Global Smooth Solutions for the Zakharov System with Quantum Effects in Two Space Dimensions*

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Abstract: We consider the Zakharov system with quantum effects in the framework of the scalar mode in two space dimensions

$$\begin{aligned} &iE_t + \Delta E = nE + \Gamma\Delta^2 E,\\ &n_t = -\nabla.\mathbf{V},\\ &\mathbf{V}_t = -\nabla n - |\nabla E|^2 + \Gamma\nabla\Delta n, \end{aligned} \qquad (ZSQ)$$

which is related to quantum corrections to the Zakharov system for Langmuir waves in plasma. We prove the existence and uniqueness of global smooth solutions to the Cauchy problem for (ZSQ) in the Sobolev space through making a priori integral esimates and utilizing the so-called continuity method.
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1 Introduction

We deal with in this paper the Zakharov system with quantum effects in the framework of the scalar model [6]:

$$iE_t + \Delta E = nE + \Gamma\Delta^2 E, \tag{1.1}$$

$$n_t = -\nabla \cdot \mathbf{V}, \tag{1.2}$$

$$\mathbf{V}_t = -\nabla n - \nabla|E|^2 + \Gamma\nabla\Delta n, \tag{1.3}$$

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where $(E,n,\mathbf{V}):\ \ (t,x)\in(\mathbb{R}\cup\{0\})\times\mathbb{R}^2\to\mathbb{C}\times\mathbb{R}\times\mathbb{R}^2$ and the initial data are taken to be:

$$E(x,0)=E^0(x),\quad n(x,0)=n^0(x),\quad \mathbf{V}(x,0)=\mathbf{V}^0(x). \tag{1.4}$$

In the recent years, special interest was devoted to quantum corrections to the Zakharov equations for Langmuir waves in a plasma [20]. The system (1.1)-(1.3) is related to the equations that govern the amplitude E of the electric field oscillations and the number density n

$$i\mathbf{E}_t-\alpha\nabla\times(\nabla\times\mathbf{E})+\nabla(\nabla\cdot\mathbf{E})=n\mathbf{E}+\Gamma\nabla\Delta(\nabla\cdot\mathbf{E}), \tag{$VZS-1$}$$

$$n_{tt}-\Delta n=\Delta|\mathbf{E}|^2-\Gamma\Delta^2 n. \tag{$VZS-2$}$$

The parameter α is defined as the square ratio of the light speed and the electron Fermi velocity and Γ measures the influence of quantum effects. As mentioned by [6], it may be suitable to abandon the vector character of the equations (VZS-1)-(VZS-2) for the sake of preserving a non-zero electric field at the center of the cavity, while keeping the rotational symmetry necessary for implementing an asymptotic analysis in the spirit of [7, 8]. We thus consider in this paper the influence of quantum effects in the frame work of the scalar model (1.1)-(1.3). Since (VZS-2) originates from the hydrodynamic system

$$\begin{cases} n_t+\nabla\cdot\mathbf{V}=0,\\ \mathbf{V}_t=-\nabla n-\nabla|\mathbf{E}|^2+\Gamma\nabla\Delta n,\end{cases}$$

governing the ion sound waves.

The system (1.1)-(1.3) describes the quantum corrections to the Zakharov equations for Langmuir waves in a plasma [20] and has been studied by many physicists and mathematicians ([12],[15]). For $\Gamma=0$, the system (1.1)-(1.3) was derived by Zakharov in [20] to model Langmuir wave in plasma. From the mathematical side, there has been considerable work on local and global well-posedness of solutions with rough data through the works of Kenig, Ponce and Vega [16], Bourgain and Colliander [4], Ginibre, Tsutsumi and Velo [13], and Bejenaru, Herr, Holmer and Tataru [2], it was shown in [1, 13, 14, 18] that for initial data small enough, the solution to the Cauchy problem (1.1)-(1.4) remains smooth for all time. In two dimensions, the smallest condition reads $\|E_0\|^2_{L^2(\mathbb{R}^2)}\le\|Q\|^2_{L^2(\mathbb{R}^2)}$ and is optimal, where Q is the ground state of the equation $\Delta u-u+u^3=0$. In three dimensions, it requires that the plasma number $\mathcal{N}$ and the Hamiltonian $\mathcal{H}$ satisfy $\mathcal{N}|\mathcal{H}|<\varepsilon$ ($\varepsilon\le 2.6\times10^{-4}$) together with $\|\nabla E_0\|^2_{L^2(\mathbb{R}^2)}<|\mathcal{H}|$, where $\mathcal{N}$ and $\mathcal{H}$ are defined as follows:

$$\mathcal{N}=\int_{\mathbb{R}^2}|E|^2dx, \tag{1.5}$$

$$\mathcal{H}=\int_{\mathbb{R}^2}\left\{|\nabla E|^2+\frac{1}{2}n^2+\frac{1}{2}|\mathbf{V}|^2+n|E|^2+\Gamma|\Delta E|^2+\frac{\Gamma}{2}|\nabla n|^2\right\}dx. \tag{1.6}$$

Recently, along with the further deep study of physical problems, certain generalized and useful system of Zakharov equations were proposed. We studied in [9, 10, 11] some generalized Zakharov system and obtained various conclusions including blow-up, existence and orbital instability of standing waves as well as the sharp threshold for global existence.

We shall be concerned with in this paper the existence and uniqueness of global smooth solutions to the Cauchy problem (1.1)-(1.4). Throughout this paper, we assume that the solution $(E(t,x), n(t,x), \mathbf{V}(t,x))$ of the Cauchy problem (1.1)-(1.4) and its derivatives tend to zero as $|x| \to +\infty$. In addition, for simplicity, we shall denote various positive constants by C and $(f,g) = \int_{\mathbb{R}^2} f(x)\overline{g(x)}dx$, where $\overline{g(x)}$ denote the complex conjugate function of $g(x)$.

2 A Priori Estimates

In this section we mainly make some a priori estimates.

Lemma 2.1 Let $E_0(x) \in H^1(\mathbb{R}^2)$. Then the solution $E(t,x)$ of the Cauchy problem (1.1)-(1.4) satisfies

$$\|E(t,x)\|_{L^2(\mathbb{R}^2)} = \|E_0(x)\|_{L^2(\mathbb{R}^2)}. \tag{2.1}$$

Proof. Taking the inner product of (1.1) with E, one gets

$$(iE_t + \Delta E - nE - \Gamma\Delta^2 E, E) = 0.. \tag{2.2}$$

Noting that

$$Im(iE_t, E) = \frac{1}{2}\frac{d}{dt}\|E\|^2_{L^2(\mathbb{R}^2)},$$

$$Im(\Delta E, E) = Im\|\nabla E\|^2_{L^2(\mathbb{R}^2)} = 0,$$

$$Im(nE, E) = Im(n\|E\|^2_{L^2(\mathbb{R}^2)}) = 0,$$

$$Im(\Gamma\Delta^2 E, E) = Im(\Gamma\|\Delta E\|^2_{L^2(\mathbb{R}^2)}) = 0,$$

we thus obtain (2.1) by (2.2). □

Lemma 2.2 [5] Let $\|u\|_{H^1(\mathbb{R}^2)} \le K$. Then

$$\|u\|_{L^\infty(\mathbb{R}^2)} \le C\left(1 + \sqrt{\log(1 + \|u\|_{H^2(\mathbb{R}^2)})}\right), \quad u \in H^2(\mathbb{R}^2), \tag{2.3}$$

