lesssim \sum_{j=1}^{\infty} \varphi\left(B_j,\, \frac{\|\lambda_j a_j\|_{L^q_\varphi(B_j)}}{\Lambda_q(\{\lambda_j a_j\}_j)}\right) \lesssim 1,
\end{aligned}$$

which, together with Lemma 3.5, further implies that

$$\|(T(f))^*_m\|_{WL^\varphi} \lesssim \Lambda_q(\{\lambda_j a_j\}_j).$$

Taking infimum for all admissible decompositions of f as above and using Lemma 3.7, we obtain that, for any $f \in H^\varphi \cap L^2$,

$$\|T(f)\|_{WH^\varphi} = \|\,(T(f))^*_m\,\|_{WL^\varphi} \lesssim \|f\|_{H^{\varphi,\,q,\,N}_{\rm at}} \sim \|f\|_{H^\varphi}. \tag{3.3}$$

Generally, suppose $f \in H^\varphi$. By Lemma 3.8, we know that there exists a sequence $\{f_j\}_{j\in\mathbb{Z}_+} \subset H^\varphi \cap L^2$ such that $f_j \to f$ as $j \to \infty$ in H^φ. Therefore, $\{f_j\}_{j\in\mathbb{Z}_+}$ is a Cauchy sequence in H^φ. From this, Lemma 3.3 and (3.3), we conclude that, for any $j, k \in \mathbb{Z}_+$,

$$\|T(f_j) - T(f_k)\|_{WH^\varphi} \le \|T(f_j - f_k)\|_{WH^\varphi} \lesssim \|f_j - f_k\|_{H^\varphi}\,.$$

Thus, $\{T(f_j)\}_{j\in\mathbb{Z}_+}$ is also a Cauchy sequence in WH^φ. According to Theorem 2.11, we conclude that there exists some $g \in WH^\varphi$ such that $T(f_j) \to g$ as $j \to \infty$ in WH^φ. Consequently, define $T(f) := g$. Below, we claim that $T(f)$ is well defined. Indeed, for any other sequence $\{f'_j\}_{j\in\mathbb{Z}_+} \subset H^\varphi \cap L^2$ satisfying $f'_j \to f$ as $j \to \infty$ in H^φ, by Lemma 3.3 and (3.3), we have

$$\begin{aligned}\left\|T(f'_j) - T(f)\right\|_{WH^\varphi} &\lesssim \left\|T(f'_j) - T(f_j)\right\|_{WH^\varphi} + \|T(f_j) - g\|_{WH^\varphi}\\ &\lesssim \left\|f'_j - f_j\right\|_{H^\varphi} + \|T(f_j) - g\|_{WH^\varphi}\\ &\lesssim \left\|f'_j - f\right\|_{H^\varphi} + \|f - f_j\|_{H^\varphi} + \|T(f_j) - g\|_{WH^\varphi} \to 0 \text{ as } j \to \infty,\end{aligned}$$

which is wished. From this, Lemma 3.9 and (3.3), it follows that

$$\|T(f)\|_{WH^\varphi} = \|g\|_{WH^\varphi} = \lim_{j\to\infty} \|T(f_j)\|_{WH^\varphi} \lesssim \lim_{j\to\infty} \|f_j\|_{H^\varphi} \sim \|f\|_{H^\varphi}.$$

This completes the proof of Theorem 2.12. □

We now recall the Musielak-Orlicz molecular Hardy space [21, Definition 2.8] as follows.

Definition 3.10. For an admissible quadruple $(\varphi, q, N, \varepsilon)$, the *Musielak-Orlicz molecular Hardy space* $H^{\varphi,q,N,\varepsilon}_{\mathrm{mol}}$ is defined as the set of all $f \in \mathcal{S}'$ which can be represented as a linear combination of $(\varphi, q, N, \varepsilon)$-molecules, that is, $f = \sum_j M_j$ in $\mathcal{S}'$, where M_j for each j is a multiple of some $(\varphi, q, N, \varepsilon)$-molecule associated with some ball B_j, with the property

$$\sum_j \varphi\left(B_j, \|M_j\|_{L^q_\varphi(B_j)}\right) < \infty.$$

Define

$$\widetilde{\Lambda_q}\left(\{M_j\}_j\right) := \inf\left\{\lambda \in (0, \infty) : \sum_j \varphi\left(B_j, \frac{\|M_j\|_{L^q_\varphi(B_j)}}{\lambda}\right) \le 1\right\}$$

and

$$\|f\|_{H^{\varphi,q,N,\varepsilon}_{\mathrm{mol}}} := \inf\left\{\widetilde{\Lambda_q}\left(\{M_j\}_j\right)\right\},$$

where the infimum is taken over all admissible decompositions of f as above.

Lemma 3.11. ([21, Theorem 2.10]) *Let $(\varphi, q, N, \varepsilon)$ be an admissible quadruple as in Definition 2.9. Then*

$$H^\varphi = H^{\varphi,q,N,\varepsilon}_{\mathrm{mol}}$$

with equivalent (quasi-)norms.

Proof of Theorem 2.13. Since the proof of Theorem 2.13 is similar to that of Theorem 2.12, we use the same notation as in the proof of Theorem 2.12. Here we just point out the necessary modifications.

