Backward uniqueness for general parabolic operators in the whole space

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Abstract

We prove the backward uniqueness for general parabolic operators of second order in the whole space under assumptions that the leading coefficients of the operator are Lipschitz and their gradients satisfy certain decay conditions. The point is that the decay rate is related to the exponential growth rate of the solution, which is quite different from the case of the half-space [12]. This result extends in some ways a classical result of Lions and Malgrange [14] and a recent result of the authors [12].

Keywords: Carleman estimate; Unique continuation; Backward uniqueness; Parabolic operator.

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1 Introduction

The backward uniqueness (BU) problem for parabolic operators is of interest in many problems, such as the control theory and the regularity theory for PDEs. In [4], Escauriaza et al. proved the critical $L^{\infty,3}_{t,x}$ regularity for 3D Navier-Stokes equations. The main idea of their proof consists in reducing the regularity problem to a BU problem for the heat operator. Their method is then used to deal with the regularity problem for some other equations, for instance, heat flow of harmonic maps [2]. The BU result also plays a crucial role in the blow-up analysis for some semi-linear heat equations [3]. On the other side, this problem is of independent interest as well.

The BU problem for the heat operator is much studied in the past. There are already many results concerning it in various domains, such as the exterior domain [5], the half-space [6] and some cones [8, 9, 10].

For general parabolic operators, the BU problem is related to a conjecture proposed by Landis and Oleinik in 1974, see [11, 13]. The authors [12] proved that BU is valid in

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the half-space under some reasonable assumptions of the leading coefficients. We state this result as follows.

Let P be a backward parabolic operator on $\mathbb{R}^n_+ \times (0,1)$,

$$P = \partial_t + \partial_i\big(a^{ij}(x,t)\partial_j\big) = \partial_t + \nabla \cdot (\mathbf{A}\nabla), \tag{1}$$

where $\mathbf{A}(x,t) = (a^{ij}(x,t))_{i,j=1}^n$ is a real symmetric matrix such that for some $\Lambda \geq \lambda > 0$,

$$\lambda|\xi|^2 \leq a^{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^n. \tag{2}$$

We note that throughout this paper ∇ always means the gradient with respect to the x variables and the Einstein sum convention is used.

Proposition 1.1 ([12]).

1. *Suppose $\{a^{ij}(x,t)\}$ satisfy (2) and*

$$|\nabla a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \leq M, \tag{3}$$

$$|\nabla a^{ij}(x,t)| \leq E|x|^{-1} \tag{4}$$

in $\mathbb{R}^n_+ \times (0,1)$. Assume that u satisfies

$$\begin{cases} |Pu| \leq N(|u| + |\nabla u|) & \text{in } \mathbb{R}^n_+ \times (0,1), \\ |u(x,t)| \leq Ne^{N|x|^2} & \text{in } \mathbb{R}^n_+ \times (0,1), \\ u(x,0) = 0 & \text{in } \mathbb{R}^n_+. \end{cases}$$

Then there exists a constant $E_0 = E_0(n,\Lambda,\lambda)$, such that when $E < E_0$, $u(x,t) \equiv 0$ in $\mathbb{R}^n_+ \times (0,1)$.

2. *Consider the following system*

$$\begin{cases} \partial_t w + \partial_i\big(b^{ij}(x,t)\partial_j w\big) = 0 & \text{in } \mathbb{R}^2_+ \times (0,1), \\ |w(x,t)| \leq N & \text{in } \mathbb{R}^2_+ \times (0,1), \\ w(x,0) = 0 & \text{in } \mathbb{R}^2_+, \end{cases}$$

with $|\nabla b^{ij}| \leq E_1$ and $|\nabla b^{ij}| \leq E_1|x|^{-1}$.
Then, if $E_1 < (\frac{\pi}{2\arccos(1/\sqrt{3})})^2 - 1$, $w(x,t) \equiv 0$; if $E_1 > 3$, this system has a nonzero solution.

However, the case of the whole space is quite different from that of the half-space. In this paper, we study the BU problem for general parabolic operators in the whole space.

Now we let P be a backward parabolic operator on $\mathbb{R}^n \times (0,1)$ having the form (1) with $\{a^{ij}(x,t)\}$ satisfying (2). Let u be a function satisfying

$$|Pu| \leq N(|u| + |\nabla u|)$$

and the growth condition

$$|u(x,t)| \leq Ne^{N|x|^\alpha}$$

for some $\alpha \in [0,2]$, or

$$e^{-N|x|^\alpha}u(x,t) \in L^2\big(\mathbb{R}^n \times (0,1)\big).$$

The BU problem is: suppose

$$u(x,0) = 0, \qquad x \in \mathbb{R}^n,$$

does u vanish identically in $\mathbb{R}^n \times (0,1)$?

Here we set $\alpha \in [0,2]$ because the classical examples of Tychonoff [1] show that BU fails when $\alpha > 2$.

First, by Proposition 1.1, we know immediately that BU in the whole space is valid if $\{a^{ij}(x,t)\}$ satisfy the Lipschitz conditions (3) and the decay at infinity conditions (4) with $E < E_0(n,\Lambda,\lambda)$ in $\mathbb{R}^n \times (0,1)$, since the whole space is the union of two half-spaces and we apply Proposition 1.1 to both of them.

Second, the classical result of Lions and Malgrange [14] showed that BU in the whole space is valid if u lies in the space

$$\mathcal{H} := H^1\big((0,1), L^2(\mathbb{R}^n)\big) \cap L^2\big((0,1), H^2(\mathbb{R}^n)\big)$$

and $\{a^{ij}(x,t)\}$ are Lipschitz.

In this paper we will prove a result which extends the above two results in some ways. We observe that in the case of the whole space there is a link between the decay rate of $|\nabla a^{ij}(x,t)|$ and the exponential growth rate of u, which is quite different from the case of the half-space. We denote $\langle x\rangle = \sqrt{1+|x|^2}$ and

$$\beta(\alpha) = \max\{0, \alpha - 1\}, \qquad \alpha \in [0,2].$$

Our main result is the following.

Theorem 1.2. *Suppose $\{a^{ij}(x,t)\}$ satisfy (2) and*

$$|\nabla a^{ij}(x,t)| + |\partial_t a^{ij}(x,t)| \le M, \qquad |\nabla a^{ij}(x,t)| \le E\langle x\rangle^{-\beta} \tag{5}$$

in $\mathbb{R}^n \times (0,1)$. Assume that u satisfies

$$|Pu| \le N(|u| + |\nabla u|) \tag{6}$$

and

$$|u(x,t)| \le N e^{N|x|^\alpha} \quad \text{or} \quad e^{-N|x|^\alpha} u(x,t) \in L^2\big(\mathbb{R}^n \times (0,1)\big). \tag{7}$$

If, further, $u(x,0) = 0$ in $\mathbb{R}^n$, then u vanishes identically in $\mathbb{R}^n \times (0,1)$.

Remark 1.3.