where the constant C depends only on K. □

Lemma 2.3 [3, 17, 19] Let $u \in H^1(\mathbb{R}^2)$. Then

$$\|u\|^4_{L^4(\mathbb{R}^2)} \le C_0\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\|u\|^2_{L^2(\mathbb{R}^2)}, \tag{2.4}$$

where $C_0 = \frac{2}{\|Q\|^2_{L^2(\mathbb{R}^2)}}$, in which $Q(x)$ is the ground state solution of the equation

$$\Delta\psi - \psi + \psi^3 = 0. \tag{2.5}$$

□

Lemma 2.4 Assume that

$$(i) \quad E_0(x) \in H^2(\mathbb{R}^2), \quad n_0(x) \in H^1(\mathbb{R}^2), \quad \mathbf{V}_0(x) \in L^2(\mathbb{R}^2);$$

$$(ii) \quad \|E_0\|^2_{L^2(\mathbb{R}^2)} \le \frac{2}{3}\|Q(x)\|^2_{L^2(\mathbb{R}^2)},$$

where $Q(x)$ is the ground state solution of (2.5). Then the solution of the Cauchy problem (1.1)-(1.4) satisfies

$$\|\Delta E\|_{L^2\times L^\infty}^2 + \|\nabla E\|_{L^2\times L^\infty}^2 + \|n\|_{L^2\times L^\infty}^2 + \|\nabla n\|_{L^2\times L^\infty}^2 + \|V\|_{L^2\times L^\infty}^2 \le E_1, \tag{2.6}$$

where the constant E_1 depends only on $\|E_0\|_{H^2(\mathbb{R}^2)}^2$, $\|n_0\|_{H^1(\mathbb{R}^2)}^2$ and $\|V_0\|_{L^2(\mathbb{R}^2)}^2$.
Proof. Taking the inner product of (1.1) with E_t, one has

$$(iE_t + \Delta E - nE - \Gamma\Delta^2 E, E_t) = 0. \tag{2.7}$$

By a direct calculation, we obtain

$$Re(iE_t, E_t) = 0,$$

$$Re(\Delta E, E_t) = -\tfrac{1}{2}\tfrac{d}{dt}\|\nabla E\|_{L^2(\mathbb{R}^2)}^2,$$

$$Re(nE, E_t) = \tfrac{1}{2}\int_{\mathbb{R}^2} n\tfrac{d}{dt}|E|^2 dx,$$

$$Re(\Gamma\Delta^2 E, E_t) = \tfrac{\Gamma}{2}\tfrac{d}{dt}\|\Delta E\|_{L^2(\mathbb{R}^2)}^2,$$

which together with (2.7) yield that

$$\frac{d}{dt}\left(\|\nabla E\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} n|E|^2 dx + \Gamma\|\Delta E\|_{L^2(\mathbb{R}^2)}^2\right) = \int_{\mathbb{R}^2} n_t|E|^2 dx. \tag{2.8}$$

Noting that by (1.2) and (1.3)

$$\begin{aligned}
\int_{\mathbb{R}^2} n_t|E|^2 dx &= \int_{\mathbb{R}^2}(-\nabla\cdot\mathbf{V})|E|^2 dx \\
&= \int_{\mathbb{R}^2} \mathbf{V}\nabla|E|^2 dx \\
&= \int_{\mathbb{R}^2} \mathbf{V}(-\nabla n + \Gamma\nabla\Delta n - \mathbf{V}_t) dx \\
&= -\tfrac{1}{2}\tfrac{d}{dt}\int_{\mathbb{R}^2}\left(n^2 + |\mathbf{V}|^2 + \Gamma|\nabla n|^2\right) dx.
\end{aligned}$$

Combining the above identity with (2.8), one concludes

$$\begin{aligned}
\mathcal{H}(t) &= \|\nabla E\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} n|E|^2 dx + \Gamma\|\Delta E\|_{L^2(\mathbb{R}^2)}^2 \\
&\quad + \tfrac{1}{2}\|n\|_{L^2(\mathbb{R}^2)}^2 + \tfrac{1}{2}\|\mathbf{V}\|_{L^2(\mathbb{R}^2)}^2 + \tfrac{1}{2}\Gamma\|\nabla n\|_{L^2(\mathbb{R}^2)}^2 \\
&= \mathcal{H}(0).
\end{aligned} \tag{2.9}$$

By Hölder's inequality, Young's inequality, Lemma 2.3 and (2.1), one sees that

$$\begin{aligned}
\int_{\mathbb{R}^2} n|E|^2 dx &\le \tfrac{1}{3}\int_{\mathbb{R}^2} n^2 dx + \tfrac{3}{4}\int_{\mathbb{R}^2}|E|^4 dx \\
&\le \tfrac{1}{3}\int_{\mathbb{R}^2} n^2 dx + \tfrac{3}{4}\tfrac{2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}\|E_0\|_{L^2(\mathbb{R}^2)}^2\|\nabla E\|_{L^2(\mathbb{R}^2)}^2 \\
&\le \tfrac{1}{3}\int_{\mathbb{R}^2} n^2 dx + \|\nabla E\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{2.10}$$

(2.9) and (2.10) conclude the estimate (2.6). $\square$

Lemma 2.5 Under the conditions of Lemma 2.4, let

$$E_0(x) \in H^3(\mathbb{R}^2), \qquad n_0(x) \in H^2(\mathbb{R}^2), \qquad \mathbf{V}_0(x) \in H^1(\mathbb{R}^2).$$

Then the solution of the Cauchy problem (1.1)-(1.4) satisfies

$$\|\nabla^3 E\|^2_{L^2\times L^\infty} + \|\Delta n\|^2_{L^2\times L^\infty} + \|E_t\|^2_{L^2\times L^\infty} + \|n_t\|^2_{L^2\times L^\infty} \leq E_2, \tag{2.11}$$

where the constant E_2 depends only on $\|E_0\|^2_{H^3(\mathbb{R}^2)}$, $\|n_0\|^2_{H^2(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^1(\mathbb{R}^2)}$.

Proof. Differentiating (1.1) with respect to t, and taking the inner product of the resulting equation with E_t, we have

$$(iE_{tt} + \Delta E_t - (nE)_t - \Gamma\Delta^2 E_t, E_t) = 0. \tag{2.12}$$

Noting that

$$Im(iE_{tt}, E_t) = Re(E_{tt}, E_t) = \tfrac{1}{2}\tfrac{d}{dt}\|E_t\|^2_{L^2(\mathbb{R}^2)},$$

$$Im(\Delta E_t, E_t) = -Im\|\nabla E_t\|^2_{L^2(\mathbb{R}^2)} = 0,$$

$$Im((nE)_t, E_t) = Im(n_t E, E_t) + Im(nE_t, E_t)$$

$$= Im \textstyle\int_{\mathbb{R}^2} n_t E \overline{E}_t dx$$

$$Im(-\Gamma\Delta^2 E_t, E_t) = Im\Gamma\|\Delta E_t\|^2_{L^2(\mathbb{R}^2)} = 0,$$

we thereby obtain

$$\frac{d}{dt}\|E_t\|^2_{L^2(\mathbb{R}^2)} \leq 2\|E\|_{L^\infty(\mathbb{R}^2)}\|n_t\|_{L^2(\mathbb{R}^2)}\|E_t\|_{L^2(\mathbb{R}^2)}. \tag{2.13}$$