We first assume that $f \in H^\varphi \cap L^2$. By Lemma 3.11, (3.2) and the assumption that $T(a_j)$ for each j is a harmless constant multiple of a $(\varphi, q, N, \varepsilon)$-molecule, we obtain that

$$\|T(f)\|_{H^\varphi} \sim \|T(f)\|_{H^{\varphi,q,N,\varepsilon}_{\mathrm{mol}}}$$

$$
\begin{aligned}
&\lesssim \max\left\{\left\|\sum_{j=1}^{\infty}|\lambda_j|T(a_j)\right\|_{H^{\varphi,\,q,\,N,\,\varepsilon}_{\mathrm{mol}}},\ \left\|\sum_{j=1}^{\infty}\lambda_j T(a_j)\right\|_{H^{\varphi,\,q,\,N,\,\varepsilon}_{\mathrm{mol}}}\right\}\\
&\lesssim \max\left\{\widetilde{\Lambda_q}\left(\{|\lambda_j|T(a_j)\}_j\right),\ \widetilde{\Lambda_q}\left(\{\lambda_j T(a_j)\}_j\right)\right\}\\
&\sim \max\left\{\Lambda_q\left(\{|\lambda_j|a_j\}_j\right),\ \Lambda_q\left(\{\lambda_j a_j\}_j\right)\right\}\\
&\sim \Lambda_q\left(\{\lambda_j a_j\}_j\right).
\end{aligned}
$$

By taking the infimum over all admissible decompositions of f as above on the both sides of the above inequality and using Lemma 3.7, we conclude that, for any $f \in H^\varphi \cap L^2$,

$$\|T(f)\|_{H^\varphi} \lesssim \|f\|_{H^{\varphi,\,q,\,N}_{at}} \sim \|f\|_{H^\varphi}.$$

Noticing that H^φ is a complete (quasi-)normed linear space (see [15, Proposition 5.2]), then the remainder of the argument is analogous to that in the proof of Theorem 2.12 and is left to the reader. This finishes the proof of Theorem 2.13. $\square$

4 Applications

In this section, as applications of our main results, we obtain the boundedness of Bochner-Riesz means from H^φ to WH^φ or from H^φ to itself.

We first recall the notion of Bochner-Riesz means. Let $\delta \in (0,\infty)$. The *Bochner-Riesz means of order* δ is defined initially for Schwartz functions f on $\mathbb{R}^n$ by setting, for any $x \in \mathbb{R}^n$,

$$T_R^\delta(f)(x) := \int_{\mathbb{R}^n} \widehat{f}(\xi)\left(1-\frac{|\xi|^2}{R^2}\right)_+^\delta e^{2\pi i x\cdot\xi}\,d\xi,\ R \in (0,\infty),$$

where $\widehat{f}$ denotes the Fourier transform of f. The Bochner-Riesz means can be also expressed as a convolution operator

$$T_R^\delta(f)(x) = (f * \phi_{1/R})(x),$$

where, for any $x \in \mathbb{R}^n$ and $\varepsilon \in (0,\infty)$, $\phi(x) := \{(1-|\cdot|^2)_+^\delta\}^\wedge(x)$ and $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(x/\varepsilon)$. The corresponding *maximal Bochner-Riesz means of order* δ is defined by setting, for any $x \in \mathbb{R}^n$,

$$T_*^\delta(f)(x) := \sup_{R\in(0,\infty)} T_R^\delta(f)(x).$$

The main results of this section are following two theorems.

Theorem 4.1. *Let φ be a growth function as in Definition 2.2 with $p \in (0,1)$, $I(\varphi) \in (0,1)$ as in* (2.2)*, and $\delta := n/p - (n+1)/2$. If $\varphi \in \mathbb{A}_1$ and $n(1/p-1) \notin \mathbb{N}$, then there exists a positive constant C independent of f such that*

$$\left\|T_R^\delta(f)\right\|_{WH^\varphi} \le C\|f\|_{H^\varphi}.$$

Theorem 4.2. *Let φ be a growth function as in Definition 2.2 and $\delta > \max\{N + (n-1)/2,\, n/p - (n+1)/2\}$, where $N := \lfloor n(1/p - 1)\rfloor$. If $\varphi \in \mathbb{A}_1$, then there exists a positive constant C independent of f such that*

$$\left\|T_R^\delta(f)\right\|_{H^\varphi} \le C\|f\|_{H^\varphi}.$$

Remark 4.3. Let ω be a classical Muckenhoupt weight and Φ an Orlicz function.

(i) When $\varphi(x,\, t) := \omega(x)\, t^p$ for all $(x,\, t) \in \mathbb{R}^n \times [0,\, \infty)$, we have $H^\varphi = H^p_\omega$. In this case, Theorems 4.1 and 4.2 are reduced to [36, Theorem 1.4] and [17, Theorem 2], respectively.

(ii) When $\varphi(x,\, t) := \omega(x)\Phi(t)$ for all $(x,\, t) \in \mathbb{R}^n \times [0,\, \infty)$, we have $H^\varphi = H^\Phi_\omega$. In this case, Theorems 4.1 and 4.2 hold true for weighted Orlicz Hardy space. Even when $\omega \equiv 1$, the above results are also new.

To prove Theorems 4.1 and 4.2, we need the following several lemmas.

Lemma 4.4. ([15, Lemma 4.5]) *Let $\varphi \in \mathbb{A}_q$ with $q \in [1,\, \infty)$. Then there exists a positive constant C such that, for any ball $B \subset \mathbb{R}^n$, $\lambda \in (1,\, \infty)$ and $t \in (0,\, \infty)$,*

$$\varphi(\lambda B,\, t) \le C\lambda^{nq}\varphi(B,\, t).$$

Lemma 4.5. ([32]) *Let $p_1 \in (0,\, 1)$, $\delta := n/p_1 - (n+1)/2$ and $\alpha \in \mathbb{N}^n$. Then there exists a positive constant $C := C_{n,\, p_1,\, \alpha}$ such that the kernel ϕ of Bochner-Riesz means of order δ satisfies the inequality*

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^{n/p_1}|\partial^\alpha \phi(x)| \le C.$$

