Cramér type moderate deviations for stationary sequences of bounded random variables

Xiequan Fan

Center for Applied Mathematics, Tianjin University, Tianjin, China

Received *****; accepted after revision +++++

Presented by

Abstract

We derive Cramér type moderate deviations for stationary sequences of bounded random variables. Our results imply the moderate deviation principles and a Berry-Esseen bound. Applications to quantile coupling inequalities, functions of ϕ-mixing sequences, and contracting Markov chains are discussed. *To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

Résumé

Déviations modérés de type Cramér pour les séquences stationnaires. Nous dérivons les déviations modérées de type Cramér pour des séquences stationnaires de variables aléatoires bornées. Nos résultats impliquent les principes de déviation modérée et un théoreme de Berry-Esseen. Les applications aux inégalités de couplage quantile, fonctions des séquences de mélange, et des chaînes de Markov contractantes sont discutées. *Pour citer cet article : A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

1. Introduction

For the stationary sequence $(X_i)_{i\in\mathbf{Z}}$ of centered random variables, define the partial sums and the normalized partial sums process by

$$S_n = \sum_{i=1}^{n} X_i \qquad \text{and} \qquad W_n = \frac{1}{\sqrt{n}} S_n,$$

Email address: `fanxiequan@hotmail.com` (Xiequan Fan).

respectively. We say that the sequence of random variables $\{W_n, n > 1\}$ satisfies the moderate deviation principle (MDP) with speed $a_n \to 0$ and good rate function $I(\cdot)$, if the level set $\{x,\, I(x) \leq t\}$ are compact for all $t \in \mathbb{R}$, and for all Borel sets B,

$$-\inf_{x \in B^o} I(x) \leq \liminf_{n\to\infty} a_n^2 \ln \mathbf{P}\big(a_n W_n \in B\big) \leq \limsup_{n\to\infty} a_n^2 \ln \mathbf{P}\big(a_n W_n \in B\big) \leq -\inf_{x\in \overline{B}} I(x), \tag{1}$$

where B^o denotes the interior of B, $\overline{B}$ the closure of B, and the infimum of a function over an empty set is interpreted as ∞. The MDP is an intermediate behavior between the central limit theorem ($a_n = O(1)$) and large deviations ($a_n \asymp \frac{1}{\sqrt{n}}$).

The MDP results have been obtained by several authors. De Acosta [2] applied Laplace approximations to prove the MDP for sums of independent random vectors. Dembo [5] showed that the MDP holds for the trajectory of a locally square integrable martingale with bounded jumps as soon as its quadratic covariation converges in probability at an exponential rate. Gao [9] and Djellout [6] obtained the MDP for martingales with non-bounded differences and ϕ-mixing sequences with summable mixing rate. Dedecker et al. [3] derived the MDP for stationary sequences of bounded random variables under martingale-type conditions. It is known that the MDP results for stationary sequences can be applied in a variety of settings. For instance, Dedecker et al. [3] showed that such type of results can be applied to functions of $\phi-$mixing sequences, contracting Markov chains, expanding maps of the interval, and symmetric random walks on the circle.

In this paper we are concerned with Cramér type moderate deviations for stationary sequences. Cramér type moderate deviations usually imply the MDP results; see Fan et al. [7] for instance. Furthermore, Cramér type moderate deviations imply Berry-Esseen bounds; see Corollary 2.2. Following the excellent work of Mason and Zhou [13] and Dedecker et al. [3], we apply our results to quantile coupling inequalities, functions of ϕ-mixing sequences, and contracting Markov chains.

Our approach is based on martingale approximation and Cramér type moderate deviations for martingales due to Fan et al. [7]. Cramér type moderate deviations for martingales have been established by Račkauskas [16,17], Grama [10] and Grama and Haeusler [11,12]. Such type of results are very useful for study of stationary sequences, for instance, Wu and Zhao [20] applied the results of Grama [10] to establish Cramér type moderate deviations for stationary sequences with physical dependence measure introduced by Wu [19], functionals of linear processes and some nonlinear time series. See also Cuny and Merlevède [1] (cf. Theorem 3.2 therein) for a result similar to Wu and Zhao [20], where Cuny and Merlevède [1] established a Cramér type moderate deviations for an adapted stationary sequence in $\mathbf{L}^p$. For relationship among our results and the last two results, we refer to point 3 of Remark 1.

The paper is organized as follows. Our main results are stated and discussed in Section 2. The applications are given in Section 3. Proofs of theorems are deferred to Section 4.

2. Main results

From now on, assume that the stationary sequence $(X_i)_{i\in\mathbf{Z}}$ is given by $X_i = X_0 \circ T^i$, where $T : \Omega \mapsto \Omega$ is a bijective bimeasurable transformation preserving the probability $\mathbf{P}$ on $(\Omega, \mathcal{F})$. For a subfield $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, let $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Our theorems and their corollaries treat the so-called adapted case, that is X_0 being $\mathcal{F}_0$-measurable and so the sequence $(X_i)_{i\in\mathbf{Z}}$ is adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$. Moreover, we denote the $\mathbf{L}^\infty$-norm by $\|X\|_\infty$, that is the smallest u such that $\mathbf{P}(|X| > u) = 0$.

Throughout the paper, let $m = m(n)$ be integers such that $1 \leq m \leq n$. For instance, we may take $m = \lfloor n^\alpha \rfloor, \alpha \in (0, \frac{1}{2})$, where $\lfloor x \rfloor$ stands for the largest integer less than x. Denote

$$\varepsilon_m = \frac{m}{n^{1/2}\sigma_n}\|X_0\|_\infty, \tag{2}$$

$$\gamma_m = \frac{1}{m^{1/2}\sigma_n}\sum_{j=1}^{\infty}\frac{1}{j^{3/2}}\Big\|\mathbf{E}[S_{mj}|\mathcal{F}_0]\Big\|_\infty \tag{3}$$

and

$$\delta_m^2 = \frac{1}{m\sigma_n^2}\Big\|\mathbf{E}[S_m|\mathcal{F}_0]\Big\|_\infty^2 + \Big\|\frac{1}{m\sigma_n^2}\mathbf{E}[S_m^2|\mathcal{F}_0]-1\Big\|_\infty, \tag{4}$$

where $\sigma_n = \sqrt{\mathbf{E}W_n^2} > 0$. The following theorem gives a Cramér type moderate deviation result for stationary sequences.

Theorem 2.1 *Assume that $\|X_0\|_\infty < \infty$, and that X_0 is $\mathcal{F}_0$-measurable. Then there exists an absolute constant $\alpha_0 > 0$ such that when $\varepsilon_m \le \frac{1}{4}$, $\gamma_m \le e^{-(80)^2}$ and $\delta_m^2 + \frac{m}{n} \le \alpha_0$, it holds for all $0 \le x \le \alpha_0\varepsilon_m^{-1}$,*

$$\left|\ln\frac{\mathbf{P}(W_n \ge x\sigma_n)}{1-\Phi(x)}\right| \le C_{\alpha_0}\bigg(x^3\varepsilon_m + x^2(\delta_m^2 + \frac{m}{n} + \gamma_m|\ln\gamma_m|)$$
$$+(1+x)\big(\varepsilon_m|\ln\varepsilon_m| + \gamma_m|\ln\gamma_m| + \delta_m + \sqrt{\frac{m}{n}}\big)\bigg),$$

where C_{α_0} depends only on α_0. In particular, the last inequality implies that

$$\frac{\mathbf{P}(W_n \ge x\sigma_n)}{1-\Phi(x)} = 1 + o(1) \tag{5}$$

uniformly for $0 \le x = o(\min\{\varepsilon_m^{-1/3}, \delta_m^{-1}, (n/m)^{1/2}, (\gamma_m|\ln\gamma_m|)^{-1/2}\})$ as $m \to \infty$. Moreover, the same results hold when replacing $\frac{\mathbf{P}(W_n \ge x\sigma_n)}{1-\Phi(x)}$ by $\frac{\mathbf{P}(W_n \le -x\sigma_n)}{\Phi(-x)}$.

Remark 1 Let us comment on the results of Theorem 2.1.

(i) Assume that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\Big\|\mathbf{E}[S_n|\mathcal{F}_0]\Big\|_\infty < \infty, \tag{6}$$

and that there exists $\sigma > 0$ such that

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\mathbf{E}[S_n^2|\mathcal{F}_0]-\sigma^2\Big\|_\infty = 0. \tag{7}$$

The conditions (6) and (7) were introduced by Dedecker et al. [3]. Assume that $m \to \infty$ and $m/\sqrt{n} \to 0$ as $n \to \infty$. By Lemma 29 of Dedecker et al. [3], the assumptions of Theorem 2.1 hold with $\max\{\varepsilon_m, \gamma_m, \delta_m\} \to 0$ as $n \to \infty$.