Integrating (2.13) with respect to t, one knows

$$\begin{aligned}\|E_t(.,t)\|^2_{L^2(\mathbb{R}^2)} &\leq \|E_t(.,0)\|^2_{L^2(\mathbb{R}^2)} \\ &\quad +2\textstyle\int_0^t \|E(.,\tau)\|_{L^\infty(\mathbb{R}^2)}\|n_t(.,\tau)\|_{L^2(\mathbb{R}^2)}\|E_t(.,\tau)\|_{L^2(\mathbb{R}^2)} d\tau.\end{aligned} \tag{2.14}$$

On the other hand, we take the inner product of (1.1) with ΔE and then obtain

$$(iE_t + \Delta E - nE - \Gamma\Delta^2 E, \Delta E) = 0, \tag{2.15}$$

where

$$|(iE_t, \Delta E)| \leq \|E_t\|_{L^2(\mathbb{R}^2)}\|\Delta E\|_{L^2(\mathbb{R}^2)},$$

$$|(\Delta E, \Delta E)| = \|\Delta E\|^2_{L^2(\mathbb{R}^2)},$$

$$\begin{aligned}|(nE, \Delta E)| &\leq \|nE\|_{L^2(\mathbb{R}^2)}\|\Delta E\|_{L^2(\mathbb{R}^2)} \\ &\leq \|n\|_{L^4(\mathbb{R}^2)}\|E\|_{L^4(\mathbb{R}^2)}\|\Delta E\|_{L^2(\mathbb{R}^2)} \\ &\leq C\|n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\Delta E\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

$$|(\Gamma\Delta^2 E, \Delta E)| = \Gamma\|\nabla^3 E\|^2_{L^2(\mathbb{R}^2)}.$$

By (2.15), Lemma 2.3 and Lemma 2.4, it follows that

$$\|\nabla^3 E\|^2_{L^2(\mathbb{R}^2)} \leq C\left(\|E_t\|^2_{L^2(\mathbb{R}^2)} + C_1\right). \tag{2.16}$$

In addition, we take the inner product of (1.1) with E, and obtain

$$(iE_t + \Delta E - nE - \Gamma\Delta^2 E, E) = 0, \tag{2.17}$$

which yields

$$\|E_t\|_{L^2(\mathbb{R}^2)} \leq C\left(\|\Delta E\|_{L^2(\mathbb{R}^2)}\|E\|_{L^2(\mathbb{R}^2)} + \|nE\|_{L^2(\mathbb{R}^2)}\|E\|_{L^2(\mathbb{R}^2)} + \|\Delta E\|^2_{L^2(\mathbb{R}^2)}\right) \leq C. \tag{2.18}$$

Combining (2.14), (2.16) with (2.18), we have by Lemma 2.2 and Lemma 2.4

$$\begin{aligned} \|\nabla^3 E(.,t)\|^2_{L^2(\mathbb{R}^2)} &\leq C + C_1\int_0^t \|E(.,\tau)\|_{L^\infty(\mathbb{R}^2)}\|n_t(.,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau \\ &\leq C + C_2\int_0^t \|n_t(.,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau. \end{aligned} \tag{2.19}$$

On the other hand, differentiating (1.2) with respect to t and noting (1.3), we have

$$n_{tt} - \Delta n - \Delta|E|^2 + \Gamma\Delta^2 n = 0. \tag{2.20}$$

By taking the inner product of (2.20) with n_t, one gets

$$(n_{tt} - \Delta n - \Delta|E|^2 + \Gamma\Delta^2 n, n_t) = 0, \tag{2.21}$$

Since

$$(n_{tt} - \Delta n + \Gamma\Delta^2 n, n_t) = \frac{1}{2}\frac{d}{dt}\left(\|\nabla n\|^2_{L^2(\mathbb{R}^2)} + \|n_t\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\Delta n\|^2_{L^2(\mathbb{R}^2)}\right),$$

$$\begin{aligned} |(\Delta|E|^2, n_t)| &\quad 2\left(\left|(\Delta E\overline{E}, n_t)\right| + |(|\nabla E|^2, n_t)|\right) \\ &\leq 2\|E\|_{L^\infty(\mathbb{R}^2)}\|n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E\|_{L^2(\mathbb{R}^2)} + \|\nabla E\|^4_{L^4(\mathbb{R}^2)} + \|n_t\|^2_{L^2(\mathbb{R}^2)} \\ &\leq C\|E\|_{L^\infty(\mathbb{R}^2)}(\|n_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta E\|^2_{L^2(\mathbb{R}^2)}) \\ &\quad + C_1\|\nabla E\|^2_{L^2(\mathbb{R}^2)}\|\Delta E\|^2_{L^2(\mathbb{R}^2)} + \|n_t\|^2_{L^2(\mathbb{R}^2)} \\ &\leq C\|n_t\|^2_{L^2(\mathbb{R}^2)} + C_1, \end{aligned}$$

noting (2.21) and Lemma 2.4, we have

$$\|\Delta n\|^2_{L^2(\mathbb{R}^2)} + \|n_t\|^2_{L^2(\mathbb{R}^2)} \leq C + C_1\int_0^t \|n_t(.,\tau)\|^2_{L^2(\mathbb{R}^2)}d\tau. \tag{2.22}$$

Let

$$J(t) = \|\nabla^3 E\|^2_{L^2(\mathbb{R}^2)} + \|\Delta n\|^2_{L^2(\mathbb{R}^2)} + \|n_t\|^2_{L^2(\mathbb{R}^2)} + 1. \tag{2.23}$$

(2.19) and (2.22) conclude

$$J(t) \leq C + C\int_0^t J(\tau)d\tau,$$

which together with Gronwall's lemma implies

$$J(t) \le C. \tag{2.24}$$

(2.18) and (2.24) thus yield the estimate (2.11). □

It is easy to obtain the following conclusion by Lemma 2.2 and Lemma 2.5.

Corollary 2.1. There holds

$$\|E\|_{L^\infty(\mathbb{R}^2)} + \|\nabla E\|_{L^\infty(\mathbb{R}^2)} + \|n\|_{L^\infty(\mathbb{R}^2)} \le E_3, \tag{2.25}$$

where the constant E_3 depends only on $\|E_0\|^2_{H^3(\mathbb{R}^2)}$, $\|n_0\|^2_{H^2(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^1(\mathbb{R}^2)}$. □

Lemma 2.6 Under the conditions of Lemma 2.4, let

$$E_0(x) \in H^4(\mathbb{R}^2), \qquad n_0(x) \in H^3(\mathbb{R}^2), \qquad \mathbf{V}_0(x) \in H^2(\mathbb{R}^2).$$

Then the solution of the Cauchy problem (1.1)-(1.4) satisfies

$$\|\Delta^2 E\|^2_{L^2\times L^\infty} + \|\nabla^3 n\|^2_{L^2\times L^\infty} + \|\nabla E_t\|^2_{L^2\times L^\infty} + \|\nabla n_t\|^2_{L^2\times L^\infty} \le E_4, \tag{2.26}$$

where the constant E_4 depends only on $\|E_0\|^2_{H^4(\mathbb{R}^2)}$, $\|n_0\|^2_{H^3(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^2(\mathbb{R}^2)}$.