Lemma 4.6. *Let φ be a growth function as in Definition 2.2 with $p \in (0,\, 1)$, and $\delta := n/p - (n+1)/2$. Suppose $b(\cdot)$ is a multiple of a $(\varphi,\, \infty,\, \lfloor n(q(\varphi)/p - 1)\rfloor)$-atom associated with some ball $B(x_0,\, r)$, where $q(\varphi)$ is as in (2.3). Then there exists a positive constant C independent of b such that, for any $x \in \mathbb{R}^n$,*

$$T_*^\delta(b)(x) \le C\|b\|_{L^\infty}\left(\frac{r}{r + |x - x_0|}\right)^{n/p}. \tag{4.1}$$

Proof. We show this lemma by borrowing some ideas from the proof of [17, Lemma 2]. It suffices to show (4.1) holds for $x_0 = \mathbf{0}$ and $r = 1$. Indeed, for any multiple of a $(\varphi,\, \infty,\, \lfloor n(q(\varphi)/p - 1)\rfloor)$-atom b associated with some ball $B(x_0,\, r)$, it is easy to see that

$$b_1(\cdot) := \|\chi_{B_1}\|^{-1}_{L^\varphi}\|b\|^{-1}_{L^\infty} b(x_0 + r\cdot)$$

is a $(\varphi,\, \infty,\, \lfloor n(q(\varphi)/p - 1)\rfloor)$-atom associated with the ball $B(\mathbf{0}, 1)$. For any $x \in \mathbb{R}^n$, we have

$$(b * \phi_\varepsilon)(x) = \varepsilon^{-n}\int_{\mathbb{R}^n} b(x - y)\phi\left(\frac{y}{\varepsilon}\right)\, dy$$

$$
\begin{aligned}
&= \|b\|_{L^\infty} \|\chi_{B_1}\|_{L^\varphi} \varepsilon^{-n} \int_{\mathbb{R}^n} b_1 \left(\frac{x - x_0}{r} - \frac{y}{r} \right) \phi\left(\frac{y}{\varepsilon}\right) dy \\
&= \|b\|_{L^\infty} \|\chi_{B_1}\|_{L^\varphi} (b_1 * \phi_{\varepsilon/r}) \left(\frac{x - x_0}{r} \right),
\end{aligned}
$$

which implies that

$$
T_*^\delta(b)(x) \le \|b\|_{L^\infty} \|\chi_{B_1}\|_{L^\varphi} T_*^\delta(b_1) \left(\frac{x - x_0}{r} \right).
$$

If we assume (4.1) holds for $x_0 = \mathbf{0}$ and $r = 1$, then, for any $x \in \mathbb{R}^n$,

$$
T_*^\delta(b)(x) \lesssim \|b\|_{L^\infty} \|\chi_{B_1}\|_{L^\varphi} \|b_1\|_{L^\infty} \left(\frac{1}{1 + |\frac{x - x_0}{r}|} \right)^{n/p} \lesssim \|b\|_{L^\infty} \left(\frac{r}{r + |x - x_0|} \right)^{n/p}.
$$

It remains to prove (4.1) holds for $x_0 = \mathbf{0}$ and $r = 1$. Let b be a multiple of a $(\varphi,\, \infty,\, \lfloor n(q(\varphi)/p - 1) \rfloor)$-atom associated with the ball $B(\mathbf{0}, 1)$. From Lemma 4.5 and $p \in (0,\, 1)$, we deduce that, for any $x \in B(\mathbf{0}, 2)$,

$$
\begin{aligned}
T_*^\delta(b)(x) &= \sup_{1/\varepsilon \in (0,\, \infty)} |(b * \phi_\varepsilon)(x)| \le \|b\|_{L^\infty} \int_{\mathbb{R}^n} |\phi(y)|\, dy \\
&\le \|b\|_{L^\infty} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|)^{n/p}}\, dy \sim \|b\|_{L^\infty} \left(\frac{1}{1 + 2} \right)^{n/p} \lesssim \|b\|_{L^\infty} \left(\frac{1}{1 + |x|} \right)^{n/p},
\end{aligned}
$$

which is wished.

By repeating the estimate of (2) in the proof of [17, Lemma 2], we know that, for any $x \in [B(\mathbf{0}, 2)]^\complement$ and $\varepsilon \in (0,\, \infty)$,

$$
|(b * \phi_\varepsilon)(x)| \lesssim \|b\|_{L^\infty} |x|^{-n/p}.
$$

From this and the inequality $|x| \sim |x| + 1$ with $x \in [B(\mathbf{0}, 2)]^\complement$, it follows that, for any $x \in [B(\mathbf{0}, 2)]^\complement$,

$$
T_*^\delta(b)(x) = \sup_{1/\varepsilon \in (0,\, \infty)} |(b * \phi_\varepsilon)(x)| \lesssim \|b\|_{L^\infty} \left(\frac{1}{1 + |x|} \right)^{n/p},
$$

which is also wished. This finishes the proof of Lemma 4.6. □

Lemma 4.7. *Let φ be a growth function as in Definition 2.2 with $p \in (0,\, 1)$, and $\delta := n/p - (n+1)/2$. Suppose $b(\cdot)$ is a multiple of a $(\varphi,\, \infty,\, N)$-atom associated with some ball $B := B(x_0,\, r) \subset \mathbb{R}^n$, where $N := \lfloor n(1/p - 1) \rfloor$ satisfying $n(1/p - 1) \notin \mathbb{N}$. If $\varphi \in \mathbb{A}_1$ and integer $m \ge N$ satisfying $(m+2)(n+1) \ge n + N + 1$. then there exists a positive constant C independent of b such that, for any $x \in (4B)^\complement$,*

$$
\left(T_R^\delta(b) \right)_m^* (x) \le C \|b\|_{L^\infty} \left(\frac{r}{|x - x_0|} \right)^{n/p}. \tag{4.2}
$$

Proof. To prove Lemma 4.7, we borrow some ideas from the proof of [36, Lemma 4.3]. We claim that, for any $x \in (4B)^{\complement}$,

$$\left|\psi_t * T_R^\delta(b)(x)\right| \lesssim \|b\|_{L^\infty} \left(\frac{r}{|x-x_0|}\right)^{n/p}, \tag{4.3}$$

where $\psi \in \mathcal{S}_m$ and $t \in (0,\infty)$. Assuming the claim for the moment, it's easy to obtain that (4.2) holds true by using (4.3). So, to end the proof, it remains to verify (4.3).