(ii) If $(X_i, \mathcal{F}_i)_{i\in\mathbf{Z}}$ is a martingale difference sequence, then Theorem 2.1 gives a Cramér type moderate deviation result with

$$\gamma_m = 0 \quad \text{and} \quad \delta_m^2 = \Big\|\frac{1}{m\sigma_n^2}\sum_{i=1}^{m}\mathbf{E}[X_i^2|\mathcal{F}_0]-1\Big\|_\infty,$$

which is similar to the main theorem of Grama and Haeusler [11] (see also Fan et al. [7]).

(iii) The range of equality (5) can be very large. For instance, if $\lim_{n\to\infty}\sigma_n^2=\sigma^2>0$, $\big\|\mathbf{E}[S_n|\mathcal{F}_0]\big\|_\infty = O(1)$ and $\big\|\frac{1}{n}\mathbf{E}[S_n^2|\mathcal{F}_0]-\sigma_n^2\big\|_\infty = O\big(\frac{1}{n}\big)$ as $n\to\infty$, then, by taking $m=\lfloor n^{2/7}\rfloor$, equality (5) holds uniformly for $0\le x=o(n^{1/14}/\sqrt{\ln n})$ as $n\to\infty$.

(iv) For stationary processes, results similar to Theorem 2.1 can be found in Wu and Zhao [20] and Cuny and Merlevède [1]. Wu and Zhao [20] showed that it is possible to prove the relative error of normal approximation tends to 0 for a certain class of stationary processes represented by functions of an i.i.d. sequence as soon as the partial sum process can be well approximated by martingales. Following the work of Wu and Zhao [20], Cuny and Merlevède (see Theorem 3.2 of [1]) proved that under certain conditions for $\mathbf{L}^p$-norm, the relative error of normal approximation tends to 0 uniformly for $0\le x=O(\sqrt{\ln n})$, that is (5) holds uniformly for $0\le x=O(\sqrt{\ln n})$. Now Theorem 2.1 shows that the last range could be as large as $0\le x=o(n^\alpha)$ for some positive constant $\alpha\in(0,\frac{1}{2})$ (cf. point (iii) of this remark) under the conditions for $\mathbf{L}^\infty$-norm (instead of $\mathbf{L}^p$-norm).

(v) The absolute constant $e^{-(80)^2}$ is very small. However, it can be improved to a larger one, provided that the absolute constant 80 in the inequality of Peligrad et al. [14] (cf. inequality (28)) can be improved to a smaller one.

(vi) Notice that the quantities γ_m and δ_m can be estimated via the quantities

$$\eta_{1,n}:=\sup_{k\ge n}\|\mathbf{E}[X_k|\mathcal{F}_0]\|_\infty \qquad \text{and} \qquad \eta_{2,n}:=\sup_{k,l\ge n}\|\mathbf{E}[X_kX_l|\mathcal{F}_0]-\mathbf{E}[X_kX_l]\|_\infty.$$

Indeed, it is easy to see that

$$\begin{aligned}\gamma_m &\le \frac{1}{m^{1/2}\sigma_n}\sum_{j=1}^\infty\frac{1}{j^{3/2}}\Big(\sum_{i=1}^{mj}\eta_{1,i}\Big)\le\frac{1}{m^{1/2}\sigma_n}\sum_{i=1}^\infty\eta_{1,i}\sum_{j\ge i/m}\frac{1}{j^{3/2}}\\ &\le\frac{C_1}{m^{1/2}\sigma_n}\Big(\sum_{i=1}^m\eta_{1,i}+\sqrt{m}\sum_{i\ge m}\frac{\eta_{1,i}}{i^{1/2}}\Big)\end{aligned}\tag{8}$$

and

$$\begin{aligned}\delta_m^2\le\frac{1}{m\sigma_n^2}\Big[\Big(\sum_{i=1}^m\eta_{1,i}\Big)^2&+\sum_{i=1}^m\|\mathbf{E}[X_i^2|\mathcal{F}_0]-\mathbf{E}[X_i^2]\|_\infty\\ &+2\sum_{i=1}^{m-1}\sum_{j=i+1}^m\|\mathbf{E}[X_iX_j|\mathcal{F}_0]-\mathbf{E}[X_iX_j]\|_\infty\Big],\end{aligned}$$

where C_1 is an absolute constant. Splitting the last sum as follows

$$\sum_{1\le i\le m/2}\sum_{i+1\le j\le 2i}+\sum_{1\le i\le m/2}\sum_{2i+1\le j\le m}+\sum_{m/2\le i\le m-1}\sum_{i+1\le j\le m},$$

we infer that

$$\delta_m^2\le\frac{C_2}{m\sigma_n^2}\Big[\Big(\sum_{i=1}^m\eta_{1,i}\Big)^2+\sum_{1\le i\le m/2}i\eta_{2,i}+\|X_0\|_\infty\sum_{1\le i\le m/2}\sum_{j\ge 2i}\eta_{1,j}+m\sum_{i\ge m/2}\eta_{2,i}\Big],\tag{9}$$

where C_2 is an absolute constant. Moreover, if

$$\lim_{n\to\infty}\sigma_n^2=\sigma^2>0 \quad \text{and} \quad \max_{i=1,2}\{\eta_{i,n}\}=O(n^{-\beta})$$

for some constant $\beta > 1$, by (8) and (9), then we have $\gamma_m = O(m^{-1/2})$ and

$$\delta_m = \begin{cases} O(m^{-1/2}), & \text{if } \beta > 2, \\ O(m^{-1/2}\sqrt{\ln m}), & \text{if } \beta = 2, \\ O(m^{-(\beta-1)/2}), & \text{if } \beta \in (1,2). \end{cases}$$

(vii) *Assume that $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$. If $\max_{i=1,2}\{\eta_{i,n}\} = O(n^{-\beta})$ for some constant $\beta \geq 3/2$, with $m = \lfloor n^{2/7} \rfloor$, then equality (5) holds uniformly for $0 \leq x = o(n^{1/14}/\sqrt{\ln n})$ as $n \to \infty$. If $\max_{i=1,2}\{\eta_{i,n}\} = O(n^{-\beta})$ for some constant $\beta \in (1, 3/2)$, with $m = \lfloor n^{1/(3\beta-1)} \rfloor$, then equality (5) holds uniformly for $0 \leq x = o(n^{(\beta-1)/(6\beta-2)})$ as $n \to \infty$.*

Theorem 2.1 implies the following Berry-Esseen bound.

Corollary 2.2 *Assume the conditions of Theorem 2.1. Then*

$$\sup_x \Big| \mathbf{P}(W_n \leq x\sigma_n) - \Phi(x) \Big| \leq C\Big(\gamma_m |\ln \gamma_m| + \varepsilon_m |\ln \varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}}\Big), \tag{10}$$

where C is an absolute constant.

Remark 2 *Let us comment on Corollary 2.2.*

(i) *Assume that $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$ and $\max_{i=1,2}\{\eta_{i,n}\} = O(n^{-\beta})$ for some constant $\beta > 1$. By point (vi) of Remark 1, if $\beta \geq 2$, then, with $m = \lfloor n^{1/3} \rfloor$, bound (10) reaches its minimum of order $n^{-1/6} \ln n$. If $\beta \in (1,2)$, then, with $m = \lfloor n^{1/(\beta+1)} \rfloor$, bound (10) gives its minimum of order $n^{-(\beta-1)/(2\beta+2)} \ln n$.*

(ii) *When $(X_i)_{i\in\mathbf{Z}}$ is a uniformly mixing sequence, we refer to Rio [18] for a result similar to Corollary 2.2. In the paper, Rio [18] gave a Berry-Esseen bound of order $n^{-1/2}$ under the condition $\sum_{k=1}^\infty k\theta_k < \infty$, where $(\theta_k)_{k\geq 1}$ is the sequence of uniformly mixing coefficients.*

(iii) *If $(X_i, \mathcal{F}_i)_{i\in\mathbf{Z}}$ is a stationary martingale difference sequence, Corollary 2.2 gives the following Berry-Esseen bound*

$$\sup_x \Big| \mathbf{P}(W_n \leq x\sigma_n) - \Phi(x) \Big| = O\Big(\frac{m}{n^{1/2}} \ln n + \Big\| \frac{1}{m\sigma_n^2} \sum_{i=1}^m \mathbf{E}[X_i^2|\mathcal{F}_0] - 1 \Big\|_\infty \Big). \tag{11}$$

When X_0 is $\mathbf{L}^p$-bounded (instead of $\mathbf{L}^\infty$-bounded), Dedecker et al. [4] have obtained some rather tight Berry-Esseen bounds. Notice that Dedecker et al. [4] assumed a martingale coboundary decomposition while we do not. On the other hand Dedecker et al. [4] worked in $\mathbf{L}^p$ and we work in $\mathbf{L}^\infty$, so the results are of independent interest. It is worth noticing that the best rates (for martingales) provided by Dedecker et al. [4] and us are the same.