Proof. Differentiating (1.1) with respect to t, and then taking the inner product of the resulting equation with ΔE_t, we have

$$(iE_{tt} + \Delta E_t - (nE)_t - \Gamma\Delta^2 E_t, \Delta E_t) = 0. \tag{2.27}$$

By a direct calculation one sees

$$Im(iE_{tt}, \Delta E_t) = -\tfrac{1}{2}\tfrac{d}{dt}\|\nabla E_t\|^2_{L^2(\mathbb{R}^2)},$$

$$Im(\Delta E_t, \Delta E_t) = Im\|\Delta E_t\|^2_{L^2(\mathbb{R}^2)} = 0,$$

$$\begin{aligned} Im(-(nE)_t, \Delta E_t) &= Im(\nabla(n_t E + nE_t), \nabla E_t) \\ &= Im\left(\nabla n_t E + n_t \nabla E + \nabla n E_t + n\nabla E_t, \nabla E_t\right), \end{aligned}$$

$$Im(-\Gamma\Delta^2 E_t, \Delta E_t) = Im\Gamma\|\nabla^3 E_t\|^2_{L^2(\mathbb{R}^2)} = 0.$$

Therefore, (2.27), Corollary 2.1, Lemma 2.5 and Young's inequality give

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla E_t\|^2_{L^2(\mathbb{R}^2)} &\leq |(\nabla n_t E, \nabla E_t)| + |(n_t\nabla E, \nabla E_t)| \\
&\quad + |(\nabla n E_t, \nabla E_t)| + |(n\nabla E_t, \nabla E_t)| \\
&\leq \|E\|_{L^\infty(\mathbb{R}^2)}\|\nabla n_t\|_{L^2(\mathbb{R}^2)}\|\nabla E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad + \|\nabla E\|_{L^\infty(\mathbb{R}^2)}\|n_t\|_{L^2(\mathbb{R}^2)}\|\nabla E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad + \|\nabla n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\Delta n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|E_t\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla E_t\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla E_t\|_{L^2(\mathbb{R}^2)} \qquad (2.28)\\
&\quad + \|n\|_{L^\infty(\mathbb{R}^2)}\|\nabla E_t\|^2_{L^2(\mathbb{R}^2)} \\
&\leq C\left(\|\nabla n_t\|_{L^2(\mathbb{R}^2)}\|\nabla E_t\|_{L^2(\mathbb{R}^2)} + \|\nabla E_t\|_{L^2(\mathbb{R}^2)} + \|\nabla E_t\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)}\right) \\
&\leq C\left(\|\nabla n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_t\|^2_{L^2(\mathbb{R}^2)} + 1\right).
\end{aligned}$$

On the other hand, taking inner product of (2.20) with Δn_t, we obtain

$$(n_{tt} - \Delta n - \Delta|E|^2 + \Gamma\Delta^2 n, \Delta n_t) = 0. \qquad (2.29)$$

In view of Corollary 2.1, Lemma 2.4 and Lemma 2.5, the following estimates hold:

$$(n_{tt} - \Delta n + \Gamma\Delta^2 n, \Delta n_t) = -\frac{1}{2}\frac{d}{dt}\left(\|\nabla n_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta n\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)}\right),$$

$$\begin{aligned}
(-\Delta|E|^2, \Delta n_t) &= \big(\nabla^3 E\overline{E} + \Delta E\nabla\overline{E} + 2\nabla^2 E\nabla\overline{E} \\
&\quad + 2\nabla E\nabla^2\overline{E} + \nabla E\Delta\overline{E} + E\nabla^3\overline{E}, \nabla n_t\big) \\
&\leq C\big(\|E\|_{L^\infty(\mathbb{R}^2)}\|\nabla^3 E\|_{L^2(\mathbb{R}^2)} + \|\Delta E\|_{L^4(\mathbb{R}^2)}\|\nabla E\|_{L^4(\mathbb{R}^2)} \\
&\quad + \|\nabla^2 E\|_{L^4(\mathbb{R}^2)}\|\nabla E\|_{L^4(\mathbb{R}^2)})\|\nabla n_t\|_{L^2(\mathbb{R}^2)}\big) \\
&\leq C\left(1 + \|\nabla n_t\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned}$$

That is,

$$\frac{d}{dt}\left(\|\nabla n_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta n\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)}\right) \leq C\left(1 + \|\nabla n_t\|^2_{L^2(\mathbb{R}^2)}\right). \qquad (2.30)$$

Let

$$G(t) = \|\nabla n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta n\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)} + 1. \qquad (2.31)$$

On account of (1.1), one gets

$$\begin{aligned}
\|\Delta^2 E\|_{L^2(\mathbb{R}^2)} &\leq C\left(\|E_t\|_{L^2(\mathbb{R}^2)} + \|\Delta E\|_{L^2(\mathbb{R}^2)} + \|nE\|_{L^2(\mathbb{R}^2)}\right) \\
&\leq C + \|n\|^{\frac{1}{2}}_{L^4(\mathbb{R}^2)}\|E\|^{\frac{1}{2}}_{L^4(\mathbb{R}^2)} \leq C.
\end{aligned}$$

Hence, by (2.28), (2.30) and (2.31), we eventually obtain

$$G(t) \leq C + C\int_0^t G(\tau)d\tau,$$

which together with Gronwall's inequality yields

$$\|\nabla E_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)} \leq C. \tag{2.33}$$

(2.32) and (2.33) conclude the estimate (2.26). □

Corollary 2.2. The following estimate holds:

$$\|\Delta E\|_{L^\infty(\mathbb{R}^2)} + \|\nabla n\|_{L^\infty(\mathbb{R}^2)} + \|\mathbf{V}\|_{L^\infty(\mathbb{R}^2)} \leq E_5, \tag{2.34}$$

where the constant E_5 depends only on $\|E_0\|^2_{H^4(\mathbb{R}^2)}$, $\|n_0\|^2_{H^3(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^2(\mathbb{R}^2)}$. □

Lemma 2.7 Under the conditions of Lemma 2.4, let

$$E_0(x) \in H^5(\mathbb{R}^2), \qquad n_0(x) \in H^4(\mathbb{R}^2), \qquad \mathbf{V}_0(x) \in H^3(\mathbb{R}^2).$$

Then the solution of the Cauchy problem (1.1)-(1.4) satisfies

$$\|\nabla^5 E\|^2_{L^2\times L^\infty} + \|\Delta^2 n\|^2_{L^2\times L^\infty} + \|\Delta E_t\|^2_{L^2\times L^\infty} + \|\Delta n_t\|^2_{L^2\times L^\infty} \leq E_6, \tag{2.26}$$

where the constant E_4 depends only on $\|E_0\|^2_{H^5(\mathbb{R}^2)}$, $\|n_0\|^2_{H^4(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^3(\mathbb{R}^2)}$.