Firstly, by [36, (4.6)], we know that, for any $\gamma \in \mathbb{N}^n$ with $|\gamma| \le N$,

$$\int_{\mathbb{R}^n} T_R^\delta(b)(y) y^\gamma \, dy = 0. \tag{4.4}$$

By this, we know that, for any $x \in (4B)^{\complement}$ and $\gamma \in \mathbb{N}^n$ with $|\gamma| \le N$,

$$\begin{aligned}
\left|\psi_t * T_R^\delta(b)(x)\right| &= \left|\int_{\mathbb{R}^n} t^{-n}\left[\psi\left(\frac{x-y}{t}\right) - \sum_{|\gamma|\le N} \frac{\partial^\gamma \psi\left(\frac{x-x_0}{t}\right)}{\gamma!}\left(\frac{x_0-y}{t}\right)^\gamma\right] T_R^\delta(b)(y)dy\right| \\
&\le t^{-n}\int_{|y-x_0|<r}\left|\psi\left(\frac{x-y}{t}\right) - \sum_{|\gamma|\le N} \frac{\partial^\gamma \psi\left(\frac{x-x_0}{t}\right)}{\gamma!}\left(\frac{x_0-y}{t}\right)^\gamma\right|\left|T_R^\delta(b)(y)\right| dy \\
&\quad + t^{-n}\int_{r\le|y-x_0|\le|x-x_0|/2}\cdots + t^{-n}\int_{|y-x_0|>|x-x_0|/2}\cdots =: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}$$

For I_1, noticing that $x \in (4B)^{\complement}$ and $|y-x_0| < r$, we have

$$|x - x_0 - \theta(y-x_0)| > |x-x_0|/2. \tag{4.5}$$

From Taylor's theorem, $T_R^\delta(b) \le T_*^\delta(b)$, $\psi \in \mathcal{S}_m$ with integer $m \ge N$, $(m+2)(n+1) \ge n+N+1$, (4.5), Lemma 4.6 and $N+1 \ge n(1/p-1)$, we deduce that, for any $x \in (4B)^{\complement}$,

$$\begin{aligned}
\mathrm{I}_1 &\le t^{-n}\left(\frac{r}{t}\right)^{N+1}\int_{|y-x_0|<r}\sum_{|\gamma|=N+1}\left|\frac{\partial^\gamma\psi\left(\frac{x-x_0-\theta(y-x_0)}{t}\right)}{\gamma!}\right|\left|T_*^\delta(b)(y)\right| dy \\
&\lesssim \frac{r^{N+1}}{t^{n+N+1}}\int_{|y-x_0|<r}\left|\frac{x-x_0-\theta(y-x_0)}{t}\right|^{-n-N-1}\left|T_*^\delta(b)(y)\right| dy \\
&\lesssim r^{N+1}\int_{|y-x_0|<r}|x-x_0|^{-n-N-1}\left|T_*^\delta(b)(y)\right| dy \\
&\lesssim \frac{r^{N+1}}{|x-x_0|^{n+N+1}}\int_{|y-x_0|<r}\|b\|_{L^\infty}\left(\frac{r}{r+|y-x_0|}\right)^{n/p} dy \\
&\lesssim \|b\|_{L^\infty}\frac{r^{N+1}}{|x-x_0|^{n+N+1}}\int_B dy \\
&\lesssim \|b\|_{L^\infty}\left(\frac{r}{|x-x_0|}\right)^{n+N+1} \lesssim \|b\|_{L^\infty}\left(\frac{r}{|x-x_0|}\right)^{n/p},
\end{aligned}$$

which is wished.

For I_2, by Taylor's theorem, $T_R^\delta(b) \le T_*^\delta(b)$, $\psi \in \mathcal{S}_m$ with integer $m \ge N$, $(m+2)(n+1) \ge n+N+1$, Lemma 4.6, the spherical coordinates transform and $-1 < n+N-n/p$, we know that, for any $x \in (4B)^\complement$,

$$\begin{aligned}
\mathrm{I}_2 &\le t^{-n} \int_{r\le|y-x_0|\le|x-x_0|/2} \sum_{|\gamma|=N+1} \left|\frac{\partial^\gamma \psi\left(\frac{x-x_0-\theta(y-x_0)}{t}\right)}{\gamma!}\right| \left|\frac{y-x_0}{t}\right|^{N+1} \left|T_*^\delta(b)(y)\right| dy \\
&\lesssim t^{-n} \int_{r\le|y-x_0|\le|x-x_0|/2} \left|\frac{x-x_0-\theta(y-x_0)}{t}\right|^{-n-N-1} \left|\frac{y-x_0}{t}\right|^{N+1} \left|T_*^\delta(b)(y)\right| dy \\
&\lesssim t^{-n} \int_{r\le|y-x_0|\le|x-x_0|/2} \left|\frac{x-x_0}{t}\right|^{-n-N-1} \left|\frac{y-x_0}{t}\right|^{N+1} \|b\|_{L^\infty} \left(\frac{r}{r+|y-x_0|}\right)^{n/p} dy \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n/p}}{|x-x_0|^{n+N+1}} \int_{r\le|y-x_0|\le|x-x_0|/2} |y-x_0|^{N+1-n/p}\, dy \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n/p}}{|x-x_0|^{n+N+1}} \int_{S^{n-1}} \int_0^{|x-x_0|/2} \rho^{N+1-n/p} \rho^{n-1} d\rho\, d\sigma(y') \\
&\lesssim \|b\|_{L^\infty} \frac{r^{n/p}}{|x-x_0|^{n+N+1}} |x-x_0|^{N+n+1-n/p} \lesssim \|b\|_{L^\infty} \left(\frac{r}{|x-x_0|}\right)^{n/p},
\end{aligned}$$

which is also wished.