Theorem 2.1 gives an alternative proof for the following moderate deviation principle (MDP) result which is implied by the functional MDP result of Dedecker et al. [3] under the conditions (6) and (7).

Corollary 2.3 *Assume the conditions of Theorem 2.1. Assume that $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$, and that $\max\{\gamma_m, \delta_m\} \to 0$ as $m \to \infty$. Let a_n be any sequence of real numbers satisfying $a_n \to 0$ and $a_n n^{1/2} \to \infty$ as $n \to \infty$. Then for each Borel set $B \subset \mathbf{R}$,*

$$-\inf_{x\in B^o} \frac{x^2}{2\sigma^2} \leq \liminf_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B\Big) \leq \limsup_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B\Big) \leq -\inf_{x\in \overline{B}} \frac{x^2}{2\sigma^2}, \tag{12}$$

where B^o and $\overline{B}$ denote the interior and the closure of B, respectively.

The following theorem gives a Bernstein type inequality for the stationary sequences. Although such type of inequalities are less precise than Cramér type moderate deviations, they are available for all positive x. Moreover, they are very useful for establishing quantile coupling inequalities ; see Theorem 2.5.

Theorem 2.4 *Assume the conditions of Theorem 2.1. Then for any* $x > 0$,

$$\mathbf{P}\Big(W_n \geq x\sigma_n\Big) \leq \exp\Bigg\{ -\frac{(1-\gamma_m|\ln\gamma_m|)^2x^2}{2\big(1+\tau_m^2+\frac{2}{3}\varepsilon_m(1-\gamma_m|\ln\gamma_m|)x\big)}\Bigg\} + 4\sqrt{e}\exp\Bigg\{-\frac{|\ln\gamma_m|^2}{2\cdot(81)^2}\,x^2\Bigg\}, \quad (13)$$

where $\tau_m^2 = \delta_m^2 + \dfrac{m}{n} + 4\varepsilon_m^2$.

Assume that $\gamma_m \to 0$ as $m \to \infty$. Then $\gamma_m|\ln\gamma_m| \to 0$ and $|\ln\gamma_m| \to \infty$ as $m \to \infty$. Thus the second term in the r.h.s. of (13) is much smaller than the first one for any $x > 0$ as $m \to \infty$. So when m satisfies $m \to \infty$ and $m/\sqrt{n} \to 0$, the bound (13) behaves like $\exp\big\{-\frac{x^2}{2(1+\delta_m^2)}\big\}$ for any $x > 0$.

Next, we apply Theorems 2.1 and 2.4 to quantile coupling inequalities for stationary sequences. We follow Mason and Zhou [13], where such type of inequalities have been established for arbitrary random variables under some Cramér type moderate deviation assumptions. Using Theorems 2.1, 2.4 and Theorem 1 of Mason and Zhou [13], we obtain the following result.

Theorem 2.5 *Assume the conditions of Theorem 2.1, and that* $\gamma_m + \varepsilon_m + \delta_m + \sqrt{\frac{m}{n}} \to 0$ *as* $n \to \infty$. *Let* $\widehat{W}_n = W_n/\sigma_n$. *Then, there exist two positive absolute constants* α *and* C_α, *a standard normal random variable* Z *and a random variable* Y_n *can be constructed on a new probability space such that* $Y_n =_d \widehat{W}_n$ *and*

$$|Y_n - Z| \leq 2C_\alpha\Big(Y_n^2+1\Big)\Big(\gamma_m|\ln\gamma_m| + \varepsilon_m\,|\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}}\Big), \quad (14)$$

whenever

$$|Y_n| \leq \alpha\Big(\gamma_m|\ln\gamma_m| + \varepsilon_m\,|\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}}\Big)^{-1} \quad (15)$$

and n *is large enough, where* $=_d$ *stands for equality in distribution. Furthermore, there exist two positive absolute constants* C *and* λ *such that for* n *large enough, we have for all* $x \geq 0$,

$$\mathbf{P}\Bigg(\frac{|Y_n - Z|}{\gamma_m|\ln\gamma_m| + \varepsilon_m\,|\ln\varepsilon_m| + \delta_m + \sqrt{m/n}} \geq x\Bigg) \leq C\exp\Big\{-\lambda\,x\Big\}. \quad (16)$$

Assume that $\lim_{n\to\infty}\sigma_n^2 = \sigma^2 > 0$ and $\max_{i=1,2}\{\eta_{i,n}\} = O(n^{-\beta})$ for some constant $\beta > 1$. By point (i) of Remark 2, if $\beta \geq 2$, then, with $m = \lfloor n^{1/3}\rfloor$, the term $\gamma_m|\ln\gamma_m| + \varepsilon_m\,|\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}}$ is of order $n^{-1/6}\ln n$. If $\beta \in (1,2)$, then, with $m = \lfloor n^{1/(\beta+1)}\rfloor$, the term $\gamma_m|\ln\gamma_m| + \varepsilon_m\,|\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}}$ is of order $n^{-(\beta-1)/(2\beta+2)}\ln n$.

3. Applications

In this section, we present some applications of our results. For more interesting applications, such as expanding map and symmetric random walk on the circle, we refer to Corollary 18 and Proposition 20 of Dedecker et al. [3]. Under their corresponding conditions, the conditions of Theorem 2.1 hold.

3.1. *ϕ-mixing sequences*

Let Y be a random variable with values in a Polish space $\mathcal{Y}$. If $\mathcal{M}$ is a σ-field, the ϕ-mixing coefficient between $\mathcal{M}$ and $\sigma(Y)$ is defined by

$$\phi(\mathcal{M}, \sigma(Y)) = \sup_{A \in \mathfrak{B}(\mathcal{Y})} \left\| \mathbf{P}_{Y|\mathcal{M}}(A) - \mathbf{P}_Y(A) \right\|_\infty . \tag{17}$$

For a sequence of random variables $(X_i)_{i\in\mathbf{Z}}$ and a positive integer m, denote

$$\phi_m(n) = \sup_{i_m > ... > i_1 \geq n} \phi(\mathcal{F}_0, \sigma(X_{i_1}, ..., X_{i_m})),$$

and let $\phi(k) = \lim_{m\to\infty} \phi_m(k)$ be the usual ϕ-mixing coefficient. Under the following conditions

$$\sum_{k\geq 1} k^{1/2} \phi_1(k) < \infty \quad \text{and} \quad \lim_{k\to\infty} \phi_2(k) = 0, \tag{18}$$

Dedecker et al. [3] obtained a MDP result for bounded random variables. See also Gao [9] for an earlier MDP result under a condition stronger than (18), that is $\sum_{k\geq 1} \phi(k) < \infty$.

When the random variables $(X_i)_{i\in\mathbf{Z}}$ are bounded, it holds $\eta_{1,n} = O(\phi_1(n))$ and $\eta_{2,n} = O(\phi_2(n))$ as $n \to \infty$. By point (vii) of Remark 1, we have the following result.

Proposition 3.1 *Assume that the random variables $(X_i)_{i\in\mathbf{Z}}$ are bounded, $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$ and*

$$\max_{i=1,2} \{\phi_i(n)\} = O(n^{-\beta}), \quad n \to \infty,$$

for some constant $\beta > 1$.

[i] *If $\beta \geq 3/2$, then (5) holds uniformly for $0 \leq x = o(n^{1/14}/\sqrt{\ln n})$ as $n \to \infty$.*

[ii] *If $\beta \in (1, 3/2)$, then (5) holds uniformly for $0 \leq x = o(n^{(\beta-1)/(6\beta-2)})$ as $n \to \infty$.*

3.2. *Functions of ϕ-mixing sequences*

Let $(\varepsilon_i)_{i\in\mathbf{Z}} = (\varepsilon_0 \circ T^i)_{i\in\mathbf{Z}}$ be a stationary sequence of ϕ-mixing random variables taking values in a subset A of a Polish space $\mathcal{X}$. Denote by $\phi_\varepsilon(n)$ the coefficient

$$\phi_\varepsilon(n) = \phi(\sigma(\varepsilon_i, i \leq 0), \sigma(\varepsilon_i, i \geq n)),$$

where ϕ is defined by (17). Let H be a function from $A^{\mathbf{N}}$ to $\mathbf{R}$ satisfying the following condition

(A) : for any $i \geq 0$, $\sup_{x\in A^{\mathbf{N}},\, y\in A^{\mathbf{N}}} \left| H(x) - H(x^{(i)}y) \right| \leq R_i$, where R_i decreases to 0,

where the sequence $x^{(i)}y$ is defined by $(x^{(i)}y)_j = x_j$ for $j < i$ and $(x^{(i)}y)_j = y_j$ for $j \geq i$. Define the stationary sequence $X_k = X_0 \circ T^k$ by

$$X_k = H((\varepsilon_{k-i})_{i\in\mathbf{N}}) - \mathbf{E}[H((\varepsilon_{k-i})_{i\in\mathbf{N}})]. \tag{19}$$

Dedecker et al. [3] gave a MDP result for $(X_k)_{k\geq 1}$, see Propositions 12 therein. From the proof of Propositions 12 of [3], it is easy to see that

$$\max_{i=1,2} \{\eta_{i,n}\} = O\Big(R_n + \sum_{i=1}^{n} R_{n-i} \phi_\varepsilon(i)\Big).$$

Notice that when $\sigma^2 := \sum_{k\in\mathbf{Z}} \mathbf{E}[X_0 X_k] > 0$, it holds $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$. By point (vii) of Remark 1, we have the following Cramér type moderate deviations.