Proof. Differentiating (1.1) with respect to t, and then taking the inner product of the resulting equation with $\Delta^2 E_t$, we have

$$(iE_{tt} + \Delta E_t - (nE)_t - \Gamma\Delta^2 E_t, \Delta^2 E_t) = 0. \tag{2.36}$$

It is easy to check

$$Im(iE_{tt}, \Delta^2 E_t) = \tfrac{1}{2}\tfrac{d}{dt}\|\Delta E_t\|^2_{L^2(\mathbb{R}^2)},$$

$$Im(\Delta E_t, \Delta^2 E_t) = -\|\nabla^3 E_t\|^2_{L^2(\mathbb{R}^2)} = 0,$$

$$(-\Gamma\Delta^2 E_t, \Delta^2 E_t) = -\Gamma\|\Delta^2 E_t\|^2_{L^2(\mathbb{R}^2)},$$

$$\begin{aligned}(-(nE)_t, \Delta^2 E_t) &= -(n_t E + nE_t, \Delta^2 E_t)\\ &= -(\Delta(n_t E + nE_t), \Delta^2 E_t)\\ &= -(\Delta n_t E + 2\nabla n_t \nabla E + n_t\Delta E + \Delta n E_t\\ &\qquad +2\nabla n\nabla E_t + n\Delta E_t, \Delta E_t).\end{aligned}$$

Thus, taking the imaginary part of (2.36), we get from Corollary 2.1, Corollary 2.2, Lemma 2.5 and Lemma 2.6,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta E_t\|^2_{L^2(\mathbb{R}^2)} &\le |(\Delta n_t E, \Delta E_t)| + 2\,|(\nabla n_t \nabla E, \Delta E_t)| \\
&\quad + |(n_t \Delta E, \Delta E_t)| + |(\Delta n E_t, \Delta E_t)| \\
&\quad +2\,|(\nabla n \nabla E_t, \Delta E_t)| + |(n\Delta E_t, \Delta E_t)| \\
&\le \|E\|_{L^\infty(\mathbb{R}^2)}\|\Delta n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +2\|\nabla E\|_{L^\infty(\mathbb{R}^2)}\|\nabla n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +\|\Delta E\|_{L^\infty(\mathbb{R}^2)}\|n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +\|\Delta n\|_{L^4(\mathbb{R}^2)}\|E_t\|_{L^4(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +2\|\nabla n\|_{L^\infty(\mathbb{R}^2)}\|\nabla E_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +\|n\|_{L^\infty(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\le \|E\|_{L^\infty(\mathbb{R}^2)}\|\Delta n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +2\|\nabla E\|_{L^\infty(\mathbb{R}^2)}\|\nabla n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +\|\Delta E\|_{L^\infty(\mathbb{R}^2)}\|n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +\|\Delta n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla^3 n\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|E_t\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla E_t\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +2\|\nabla n\|_{L^\infty(\mathbb{R}^2)}\|\nabla E_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\quad +\|n\|_{L^\infty(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} \\
&\le C\left(\|\Delta n_t\|_{L^2(\mathbb{R}^2)}\|\Delta E_t\|_{L^2(\mathbb{R}^2)} + \|\Delta E_t\|_{L^2(\mathbb{R}^2)} + \|\Delta E_t\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\le C\left(\|\Delta n_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta E_t\|^2_{L^2(\mathbb{R}^2)} + 1\right).
\end{aligned}
\tag{2.37}
$$

We further obtain through taking inner product of (2.20) with $\Delta^2 n_t$ that

$$
(n_{tt} - \Delta n - \Delta|E|^2 + \Gamma\Delta^2 n, \Delta^2 n_t) = 0. \tag{2.38}
$$

According to Corollary 2.1, Corollary 2.2, Lemma 2.4, Lemma 2.5 and Lemma 2.6, the following estimates hold:

$$
(n_{tt} - \Delta n + \Gamma\Delta^2 n, \Delta^2 n_t) = \frac{1}{2}\frac{d}{dt}\left(\|\Delta n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\Delta^2 n\|^2_{L^2(\mathbb{R}^2)}\right),
$$

$$
\begin{aligned}
(\Delta|E|^2, \Delta^2 n_t) &= (\Delta^2|E|^2, \Delta n_t) \\
&= (\Delta^2 E\overline{E} + E\Delta^2\overline{E} + 4\nabla^3 E\nabla\overline{E} + 4\nabla E\nabla^3\overline{E} \\
&\qquad +2\Delta E\Delta\overline{E} + 4\nabla^2 E\nabla^2\overline{E}, \Delta n_t) \\
&\le \big(\|E\|_{L^\infty(\mathbb{R}^2)}\|\Delta^2 E\|_{L^2(\mathbb{R}^2)} + \|\nabla E\|_{L^\infty(\mathbb{R}^2)}\|\nabla^3 E\|_{L^2(\mathbb{R}^2)} \\
&\qquad + \|\Delta E\|_{L^\infty(\mathbb{R}^2)}\|\Delta E\|_{L^2(\mathbb{R}^2)} + \|\nabla^2 E\|^2_{L^4(\mathbb{R}^2)}\big)\,\|\Delta n_t\|_{L^2(\mathbb{R}^2)} \\
&\le \left(C + \|\nabla^2 E\|_{L^2(\mathbb{R}^2)}\|\nabla^3 E\|_{L^2(\mathbb{R}^2)}\right)\|\Delta n_t\|_{L^2(\mathbb{R}^2)} \\
&\le C(1 + \|\Delta n_t\|^2_{L^2(\mathbb{R}^2)}).
\end{aligned}
$$

We thus obtain

$$
\frac{d}{dt}\left(\|\Delta n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\Delta^2 n\|^2_{L^2(\mathbb{R}^2)}\right) \le C(1 + \|\Delta n_t\|^2_{L^2(\mathbb{R}^2)}). \tag{2.39}
$$

Let

$$J(t) = \|\Delta E_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\Delta^2 n\|^2_{L^2(\mathbb{R}^2)}. \tag{2.40}$$

Combining (2.37) and (2.39) with (2.40), one has

$$J(t) \le C + C\int_0^t J(\tau)d\tau,$$

which together with Gronwall's inequality and Lemma 2.6 yields

$$\|\Delta E_t\|^2_{L^2(\mathbb{R}^2)} + \|\Delta n_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^3 n\|^2_{L^2(\mathbb{R}^2)} + \|\Delta^2 n\|^2_{L^2(\mathbb{R}^2)} \le C. \tag{2.41}$$

On the other hand, from (1.1) it follows that

$$i\nabla E_t + \nabla^3 E = \nabla(nE) + \Gamma\nabla^5 E.$$

Furthermore, Lemma 2.5 and Lemma 2.6 show that

$$\|\nabla^5 E\|_{L^2(\mathbb{R}^2)} \le \|\nabla E_t\|_{L^2(\mathbb{R}^2)} + \|\nabla^3 E\|_{L^2(\mathbb{R}^2)} + \|\nabla n E\|_{L^2(\mathbb{R}^2)} + \|n\nabla E\|_{L^2(\mathbb{R}^2)} \le C. \tag{2.42}$$

Hence (2.35) is valid by (2.41) and (2.42). □

By Lemma 2.2, Lemma 2.7 implies the following estimates:

Corollary 2.3.

$$\begin{aligned} &\|\nabla^3 E\|_{L^\infty(\mathbb{R}^2)} + \|\Delta n\|_{L^\infty(\mathbb{R}^2)} + \|E_t\|_{L^\infty(\mathbb{R}^2)} + \|n_t\|_{L^\infty(\mathbb{R}^2)} \le E_7, \\ &\|n_{tt}\|_{L^2(\mathbb{R}^2)} \le E_8, \end{aligned} \tag{2.43}$$

where E_7 and E_8 depend only on $\|E_0\|^2_{H^5(\mathbb{R}^2)}$, $\|n_0\|^2_{H^4(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^3(\mathbb{R}^2)}$. □

Lemma 2.8 Under the conditions of Lemma 2.4, let

$$E_0(x) \in H^{m+3}(\mathbb{R}^2), \qquad n_0(x) \in H^{m+2}(\mathbb{R}^2), \qquad \mathbf{V}_0(x) \in H^{m+1}(\mathbb{R}^2).$$

Then the solution of the Cauchy problem (1.1)-(1.4) satisfies

$$\begin{aligned} &\|\nabla^m E_t\|^2_{L^2\times L^\infty} + \|\nabla^m n_t\|^2_{L^2\times L^\infty} + \|\nabla^{m+3} E\|^2_{L^2\times L^\infty} \\ &\quad + \|\nabla^{m+2} n\|^2_{L^2\times L^\infty} + \|\nabla^{m+1}\mathbf{V}\|^2_{L^2\times L^\infty} \le E_8, \end{aligned} \tag{2.44}$$

where the constant E_9 depends only on $\|E_0\|^2_{H^{m+3}(\mathbb{R}^2)}$, $\|n_0\|^2_{H^{m+2}(\mathbb{R}^2)}$ and $\|\mathbf{V}_0\|^2_{H^{m+1}(\mathbb{R}^2)}$.