For I_3, noticing that $n/p-n > N$, we see that, for any $j \in [0,\, N] \cap \mathbb{N}$,

$$n/p - n - j > 0. \tag{4.6}$$

From $T_R^\delta(b) \le T_*^\delta(b)$, $\psi \in \mathcal{S}_m$ with integer $m \ge N$, $(m+2)(n+1) \ge n+N+1$, Lemma 4.6, the spherical coordinates transform and (4.6), it follows that, for any $x \in (4B)^\complement$,

$$\begin{aligned}
\mathrm{I}_3 &\le t^{-n} \int_{|y-x_0|>|x-x_0|/2} \left(\left|\psi\left(\frac{x-y}{t}\right)\right| + \sum_{j=0}^N \sum_{|\gamma|=j} \left|\frac{\partial^\gamma \psi\left(\frac{x-x_0}{t}\right)}{\gamma!}\right| \left|\frac{y-x_0}{t}\right|^j\right) \left|T_*^\delta(b)(y)\right| dy \\
&\lesssim \int_{|y-x_0|>|x-x_0|/2} \left(|\psi_t\,(x-y)\,| + \; t^{-n} \sum_{j=0}^N \left|\frac{x-x_0}{t}\right|^{-n-j} \left|\frac{y-x_0}{t}\right|^j\right) \left|T_*^\delta(b)(y)\right| dy \\
&\lesssim \|b\|_{L^\infty} \int_{|y-x_0|>|x-x_0|/2} |\psi_t(x-y)| \left(\frac{r}{|y-x_0|}\right)^{n/p} dy \\
&\quad + \|b\|_{L^\infty} \int_{|y-x_0|>|x-x_0|/2} \sum_{j=0}^N \frac{1}{|x-x_0|^{n+j}} \frac{r^{n/p}}{|y-x_0|^{n/p-j}} dy \\
&\lesssim \|\psi\|_{L^1} \|b\|_{L^\infty} \left(\frac{r}{|x-x_0|}\right)^{n/p} + \|b\|_{L^\infty} \sum_{j=0}^N \frac{r^{n/p}}{|x-x_0|^{n+j}} \int_{|y-x_0|>|x-x_0|/2} \frac{1}{|y-x_0|^{n/p-j}} dy \\
&\sim \|b\|_{L^\infty} \left(\frac{r}{|x-x_0|}\right)^{n/p} + \|b\|_{L^\infty} \sum_{j=0}^N \frac{r^{n/p}}{|x-x_0|^{n+j}} \int_{S^{n-1}} \int_{|x-x_0|/2}^\infty \frac{1}{\rho^{n/p-j}} \rho^{n-1} d\rho\, d\sigma(y')
\end{aligned}$$

$$\sim \|b\|_{L^\infty}\left(\frac{r}{|x-x_0|}\right)^{n/p}+\|b\|_{L^\infty}\sum_{j=0}^{N}\frac{r^{n/p}}{|x-x_0|^{n+j}}\frac{1}{|x-x_0|^{n/p-n-j}}$$
$$\sim \|b\|_{L^\infty}\left(\frac{r}{|x-x_0|}\right)^{n/p}.$$

Finally, combining the estimates of I_1, I_2 and I_3, we obtain that (4.3) holds true. This finishes the proof of Lemma 4.7. □

Lemma 4.8. *Let φ be a growth function as in Definition 2.2 with $p\in(0,1)$, and $\delta:=n/p-(n+1)/2$. Suppose $b(\cdot)$ is a multiple of a (φ,∞,N)-atom associated with some ball $B:=B(x_0,r)\subset\mathbb{R}^n$, where $N:=\lfloor n(1/p-1)\rfloor$. If $\varphi\in\mathbb{A}_1$ and $n(1/p-1)\notin\mathbb{N}$, then there exists a positive constant C independent of b such that, for any $\lambda\in(0,\infty)$,*

$$\sup_{\alpha\in(0,\infty)}\varphi\left(\left\{\left(T_R^\delta(b)\right)_m^*>\alpha\right\},\frac{\alpha}{\lambda}\right)\le C\varphi\left(B,\frac{\|b\|_{L^\infty}}{\lambda}\right).$$

Proof. We show this lemma by borrowing some ideas from the proof of [26, Theorem 5.2]. Write

$$\begin{aligned}\sup_{\alpha\in(0,\infty)}\varphi\left(\left\{\left(T_R^\delta(b)\right)_m^*>\alpha\right\},\frac{\alpha}{\lambda}\right)&\le\sup_{\alpha\in(0,\infty)}\varphi\left(\left\{x\in 4B:\ \left(T_R^\delta(b)\right)_m^*(x)>\alpha\right\},\frac{\alpha}{\lambda}\right)\\&\quad+\sup_{\alpha\in(0,\infty)}\varphi\left(\left\{x\in (4B)^\complement:\ \left(T_R^\delta(b)\right)_m^*(x)>\alpha\right\},\frac{\alpha}{\lambda}\right)\\&=:\mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