Proposition 3.2 *Let* $(X_k)_{k\in\mathbf{Z}}$ *be defined by (19), for a function* H *satisfying condition (A). Assume*

$$R_n + \sum_{i=1}^{n} R_{n-i}\phi_\varepsilon(i) = O(n^{-\beta}), \qquad n\to\infty, \tag{20}$$

for some constant $\beta > 1$, *and* $\sigma^2 := \sum_{k\in\mathbf{Z}} \mathbf{E}[X_0X_k] > 0$.

[i] *If* $\beta \geq 3/2$, *then (5) holds uniformly for* $0 \leq x = o(n^{1/14}/\sqrt{\ln n})$ *as* $n\to\infty$.

[ii] *If* $\beta \in (1, 3/2)$, *then (5) holds uniformly for* $0 \leq x = o(n^{(\beta-1)/(6\beta-2)})$ *as* $n\to\infty$.

3.3. *Contracting Markov chains*

Let $(Y_n)_{n\geq 0}$ be a stationary Markov chain of bounded random variables with invariant measure μ and transition kernel K. Denote by $\|\cdot\|_{\infty,\mu}$ the essential norm with respect to μ. Let Λ_1 be the set of 1-Lipschitz functions. Assume that the chain satisfies the following condition :

(B) : there exist two constants $C > 0$ and $\rho \in (0,1)$ such that

$$\sup_{g\in\Lambda_1} \|K^n(g) - \mu(g)\|_{\infty,\mu} \leq C\rho^n,$$

and for any $m \geq 0$,

$$\sup_{f,g\in\Lambda_1} \Big\| K^n\big(fK^m(g)\big) - \mu\big(fK^m(g)\big)\Big\|_{\infty,\mu} \leq C\rho^n.$$

We shall see in the next proposition that MDP result holds for the sequence

$$X_n = f(Y_n) - \mu(f) \tag{21}$$

as soon as the function f belongs to the class $\mathcal{L}$ introduced by Dedecker et al. [3]. Let $\mathcal{L}$ be the class of functions $f : \mathbf{R} \mapsto \mathbf{R}$ such that $|f(x) - f(y)| \leq g(|x-y|)$, where g is a concave and non-decreasing function and satisfies

$$\int_0^1 \frac{g(t)}{t\sqrt{|\ln t|}} dt < \infty. \tag{22}$$

Clearly, (22) holds if $g(t) \leq c|\ln(t)|^{-\gamma}$ for some constants $c > 0$ and $\gamma > 1/2$. In particular, $\mathcal{L}$ contains the class of α-Hölder continuous functions from $[0,1]$ to $\mathbf{R}$, where $\alpha \in (0,1]$.

Dedecker et al. [3] gave a MDP result for $(Y_n)_{n\geq 0}$, see Propositions 14 therein. From the proof of Propositions 14 of [3], it is easy to see that

$$\max_{i=1,2}\{\eta_{i,n}\} = O\big(g(C\rho^n)\big),$$

where C is given by condition (B).

Proposition 3.3 *Assume that the stationary Markov chain* $(Y_n)_{n\geq 0}$ *satisfies condition (B), and let* X_n *be defined by (21). Assume* $f \in \mathcal{L}$,

$$\sigma^2 := \sigma^2(f) = \mu\Big((f-\mu(f))^2\Big) + 2\sum_{n>0}\mu\Big(K^n(f)\cdot(f-\mu(f))\Big) > 0$$

and

$$g(C\rho^n) = O(n^{-\beta}), \qquad n\to\infty, \tag{23}$$

for some constant $\beta > 1$.

[i] *If* $\beta \geq 3/2$*, then (5) holds uniformly for* $0 \leq x = o(n^{1/14}/\sqrt{\ln n})$ *as* $n \to \infty$.

[ii] *If* $\beta \in (1, 3/2)$*, then (5) holds uniformly for* $0 \leq x = o(n^{(\beta-1)/(6\beta-2)})$ *as* $n \to \infty$.

Notice that if $g(t) \leq D|\ln(t)|^{-\beta}$ for some constants $D > 0$ and $\beta > 1$, then (23) is satisfied.

4. Proofs of Theorems and Corollaries

The proofs of our results are mainly based on the following lemmas, which give some exponential deviation inequalities for the partial sums of dependent random variables.

4.1. *Preliminary lemmas*

Let $(\xi_i, \mathcal{F}_i)_{i=0,...,n}$ be a sequence of martingale differences, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\xi_0 = 0$, $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq ... \subseteq \mathcal{F}_n \subseteq \mathcal{F}$ are increasing σ-fields. Set

$$M_0 = 0, \qquad M_k = \sum_{i=1}^{k} \xi_i, \quad k = 1, ..., n. \tag{24}$$

Then $M = (M_k, \mathcal{F}_k)_{k=0,...,n}$ is a martingale. Denote $\langle M \rangle$ the quadratic characteristic of the martingale M, that is

$$\langle M \rangle_0 = 0, \qquad \langle M \rangle_k = \sum_{i=1}^{k} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}], \quad k = 1, ..., n. \tag{25}$$

Assume the following two conditions :

(C1) There exists $\epsilon_n \in (0, \frac{1}{2}]$ such that

$$\Big|\mathbf{E}[\xi_i^k | \mathcal{F}_{i-1}]\Big| \leq \frac{1}{2} k! \epsilon_n^{k-2} \mathbf{E}[\xi_i^2 | \mathcal{F}_{i-1}], \qquad \text{for all } k \geq 3 \text{ and } 1 \leq i \leq n;$$

(C2) There exists $\iota_n \in [0, \frac{1}{2}]$ such that $\| \langle M \rangle_n - 1 \|_\infty \leq \iota_n^2$.

Clearly, condition (C1) is satisfied for bounded martingale differences $\|\xi_i\|_\infty \leq \epsilon_n$.

In the proof of Theorem 2.1, we need the following Cramér moderate deviation expansions for martingales, which is a simple consequence of Theorems 2.1 and 2.2 of Fan et al. [7].

Lemma 4.1 *Assume conditions (C1) and (C2). Then there is an absolute constant* $\alpha_0 > 0$ *such that for all* $0 \leq x \leq \alpha_0 \epsilon_n^{-1}$ *and* $\iota_n \leq \alpha_0$,

$$\left| \ln \frac{\mathbf{P}(M_n \geq x)}{1 - \Phi(x)} \right| \leq C_{\alpha_0} \Big(x^3 \epsilon_n + x^2 \iota_n^2 + (1 + x) (\epsilon_n |\ln \epsilon_n| + \iota_n) \Big), \tag{26}$$

where C_{α_0} *depends only on* α_0*. Moreover, the same equality remains true when* $\frac{\mathbf{P}(M_n \geq x)}{1-\Phi(x)}$ *is replaced by* $\frac{\mathbf{P}(M_n \leq -x)}{\Phi(-x)}$.

In the proof of Theorem 2.4, we make use of the following Freedman inequality [8].

Lemma 4.2 *Assume that* $\xi_i \leq a$ *for some constant* a *and all* $1 \leq i \leq n$*. Then for all* $x \geq 0$ *and* $v_n > 0$,

$$\mathbf{P}\left(M_n \geq x \ \textit{and}\ \langle M\rangle_n \leq v_n^2\ \right) \leq \exp\left\{-\frac{x^2}{2(v_n^2+\frac{a}{3}x)}\right\}. \tag{27}$$

We also use the following exponential inequality of Peligrad et al. [14] (cf. Proposition 2 therein), which plays an important role in the proof of Theorem 2.4.