1. *In the case of the whole space, the decay rate of $|\nabla a^{ij}(x,t)|$ is related to the exponential growth rate of u, i.e. $\beta(\alpha) = \max\{0, \alpha - 1\}$, while in the case of the half-space, the decay rate of $|\nabla a^{ij}(x,t)|$ is always $E|x|^{-1}$, where E is small.*

2. *When $\alpha \in [0,1]$, $\beta = 0$. Theorem 1.2 tells us that the Lipschitz conditions are sufficient for BU and we don't need to assume that $u \in \mathcal{H}$. This extends the result of Lions and Malgrange [14] in some ways.*

3. *When $\alpha = 2$, the smallness of E is required for BU in the half-space. However as for the whole space, we don't require such a condition.*

A favourite method to prove BU is the so-called Carleman estimate, which was first introduced by Carleman [15] to prove the unique continuation for elliptic equations. S-ince then Carleman estimate has caused great concern and been applied to solve many problems. For Carleman estimate for parabolic operators and its applications, one could refer to [16] and references therein for more information.

To prove our result we need the following Carleman estimate.

Proposition 1.4. *Suppose* $\{a^{ij}\}$ *satisfy (2) and (5). For any* $v \in C_0^\infty\big(\mathbb{R}^n \times (0,1)\big)$ *and any* $\gamma > 0$*, we have*

$$\begin{aligned}\int_{\mathbb{R}^n\times(0,1)} & e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}(|v|^2+|\nabla v|^2)dxdt \\ &\le \int_{\mathbb{R}^n\times(0,1)} e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}|Pv|^2dxdt,\end{aligned} \tag{8}$$

where $b = \frac{1}{8\Lambda}$ *and* $K = K(n, \Lambda, \lambda, M, E, \alpha)$.

It is worthwhile to mention [17, 18] and related results, which discuss BU problem when $u \in \mathcal{H}$ and $\{a^{ij}(x,t)\}$ are non-Lipschitz. However, here we just assume that u satisfies (7).

The paper is organized as follows. First we use Carleman inequality (8) to prove Theorem 1.2, then we prove this Carleman inequality.

2 Proof of the main result

In this section, we prove Theorem 1.2. First, we extend u and a^{ij} as follows:

$$\begin{aligned} u(x,t) =& 0, \qquad\qquad if \quad t<0; \\ a^{ij}(x,t) =& a^{ij}(x,0), \qquad if \quad t<0. \end{aligned}$$

The proof of Theorem 1.2 is based on the following lemma.

Lemma 2.1. *Suppose* $\{a^{ij}\}$ *and* u *are the same as those in Theorem 1.2. Then there exists* $T_1 = T_1(\Lambda, N) > 0$*, such that* $u(x,t) \equiv 0$ *in* $\mathbb{R}^n \times (0, T_1)$.

Proof. We use Carleman inequality (8) to prove this lemma, mainly following the arguments of the corresponding parts in [5] and [12]. We just give the proof for the case that $|u(x,t)| \le Ne^{N|x|^\alpha}$, since the proof of the other case is similar.

Without loss of generality we assume that $\alpha \in [1,2]$, for when $\alpha \in [0,1)$, $|u(x,t)| \le Ne^{N|x|^\alpha} \le \widetilde{N}e^{N|x|}$ and $\beta(\alpha) = \beta(1) = 0$.

Step 1. By the regularity theory for solutions of parabolic equations (see, for example, [19]), we have

$$|u(x,t)| + |\nabla u(x,t)| \le C(n, \Lambda, \lambda, M, N)e^{2N|x|^\alpha} \tag{9}$$

when $(x,t) \in \mathbb{R}^n \times (0, \frac{1}{2})$. In the following, we always denote $C = C(n, \Lambda, \lambda, M, N)$. Let

$$\tau = \min\{1, \frac{1}{2N}, \frac{b}{8N}\}. \tag{10}$$

We denote

$$\tilde{u}(x,t) = u\big(\tau x, \tau^2(t-\frac{1}{2})\big)$$

and

$$\tilde{a}^{ij}(x,t) = a^{ij}\big(\tau x, \tau^2(t-\frac{1}{2})\big)$$

for $(x,t) \in \mathbb{R}^n \times (0,1)$. Then it is easy to see that

$$|\nabla \tilde{a}^{ij}(x,t)| + |\partial_t \tilde{a}^{ij}(x,t)| \le \tau M \le M,$$

and

$$|\nabla \tilde{a}^{ij}(x,t)| = \tau |\nabla a^{ij}\big(\tau x, \tau^2(t-\frac{1}{2})\big)| \le \tau E \langle \tau x\rangle^{-\beta} \le E\tau^{1-\beta}\langle x\rangle^{-\beta} \le E\langle x\rangle^{-\beta}.$$

We denote

$$\tilde{P} = \partial_t + \partial_i(\tilde{a}^{ij}\partial_j),$$

then by (6) and (10) we have

$$|\tilde{P}\tilde{u}| \le \tau N(|\tilde{u}| + |\nabla \tilde{u}|) \le \frac{1}{2}(|\tilde{u}| + |\nabla \tilde{u}|). \tag{11}$$

By (9) and (10) we have

$$|\tilde{u}(x,t)| + |\nabla \tilde{u}(x,t)| \le Ce^{2N\tau^\alpha |x|^\alpha} \le Ce^{2N\tau |x|^\alpha} \le Ce^{\frac{b}{4}\langle x\rangle^\alpha} \tag{12}$$

when $(x,t) \in \mathbb{R}^n \times (0,1)$, and

$$\tilde{u}(x,t) = 0 \tag{13}$$

when $(x,t) \in \mathbb{R}^n \times (0, \frac{1}{2}]$.

Step 2. In order to apply Carleman inequality (8), we choose two smooth cut-off functions such that

$$\eta_1(r) = \begin{cases} 1, & \text{if } r < R; \\ 0, & \text{if } r > R+1, \end{cases}$$

where R is large enough, and

$$\eta_2(t) = \begin{cases} 1, & \text{if } t < \frac{3}{4}; \\ 0, & \text{if } t > \frac{7}{8}. \end{cases}$$

Furthermore, $0 \le \eta_1, \eta_2 \le 1$; $|\eta_1'|$, $|\eta_1''|$, $|\eta_2'|$ and $|\eta_2''|$ are all bounded.

Let $\eta(x,t) = \eta_1(|x|)\eta_2(t)$ and $v = \eta\tilde{u}$. Then *supp* $v \subset \mathbb{R}^n \times (0,1)$. By (11) we have

$$\begin{aligned} |\tilde{P}v| =& |\eta \tilde{P}\tilde{u} + \tilde{u}\tilde{P}\eta + 2\tilde{a}^{ij}\partial_i\eta\partial_j\tilde{u}| \\ \le& \frac{1}{2}\eta(|\tilde{u}| + |\nabla\tilde{u}|) + C(|\tilde{u}| + |\nabla \tilde{u}|)(|\partial_t\eta| + |\nabla\eta| + |\nabla^2\eta|) \\ \le& \frac{1}{2}(|v| + |\nabla v|) + C\chi_\Omega(|\tilde{u}| + |\nabla\tilde{u}|), \end{aligned} \tag{14}$$

where χ_Ω is the characteristic function of Ω and

$$\Omega = \{0 < \eta < 1,\ \frac{1}{2} < t < 1\}.$$

Moreover,

$$\begin{aligned}\Omega =&\{0<\eta_1<1,\ \eta_2>0,\ \frac{1}{2}<t<1\}\cup\{\ \eta_1=1,\ 0<\eta_2<1,\ \frac{1}{2}<t<1\}\\ =&\{R<|x|<R+1,\ \frac{1}{2}<t<\frac{7}{8}\}\cup\{|x|<R,\ \frac{3}{4}<t<\frac{7}{8}\}.\end{aligned}$$