Proof. We shall show this lemma by induction on m.

For $m = 0, 1$, (2.44) follows from Lemma 2.5 and Lemma 2.6.

For $m = 2$, (2.44) is also true by Lemma 2.7.

Now, we suppose that the estimate (2.44) is true for $m = k \ge 2$, that is,

$$\begin{aligned} &\|\nabla^k E_t\|^2_{L^2\times L^\infty} + \|\nabla^k n_t\|^2_{L^2\times L^\infty} + \|\nabla^{k+3} E\|^2_{L^2\times L^\infty} \\ &\quad + \|\nabla^{k+2} n\|^2_{L^2\times L^\infty} + \|\nabla^{k+1}\mathbf{V}\|^2_{L^2\times L^\infty} \le E_9. \end{aligned} \tag{2.45}$$

In the following, we show the estimate (2.44) is also true for $m = k + 1$.

Differentiating (1.1) with respect to t, and then taking the inner product of the resulting equation with $\Delta^{k+1} E_t$, we have

$$(iE_{tt} + \Delta E_t - (nE)_t - \Gamma\Delta^2 E_t, \Delta^{k+1} E_t) = 0. \tag{2.46}$$

By a direct calculation, one has

$$Im(iE_{tt}, \Delta^{k+1}E_t) = \tfrac{1}{2}(-1)^{k+1}\tfrac{d}{dt}\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)},$$

$$Im(\Delta E_t, \Delta^{k+1}E_t) = Im(-1)^k\|\nabla^{k+2}E_t\|^2_{L^2(\mathbb{R}^2)} = 0,$$

$$Im(-\Gamma\Delta^2 E_t, \Delta^{k+1}E_t) = Im(-1)^{k-1}\|\nabla^{k+3}E_t\|^2_{L^2(\mathbb{R}^2)} = 0.$$

(2.46) immediately thereby yields that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)} \leq \big|((nE)_t, \Delta^{k+1}E_t)\big| \\
&= \big|(n_tE + nE_t, \Delta^{k+1}E_t)\big| \\
&\leq \big|(\nabla^{k+1}(n_tE + nE_t), \nabla^{k+1}E_t)\big| \\
&\leq C\left(\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^{k+1}(n_tE)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla^{k+1}(nE_t)\|^2_{L^2(\mathbb{R}^2)}\right) \\
&\leq C\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)} + C\|E\|_{L^\infty(\mathbb{R}^2)}\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)} + C\|\nabla^{k+1}E\|^2_{L^4(\mathbb{R}^2)}\|n_t\|^2_{L^4(\mathbb{R}^2)} \\
&\quad + C\|n\|_{L^\infty(\mathbb{R}^2)}\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)} + C\|\nabla^{k+1}n\|^2_{L^4(\mathbb{R}^2)}\|E_t\|^2_{L^4(\mathbb{R}^2)} \\
&\quad + C\sum_{i+j=k+1}\|\nabla^j E\|_{L^\infty(\mathbb{R}^2)}\|\nabla^i n_t\|^2_{L^4(\mathbb{R}^2)} \\
&\quad + C\sum_{i+j=k+1}\|\nabla^i n\|_{L^\infty(\mathbb{R}^2)}\|\nabla^j E_t\|^2_{L^4(\mathbb{R}^2)},
\end{aligned} \tag{2.47}$$

in which $i, j \in \mathbb{Z}^+$.

Whereas, (2.45) and Lemma 2.2 conclude that for $i \leq k+1$, $j \leq k$,

$$\|\nabla^i E\|_{L^\infty(\mathbb{R}^2)} + \|\nabla^j n\|_{L^\infty(\mathbb{R}^2)} \leq E_{10}. \tag{2.48}$$

Combining (2.45) with (2.47) and (2.48), noting that Lemma 2.4-Lemma 2.7, Corollary 2.1-Corollary 2.3, one gets

$$\frac{d}{dt}\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)} \leq C(1 + \|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)}), \tag{2.49}$$

which together with Gronwall's inequality yields

$$\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)} \leq C. \tag{2.50}$$

On the other hand, we take inner product of (2.21) with $\Delta^{k+1}n_t$, and then obtain

$$(n_{tt} - \Delta n - \Delta|E|^2 + \Gamma\Delta^2 n, \Delta^{k+1}n_t) = 0, \tag{2.51}$$

which manifests

$$
\begin{aligned}
&\tfrac{d}{dt}\left[\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+2}n\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+3}n\|^2_{L^2(\mathbb{R}^2)}\right]\\
&\quad=(\nabla^{k+3}|E|^2,\nabla^{k+1}n_t)\\
&\quad\le\|\nabla^{k+3}|E|^2\|_{L^2(\mathbb{R}^2)}\|\nabla^{k+1}n_t\|_{L^2(\mathbb{R}^2)}\\
&\quad\le C\left(\|\nabla^{k+3}|E|^2\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\quad\le C\Bigg(\|\nabla^{k+3}E\|^2_{L^4(\mathbb{R}^2)}\|E\|^2_{L^4(\mathbb{R}^2)}\\
&\qquad\qquad+\sum_{i,j\in\mathbb{Z}^+ i+j=k+3}\|\nabla^{i}E\|^2_{L^4(\mathbb{R}^2)}\|\nabla^{j}E\|^2_{L^4(\mathbb{R}^2)}+\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}\Bigg)\\
&\quad\le C\left(1+\|\nabla^{k+4}E\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned}
\tag{2.52}
$$

Similarly, on account of (1.1), we can show

$$
\begin{aligned}
\|\nabla^{k+1}E\|^2_{L^2(\mathbb{R}^2)}\ &\le C\Bigg(\|\nabla^{k}E_t\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+2}E\|^2_{L^2(\mathbb{R}^2)}\\
&\qquad+\|E\|^2_{L^\infty(\mathbb{R}^2)}\|\nabla^{k}n\|^2_{L^2(\mathbb{R}^2)}+\|n\|_{L^\infty(\mathbb{R}^2)}\|\nabla^{k}E\|^2_{L^2(\mathbb{R}^2)}\\
&\qquad+\sum_{i,j\in\mathbb{Z}^+,i+j=k}\|\nabla n\|^2_{L^2(\mathbb{R}^2)}\|\nabla^{j}E\|_{L^4(\mathbb{R}^2)}\Bigg),
\end{aligned}
\tag{2.53}
$$

which together with (2.45) concludes

$$
\|\nabla^{k+4}E\|^2_{L^2(\mathbb{R}^2)}\le C. \tag{2.54}
$$

Combining (2.52) with (2.54), it is easy to check that

$$
\begin{aligned}
&\tfrac{d}{dt}\left[\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+2}n\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+3}n\|^2_{L^2(\mathbb{R}^2)}\right]\\
&\quad\le C\left(1+\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned}
\tag{2.55}
$$

Let

$$
H(t)=1+\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+2}n\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+3}n\|^2_{L^2(\mathbb{R}^2)}. \tag{2.56}
$$

Gronwall's inequality manifests that

$$
H(t)\le C. \tag{2.57}
$$

(2.50), (2.54), (2.56) and (2.57) thereby conclude the estimate:

$$
\begin{aligned}
&\|\nabla^{k+1}E_t\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+1}n_t\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+4}E\|^2_{L^2(\mathbb{R}^2)}\\
&\quad+\|\nabla^{k+3}n\|^2_{L^2(\mathbb{R}^2)}+\|\nabla^{k+1}\mathbf{V}\|^2_{L^2(\mathbb{R}^2)}\le C.
\end{aligned}
\tag{2.58}
$$

(2.44) hence is true for $m = k + 1$.