Lemma 4.3 *Let $(X_i)_{i\in\mathbf{Z}}$ be a stationary sequence of random variables adapted to the filtration $(\mathcal{F}_i)_{i\in\mathbf{Z}}$. Then for all $x \geq 0$,*

$$\mathbf{P}\left(\max_{1\leq i\leq n}|S_i| \geq x\right) \leq 4\sqrt{e}\exp\left\{-\frac{x^2}{2n(\|X_1\|_\infty + 80\sum_{j=1}^n j^{-3/2}\|\mathbf{E}[S_j|\mathcal{F}_0]\|_\infty)^2}\right\}. \tag{28}$$

4.2. *Proof of Theorem 2.1*

Let $k = k(n,m) = \lfloor n/m \rfloor$ be the integer part of n/m. The initial step of the proof is to divide the random variables into blocks of size m and to make the sums in each block

$$X_{i,m} = \sum_{j=(i-1)m+1}^{im} X_j, \quad 1 \leq i \leq k, \quad \text{and} \quad X_{k+1,m} = \sum_{j=km+1}^{n} X_j.$$

It is easy to see that $S_n = \sum_{i=1}^{k+1} X_{i,m}$. Define

$$D_{i,m} = X_{i,m} - \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}], \quad 1 \leq i \leq k.$$

Then $(D_{i,m}, \mathcal{F}_{im})_{1\leq i\leq k}$ is a stationary sequence of bounded martingale differences, that is

$$\|D_{i,m}\|_\infty \leq 2m\|X_0\|_\infty.$$

Notice that

$$\mathbf{E}[D_{i,m}^2|\mathcal{F}_{(i-1)m}] = \mathbf{E}[X_{i,m}^2|\mathcal{F}_{(i-1)m}] - (\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}])^2,$$

and that, by stationarity, it follows that

$$\frac{1}{n}\Big\|\sum_{i=1}^k (\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}])^2\Big\|_\infty \leq \frac{1}{m}\Big\|\mathbf{E}[S_m|\mathcal{F}_0]\Big\|_\infty^2.$$

Moreover,

$$\begin{aligned}\Big\|\frac{1}{n}\sum_{i=1}^k \mathbf{E}[X_{i,m}^2|\mathcal{F}_{(i-1)m}] - \sigma_n^2\Big\|_\infty &\leq \frac{1}{n}\sum_{i=1}^k \Big\|\mathbf{E}[X_{i,m}^2|\mathcal{F}_{(i-1)m}] - m\sigma_n^2\Big\|_\infty^2 + \frac{n-mk}{n}\sigma_n^2 \\ &\leq \Big\|\frac{1}{m}\mathbf{E}[S_m^2|\mathcal{F}_0] - \sigma_n^2\Big\|_\infty^2 + \frac{m}{n}\sigma_n^2.\end{aligned}$$

Consequently, it holds

$$\begin{aligned}&\Big\|\frac{1}{n}\sum_{i=1}^k \mathbf{E}[D_{i,m}^2|\mathcal{F}_{(i-1)m}] - \sigma_n^2\Big\|_\infty \\ &\leq \Big\|\frac{1}{n}\sum_{i=1}^k \mathbf{E}[X_{i,m}^2|\mathcal{F}_{(i-1)m}] - \sigma_n^2\Big\|_\infty^2 \ + \ \frac{1}{n}\Big\|\sum_{i=1}^k (\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}])^2\Big\|_\infty\end{aligned}$$

$$\leq \Big\|\frac{1}{m}\mathbf{E}[S_m^2|\mathcal{F}_0]-\sigma_n^2\Big\|_\infty^2+\frac{m}{n}\sigma_n^2 \ + \ \frac{1}{m}\Big\|\mathbf{E}[S_m|\mathcal{F}_0]\Big\|_\infty^2$$
$$= \big(\delta_m^2+\frac{m}{n}\big)\sigma_n^2$$

and

$$\|n^{-1/2}D_{i,m}\|_\infty \leq 2\sigma_n\varepsilon_m.$$

Denote $\xi_i = D_{i,m}/(n^{1/2}\sigma_n)$ and $M_k=\sum_{i=1}^k \xi_i$. Then it is obvious that

$$|\xi_i|\leq 2\varepsilon_m \quad \text{and} \quad \|\langle M\rangle_k-1\|_\infty \leq \delta_m^2+\frac{m}{n}.$$

Assume $\varepsilon_m\leq \frac{1}{4}$ and $\delta_m^2+\frac{m}{n}\leq \alpha_0$, where $\alpha_0\in(0,\frac{1}{2}]$ is given by Lemma 4.1. By Lemma 4.1, we have for all $0\leq x\leq \alpha_0\varepsilon_m^{-1}$,

$$\Bigg|\ln\frac{\mathbf{P}(M_k\geq x)}{1-\Phi\left(x\right)}\Bigg| \leq C_{\alpha_0}'\left(x^3\varepsilon_m+x^2(\delta_m^2+\frac{m}{n})+(1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\,\big)\right), \tag{29}$$

where C_{α_0}' depends only on α_0. Notice that for all $x\geq 0$ and $|\varepsilon|\leq\frac{1}{2}$,

$$\frac{1-\Phi\left(x+\varepsilon\right)}{1-\Phi\left(x\right)}=\exp\Big\{\theta\sqrt{2\pi}(1+x)|\varepsilon|\Big\} \tag{30}$$

and

$$\frac{1}{\sqrt{n}\sigma_n}\big\|X_{k+1,m}\big\|_\infty\leq \frac{1}{\sqrt{n}\sigma_n}(n-km)\big\|X_0\big\|_\infty\leq \varepsilon_m,$$

where $|\theta|\leq 1$. It is obvious that

$$M_k+\frac{1}{\sqrt{n}\sigma_n}X_{k+1,m}=\frac{1}{\sqrt{n}\sigma_n}\Big(S_n-\sum_{i=1}^k\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}]\Big).$$

Therefore, by (29) and (30), for all $0\leq x\leq \alpha_0\varepsilon_m^{-1}$,

$$\frac{\mathbf{P}(S_n-\sum_{i=1}^k\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}]\geq x\sigma_n n^{1/2})}{1-\Phi\left(x\right)}$$
$$\leq \ \frac{\mathbf{P}(M_k\geq x+\varepsilon_m)}{1-\Phi\left(x+\varepsilon_m\right)}\cdot\frac{1-\Phi\left(x+\varepsilon_m\right)}{1-\Phi\left(x\right)}$$
$$\leq \ \exp\left\{C_{\alpha_0}\left(x^3\varepsilon_m+x^2(\delta_m^2+\frac{m}{n})+(1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\big)\right)\right\}.$$

Similarly, we have for all $0\leq x\leq \alpha_0\varepsilon_m^{-1}$,

$$\frac{\mathbf{P}(S_n-\sum_{i=1}^k\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}]\geq x\sigma_n n^{1/2})}{1-\Phi\left(x\right)}$$
$$\geq \ \exp\left\{-C_{\alpha_0}\left(x^3\varepsilon_m+x^2(\delta_m^2+\frac{m}{n})+(1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\big)\right)\right\}.$$

The last two inequalities imply that for all $0\leq x\leq \alpha_0\varepsilon_m^{-1}$,

$$\Big| \ln \frac{\mathbf{P}(S_n - \sum_{i=1}^k \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq x\sigma_n n^{1/2})}{1-\Phi\left(x\right)} \Big|$$
$$\leq\ C_{\alpha_0} \left(x^3 \varepsilon_m + x^2 (\delta_m^2 + \frac{m}{n}) + (1+x)\big(\varepsilon_m \left|\ln \varepsilon_m\right| + \delta_m + \sqrt{m/n}\big) \right). \tag{31}$$

By Lemma 4.3, we derive that for all $x \geq 0$,

$$\mathbf{P}\Big(\Big|\sum_{i=1}^k \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}]\Big| \geq x\sigma_n n^{1/2}\Big)$$
$$\leq 4\sqrt{e}\exp\bigg\{ - \frac{n\sigma_n^2 x^2}{2k(\big\|\mathbf{E}[S_m|\mathcal{F}_0]\big\|_\infty + 80\sum_{j=1}^k j^{-3/2}\|\mathbf{E}[S_{jm}|\mathcal{F}_0]\|_\infty)^2} \bigg\}$$
$$\leq 4\sqrt{e}\exp\bigg\{ - \frac{x^2}{2\cdot(81)^2\gamma_m^2} \bigg\}. \tag{32}$$

It is easy to see that for all $x \geq 0$,

$$\mathbf{P}\Big(W_n \geq x\sigma_n\Big) \leq \mathbf{P}\bigg(S_n - \sum_{i=1}^k \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq (1-\gamma_m|\ln\gamma_m|)x\sigma_n n^{1/2}\bigg)$$
$$+ \mathbf{P}\bigg(\sum_{i=1}^k \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq \gamma_m|\ln\gamma_m|x\sigma_n n^{1/2}\bigg). \tag{33}$$