Step 3. We apply Carleman inequality (8) for $\tilde{P}$ and v, then

$$\begin{aligned}J\equiv&\int_{\mathbb{R}^n\times(0,1)} e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}(|v|^2+|\nabla v|^2)dxdt\\ \leq&\int_{\mathbb{R}^n\times(0,1)} e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}|\tilde{P}v|^2dxdt.\end{aligned}$$

By (14) we have

$$J\leq\frac{3}{4}J+C\int_\Omega e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}(|\tilde{u}|+|\nabla\tilde{u}|)^2dxdt,$$

thus

$$J\leq C\int_\Omega e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}(|\tilde{u}|+|\nabla\tilde{u}|)^2dxdt.$$

By (12) we obtain

$$\begin{aligned}J\leq&C\int_\Omega e^{2\gamma(t^{-K}-1)-\frac{b}{2}\langle x\rangle^\alpha}dxdt\\ =&C\Big(\int_{\{R<|x|<R+1,\ \frac{1}{2}<t<\frac{7}{8}\}}+\int_{\{|x|<R,\ \frac{3}{4}<t<\frac{7}{8}\}}\Big)e^{2\gamma(t^{-K}-1)-\frac{b}{2}\langle x\rangle^\alpha}dxdt\\ =:&J_1+J_2.\end{aligned}\tag{15}$$

Step 4. Now we estimate both sides of the above inequality.

Estimate of J_1.

$$\begin{aligned}J_1\leq&Ce^{2\gamma(2^K-1)}\int_{\{R<|x|<R+1\}}e^{-\frac{b}{2}\langle x\rangle^\alpha}dx\\ \leq&Ce^{2^{K+1}\gamma-\frac{b}{4}R^\alpha}\int_{\{R<|x|<R+1\}}e^{-\frac{b}{4}\langle x\rangle^\alpha}dx\\ \leq&Ce^{2^{K+1}\gamma-\frac{b}{4}R^\alpha}.\end{aligned}\tag{16}$$

Estimate of J_2.

$$J_2\leq Ce^{2\gamma[(\frac{3}{4})^{-K}-1]}\int_{\{|x|<R\}}e^{-\frac{b}{2}\langle x\rangle^\alpha}dx\leq Ce^{2\gamma[(\frac{3}{4})^{-K}-1]}.\tag{17}$$

Estimate of J. For an arbitrary $l\in(\frac{1}{2},\frac{3}{4})$,

$$\begin{aligned}J\geq&\int_{\{|x|<R,\ \frac{1}{2}<t<l\}} e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}}(|\tilde{u}|^2+|\nabla\tilde{u}|^2)dxdt\\ \geq&e^{2\gamma(l^{-K}-1)}\int_{\{|x|<R,\ \frac{1}{2}<t<l\}} e^{-\frac{b\langle x\rangle^\alpha+K}{t}}(|\tilde{u}|^2+|\nabla\tilde{u}|^2)dxdt.\end{aligned}\tag{18}$$

We combine (15)-(18), then we have

$$\begin{aligned}&\int_{\{|x|<R,\ \frac{1}{2}<t<l\}} e^{-\frac{b\langle x\rangle^{\alpha}+K}{t}}(|\tilde{u}|^2+|\nabla\tilde{u}|^2)dxdt\\ \leq & Ce^{2\gamma(1-l^{-K})}\big(e^{2^{K+1}\gamma-\frac{b}{4}R^{\alpha}}+e^{2\gamma[(\frac{3}{4})^{-K}-1]}\big).\end{aligned}$$

In the above inequality, we fix γ and let $R\to\infty$, then we obtain

$$\int_{\mathbb{R}^n\times(\frac{1}{2},l)} e^{-\frac{b\langle x\rangle^{\alpha}+K}{t}}(|\tilde{u}|^2+|\nabla\tilde{u}|^2)dxdt \leq Ce^{2\gamma[(\frac{3}{4})^{-K}-l^{-K}]}.$$

Now we fix l and let $\gamma\to\infty$, then we have $\tilde{u}(x,t)\equiv 0$ in $\mathbb{R}^n\times(\frac{1}{2},l)$.

Since l is an arbitrary number in $(\frac{1}{2},\frac{3}{4})$, then $\tilde{u}(x,t)\equiv 0$ in $\mathbb{R}^n\times(\frac{1}{2},\frac{3}{4})$. That is, $u(x,t)\equiv 0$ in $\mathbb{R}^n\times(0,\frac{\tau^2}{4})$.

Finally we let

$$T_1=\frac{\tau^2}{4}=\min\{\frac{1}{4},\frac{1}{16N^2},\frac{b^2}{256N^2}\},$$

then $T_1=T_1(\Lambda,N)$ and $u(x,t)\equiv 0$ in $\mathbb{R}^n\times(0,T_1)$.

Thus we proved this lemma. □

Next we give the complete proof of Theorem 1.2. The argument is similar to that in the proof of Lemma 5.4 in [4].

Proof of Theorem 1.2. By Lemma 2.1, we know that $u(x,t)=0$ in $\mathbb{R}^n\times(0,T_1)$. Now we define

$$\begin{aligned}u^{(1)}(x,t)=&u(\sqrt{1-T_1}\ x,(1-T_1)t+T_1),\\ a^{ij(1)}(x,t)=&a^{ij}(\sqrt{1-T_1}\ x,(1-T_1)t+T_1),\\ P^{(1)}=&\partial_t+\partial_i(a^{ij(1)}\partial_j).\end{aligned}$$

for $(x,t)\in\mathbb{R}^n\times(0,1)$. It is easy to verify that $\{a^{ij(1)}\}$ satisfy (2) and (5), $u^{(1)}$ satisfies

$$|P^{(1)}u^{(1)}|\leq N(|u^{(1)}|+|\nabla u^{(1)}|)$$

and (7). Using again Lemma 2.1, we have $u^{(1)}(x,t)=0$ in $\mathbb{R}^n\times(0,T_1)$, i.e, $u(x,t)=0$ in $\mathbb{R}^n\times(0,T_2)$, where

$$T_2=(1-T_1)T_1+T_1.$$

Next we define

$$u^{(2)}(x,t)=u(\sqrt{1-T_2}\ x,(1-T_2)t+T_2)$$

....

After iterating k steps, we obtain that $u(x,t)=0$ in $\mathbb{R}^n\times(0,T_{k+1})$, where

$$T_{k+1}=(1-T_k)T_1+T_k.$$

It is easy to see that $T_k\to 1$ as $k\to\infty$. This proves Theorem 1.2. □

3 Proof of the Carleman inequality

In this section, we prove Carleman inequality (8). We need two lemmas in our proof. The first one is due to Escauriaza and Fernández [7, Lemma 1], however we make some changes here, see also [11, Lemma 4.5] and [13, Corollary 3.2].

In the following, we write

$$\tilde{\Delta} = \partial_i(a^{ij}\partial_j)$$

and denote by $\cdot$ the inner product on $\mathbb{R}^n$.