This completes the proof of Lemma 2.8. $\Box$

3 Existence and Uniqueness of Global Smooth solutions

In this section, we shall consider the existence and uniqueness of global smooth solutions to the Cauchy problem (1.1)-(1.4) by using those estimates obtained in Section 2.

Theorem 3.1 Under the conditions of Lemma 2.4, let

$$E_0(x) \in H^{m+3}(\mathbb{R}^2), \qquad n_0(x) \in H^{m+2}(\mathbb{R}^2), \qquad \mathbf{V}_0(x) \in H^{m+1}(\mathbb{R}^2), \quad m \geq 0.$$

The Cauchy problem (1.1)-(1.4) then admits a global smooth solution such that

$$\begin{aligned}
&E(x,t) \in L^\infty\left(0,T; H^{m+3}(\mathbb{R}^2)\right), \qquad &&E_t(x,t) \in L^\infty\left(0,T; H^{m}(\mathbb{R}^2)\right), \\
&n(x,t) \in L^\infty\left(0,T; H^{m+2}(\mathbb{R}^2)\right), \qquad &&n_t(x,t) \in L^\infty\left(0,T; H^{m}(\mathbb{R}^2)\right), \\
&\mathbf{V}(x,t) \in L^\infty\left(0,T; H^{m+1}(\mathbb{R}^2)\right), \qquad &&\mathbf{V}_t(x,t) \in L^\infty\left(0,T; H^{m-1}(\mathbb{R}^2)\right).
\end{aligned} \tag{3.1}$$

Theorem 3.2 The global smooth solution of the Cauchy problem (1.1)-(1.4) is unique.

We now begin to prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1.

By using Galerkin method, we choose the basic periodic functions $w_s(x)$ as follows:

$$-\Delta w_s(x) = \lambda_s w_s(x), \qquad w_s(x) \in H^{m+3}(\Omega), \quad s = 1, 2, \cdots, l.$$

Then the approximate solution of problem (1.1)-(1.4) can be written as

$$\begin{aligned}
E_l(x,t) &= \sum_{s=1}^{l} \alpha_{sl}(t) w_s(x), \\
n_l(x,t) &= \sum_{s=1}^{l} \beta_{sl}(t) w_s(x), \\
\mathbf{V}_l(x,t) &= \sum_{s=1}^{l} \gamma_{sl}(t) w_s(x),
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
\mathbf{V}_l(x,t) &= (V_{l1}(x,t), V_{l2}(x,t)), \\
\gamma_{sl}(x,t) &= (\gamma_{sl1}(x,t), \gamma_{sl2}(x,t)),
\end{aligned} \tag{3.3}$$

and Ω is a two-dimensional cube with $2R$ in each direction, that is,

$$\overline{\Omega} = \left\{ x = (x_1, x_2) : \quad |x_i| \leq 2R, \quad i = 1, 2 \right\}.$$

The above undetermined coefficients α_{sl}, β_{sl} and γ_{sl} need to satisfy the following initial value problem:

$$i(E_{lt}, w_s) + (\Delta E_l, w_s) - (n_l E_l, w_s) - \Gamma(\Delta^2 E_l, w_s) = 0, \tag{3.4}$$

$$(n_{lt} + \nabla \cdot \mathbf{V}_l, w_s) = 0, \tag{3.5}$$

$$(\mathbf{V}_{lt}, w_s) + (\nabla n_l, w_s) + (\nabla |E_l|^2, w_s) - \Gamma(\nabla \Delta n_l, w_s) = 0, \tag{3.6}$$

$$E_l(x,0) = E_l^0(x), \quad n_l(x,0) = n_l^0(x), \quad \mathbf{V}_l(x,0) = \mathbf{V}_l^0(x), \tag{3.7}$$

where $\mathbf{V}_l(x,0) = (V_{l1}(x,0), V_{l2}(x,0)) = (V_{l1}^0(x), V_{l2}^0(x)) = \mathbf{V}_l^0(x)$. $s = 1, \cdots, l$, $(u,v) = \int_\Omega u(x)v(x)dx$, $V_{lk}(x,t) = \sum\limits_{s=1}^{l} \gamma_{slk}(t) w_s(x)$, $k = 1,2$, $|\mathbf{V}_l|^2 = \sum\limits_{k=1}^{2} |V_{lk}|^2$.
Suppose that as $l \to \infty$,

$$E_l^0(x) \to E^0(x), \quad n_l^0(x) \to n^0(x), \quad \mathbf{V}_l^0(x) \to \mathbf{V}^0(x) \quad in \quad H^{m+3}(\Omega). \tag{3.8}$$

Making the similar estimates to Lemma 2.4-Lemma 2.8 and Corollary 2.1-Corollary 2.3, we establish the following estimates for the solution $(E_l(x,t), n_l(x,t), \mathbf{V}_l(x,t))$ of the Cauchy problem (3.4)-(3.7):

$$\sup_{0 \le t \le T_0} \left[\|E_l(x,t)\|_{H^{m+3}(\Omega)} + \|n_l(x,t)\|_{H^{m+2}(\Omega)} + \|\mathbf{V}_l(x,t)\|_{H^{m+1}(\Omega)} \right] \le E_{11}, \tag{3.9}$$

$$\sup_{0 \le t \le T_0} \left[\|E_{lt}(x,t)\|_{H^{m}(\Omega)} + \|n_{lt}(x,t)\|_{H^{m}(\Omega)} + \|\mathbf{V}_{lt}(x,t)\|_{H^{m+1}(\Omega)} \right] \le E_{12}, \tag{3.10}$$

where T_0 depends only on $\|E_l^0(x)\|_{H^{m+3}(\Omega)}$, $\|n_l^0(x)\|_{H^{m+2}(\Omega)}$, $\|\mathbf{V}_l^0(x)\|_{H^{m+1}(\Omega)}$, E_{11} and E_{12} are independent of l and R.