By the inequalities (31)-(33), it follows that for all $0 \leq x \leq \alpha_0\varepsilon_m^{-1}$,

$$\frac{\mathbf{P}(W_n \geq x\sigma_n)}{1-\Phi\left(x\right)} \leq \frac{1-\Phi\left((1-\gamma_m|\ln\gamma_m|)x\right)}{1-\Phi\left(x\right)}$$
$$\times \exp\bigg\{ C_{\alpha_0} \left(x^3\varepsilon_m + x^2(\delta_m^2 + \frac{m}{n}) + (1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}\big) \right) \bigg\}$$
$$+ \frac{4\sqrt{e}}{1-\Phi\left(x\right)}\exp\bigg\{ - \frac{1}{2\cdot(81)^2}(\ln\gamma_m)^2 x^2 \bigg\}.$$

Using the following two-sided bound on tail probabilities of the standard normal random variable

$$\frac{1}{\sqrt{2\pi}(1+x)}e^{-x^2/2} \leq 1-\Phi(x) \leq \frac{1}{\sqrt{\pi}(1+x)}e^{-x^2/2}, \quad x \geq 0, \tag{34}$$

we deduce that for all $\gamma_m \leq e^{-(80)^2}$ and $1 \leq x \leq \alpha_0\varepsilon_m^{-1}$,

$$\frac{\mathbf{P}(W_n \geq x\sigma_n)}{1-\Phi\left(x\right)} \leq \exp\bigg\{ C_{\alpha_0}\Big(x^3\varepsilon_m + x^2(\delta_m^2 + \frac{m}{n} + \gamma_m|\ln\gamma_m|) + (1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}\big)\Big) \bigg\}$$
$$+ C_1 \times \exp\bigg\{ - \frac{1}{4\cdot(81)^2}|\ln\gamma_m|x^2 \bigg\}$$
$$\leq \exp\bigg\{ C_{\alpha_0}\Big(x^3\varepsilon_m + x^2(\delta_m^2 + \frac{m}{n} + \gamma_m|\ln\gamma_m|) + (1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}\big)\Big) \bigg\}$$
$$+ C_2\gamma_m|\ln\gamma_m|x^2$$
$$\leq \exp\bigg\{ C'_{\alpha_0}\Big(x^3\varepsilon_m + x^2(\delta_m^2 + \frac{m}{n} + \gamma_m|\ln\gamma_m|) + (1+x)\big(\varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}\big)\Big) \bigg\}.$$

(35)

Notice that for $x \geq 0$,

$$\mathbf{P}\Big(W_n \geq x\sigma_n\Big) \geq \mathbf{P}\Big(S_n - \sum_{i=1}^{k} \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq (1+\gamma_m|\ln\gamma_m|)x\sigma_n n^{1/2}\Big) - \mathbf{P}\Big(\sum_{i=1}^{k} \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \leq -\gamma_m|\ln\gamma_m|x\sigma_n n^{1/2}\Big). \tag{36}$$

By an argument similar to the proof of (35), we deduce that for all $1 \leq x \leq \alpha_0\varepsilon_m^{-1}$,

$$\frac{\mathbf{P}(W_n \geq x\sigma_n)}{1-\Phi(x)} \geq \exp\bigg\{ - C'_{\alpha_0}\bigg(x^3\varepsilon_m + x^2(\delta_m^2 + \frac{m}{n} + \gamma_m|\ln\gamma_m|) + (1+x)\big(\varepsilon_m\,|\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}})\bigg)\bigg\}. \tag{37}$$

Combining (35) and (37) together, we obtain the desired equality for all $1 \leq x \leq \alpha_0\varepsilon_m^{-1}$. Next, we consider the case where $x \in [0,1]$. Notice that (31) holds also for $(-X_i)_{i\in\mathbf{Z}}$. Thus, from (31), we have

$$\sup_{|x|\leq 2}\Big|\mathbf{P}\Big(S_n - \sum_{i=1}^{k}\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq x\sigma_n n^{1/2}\Big) - \Big(1-\Phi(x)\Big)\Big| \leq C_{\alpha_0}\Big(\varepsilon_m|\ln\varepsilon_m| + \delta_m + \sqrt{m/n}\Big). \tag{38}$$

For all $x \in [0,1]$, we deduce that

$$\begin{aligned}
&\mathbf{P}\Big(W_n \geq x\sigma_n\Big) - \Big(1-\Phi(x)\Big)\\
&\geq \mathbf{P}\Big(S_n - \sum_{i=1}^{k}\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq (x-\gamma_m|\ln\gamma_m|)\sigma_n n^{1/2}\Big) - \Big(1-\Phi(x)\Big)\\
&\quad - \mathbf{P}\Big(\sum_{i=1}^{k}\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq \gamma_m|\ln\gamma_m|\sigma_n n^{1/2}\Big)\\
&\geq -C_{\alpha_0}\Big(\varepsilon_m|\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}}\Big) - \Big|\Big(1-\Phi(x-\gamma_m|\ln\gamma_m|)\Big) - \Big(1-\Phi(x)\Big)\Big|\\
&\quad - \mathbf{P}\Big(\sum_{i=1}^{k}\mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq \gamma_m|\ln\gamma_m|\sigma_n n^{1/2}\Big)\\
&\geq -C_{\alpha_0}\Big(\varepsilon_m|\ln\varepsilon_m| + \gamma_m|\ln\gamma_m| + \delta_m + \sqrt{m/n}\Big),
\end{aligned}$$

where the last line follows by (32). Similarly, we have for all $x \in [0,1]$,

$$\mathbf{P}\Big(W_n \geq x\sigma_n\Big) - \Big(1-\Phi(x)\Big) \leq C_{\alpha_0}\Big(\varepsilon_m|\ln\varepsilon_m| + \gamma_m|\ln\gamma_m| + \delta_m + \sqrt{m/n}\Big).$$

The last two inequalities imply that for all $x \in [0,1]$,

$$\Big|\mathbf{P}\Big(W_n \geq x\sigma_n\Big) - \Big(1-\Phi(x)\Big)\Big| \leq C_{\alpha_0}\Big(\varepsilon_m|\ln\varepsilon_m| + \gamma_m|\ln\gamma_m| + \delta_m + \sqrt{m/n}\Big).$$

The last inequality implies the desired equality for all $x \in [0,1]$.

Since $(-X_i)_{i\in\mathbf{Z}}$ also satisfies the conditions of Theorem 2.1, the same equalities remain true when $\frac{\mathbf{P}(W_n\geq x\sigma_n)}{1-\Phi(x)}$ is replaced by $\frac{\mathbf{P}(W_n\leq -x\sigma_n)}{\Phi(-x)}$.

4.3. *Proof of Corollary 2.2*

We only need to consider the case where $\max\{\gamma_m,\varepsilon_m,\delta_m,m/n\}\leq 1/10$. Otherwise, Corollary 2.2 holds obviously for C large enough. Denote

$$\kappa_n=\alpha_0\min\{\gamma_m^{-1/4},\,\varepsilon_m^{-1/4},\delta_m^{-1/4},(m/n)^{-1/4}\},$$

where α_0 is the absolute constant given by Theorem 2.1. It is easy to see that

$$\begin{aligned}\sup_x\Big|\mathbf{P}(W_n\leq x\sigma_n)-\Phi\left(x\right)\Big| &\leq \sup_{|x|\leq\kappa_n}\Big|\mathbf{P}(W_n\leq x\sigma_n)-\Phi\left(x\right)\Big|\\ &\quad+\sup_{|x|>\kappa_n}\Big|\mathbf{P}(W_n\leq x\sigma_n)-\Phi\left(x\right)\Big|\\ &=\sup_{|x|\leq\kappa_n}\Big|\mathbf{P}(W_n\leq x\sigma_n)-\Phi\left(x\right)\Big|\\ &\quad+\sup_{x<-\kappa_n}\mathbf{P}(W_n\leq x\sigma_n)+\sup_{x<-\kappa_n}\Phi\left(x\right)\\ &\quad+\sup_{x>\kappa_n}\mathbf{P}(W_n> x\sigma_n)+\sup_{x>\kappa_n}(1-\Phi\left(x\right)).\end{aligned}\tag{39}$$

By Theorem 2.1 and the inequality $|e^x-1|\leq|x|e^{|x|}$, we have

$$\begin{aligned}&\sup_{|x|\leq\kappa_n}\Big|\mathbf{P}(W_n\leq x\sigma_n)-\Phi\left(x\right)\Big|\\ &\leq\sup_{|x|\leq\kappa_n}\Big(1-\Phi(|x|)\Big)\Big|e^{C_{\alpha_0}\left(x^3\varepsilon_m+x^2(\delta_m^2+\frac{m}{n}+\gamma_m|\ln\gamma_m|)+(1+x)(\varepsilon_m|\ln\varepsilon_m|+\gamma_m|\ln\gamma_m|+\delta_m+\sqrt{m/n})\right)}-1\Big|\\ &\leq C_{\alpha_0,1}\Big(\gamma_m|\ln\gamma_m|+\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\Big).\end{aligned}\tag{40}$$