To estimate I_1, we claim that

$$\left(T_R^\delta(b)\right)_m^*\lesssim M\left(M(b)\right),\tag{4.7}$$

where M denotes the Hardy-Littlewood maximal operator as usual. Indeed, since $0<p<1$ and $\delta=n/p-(n+1)/2$, then $\delta>(n-1)/2$. In this case, it is well known that

$$T_*^\delta(b)\lesssim M(b)\ \text{(see also [34])}.$$

In addition, it is well known that, for any $g\in L^q$ with $q\in[1,\infty)$, $g_m^*\lesssim M(g)$. Consequently, we infer that

$$\left(T_*^\delta(b)\right)_m^*\lesssim M\left(M(b)\right),$$

which, together with $T_R^\delta(b)\le T_*^\delta(b)$, implies that (4.7) holds. By the uniformly upper type 1 property of φ, (4.7), the boundedness on $L^2(\mathbb{R}^n,\varphi(\cdot,t))$, uniformly in $t\in(0,\infty)$, of the Hardy-Littlewood maximal operator M, and Lemma 4.4 with $\varphi\in\mathbb{A}_1$, we know that

$$\mathrm{I}_1=\sup_{\alpha\in(0,\infty)}\int_{\left\{x\in 4B:\ (T_R^\delta(b))_m^*(x)>\alpha\right\}}\varphi\left(x,\frac{\alpha}{\lambda}\right)dx$$

$$
\begin{aligned}
&\le \int_{4B} \varphi\left(x,\, \frac{\left(T_R^\delta(b)\right)_m^*(x)}{\lambda}\right) dx \\
&\lesssim \int_{4B} \left(1+\frac{\left(T_R^\delta(b)\right)_m^*(x)}{\|b\|_{L^\infty}}\right)^2 \varphi\left(x,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) dx \\
&\lesssim \int_{4B} \varphi\left(x,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) dx + \frac{1}{\|b\|_{L^\infty}^2} \int_{4B} \left[\left(T_R^\delta(b)\right)_m^*(x)\right]^2 \varphi\left(x,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) dx \\
&\lesssim \varphi\left(4B,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) + \frac{1}{\|b\|_{L^\infty}^2} \int_{4B} [M(M(b))(x)]^2 \varphi\left(x,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) dx \\
&\lesssim \varphi\left(4B,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) + \frac{1}{\|b\|_{L^\infty}^2} \int_{B} |b(x)|^2 \varphi\left(x,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) dx \\
&\lesssim \varphi\left(4B,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) + \int_{B} \varphi\left(x,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) dx \\
&\lesssim \varphi\left(B,\, \frac{\|b\|_{L^\infty}}{\lambda}\right),
\end{aligned}
$$

which is wished.

For I_2, from Lemma 4.7, Lemma 4.4 with $\varphi \in \mathbb{A}_1$, and the uniformly lower type p property of φ, we deduce that, for any $\lambda \in (0, \infty)$,

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in (4B)^\complement :\ \|b\|_{L^\infty}\left(\frac{r}{|x-x_0|}\right)^{n/p} > \alpha\right\},\, \frac{\alpha}{\lambda}\right) \\
&\lesssim \sup_{\alpha\in(0,\infty)} \varphi\left(\left\{x \in \mathbb{R}^n :\ r \le |x-x_0| < \left(\frac{\|b\|_{L^\infty}}{\alpha}\right)^{p/n} r\right\},\, \frac{\alpha}{\lambda}\right) \\
&\lesssim \sup_{\alpha\in(0,\|b\|_{L^\infty})} \varphi\left(\left\{x \in \mathbb{R}^n :\ |x-x_0| < \left(\frac{\|b\|_{L^\infty}}{\alpha}\right)^{p/n} r\right\},\, \frac{\alpha}{\lambda}\right) \\
&\sim \sup_{\alpha\in(0,\|b\|_{L^\infty})} \varphi\left(\left[\frac{\|b\|_{L^\infty}}{\alpha}\right]^{p/n} B,\, \frac{\alpha}{\lambda}\right) \\
&\lesssim \sup_{\alpha\in(0,\|b\|_{L^\infty})} \left(\frac{\|b\|_{L^\infty}}{\alpha}\right)^p \varphi\left(B,\, \frac{\alpha}{\lambda}\right) \\
&\lesssim \sup_{\alpha\in(0,\|b\|_{L^\infty})} \left(\frac{\|b\|_{L^\infty}}{\alpha}\right)^p \left(\frac{\alpha}{\|b\|_{L^\infty}}\right)^p \varphi\left(B,\, \frac{\|b\|_{L^\infty}}{\lambda}\right) \\
&\sim \varphi\left(B,\, \frac{\|b\|_{L^\infty}}{\lambda}\right),
\end{aligned}
$$

which is also wished.

Combining the estimates of I_1 and I_2, we obtain the desired inequality. This finishes the proof of Lemma 4.8. $\square$

Proof of Theorem 4.1. It is well known that T_R^δ is a linear operator and is bounded on L^2 (see [8, p. 354]). By Lemma 4.8, applying Theorem 2.12 with $q = \infty$ and $N = \lfloor n(1/p-1)\rfloor$, we know that, T_R^δ extends uniquely to a bounded operator from H^φ to WH^φ. This finishes the proof of Theorem 4.1. $\square$

Lemma 4.9. *Let φ be a growth function as in Definition 2.2 with additional assumption that $\varphi \in \mathbb{A}_1$, and $\delta > \max\{N + (n-1)/2,\, n/p - (n+1)/2\}$, where $N := \lfloor n(1/p-1) \rfloor$. If $a(\cdot)$ is a $(\varphi,\, \infty,\, N)$-atom associated with some ball $B := B(x_0,\, r) \subset \mathbb{R}^n$, then $T_R^\delta(a)$ is a harmless constant multiple of a $(\varphi,\, \infty,\, N,\, \varepsilon)$-molecule.*