Lemma 3.1. *Suppose F is differentiable, F_0 and G are twice differentiable and $G > 0$. Then the following identity holds for any $v \in C_0^\infty\big(\mathbb{R}^n \times [0,T]\big)$:*

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^n\times[0,T]} M_0 v^2 G dxdt + \int_{\mathbb{R}^n\times[0,T]} \Big(2\mathbf{D}_G + (\frac{\partial_t G - \tilde{\Delta}G}{G} - F)\mathbf{A}\Big)\nabla v\cdot\nabla v G dxdt \\ &- \int_{\mathbb{R}^n\times[0,T]} v\mathbf{A}\nabla v\cdot\nabla(F-F_0) G dxdt = 2\int_{\mathbb{R}^n\times[0,T]} Lv(Pv - Lv) G dxdt \\ &\qquad + \int_{\mathbb{R}^n} \mathbf{A}\nabla v\cdot\nabla v G dx\Big|_0^T + \frac{1}{2}\int_{\mathbb{R}^n} v^2 F G dx\Big|_0^T,\end{aligned} \tag{19}$$

where

$$Lv = \partial_t v - \mathbf{A}\nabla v\cdot\nabla\log G + \frac{F}{2}v,$$

$$M_0 = \partial_t F + F(\frac{\partial_t G - \tilde{\Delta}G}{G} - F) + \tilde{\Delta}F_0 - \mathbf{A}\nabla(F-F_0)\cdot\nabla\log G,$$

and

$$\mathbf{D}_G^{ij} = a^{ik}\partial_{kl}(\log G)a^{lj} + \frac{\partial_l(\log G)}{2}(a^{ik}\partial_k a^{lj} + a^{jk}\partial_k a^{li} - a^{kl}\partial_k a^{ij}) + \frac{1}{2}\partial_t a^{ij}.$$

Proof. We prove this identity by the method of integration by parts. For simplicity of notations, during this proof we denote

$$Q = \mathbb{R}^n\times[0,T], \quad \partial_i v = v_i, \quad \partial_t v = v_t, \quad \partial_{ti} v = v_{ti}.$$

First,

$$\begin{aligned}&2\int_Q Lv(Pv - Lv) G dxdt \\ =&2\int_Q (v_t - \mathbf{A}\nabla v\cdot\nabla\log G + \frac{F}{2}v)(\tilde{\Delta}v + \mathbf{A}\nabla v\cdot\nabla\log G - \frac{F}{2}v) G dxdt \\ =:&\sum_{i=1}^{8} I_i,\end{aligned} \tag{20}$$

where

$$\begin{aligned}
I_1 =& 2\int_Q v_t\tilde{\Delta} v G dxdt,\\
I_2 =& 2\int_Q v_t \mathbf{A}\nabla v\cdot\nabla G dxdt,\\
I_3 =& -\int_Q FGvv_t dxdt,\\
I_4 =& -2\int_Q \mathbf{A}\nabla v\cdot\nabla G\tilde{\Delta} v dxdt,\\
I_5 =& -2\int_Q \frac{\left(\mathbf{A}\nabla v\cdot\nabla G\right)^2}{G} dxdt,\\
I_6 =& 2\int_Q Fv\mathbf{A}\nabla v\cdot\nabla G dxdt,\\
I_7 =& \int_Q FGv\tilde{\Delta} v dxdt,\\
I_8 =& -\frac{1}{2}\int_Q F^2Gv^2 dxdt.
\end{aligned} \tag{21}$$

Next we compute these integrals.

$$\begin{aligned}
I_1 =& 2\int_Q v_t\ (a^{ij}v_j)_i G dxdt,\\
=& -2\int_Q v_{ti}\ a^{ij}v_j G dxdt - 2\int_Q v_t a^{ij}v_j G_i dxdt,\\
=:& I_{1,1} - 2\int_Q v_t \mathbf{A}\nabla v\cdot\nabla G dxdt,
\end{aligned} \tag{22}$$

where

$$I_{1,1} = -2\int_Q v_{ti}\ a^{ij}v_j G dxdt.$$

Using integration by parts with respect to t,

$$\begin{aligned}
I_{1,1} =& -2\int_{\mathbb{R}^n} v_i a^{ij} v_j G dx\Big|_0^T + 2\int_Q v_i a_t^{ij} v_j G dxdt\\
&+ 2\int_Q v_i a^{ij} v_j G_t dxdt + 2\int_Q v_i a^{ij} v_{tj} G dxdt\\
=& -2\int_{\mathbb{R}^n} \mathbf{A}\nabla v\cdot\nabla v G dx\Big|_0^T + 2\int_Q a_t^{ij} v_i v_j G dxdt\\
&+ 2\int_Q \mathbf{A}\nabla v\cdot\nabla v G_t dxdt - I_{1,1},
\end{aligned}$$

then

$$I_{1,1} = -\int_{\mathbb{R}^n} \mathbf{A}\nabla v\cdot\nabla v G dx\Big|_0^T + \int_Q a_t^{ij} v_i v_j G dxdt + \int_Q \mathbf{A}\nabla v\cdot\nabla v G_t dxdt. \tag{23}$$

Combining (22) and (23), we have

$$I_1 = -\int_{\mathbb{R}^n} \mathbf{A}\nabla v \cdot \nabla v G dx\Big|_0^T + \int_Q a_t^{ij} v_i v_j G dxdt + \int_Q \mathbf{A}\nabla v \cdot \nabla v G_t dxdt - I_2. \tag{24}$$

For I_3, we use integration by parts with respect to t, then

$$I_3 = -\frac{1}{2}\int_Q (v^2)_t FG dxdt = -\frac{1}{2}\int_{\mathbb{R}^n} v^2 FG dx\Big|_0^T + \frac{1}{2}\int_Q (F_t G + FG_t) v^2 dxdt. \tag{25}$$

In order to compute I_4, we need to use integration by parts three times.

$$\begin{aligned} I_4 =& -2\int_Q a^{lj} G_l v_j \ (a^{ik} v_i)_k dxdt \\ =& 2\int_Q (a^{lj} G_l)_k \ a^{ik} v_i v_j dxdt + 2\int_Q a^{ik} a^{lj} G_l v_i v_{jk} dxdt. \end{aligned} \tag{26}$$

For the second term of the above line, we use integration by parts with respect to x_j, then

$$2\int_Q a^{ik} a^{lj} G_l \ v_i v_{jk} dxdt = -2\int_Q (a^{ik} a^{lj} G_l)_j v_i v_k dxdt - 2\int_Q a^{ik} a^{lj} G_l v_k v_{ij} dxdt. \tag{27}$$

In the right-hand side of (27), for the first term we interchange subscript k with j, and for the second term we use integration by parts with respect to x_i, then

$$-2\int_Q (a^{ik} a^{lj} G_l)_j v_i v_k dxdt = -2\int_Q (a^{ij} a^{kl} G_l)_k v_i v_j dxdt; \tag{28}$$

$$\begin{aligned} -2\int_Q a^{ik} a^{lj} G_l v_k \ v_{ij} dxdt =& 2\int_Q (a^{lj} G_l \ a^{ik} v_k)_i v_j dxdt \\ =& 2\int_Q (a^{lj} G_l)_i a^{ik} v_k v_j dxdt + 2\int_Q a^{lj} G_l v_j \ (a^{ik} v_k)_i dxdt. \end{aligned} \tag{29}$$

In the right-hand side of (29), for the first term we interchange subscript k with i, and for the second term we notice that it is exactly $-I_4$, then we can rewrite (29) as

$$-2\int_Q a^{ik} a^{lj} G_l v_k v_{ij} dxdt = 2\int_Q (a^{lj} G_l)_k a^{ik} v_i v_j dxdt - I_4. \tag{30}$$