By utilizing the Galerkin method, and making the similar procedures to that in [21], the periodic boundary value problem of the system (1.1)-(1.4) admits a local smooth solution

$$E(x,t) \in L^\infty\left(0, T_0; H^{m+3}(\Omega)\right), \quad E_t(x,t) \in L^\infty\left(0, T_0; H^{m}(\Omega)\right),$$

$$n(x,t) \in L^\infty\left(0, T_0; H^{m+2}(\Omega)\right), \quad n_t(x,t) \in L^\infty\left(0, T_0; H^{m}(\Omega)\right),$$

$$\mathbf{V}(x,t) \in L^\infty\left(0, T_0; H^{m+1}(\Omega)\right), \quad \mathbf{V}_t(x,t) \in L^\infty\left(0, T_0; H^{m-1}(\Omega)\right).$$

Therefore, by the uniform boundedness of the initial data in the relative Hilbert spaces, and the independence of the estimates for the approximate solution with respect to R, letting $R \to \infty$, we get the existence of local smooth solution for the Cauchy problem (1.1)-(1.4). In addition, noting that Lemma 2.1-Lemma 2.8 and by using continuity argument, we obtain the global existence of a smooth solution to the Cauchy problem (1.1)-(1.4).

This completes the proof of Theorem 3.1. □

We then prove Theorem 3.2.

Proof of Theorem 3.2. Suppose that $\{E_i(x,t), \ n_i(x,t), \ \mathbf{V}_i(x,t),\}$ $(i = 1,2)$ is the global solution to the Cauchy problem (1.1)-(1.4). Set

$$\begin{aligned} \psi(x,t) &= E_1(x,t) - E_2(x,t), \\ \varphi(x,t) &= n_1(x,t) - n_2(x,t), \\ \mathbf{\Phi}(x,t) &= \mathbf{V}_1(x,t) - \mathbf{V}_2(x,t). \end{aligned} \tag{3.11}$$

$(\psi, \varphi, \mathbf{\Phi})$ then solves the following Cauchy problem according to (1.1)-(1.4):

$$i\psi_t + \Delta\psi = n_1 E_1 - n_2 E_2 + \Gamma \Delta^2 \psi, \tag{3.12}$$

$$\varphi_t = -\nabla \cdot \mathbf{\Phi}, \tag{3.13}$$

$$\mathbf{\Phi}_t = -\nabla\varphi - (\nabla|E_1|^2 - \nabla|E_2|^2) + \Gamma\nabla\Delta\varphi, \tag{3.14}$$

$$\psi(x,0) = 0, \quad \varphi(x,0) = 0, \quad \mathbf{\Phi}(x,0) = 0. \tag{3.15}$$

Differentiating (3.12) with respect to t, and then taking the inner product of the resulting equation with ψ_t, we have

$$(i\psi_{tt} + \Delta\psi_t - (n_1E_1 - n_2E_2)_t - \Gamma\Delta^2\psi_t, \psi_t) = 0. \tag{3.16}$$

Taking the imaginary part for (3.16), one gets

$$\begin{aligned} \tfrac{d}{dt}\|\psi_t\|^2_{L^2(\mathbb{R}^2)} &= Im\left((n_1E_1 - n_2E_2)_t, \psi_t\right) \\ &= Im\left((n_1\psi + \varphi E_2)_t, \psi_t\right) \\ &= Im\left((n_{1t}\psi + n_1\psi_t + \varphi_t E_2 + \varphi E_{2t}, \psi_t\right). \end{aligned} \tag{3.17}$$

In addition, the following estimates hold immediately through a direct calculation:

$$\begin{aligned} |Im\,(n_1\psi_t, \psi_t)| &\le \left|Im \textstyle\int_{\mathbb{R}^2} n_1|\psi_t|^2 dx\right| = 0, \\ |Im\,(n_{1t}\psi, \psi_t)| &= \left|Im \textstyle\int_{\mathbb{R}^2} n_{1t}\psi\psi_t dx\right| \\ &\le \|n_{1t}\|_{L^\infty(\mathbb{R}^2)}\|\psi\|_{L^2(\mathbb{R}^2)}\|\psi_t\|_{L^2(\mathbb{R}^2)} \\ &\le C\|\psi\|_{L^2(\mathbb{R}^2)}\|\psi_t\|_{L^2(\mathbb{R}^2)}, \\ |Im\,(\varphi_t E_2, \psi_t)| &\le \|E_2\|_{L^\infty(\mathbb{R}^2)}\|\varphi_t\|_{L^2(\mathbb{R}^2)}\|\psi_t\|_{L^2(\mathbb{R}^2)} \\ &\le C\|\varphi_t\|_{L^2(\mathbb{R}^2)}\|\psi_t\|_{L^2(\mathbb{R}^2)}, \\ |Im\,(\varphi E_{2t}, \psi_t)| &\le C\|E_{2t}\|_{L^\infty(\mathbb{R}^2)}\|\varphi\|_{L^2(\mathbb{R}^2)}\|\psi_t\|_{L^2(\mathbb{R}^2)} \\ &\le C\|\varphi\|_{L^2(\mathbb{R}^2)}\|\psi_t\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.18}$$

(3.17) and (3.18) thereby conclude

$$\frac{d}{dt}\|\psi_t\|^2_{L^2(\mathbb{R}^2)} \le C\left(\|\psi\|^2_{L^2(\mathbb{R}^2)} + \|\psi_t\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^2)}\right). \tag{3.19}$$

Similarly, (3.12) manifests

$$(i\psi_t + \Delta\psi_t - (n_1E_1 - n_2E_2) - \Gamma\Delta^2\psi, \psi) = 0. \tag{3.20}$$

Take the imaginary part of (3.20), it is easy to obtain

$$\frac{d}{dt}\|\psi\|^2_{L^2(\mathbb{R}^2)} \le C\left(\|\psi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)}\right). \tag{3.21}$$

On the other hand, (3.13) and (3.14) yield

$$(\varphi_{tt} - \Delta\varphi - (\Delta|E_1|^2 - \Delta|E_2|^2) + \Gamma\Delta^2\varphi, \varphi_t) = 0 \tag{3.22}$$

which implies

$$\begin{aligned} &\frac{d}{dt}\left(\|\varphi_t\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} + \Gamma\|\Delta\varphi\|^2_{L^2(\mathbb{R}^2)}\right) \\ &\quad \leq C\left(\|\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\psi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^2)}\right). \end{aligned} \tag{3.23}$$

It is easy to check

$$\begin{aligned} \tfrac{d}{dt}\|\varphi\|^2_{L^2(\mathbb{R}^2)} &\leq 2\|\varphi\|_{L^2(\mathbb{R}^2)}\|\varphi_t\|_{L^2(\mathbb{R}^2)} \\ &\leq \|\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.24}$$

Thus, (3.19), (3.21), (3.23) and (3.24) conclude

$$\begin{aligned} &\tfrac{d}{dt}\Big(\|\psi_t\|^2_{L^2(\mathbb{R}^2)} + \|\psi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^2)} \\ &\qquad + \|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\Delta\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)}\Big) \\ &\leq C\Big(\|\psi_t\|^2_{L^2(\mathbb{R}^2)} + \|\psi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^2)} \\ &\qquad + \|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\Delta\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi\|^2_{L^2(\mathbb{R}^2)}\Big). \end{aligned} \tag{3.25}$$

Gronwall's inequality together with the zero initial data conditions (3.15) deduces that

$$\psi \equiv \varphi \equiv \varphi_t \equiv \psi_t \equiv \nabla\varphi \equiv 0. \tag{3.26}$$

It is also easy to check that $\mathbf{\Phi} \equiv 0$ by $\nabla.\mathbf{\Phi} = 0$, $\mathbf{\Phi}_t = 0$ and $\mathbf{\Phi}(0, x) = 0$. Furthermore, (3.26) and (3.27) conclude the uniqueness of global smooth solution to the Cauchy problem (1.1)-(1.4). □

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