Proof. First, we need to verify the size condition of $T_R^\delta(a)$. Let $p_1 := 2n/(n+1+2\delta) < p$. For any $(x,\, t) \in \mathbb{R}^n \times (0,\, \infty)$, set $\varphi_1(x,\, t) := \varphi(x,\, t)t^{p_1 - p}$. Then φ_1 is a Musieelak-Orlicz function of uniformly lower type p_1 and of uniformly upper type $1 + p_1 - p$. It is easy to see that

$$a_1 := \|\chi_B\|_{L^{\varphi_1}}^{-1} \|a\|_{L^\infty}^{-1} a$$

is a $(\varphi_1,\, \infty,\, N)$-atom associated with the ball B. By $T_R^\delta(a) \le T_*^\delta(a)$ and Lemma 4.6, we know that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\left|T_R^\delta(a)(x)\right| \le \left|T_*^\delta(a)(x)\right| &= T_*^\delta\left(\|a\|_{L^\infty} \|\chi_B\|_{L^{\varphi_1}} a_1\right)(x)\\ &\lesssim \|a\|_{L^\infty} \left(\frac{r}{r + |x - x_0|}\right)^{n/p_1} \lesssim \|a\|_{L^\infty},\end{aligned}$$

which, together with (b) of Definition 2.8(ii), implies that

$$\left\|T_R^\delta(a)\right\|_{L^\infty} \lesssim \|\chi_B\|_{L^\varphi}^{-1}. \tag{4.8}$$

The next thing is to check the pointwise estimates of $T_R^\delta(a)$. Let $E_j := (2^{j+1}B) \setminus (2^j B)$ with $j \in \mathbb{N}$ and $\varepsilon := 1/p_1$. By $\delta > \max\{N + (n-1)/2,\, n/p - (n+1)/2\}$, it is easy to check that $(\varphi,\, \infty,\, N,\, \varepsilon)$ is an admissible quadruple. From $T_R^\delta(a) \le T_*^\delta(a)$, the size condition of a and Lemma 4.6, it follows that, for any $x \in E_j$ with $j \in \mathbb{N}$,

$$\left|T_R^\delta(a)(x)\right| \lesssim \|a\|_{L^\infty} \left(\frac{r}{r + |x - x_0|}\right)^{n/p_1} \lesssim \|\chi_B\|_{L^\varphi}^{-1} \left(\frac{1}{2^j}\right)^{n/p_1} \sim 2^{-nj\varepsilon} \|\chi_B\|_{L^\varphi}^{-1}. \tag{4.9}$$

Finally, by (4.8), (4.9) and the cancellation moment condition of $T_R^\delta(a)$ (which is guaranteed by (4.4)), we know that $T_R^\delta(a)$ is a harmless constant multiple of a $(\varphi,\, \infty,\, N,\, \varepsilon)$-molecule. This finishes the proof of Lemma 4.9. $\square$

Proof of Theorem 4.2. It is well known that T_R^δ is a linear operator and bounded on L^2 (see [8, p. 354]). By Lemma 4.9, applying Theorem 2.13 with $q = \infty$, we know that T_R^δ extends uniquely to a bounded operator from H^φ to H^φ. This finishes the proof of Theorem 4.2. $\square$

5 Conclusions

What we have seen from the above are the completeness of weak Musielak-Orlicz Hardy space and two boundedness criterions for some operators from H^φ to WH^φ or from H^φ to H^φ. As applications, we establish the boundedness of Bochner-Riesz means from H^φ to WH^φ or from H^φ to H^φ, which generalizes the corresponding results under the setting of both of the weighted Hardy space (see, for example, [33]) and the Orlicz Hardy space (see, for example, [10, 12]), and hence has a wide generality.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

Li Bo conceived of the study. Qiu Xiaoli, Li Baode, Liu Xiong and Li Bo carried out the main results, participated in the sequence alignment and drafted the manuscript. Moreover, all authors read and approved the final manuscript.

Acknowledgements

The authors thank the referee for careful reading and checking, and for all the helpful suggestions and comments, which helped to make this paper more readable.

References

[1] Álvarez, Jorge; Milman, Mario, H^p continuity properties of Calderón-Zygmund-type operators, J. Math. Anal. Appl. 118 (1986), no. 1, 63-79.

[2] Bui, The Anh; Cao, Jun; Ky, Luong Dang; Yang, Dachun; Yang, Sibei, Musielak-Orlicz-Hardy spaces associated with operators satisfying reinforced off-diagonal estimates, Anal. Geom. Metr. Spaces 1 (2013), 69-129.

[3] Diening, Lars, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, Bull. Sci. Math. 129 (2005), no. 8, 657-700.

[4] Diening, Lars; Hästö, Peter A.; Roudenko, Svetlana, Function spaces of variable smoothness and integrability, J. Funct. Anal. 256 (2009), no. 6, 1731-1768.

[5] Fan, Xingya; He, Jianxun; Li, Baode; Yang, Dachun, Real-variable characterizations of anisotropic product Musielak-Orlicz Hardy spaces, Sci. China Math. 60 (2017), no. 11, 2093-2154.

[6] Fefferman, Robert A.; Soria, Fernando, The space weak H^1, Studia Math. 85 (1986), no. 1, 1-16 (1987).

[7] Fefferman, Charles Louis; Stein, Elias M., H^p spaces of several variables, Acta Math, 129 (1972), no. 3-4, 137-193.

[8] Grafakos, Loukas, Modern Fourier Analysis, Second edition, Graduate Texts in Mathematics Vol. 250 (Springer, New York, 2009).

[9] Hou, Shaoxiong; Yang, Dachun; Yang, Sibei, Lusin area function and molecular characterizations of Musielak-Orlicz Hardy spaces and their applications, Commun. Contemp. Math. 15 (2013), no. 6, 1350029, 37 pp.

[10] Janson, Svante, Generalizations of Lipschitz spaces and an applications to Hardy spaces and bounded mean oscillation, Duke Math. J. 47 (1980), no. 4, 959-982.

[11] Johnson, Raymond L.; Neugebauer, Christoph Johannes, Homeomorphisms preserving A_p, Rev. Mat. Iberoamericana 3 (1987), no. 2, 249-273.

[12] Jiang, Renjin; Yang, Dachun, New Orlicz-Hardy spaces associated with divergence form elliptic operators, J. Funct. Anal. 258 (2010), no. 4, 1167-1224.