Now we combine (26), (27), (28) and (30), then we obtain

$$I_4 = \int_Q \Big(4(a^{lj} G_l)_k a^{ik} - 2(a^{ij} a^{kl} G_l)_k\Big) v_i v_j dxdt - I_4.$$

Hence

$$\begin{aligned} I_4 =& \int_Q \Big(2(a^{lj} G_l)_k a^{ik} - (a^{ij} \ a^{kl} G_l)_k\Big) v_i v_j dxdt \\ =& \int_Q \Big(2a^{ik} G_{kl} a^{lj} + (2a^{ik} a_k^{lj} - a^{kl} a_k^{ij}) G_l - a^{ij} (a^{kl} G_l)_k\Big) v_i v_j dxdt. \end{aligned}$$

Notice that

$$\int_Q a^{ik}a^{lj}v_iv_jdxdt = \int_Q a^{jk}a^{li}v_iv_jdxdt$$

and

$$-\int_Q a^{ij}(a^{kl}G_l)_kv_iv_jdxdt = -\int_Q a^{ij}\tilde{\Delta}Gv_iv_jdxdt = -\int_Q \mathbf{A}\nabla v\cdot\nabla v\tilde{\Delta}Gdxdt,$$

then

$$I_4 = \int_Q \Big(2a^{ik}G_{kl}a^{lj} + (a^{ik}a^{lj}_k + a^{jk}a^{li}_k - a^{kl}a^{ij}_k)G_l\Big)v_iv_jdxdt - \int_Q \mathbf{A}\nabla v\cdot\nabla v\tilde{\Delta}Gdxdt. \quad (31)$$

Next we compute I_6 and I_7.

$$\begin{aligned} I_6 &= \int_Q F\ a^{ij}G_i\ (v^2)_jdxdt, \\ &= -\int_Q \big(F_ja^{ij}G_i + F(a^{ij}G_i)_j\big)v^2dxdt, \\ &= -\int_Q \big(\mathbf{A}\nabla F\cdot\nabla G + F\tilde{\Delta}G\big)v^2dxdt. \end{aligned} \quad (32)$$

$$\begin{aligned} I_7 &= \int_Q FGv(a^{ij}v_j)_idxdt \\ &= -\int_Q F_iGva^{ij}v_jdxdt - \int_Q FG_iva^{ij}v_jdxdt - \int_Q FGv_ia^{ij}v_jdxdt \\ &= -\int_Q v\mathbf{A}\nabla v\cdot\nabla FGdxdt - \frac{1}{2}\int_Q Fa^{ij}G_i(v^2)_jdxdt - \int_Q FG\mathbf{A}\nabla v\cdot\nabla vdxdt. \end{aligned}$$

For the second term of the above line, integrating by parts we have

$$\begin{aligned} -\frac{1}{2}\int_Q F\ a^{ij}G_i(v^2)_jdxdt =& \frac{1}{2}\int_Q \big(F_ja^{ij}G_i + F(a^{ij}G_i)_j\big)v^2dxdt \\ =& \frac{1}{2}\int_Q \big(\mathbf{A}\nabla F\cdot\nabla G + F\tilde{\Delta}G\big)v^2dxdt, \end{aligned}$$

then

$$\begin{aligned} I_7 =& -\int_Q v\mathbf{A}\nabla v\cdot\nabla FGdxdt - \int_Q FG\mathbf{A}\nabla v\cdot\nabla vdxdt \\ &+ \frac{1}{2}\int_Q \big(\mathbf{A}\nabla F\cdot\nabla G + F\tilde{\Delta}G\big)v^2dxdt. \end{aligned} \quad (33)$$

Now we combine (20), (21), (24), (25), (31), (32) and (33), then we obtain

$$
\begin{aligned}
&2\int_Q Lv(Pv-Lv)Gdxdt+\int_{\mathbb{R}^n}\mathbf{A}\nabla v\cdot\nabla vGdx\Big|_0^T+\frac{1}{2}\int_{\mathbb{R}^n}v^2FGdx\Big|_0^T\\
=&\frac{1}{2}\int_Q\Big(F_tG+F(G_t-\tilde{\Delta}G-FG)-\mathbf{A}\nabla F\cdot\nabla G\Big)v^2dxdt\\
&+\int_Q\Big(2a^{ik}G_{kl}a^{lj}+(a^{ik}a^{lj}_k+a^{jk}a^{li}_k-a^{kl}a^{ij}_k)G_l+a^{ij}_tG\Big)v_iv_jdxdt\\
&-2\int_Q\frac{\big(\mathbf{A}\nabla v\cdot\nabla G\big)^2}{G}dxdt+\int_Q(G_t-\tilde{\Delta}G-FG)\mathbf{A}\nabla v\cdot\nabla vdxdt\\
&-\int_Q v\mathbf{A}\nabla v\cdot\nabla FGdxdt.
\end{aligned}
$$

Notice that

$$
a^{ik}G(\log G)_{kl}a^{lj}v_iv_j=a^{ik}G_{kl}a^{lj}v_iv_j-\frac{\big(\mathbf{A}\nabla v\cdot\nabla G\big)^2}{G},
$$

then

$$
\begin{aligned}
&2\int_Q Lv(Pv-Lv)Gdxdt+\int_{\mathbb{R}^n}\mathbf{A}\nabla v\cdot\nabla vGdx\Big|_0^T+\frac{1}{2}\int_{\mathbb{R}^n}v^2FGdx\Big|_0^T\\
=&\frac{1}{2}\int_Q\Big(F_tG+F(G_t-\tilde{\Delta}G-FG)-\mathbf{A}\nabla F\cdot\nabla G\Big)v^2dxdt\\
&+\int_Q\Big(2a^{ik}G(\log G)_{kl}a^{lj}+(a^{ik}a^{lj}_k+a^{jk}a^{li}_k-a^{kl}a^{ij}_k)G_l+a^{ij}_tG\Big)v_iv_jdxdt\\
&+\int_Q(G_t-\tilde{\Delta}G-FG)\mathbf{A}\nabla v\cdot\nabla vdxdt-\int_Q v\mathbf{A}\nabla v\cdot\nabla FGdxdt\\
=&\frac{1}{2}\int_Q\Big(F_tG+F(G_t-\tilde{\Delta}G-FG)-\mathbf{A}\nabla F\cdot\nabla G\Big)v^2dxdt\\
&+\int_Q\Big(2\mathbf{D}_G+(\frac{\partial_tG-\tilde{\Delta}G}{G}-F)\mathbf{A}\Big)\nabla v\cdot\nabla vGdxdt-\int_Q v\mathbf{A}\nabla v\cdot\nabla FGdxdt
\end{aligned}
\tag{34}
$$

Using again integration by parts, we have

$$
\int_Q v\mathbf{A}\nabla v\cdot\nabla F_0Gdxdt=-\frac{1}{2}\int_Q\Big(\mathbf{A}\nabla F_0\cdot\nabla G+\tilde{\Delta}F_0G\Big)v^2dxdt. \tag{35}
$$

Finally, combining (34) and (35), we obtain (19). □

The second one is concerned with the properties of mollified $\{a^{ij}\}$.

Lemma 3.2. *Suppose $\{a^{ij}\}$ satisfy (2) and (5). Let*

$$
a^{ij}_\epsilon(x,t)=\int_{\mathbb{R}^n}a^{ij}(x-y,t)\phi_\epsilon(y)dy,
$$

where ϕ is a smooth function satisfying $\phi\geq 0$, supp $\phi=\{x,\ |x|\leq 1\}$ and $\|\phi\|_{L^1}=1$; $\epsilon=\frac{1}{2}$ and $\phi_\epsilon(y)=\epsilon^{-n}\phi(\frac{y}{\epsilon})$.