Using the last inequality, we deduce that

$$\begin{aligned}\sup_{x<-\kappa_n}\mathbf{P}(W_n\leq x\sigma_n)&=\mathbf{P}(W_n\leq-\kappa_n\sigma_n)\\ &\leq C_{\alpha_0,2}\Big(\gamma_m|\ln\gamma_m|+\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\Big)+\Phi\left(-\kappa_n\right)\\ &\leq C_{\alpha_0,3}\Big(\gamma_m|\ln\gamma_m|+\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\Big).\end{aligned}\tag{41}$$

Similarly, it holds that

$$\sup_{x>\kappa_n}\mathbf{P}(W_n> x\sigma_n)\leq C_{\alpha_0,4}\Big(\gamma_m|\ln\gamma_m|+\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\Big).\tag{42}$$

It is obvious that

$$\sup_{x>\kappa_n}(1-\Phi\left(x\right))=\sup_{x<-\kappa_n}\Phi\left(x\right)=\Phi\left(-\kappa_n\right)\leq C_{\alpha_0,5}\Big(\gamma_m|\ln\gamma_m|+\varepsilon_m\left|\ln\varepsilon_m\right|+\delta_m+\sqrt{m/n}\Big).\tag{43}$$

Combining the inequalities (39)-(43) together, we obtain the desired inequality.

4.4. *Proof of Corollary 2.3*

Let $m = \sqrt{a_n \sqrt{n}}$. Then it holds that $m \to \infty$ as $n \to \infty$. Thus $\max\{\gamma_m, \delta_m\} \to 0$ as $n \to \infty$.
First, we prove that

$$\limsup_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B\Big) \leq -\inf_{x \in \overline{B}} \frac{x^2}{2\sigma^2}. \tag{44}$$

For any given Borel set $B \subset \mathbf{R}$, let $x_0 = \inf_{x\in B} |x|$. Then, it is obvious that $x_0 \geq \inf_{x\in \overline{B}} |x|$. Therefore, by Theorem 2.1,

$$\begin{aligned}
&\mathbf{P}\Big(a_n W_n \in B\Big) \\
&\leq \mathbf{P}\Big(\Big|\frac{W_n}{\sigma}\Big| \geq \frac{x_0}{a_n \sigma_n}\Big) \\
&\leq 2\Big(1 - \Phi\big(\frac{x_0}{a_n\sigma_n}\big)\Big) \exp\Big\{C\Big((\frac{x_0}{a_n\sigma_n})^3 \varepsilon_m + (\frac{x_0}{a_n\sigma_n})^2\big(\delta_m^2 + \frac{m}{n} + \gamma_m |\ln \gamma_m|\big) \\
&\qquad + (1 + \frac{x_0}{a_n\sigma_n})\big(\varepsilon_m |\ln \varepsilon_m| + \gamma_m |\ln \gamma_m| + \delta_m + \sqrt{\frac{m}{n}}\big)\Big)\Big\}.
\end{aligned}$$

Notice that

$$\varepsilon_m / a_n = \|X_0\|_\infty / \sqrt{m} \to 0$$

as $n \to \infty$. Using (34) and the fact $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$, we deduce that

$$\limsup_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B\Big) \;\leq\; -\frac{x_0^2}{2\sigma^2} \;\leq\; -\inf_{x\in\overline{B}} \frac{x^2}{2\sigma^2},$$

which gives (44).
Next, we prove that

$$\liminf_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B\Big) \geq -\inf_{x\in B^o} \frac{x^2}{2\sigma^2}. \tag{45}$$

We may assume that $B^o \neq \emptyset$, otherwise the last inequality holds obviously because the infimum of a function over an empty set is interpreted as ∞. For any $\varepsilon_1 > 0$, there exists an $x_0 \in B^o$, such that

$$0 < \frac{x_0^2}{2\sigma^2} \leq \inf_{x\in B^o} \frac{x^2}{2\sigma^2} + \varepsilon_1. \tag{46}$$

Without loss of generality, we may assume that $x_0 > 0$. For $x_0 \in B^o$, there exists small $\varepsilon_2 \in (0, x_0)$, such that $(x_0 - \varepsilon_2, x_0 + \varepsilon_2] \subset B$. Then it is obvious that $x_0 \geq \inf_{x\in\overline{B}} x$. By Theorem 2.1, we deduce that

$$\begin{aligned}
\mathbf{P}\Big(a_n W_n \in B\Big) &\geq \mathbf{P}\Big(W_n \in (a_n^{-1}(x_0 - \varepsilon_2), a_n^{-1}(x_0 + \varepsilon_2)]\Big) \\
&\geq \mathbf{P}\Big(W_n > a_n^{-1}(x_0 - \varepsilon_2)\Big) - \mathbf{P}\Big(W_n > a_n^{-1}(x_0 + \varepsilon_2)\Big)
\end{aligned}$$

Using Theorem 2.1, (34) and the fact $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$ again, it follows that

$$\liminf_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B\Big) \geq -\frac{1}{2\sigma^2}(x_0 - \varepsilon_2)^2.$$

Letting $\varepsilon_2 \to 0$, we get

$$\liminf_{n\to\infty} a_n^2 \ln \mathbf{P}\Big(a_n W_n \in B \Big) \geq -\frac{x_0^2}{2\sigma^2} \ \geq \ -\inf_{x\in B^o} \frac{x^2}{2\sigma^2} - \varepsilon_1 .$$

Because ε_1 can be arbitrarily small, we obtain (45). This completes the proof of Corollary 2.3.

4.5. *Proof of Theorem 2.4*

Recall the notations in the proof of Theorem 2.1. It is easy to see that

$$\|D_{i,m}/(n^{1/2}\sigma_n)\|_\infty \leq 2\varepsilon_m$$

and

$$\begin{aligned}\Big\| \frac{1}{n\sigma_n^2} \sum_{i=1}^{k+1} \mathbf{E}[D_{i,m}^2|\mathcal{F}_{(i-1)m}] - 1 \Big\|_\infty &\leq \Big\| \frac{1}{n\sigma_n^2} \sum_{i=1}^{k} \mathbf{E}[D_{i,m}^2|\mathcal{F}_{(i-1)m}] - 1 \Big\|_\infty + \Big\| \frac{1}{n\sigma_n^2} \mathbf{E}[D_{k+1,m}^2|\mathcal{F}_{km}] \Big\|_\infty \\ &\leq \delta_m^2 + \frac{m}{n} + 4\varepsilon_m^2 = \tau_m^2 .\end{aligned}$$

Applying Lemma 4.2 to $\xi_i = D_{i,m}/(\sigma_n n^{1/2})$, we have for all $x \geq 0$,

$$\mathbf{P}\Big(W_n - \frac{1}{\sqrt{n}} \sum_{i=1}^{k+1} \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \geq x\sigma_n \Big) \leq \exp\Big\{ -\frac{x^2}{2(1+\tau_m^2+\frac{2}{3}x\varepsilon_m)} \Big\} .$$

By an argument similar to the proof of (32), we obtain for all $x \geq 0$,

$$\mathbf{P}\Big(\Big| \sum_{i=1}^{k+1} \mathbf{E}[X_{i,m}|\mathcal{F}_{(i-1)m}] \Big| \geq x\sigma_n n^{1/2} \Big) \leq 4\sqrt{e}\exp\Big\{ -\frac{x^2}{2\cdot(81)^2\gamma_m^2} \Big\} . \tag{47}$$

Using (33) again, we obtain the desired inequality.