[13] Jiang, Renjin; Yang, Dachun, Predual spaces of Banach completions of Orlicz-Hardy spaces associated with operators, J. Fourier Anal. Appl. 17 (2011), no. 1, 1-35.

[14] Jiang, Renjin; Yang, Dachun, Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates, Commun. Contemp. Math. 13 (2011), no. 2, 331-373.

[15] Ky, Luong Dang, New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators, Integral Equations Operator Theory 78 (2014), no. 1, 115-150.

[16] Latter, Robert H., A characterization of H^p in terms of atoms, Studia Math. 62 (1978), no. 1, 93-101.

[17] Lee, Ming-Yi, Weighted norm inequalities of Bochner-Riesz means, J. Math. Anal. Appl. 324 (2006), no. 2, 1274-1281.

[18] Lu, Shanzhen, Four Lectures on Real H^p Spaces, (World Scientific Publishing Co., Inc., River Edge, NJ, 1995).

[19] Liu Heping, The weak H^p spaces on homogenous groups, in: Harmonic Analysis (Tianjin, 1988), Lecture Notes in Math. Vol. 1984 (Springer, Berlin, 1991), pp. 113-118.

[20] Li, Bo; Liao, Minfeng; Li, Baode, Boundedness of Marcinkiewicz integrals with rough kernels on Musielak-Orlicz Hardy spaces, J. Inequal. Appl. 2017, Paper No. 228, 14 pp.

[21] Li, Baode; Fan, Xingya; Fu, Zunwei; Yang, Dachun, Molecular characterization of anisotropic Musielak-Orlicz Hardy spaces and their applications, Acta Math. Sin. (Engl. Ser.) 32 (2016), no. 11, 1391-1414.

[22] Li, Baode; Fan, Xingya; Yang, Dachun, Littlewood-Paley characterizations of anisotropic Hardy spaces of Musielak-Orlicz type, Taiwanese J. Math. 19 (2015), no. 1, 279-314.

[23] Liang, Yiyu; Huang, Jizheng; Yang, Dachun, New real-variable characterizations of Musielak-Orlicz Hardy spaces, J. Math. Anal. Appl. 395 (2012), no. 1, 413-428.

[24] Liang, Yiyu; Nakai, Eiichi; Yang, Dachun; Zhang, Junqiang, Boundedness of intrinsic Littlewood-Paley functions on Musielak-Orlicz Morrey and Campanato spaces, Banach J. Math. Anal. 8 (2014), no. 1, 221-268.

[25] Liang, Yiyu; Yang, Dachun, Musielak-Orlicz Campanato spaces and applications, J. Math. Anal. Appl. 406 (2013), no. 1, 307-322.

[26] Liang, Yiyu; Yang, Dachun; Jiang, Renjin, Weak Musielak-Orlicz Hardy spaces and applications, Math. Nachr. 289 (2016), no. 5-6, 634-677.

[27] Liu, Jun; Yang, Dachun; Yuan, Wen, Anisotropic Hardy-Lorentz spaces and their applications, Sci. China Math. 59 (2016), no. 9, 1669-1720.

[28] Peloso, Marco M.; Secco, Silvia, Boundedness of Fourier integral operators on Hardy spaces. Proc. Edinb. Math. Soc. (2) 51 (2008), no. 2, 443-463.

[29] Quek, Tongseng; Yang, Dachun, Calderón-Zygmund-type operators on weighted weak Hardy spaces over $\mathbb{R}^n$, Acta Math. Sin. (Engl. Ser.) 16 (2000), no. 1, 141-160.
[30] Rolewicz, Stefan Metric, linear spaces, Second edition. PWNPolish Scientific Publishers, Warsaw; (D. Reidel Publishing Co., Dordrecht, 1984).
[31] Sato, Shuichi, Divergence of the Bochner-Riesz means in the weighted Hardy spaces. Studia Math. 118 (1996), no. 3, 261-275.
[32] Sato, Shuichi, Weak type estimates for some maximal operators on the weighted Hardy spaces, Ark. Mat. 33 (1995), no. 2, 377-384.
[33] Strömberg, Jan-Olov; Torchinsky, Alberto, Weighted Hardy spaces, Lecture Notes in Mathematics Vol. 1381 (Springer-Verlag, Berlin, 1989).
[34] Stein, Elias M.; Weiss, Guido, Introduction to Fourier analysis on Euclidean spaces, Princeton Mathematical Series, No. 32. (Princeton University Press, Princeton, N.J., 1971).
[35] Stein, Elias M.; Weiss, Guido, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, Acta Math. 103 (1960), 25-62.
[36] Wang, Hua, A new estimate for Bochner-Riesz operators at the critical index on weighted Hardy spaces, Anal. Theory Appl. 29 (2013), no. 3, 221-233.
[37] Yang, Dachun; Liang, Yiyu; Ky, Luong Dang, Real-Variable Theory of Musielak-Orlicz Hardy spaces, Lecture Notes in Mathematics Vol. 2182 (Springer-Verlag, Cham, 2017).
[38] Yang, Dachun; Yuan, Wen; Zhuo, Ciqiang, Musielak-Orlicz Besov-type and Triebel-Lizorkin-type spaces, Rev. Mat. Complut. 27 (2014), no. 1, 93-157.
[39] Zhang, Hui; Qi, Chunyan; Li, Baode, Anisotropic weak Hardy spaces of Musielak-Orlicz type and their applications, Front. Math. China 12 (2017), no. 4, 993-1022.

Qiu Xiaoli, Li Baode and Liu Xiong

College of Mathematics and System Sciences
Xinjiang University
Urumqi 830046
P. R. China

E-mail :
2237424863@qq.com (Qiu Xiaoli)
baodeli@xju.edu.cn (Li Baode)
1394758246@qq.com (Liu Xiong)

Li Bo

Center for Applied Mathematics
Tianjin University
Tianjin 300072
P. R. China

E-mail : bli.math@outlook.com