Then $\{a_\epsilon^{ij}\}$ have the following properties:

$$\begin{aligned}
&1)\ \lambda|\xi|^2 \le a_\epsilon^{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall \xi\in\mathbb{R}^n;\\
&2)\ |\nabla a_\epsilon^{ij}(x,t)| \le M; \qquad |\nabla a_\epsilon^{ij}(x,t)| \le 2E\langle x\rangle^{-\beta} \quad \textit{when } |x|\ge 1;\\
&3)\ |a_\epsilon^{ij}(x,t)-a^{ij}(x,t)| \le 2\Lambda; \qquad |a_\epsilon^{ij}(x,t)-a^{ij}(x,t)| \le E\langle x\rangle^{-\beta} \quad \textit{when } |x|\ge 1;\\
&4)\ |\partial_{kl}a_\epsilon^{ij}(x,t)| \le c(n)M; \qquad |\partial_{kl}a_\epsilon^{ij}(x,t)| \le c(n)E\langle x\rangle^{-\beta} \quad \textit{when } |x|\ge 1.
\end{aligned} \tag{36}$$

Lemma 3.2 can be proved with only minor changes to the proof in Appendix A of [12].

Now we begin to prove Proposition 1.4.

Proof of Proposition 1.4. We use identity (19) to prove Carleman inequality (8).

In (19), we let

$$G = e^{2\gamma(t^{-K}-1)-\frac{b\langle x\rangle^\alpha+K}{t}},$$

then

$$\frac{\partial_t G-\tilde{\Delta}G}{G} = \frac{b\langle x\rangle^\alpha-\alpha^2b^2\langle x\rangle^{2\alpha-4}a^{ij}x_ix_j+K}{t^2}+\frac{\alpha b\langle x\rangle^{\alpha-2}(a^{ii}+\partial_k a^{kl}x_l)}{t}-2\gamma Kt^{-K-1}.$$

Let

$$F = \frac{b\langle x\rangle^\alpha-\alpha^2b^2\langle x\rangle^{2\alpha-4}a^{ij}x_ix_j+K}{t^2}+\frac{\alpha b\langle x\rangle^{\alpha-2}a^{ii}-d}{t}-2\gamma Kt^{-K-1},$$

where d is a positive constant to be determined, and

$$F_0 = \frac{b\langle x\rangle^\alpha-\alpha^2b^2\langle x\rangle^{2\alpha-4}a_\epsilon^{ij}x_ix_j+K}{t^2}+\frac{\alpha b\langle x\rangle^{\alpha-2}a_\epsilon^{ii}-d}{t}-2\gamma Kt^{-K-1}.$$

In the following arguments, we denote by $\mathbf{I}_n$ the identity matrix of $\mathbb{R}^n$, C are generic constants depending on n,Λ,λ,M,E and α. We need some estimates which we list in the following lemma.

Lemma 3.3. *Set $b=\frac{1}{8\Lambda}$ and $d=\frac{K}{4}$. For $K\ge K_0(n,\Lambda,\lambda,M,E,\alpha)$, we have*

$$2\mathbf{D}_G+\left(\frac{\partial_t G-\tilde{\Delta}G}{G}-F\right)\mathbf{A} \ge \frac{\lambda K}{8t}\mathbf{I}_n; \tag{37}$$

$$\partial_t F+F\left(\frac{\partial_t G-\tilde{\Delta}G}{G}-F\right) \ge \frac{bK\langle x\rangle^\alpha}{16t^3}; \tag{38}$$

$$|\tilde{\Delta}F_0| \le \frac{C\langle x\rangle^\alpha}{t^2}; \tag{39}$$

$$|\nabla(F-F_0)| \le \frac{C\langle x\rangle^{\alpha-1}}{t^2}. \tag{40}$$

We will prove this lemma later.

First by (37) we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\times(0,1)}\Big(2\mathbf{D}_G+\left(\frac{\partial_t G-\tilde{\Delta}G}{G}-F\right)\mathbf{A}\Big)\nabla v\cdot\nabla vG\,dxdt\\
\ge&\frac{\lambda K}{8}\int_{\mathbb{R}^n\times(0,1)}\frac{|\nabla v|^2}{t}G\,dxdt.
\end{aligned} \tag{41}$$

Next we estimate M_0. By (40) and

$$\nabla \log G = -\frac{\alpha b}{t}\langle x\rangle^{\alpha-2}x$$

we have

$$|\mathbf{A}\nabla(F-F_0)\cdot\log G| \leq \Lambda|\nabla(F-F_0)||\nabla\log G| \leq \frac{C\langle x\rangle^{2\alpha-2}}{t^3} \leq \frac{C\langle x\rangle^{\alpha}}{t^3}. \tag{42}$$

Then by (38), (39) and (42) we have

$$\begin{aligned} M_0 &= \partial_t F + F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) + \tilde{\Delta} F_0 - \langle A\nabla(F-F_0), \nabla \log G\rangle \\ &\geq (\frac{bK}{16} - C)\frac{\langle x\rangle^{\alpha}}{t^3}, \end{aligned}$$

thus

$$\frac{1}{2}\int_{\mathbb{R}^n\times(0,1)} M_0 v^2 G dxdt \geq (\frac{bK}{32} - C)\int_{\mathbb{R}^n\times(0,1)} \frac{\langle x\rangle^{\alpha}}{t^3} v^2 G dxdt. \tag{43}$$

By the Cauchy inequality and (40) we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}^n\times(0,1)} v\mathbf{A}\nabla v\cdot\nabla(F-F_0)Gdxdt\Big| \\ \leq&\Lambda\int_{\mathbb{R}^n\times(0,1)} |\nabla(F-F_0)||v||\nabla v|Gdxdt \\ \leq&C\int_{\mathbb{R}^n\times(0,1)} \frac{\langle x\rangle^{\alpha-1}}{t^2}|v||\nabla v|Gdxdt \\ \leq&C\int_{\mathbb{R}^n\times(0,1)} \frac{\langle x\rangle^{2\alpha-2}}{t^3}v^2Gdxdt + C\int_{\mathbb{R}^n\times(0,1)} \frac{|\nabla v|^2}{t}Gdxdt \\ \leq&C\int_{\mathbb{R}^n\times(0,1)} \frac{\langle x\rangle^{\alpha}}{t^3}v^2Gdxdt + C\int_{\mathbb{R}^n\times(0,1)} \frac{|\nabla v|^2}{t}Gdxdt. \end{aligned} \tag{44}$$

Finally, by (19), (41), (43), (44) and the Cauchy inequality, we have

$$\int_{\mathbb{R}^n\times(0,1)} |Pu|^2 G dxdt \geq (\frac{bK}{32} - C)\int_{\mathbb{R}^n\times(0,1)} \frac{\langle x\rangle^{\alpha}}{t^3}v^2Gdxdt + (\frac{\lambda K}{8} - C)\int_{\mathbb{R}^n\times(0,1)} \frac{|\nabla v|^2}{t}Gdxdt,$$

if we choose $K \geq K_0(n, \Lambda, \lambda, M, E, \alpha)$ large enough, we obtain

$$\int_{\mathbb{R}^n\times(0,1)} |Pv|^2 G dxdt \geq \int_{\mathbb{R}^n\times(0,1)} (v^2 + |\nabla v|^2)Gdxdt.$$

Thus we proved Carleman inequality (8). □

There is only Lemma 3.3 left to be proven.