4.6. *Proof of Proposition 2.5*

For each integer $n \geq 1$, let

$$F_n(x) = \mathbf{P}(\widehat{W}_n \leq x), \quad x \in \mathbf{R},$$

be the cumulative distribution function of $\widehat{W}_n$. Then its quantile function is define by

$$H_n(s) = \inf\{x : F_n(x) \geq s\}, \quad s \in (0,1).$$

Let Z be a standard normal random variable. Denote

$$Y_n = H_n(\Phi(Z)). \tag{48}$$

Then $Y_n =_d \widehat{W}_n$; see Mason and Zhou [13]. Denote

$$K_n = n^{1/2}\Big(\gamma_m |\ln\gamma_m| + \varepsilon_m |\ln\varepsilon_m| + \delta_m + \sqrt{\frac{m}{n}} \Big). \tag{49}$$

By Theorem 2.1, there exist an absolute constants $\beta \in (0,1]$ and $C_\beta \geq 1$ such that when n is large enough, we have for all $0 \leq x \leq \beta n^{1/2}\sigma_n/(m\|X_0\|_\infty)$,

$$\ln\left|\frac{\mathbf{P}\big(Y_n > x\big)}{1-\Phi\left(x\right)}\right| \le C_\beta(1+x^3)\frac{K_n}{n^{1/2}} \tag{50}$$

and

$$\ln\left|\frac{\mathbf{P}\big(Y_n < -x\big)}{\Phi\left(-x\right)}\right| \le C_\beta(1+x^3)\frac{K_n}{n^{1/2}}, \tag{51}$$

where C_β depends only on β. By Theorem 1 of Mason and Zhou [13], then whenever $n \ge 64C_\beta^2K_n^2$ and

$$|Y_n| \le \Big(\frac{\beta\sigma_n}{m\|X_0\|_\infty} \wedge \frac{1}{8C_\beta K_n}\Big)n^{1/2} \tag{52}$$

$$\le \Big(\beta \wedge \frac{1}{8C_\beta}\Big)\Big(\gamma_m|\ln\gamma_m| + \varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}\,\Big)^{-1}, \tag{53}$$

we have

$$|Y_n - Z| \le 2C_\beta\Big(Y_n^2\ +1\Big)\Big(\gamma_m|\ln\gamma_m| + \varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}\,\Big), \tag{54}$$

which gives (14) with $\alpha = \beta \wedge \frac{1}{8C_\beta}$ and $C_\alpha = C_\beta$. Notice that there exists an integer n_0 such that $n \ge 64C_\beta^2K_n^2$ for all $n \ge n_0$.

Next we give the proof of (16). Set for brevity

$$\varsigma_n = \gamma_m|\ln\gamma_m| + \varepsilon_m\left|\ln\varepsilon_m\right| + \delta_m + \sqrt{\frac{m}{n}}.$$

By (14), we have for all $0 \le x \le \frac{1}{4C_\alpha}\varsigma_n^{-2}$,

$$\begin{aligned}\mathbf{P}\Big(|Y_n - Z| > x\,\varsigma_n\Big) &\le \mathbf{P}\Big(|Y_n - Z| > x\,\varsigma_n, |Y_n| \le \alpha\,\varsigma_n^{-1}\Big) + \mathbf{P}\Big(|Y_n| > \alpha\,\varsigma_n^{-1}\Big)\\ &\le \mathbf{P}\Big(2C_\alpha\big(Y_n^2\ +1\big) > x\Big) + \mathbf{P}\Big(|Y_n| > \alpha\,\varsigma_n^{-1}\Big),\end{aligned} \tag{55}$$

Notice that

$$1-\Phi\left(x\right) \le \exp\{-x^2/2\}, \qquad x \ge 0.$$

When $0 \le x \le \frac{1}{8C_\alpha}\varsigma_n^{-2}$, by the inequalities (50) and (51), it holds that

$$\begin{aligned}\mathbf{P}\Big(2C_\alpha\big(Y_n^2\ +1\big) > x\Big) &\le 2\exp\Big\{-\frac{1}{4}(\frac{x}{2C_\alpha}-1)\Big\}\\ &\le \exp\Big\{1-\frac{x}{8C_\alpha}\Big\},\end{aligned} \tag{56}$$

and that

$$\begin{aligned}\mathbf{P}\Big(|Y_n| > \alpha\varsigma_n^{-1}\Big) &\le 2\exp\Big\{-\frac{1}{4}(\alpha\varsigma_n^{-1})^2\Big\}\\ &\le 2\exp\Big\{-2C_\alpha\alpha^2x\Big\}.\end{aligned} \tag{57}$$

Returning to (55), we obtain for all $0 \le x \le \frac{1}{8C_\alpha}\varsigma_n^{-2}$,

$$\mathbf{P}\Big(|Y_n - Z| > x\varsigma_n\Big) \leq 2\exp\Big\{1 - c'x\Big\}, \tag{58}$$

where $c' = \min\{\frac{1}{8C_\alpha}, 2C_\alpha\alpha^2\}$. For $x > 0$, it is easy to see that

$$\mathbf{P}\Big(|Y_n - Z| > x\varsigma_n\Big) \leq \mathbf{P}\Big(|Y_n| > \frac{1}{2}x\varsigma_n\Big) + \mathbf{P}\Big(|Z| > \frac{1}{2}x\varsigma_n\Big). \tag{59}$$

Clearly, it holds for all $x > \frac{1}{8C_\alpha}\varsigma_n^{-2}$,

$$\mathbf{P}\Big(|Z| > \frac{1}{2}x\varsigma_n\Big) \leq 2\exp\Big\{-\frac{1}{8}x^2\varsigma_n^2\Big\} \leq 2\exp\Big\{-\frac{1}{64C_\alpha}x\Big\}.$$

By Theorem 2.4, there exists a positive constant λ such that for all $x > \frac{1}{8C_\alpha}\varsigma_n^{-2}$,

$$\mathbf{P}\Big(|Y_n| > \frac{1}{2}x\varsigma_n\Big) \leq (1 + 4\sqrt{e})\exp\Big\{-\lambda x\Big\}.$$

Returning to (59), we have for all $x > \frac{1}{8C_\alpha}\varsigma_n^{-2}$,

$$\mathbf{P}\bigg(|Y_n - Z| > x\varsigma_n\bigg) \leq (3 + 4\sqrt{e})\exp\Big\{-c''x\Big\}, \tag{60}$$

where $c'' = \min\{\lambda, \frac{1}{64C_\alpha}\}$. Combining (58) and (60), we get the desired inequality.

References

[1] Cuny, C., Merlevède, F., 2014. On martingale approximations and the quenched weak inviariance principle. *Ann. Probab.* **42**(2), 760–793.

[2] De Acosta, A., 1992. Moderate deviations and associated Laplace approximations for sums of independent random vectors. *Trans. Amer. Math. Soc.* **329**(1): 357-375.

[3] Dedecker, J., Merlevède, F., Peligrad, M., Utev, S., 2009. Moderate deviations for stationary sequences of bounded random variables. *Ann. Inst. H. Poincaré Probab. Statist.* **45**(2), 453–476.

[4] Dedecker, J., Merlevède, F., Rio, E., 2009. Rates of convergence for minimal distances in the central limit theorem under projective criteria. *Electron. J. Probab.*, **14**, 978-1011.

[5] Dembo, A., 1996. Moderate deviations for martingales with bounded jumps. *Electron. Comm. Proba.* **1**, 11–17.

[6] Djellout, H., 2002. Moderate deviations for martingale differences and applications to $\phi-$mixing sequences. *Stochastic Stochastic Rep.* **73**, 37–63.

[7] Fan X, Grama I, Liu Q., 2013. Cramér large deviation expansions for martingales under Bernstein's condition. *Stochastic Process. Appl.* **123**(11): 3919–3942.

[8] Freedman, D. A., 1975. On tail probabilities for martingales. *Ann. Probab.* **3**(1): 100–118.

[9] Gao, F. Q., 1996. Moderate deviations for martingales and mixing random processes. *Stochastic Process. Appl.* **61**, 263–275.

[10] Grama, I., 1997. On moderate deviations for martingales. *Ann. Probab.* **25**, 152–184.

[11] Grama, I., Haeusler, E., 2000. Large deviations for martingales via Cramér's method. *Stochastic Process. Appl.* **85**, 279–293.

[12] Grama, I., Haeusler, E., 2006. An asymptotic expansion for probabilities of moderate deviations for multivariate martingales. *J. Theoret. Probab.* **19**, 1–44.

[13] Mason, D.A., Zhou, H.H., 2012. Quantile coupling inequalities and their applications. *Probab. Surveys* **9**: 439–479.

[14] Peligrad, M., Utev, S., Wu, W.B., 2007. A maximal L_p-inequality for stationary sequecens and its applications. *Proc. Amer. Math. Soc.* **135**, 541–550.

[15] Peligrad, M., Utev, S., 2005. A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.* **33**: 798–815.

[16] Račkauskas, A., 1990. On probabilities of large deviations for martingales. *Liet. Mat. Rink.* **30**, 784–795.

[17] Račkauskas, A., 1995. Large deviations for martingales with some applications. *Acta Appl. Math.* **38**, 109–129.

[18] Rio, E., 1996. Sur le théoreme de Berry-Esseen pour les suites faiblement dépendantes. *Probab. Theory Relat. Fields* **104**(2), 255–282.

[19] Wu, W.B., 2005. Nonlinear system theorey: Another look at dependence. *Proc. Natl. Acad. Sci. USA* **102**, 14150–14154.

[20] Wu, W.B., Zhao, Z., 2008. Moderate deviations for stationary processes. *Statist. Sinica* **18**, 769–782.