Proof of Lemma 3.3. We estimate them one by one.

Estimate of $2\mathbf{D}_G + (\frac{\partial_t G-\tilde{\Delta}G}{G} - F)\mathbf{A}$.

By direct computations we have

$$\begin{aligned}
&2\mathbf{D}_G + (\frac{\partial_t G-\tilde{\Delta}G}{G} - F)\mathbf{A}\\
=&-\frac{2\alpha b}{t}\langle x\rangle^{\alpha-2}\mathbf{A}^2+\frac{2\alpha(2-\alpha)b}{t}\langle x\rangle^{\alpha-4}\mathbf{A}x(\mathbf{A}x)'\\
&-\frac{\alpha b}{t}\langle x\rangle^{\alpha-2}x_l(a^{ki}\partial_k a^{lj}+a^{kj}\partial_k a^{li}-a^{kl}\partial_k a^{ij}-a^{ij}\partial_k a^{kl})+\partial_t a^{ij}+\frac{d}{t}\mathbf{A}\\
\geq&-\frac{2\alpha b\Lambda^2}{t}\langle x\rangle^{\alpha-2}\mathbf{I}_n-\frac{\alpha b}{t}\langle x\rangle^{\alpha-2}x_l(a^{ki}\partial_k a^{lj}+a^{kj}\partial_k a^{li}-a^{kl}\partial_k a^{ij}-a^{ij}\partial_k a^{kl})+\partial_t a^{ij}+\frac{\lambda d}{t}\mathbf{I}_n.
\end{aligned}$$

Next we estimate the lower bounds of the matrices in the above line. We just need to estimate matrix $x_l a^{ki}\partial_k a^{lj}$ and $\partial_t a^{ij}$. For any $\xi\in\mathbb{R}^n$,

$$|x_l a^{ki}\partial_k a^{lj}\xi_i\xi_j|\leq n^2\Lambda E|x|\langle x\rangle^{-\beta}\sum_{i,j}|\xi_i||\xi_j|\leq n^3\Lambda E\langle x\rangle^{1-\beta}|\xi|^2,$$

then

$$-n^3\Lambda E\langle x\rangle^{1-\beta}\mathbf{I}_n\leq x_l a^{ki}\partial_k a^{lj}\leq n^3\Lambda E\langle x\rangle^{1-\beta}\mathbf{I}_n.$$

Similarly,

$$|\partial_t a^{ij}\xi_i\xi_j|\leq M\sum_{i,j}|\xi_i||\xi_j|\leq Mn|\xi|^2,$$

then

$$-Mn\mathbf{I}_n\leq\partial_t a^{ij}\leq Mn\mathbf{I}_n.$$

Thus we have

$$2\mathbf{D}_G + (\frac{\partial_t G-\tilde{\Delta}G}{G} - F)\mathbf{A}\geq\big(-\frac{2\alpha b\Lambda^2}{t}\langle x\rangle^{\alpha-2}-\frac{4\alpha b n^3\Lambda E}{t}\langle x\rangle^{\alpha-\beta-1}-Mn+\frac{\lambda d}{t}\big)\mathbf{I}_n.$$

Notice that $\alpha-2\leq 0$ and $\alpha-\beta-1\leq 0$, and if we choose $d=d(n,\Lambda,\lambda,M,E,\alpha)$ large enough, then

$$2\mathbf{D}_G + (\frac{\partial_t G-\tilde{\Delta}G}{G} - F)\mathbf{A}\geq\frac{\lambda d}{2t}\mathbf{I}_n.$$

Estimate of $\partial_t F + F(\frac{\partial_t G-\tilde{\Delta}G}{G} - F)$.

Direct computations give us

$$\begin{aligned}
&\partial_t F + F(\frac{\partial_t G-\tilde{\Delta}G}{G} - F)\\
=&\frac{(d+\alpha b\langle x\rangle^{\alpha-2}\partial_i a^{ij}x_j-2)(b\langle x\rangle^{\alpha}-\alpha^2b^2\langle x\rangle^{2\alpha-4}a^{ij}x_ix_j+K)}{t^3}\\
&-\frac{\alpha^2b^2\langle x\rangle^{2\alpha-4}\partial_t a^{ij}x_ix_j+(d-\alpha b\langle x\rangle^{\alpha-2}a^{ii})(d+\alpha b\langle x\rangle^{\alpha-2}\partial_i a^{ij}x_j-1)}{t^2}\\
&+\frac{\alpha b\langle x\rangle^{\alpha-2}\partial_t a^{ii}}{t}+2\gamma Kt^{-K-2}\big(K+1-(d+\alpha b\langle x\rangle^{\alpha-2}\partial_i a^{ij}x_j)\big).
\end{aligned}$$

Notice that

$$\begin{aligned} \langle x\rangle^{\alpha-2}|\partial_i a^{ij}x_j| \leq & C\langle x\rangle^{\alpha-\beta-2}|x| \leq C\langle x\rangle^{\alpha-\beta-1} \leq C, \\ \langle x\rangle^{2\alpha-4}a^{ij}x_ix_j \leq & \Lambda\langle x\rangle^{2\alpha-4}|x|^2 \leq \Lambda\langle x\rangle^{2\alpha-2} \leq \Lambda\langle x\rangle^{\alpha}, \\ \langle x\rangle^{2\alpha-4}|\partial_t a^{ij}x_ix_j| \leq & C\langle x\rangle^{2\alpha-4}|x|^2 \leq C\langle x\rangle^{\alpha}, \end{aligned}$$

then we have

$$\begin{aligned} & \partial_t F + F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) \\ \geq & \frac{(d-C)\big((b-\alpha^2 b^2\Lambda)\langle x\rangle^{\alpha} + K\big)}{t^3} - \frac{C\langle x\rangle^{\alpha} + (d+C)^2}{t^2} \\ & - \frac{C}{t} + 2\gamma K t^{-K-2}(K-d-C). \end{aligned}$$

Recall that $b=\frac{1}{8\Lambda}$, and thus $\alpha^2b^2\Lambda \leq 4b^2\Lambda = \frac{b}{2}$. If we choose d large enough, then

$$\begin{aligned} & \partial_t F + F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) \\ \geq & \frac{(d-C)\big(\frac{b}{2}\langle x\rangle^{\alpha} + K\big) - C\langle x\rangle^{\alpha} - (d+C)^2 - C}{t^3} + 2\gamma K t^{-K-2}(K-2d) \\ \geq & \frac{(\frac{bd}{2} - C)\langle x\rangle^{\alpha} + (d-C)K - 2d^2}{t^3} + 2\gamma K t^{-K-2}(K-2d). \end{aligned}$$

We choose $d=\frac{K}{4}$, then

$$\partial_t F + F(\frac{\partial_t G - \tilde{\Delta} G}{G} - F) \geq (\frac{bK}{8} - C)\frac{\langle x\rangle^{\alpha}}{t^3} + \gamma K^2 t^{-K-2} \geq \frac{bK\langle x\rangle^{\alpha}}{16t^3}.$$

Estimate of $\tilde{\triangle} F_0$.

Direct computations show that

$$\tilde{\triangle} F_0 = \frac{b}{t^2}\tilde{\triangle}(\langle x\rangle^{\alpha}) - \frac{\alpha^2 b^2}{t^2}\tilde{\triangle}(\langle x\rangle^{2\alpha-4}a_{\epsilon}^{ij}x_ix_j) + \frac{\alpha b}{t}\tilde{\triangle}(\langle x\rangle^{\alpha-2}a_{\epsilon}^{ii}), \tag{45}$$

and

$$\begin{aligned} \tilde{\triangle}(\langle x\rangle^{\alpha}) = & \alpha\langle x\rangle^{\alpha-2}(a^{ii} + \partial_i a^{ij}x_j) + \alpha(\alpha-2)\langle x\rangle^{\alpha-4}a^{ij}x_ix_j, \\ \tilde{\triangle}(\langle x\rangle^{2\alpha-4}a_{\epsilon}^{ij}x_ix_j) = & (2\alpha-4)(2\alpha-6)\langle x\rangle^{2\alpha-8}a^{kl}a_{\epsilon}^{ij}x_ix_jx_kx_l \\ & + (2\alpha-4)\langle x\rangle^{2\alpha-6}\Big((\partial_l a^{kl}a_{\epsilon}^{ij} + 2a^{kl}\partial_l a_{\epsilon}^{ij})x_ix_jx_k + (4a^{ki}a_{\epsilon}^{kj} + a^{kk}a_{\epsilon}^{ij})x_ix_j\Big) \\ & + \langle x\rangle^{2\alpha-4}\Big((a^{kl}\partial_{kl}a_{\epsilon}^{ij} + \partial_k a^{kl}\partial_l a_{\epsilon}^{ij})x_ix_j + (2\partial_k a^{kj}a_{\epsilon}^{ij} + 4a^{kj}\partial_k a_{\epsilon}^{ij})x_i + 2a^{ij}a_{\epsilon}^{ij}\Big), \\ \tilde{\triangle}(\langle x\rangle^{\alpha-2}a_{\epsilon}^{ii}) = & (\alpha-2)(\alpha-4)\langle x\rangle^{\alpha-6}a^{ij}a_{\epsilon}^{kk}x_ix_j \\ & + (\alpha-2)\langle x\rangle^{\alpha-4}\Big((\partial_j a^{ij}a_{\epsilon}^{kk} + 2a^{ij}\partial_j a_{\epsilon}^{kk})x_i + a^{ii}a_{\epsilon}^{kk}\Big) \\ & + \langle x\rangle^{\alpha-2}(a^{ij}\partial_{ij}a_{\epsilon}^{kk} + \partial_i a^{ij}\partial_j a_{\epsilon}^{kk}). \end{aligned}$$

By Lemma 3.2 we know that $a^{ij}, \nabla a^{ij}, a_\epsilon^{ij}, \nabla a_\epsilon^{ij}$ and $\nabla^2 a_\epsilon^{ij}$ are all bounded, then it is easy to verify that

$$\begin{aligned} |\tilde{\triangle}(\langle x\rangle^\alpha)| \leq& C(\langle x\rangle^{\alpha-1} + \langle x\rangle^{\alpha-2}) \leq C\langle x\rangle^{\alpha-1}; \\ |\tilde{\triangle}(\langle x\rangle^{\alpha-4} a_\epsilon^{ij} x_i x_j)| \leq& C(\langle x\rangle^{2\alpha-4} + \langle x\rangle^{2\alpha-3} + \langle x\rangle^{2\alpha-2}) \leq C\langle x\rangle^{2\alpha-2}; \\ |\tilde{\triangle}(\langle x\rangle^{\alpha-2} a_\epsilon^{ii})| \leq& C(\langle x\rangle^{\alpha-4} + \langle x\rangle^{\alpha-3} + \langle x\rangle^{\alpha-2}) \leq C\langle x\rangle^{\alpha-2}. \end{aligned} \tag{46}$$

Finally by (45) and (46) we have

$$|\tilde{\triangle} F_0| \leq \frac{C}{t^2}(\langle x\rangle^{\alpha-1} + \langle x\rangle^{2\alpha-2} + \langle x\rangle^{\alpha-2}) \leq \frac{C\langle x\rangle^\alpha}{t^2}.$$

Estimate of $|\nabla(F - F_0)|$.

Since

$$F - F_0 = \frac{\alpha^2 b^2}{t^2}\langle x\rangle^{2\alpha-4}(a_\epsilon^{ij} - a^{ij})x_i x_j - \frac{\alpha b}{t}\langle x\rangle^{\alpha-2}(a_\epsilon^{ii} - a^{ii}),$$

then

$$\begin{aligned} \nabla(F - F_0) =& \frac{\alpha^2 b^2}{t^2}\Big((2\alpha - 4)\langle x\rangle^{2\alpha-6}(a_\epsilon^{ij} - a^{ij})x_i x_j x + 2\langle x\rangle^{2\alpha-4}(a_\epsilon^{ij} - a^{ij})x_i \nabla x_j \\ &+ \langle x\rangle^{2\alpha-4}(\nabla a_\epsilon^{ij} - \nabla a^{ij})x_i x_j\Big) \\ &- \frac{\alpha b}{t}\Big((\alpha - 2)\langle x\rangle^{\alpha-4}(a_\epsilon^{ii} - a^{ii})x + \langle x\rangle^{\alpha-2}(\nabla a_\epsilon^{ii} - \nabla a^{ii})\Big). \end{aligned}$$

Notice that $a^{ij}, \nabla a^{ij}, a_\epsilon^{ij}$ and ∇a_ϵ^{ij} are all bounded, then

$$|\nabla(F - F_0)| \leq \frac{C}{t^2}(\langle x\rangle^{2\alpha-3} + \langle x\rangle^{2\alpha-4}|\nabla a_\epsilon^{ij} - \nabla a^{ij}||x|^2) + \frac{C}{t}(\langle x\rangle^{\alpha-3} + \langle x\rangle^{\alpha-2}). \tag{47}$$

By 2) of (36), when $|x| < 1$,

$$|\nabla a_\epsilon^{ij} - \nabla a^{ij}||x|^2 \leq 2M|x|^2 \leq 2M,$$

and when $|x| \geq 1$,

$$|\nabla a_\epsilon^{ij} - \nabla a^{ij}||x|^2 \leq (2E\langle x\rangle^{-\beta} + E\langle x\rangle^{-\beta})|x|^2 \leq 3E\langle x\rangle^{2-\beta}.$$

In both cases we have

$$|\nabla a_\epsilon^{ij} - \nabla a^{ij}||x|^2 \leq C\langle x\rangle^{2-\beta}. \tag{48}$$

By (47) and (48) we have

$$|\nabla(F - F_0)| \leq \frac{C}{t^2}(\langle x\rangle^{2\alpha-3} + \langle x\rangle^{2\alpha-\beta-2}) + \frac{C\langle x\rangle^{\alpha-2}}{t}$$

Since $2\alpha - \beta - 2 \leq \alpha - 1$, then

$$|\nabla(F - F_0)| \leq \frac{C}{t^2}(\langle x\rangle^{2\alpha-3} + \langle x\rangle^{\alpha-1}) + \frac{C\langle x\rangle^{\alpha-2}}{t} \leq \frac{C\langle x\rangle^{\alpha-1}}{t^2}.$$

Thus we proved Lemma 3.3. □

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