Variational formula for the stability of regime-switching diffusion processes *

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Abstract

The asymptotical stability in probability is studied for diffusion processes and regime-switching diffusion processes in this work. For diffusion processes, some criteria based on the integrability of the functionals of the coefficients are given, which yields a useful comparison theorem on stability with respect to some nonlinear systems. For regime-switching diffusion processes, some criteria based on the idea of a variational formula are given. Both state-independent and state-dependent regime-switching diffusion processes are investigated in this work. These conditions are easily verified and are shown to be sharp by examples.

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1 Introduction

This work is devoted to the study of stability of diffusion processes and regime-switching diffusion processes. Both state-independent and state-dependent regime-switching diffusion processes are studied in this work. Stability of stochastic processes is an important

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subjects and has attracted considerable attention. On the other hand, in the past two decades, a great deal of mathematical effort has been devoted to the study of regime-switching diffusion processes. The regime-switching diffusion processes can provide much realistic models for many applications such as mathematical finance, wireless communication, biology and etc. (cf. [26] and references therein). For example, these models can describe the random change of environment (such as dry and rainy weather) on the birth rates and death rates of the species. The coexistence of diffusion processes and Markov chains cause a lot of difficulties in the study of regime-switching diffusion processes. And a lot of issues are still unclear. For instance, we have a dynamical system, which is stable in warm environment, but is unstable in cold environment. Now, put this system into an environment changing randomly between warm weather and cold weather. It is known that this system could be stable or unstable, which depends on many factors. But, to determine whether it is stable or not is of great importance in application. Analogous problems exist in the study of population dynamics (cf. [2, 12] and references therein).

We shall not mention all contributions to this intensively studied topic, but refer to the books [11, 13, 14, 26] for more details on the stability of diffusion processes or diffusion processes with regime-switching. Moreover, we mention some works which are closely related to our present work, i.e. [8, 20, 23, 25, 22, 27]. Especially, [22] provides a good survey on the recent advancements of hybrid/switched systems, which focuses on the state-dependent switching systems and shows their important application in the control engineering. Recently, there are also some development on the study of recurrent property and long time behavior of regime-switching diffusion processes. See, for instance, [1, 6, 7, 15, 16, 17, 19]. As an application, a sharp criterion on the persistence and extinction of preys and predators is presented in [2] for the predator-prey model with Beddington-DeAngelsis functional response.

Usually, the stability in probability is justified by the existence of certain Lyapunov functions. But, the construction of Lyapunov function is known to be difficult. So, it is better to obtain some explicit conditions in terms of the coefficients of the processes and to be easily checked. Moreover, the stability probability of regime-switching diffusion processes is much more complicated than that of diffusion processes. It is more difficult to construct Lyapunov functions for these processes due to the coexistence of infinitesimal generators of diffusion processes and jump processes. Due to the remarkable works [4, 9], the recurrent properties of diffusion processes can be justified explicitly by their coefficients, and these criteria could be very sharp (cf. A. Friedman [10, Chapter 9]). Corresponding results also hold for the stability of diffusion processes. We aim to provide explicit criteria for stability in probability of regime-switching diffusion processes in this work.

Consider the stochastic differential equation (SDE):

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}^n, \tag{1.1}$$

where (B_t) is a Brownian motion in $\mathbb{R}^n$, and $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$. We use $(X_t^{x_0})$ to denote the solution of (1.1) with the initial condition $X_0 = x_0$. Throughout this work, we assume the following conditions hold for the coefficients b and σ:

(H) $b(0) = 0$, $\sigma(0) = 0$, *and there exists a constant K such that*

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| \le K|x - y|, \quad \forall\, x, y \in \mathbb{R}^n,$$

where $|b|$ denotes the absolute value of b and $\|\sigma\|$ denotes the operator norm of matrix σ. Moreover, there exists a function $m(x)$ with $m(x) > 0$ for $x \neq 0$ such that

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j > m(x)\sum_i \xi_i^2, \quad \forall\,(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,\ x \in \mathbb{R}^n,$$

where $(a_{ij}(x)) = \frac{1}{2}\sigma(x)\sigma^*(x)$ and $\sigma^*(x)$ stands for the transpose of $\sigma(x)$.

The infinitesimal generator of $(X_t^{x_0})$ is hence given by

$$L = \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}.$$

We focus on the stability of the trivial solution $X_t \equiv 0$, and adopt the notations of Khasminskii [11]. Namely, the solution $X_t \equiv 0$ of (1.1) is said to be *stable in probability* if for any $\varepsilon > 0$,

$$\lim_{x\to 0} \mathbb{P}\big(\sup_{t\ge 0} |X_t^x| > \varepsilon\big) = 0,$$

and $X_t \equiv 0$ is said to be *unstable in probability* if it is not stable in probability. The solution $X_t \equiv 0$ of (1.1) is said to *asymptotically stable in probability* if it is stable in probability and moreover

$$\lim_{x\to 0} \mathbb{P}\big(\lim_{t\to\infty} X_t^x = 0\big) = 1.$$

In this work, as a preparation we first provide some easily verifiable conditions for the asymptotical stability in probability of SDE (1.1). To get some impression on these results, let us consider the one dimensional diffusion process $(X_t^{x_0})$ with $x_0 > 0$. Under the condition (H), the point 0 is inaccessible (cf. [11, Lemma 5.3]), which implies that $X_t > 0$ for all $t > 0$ a.e. Given some constant $r_0 > 0$, define

$$C(x) = \int_x^{r_0} \frac{b(u)}{a(u)} \mathrm{d}u, \quad 0 < x \leq r_0. \tag{1.2}$$

We shall use $C(x)$ to study the stability of diffusion processes. It is well known that $C(x)$ is also used to study the recurrence of diffusion processes (cf. for instance, [11, Example 3.10]). We show that if $\int_0^{r_0} e^{C(x)}\mathrm{d}x < +\infty$, then $X_t \equiv 0$ is asymptotically stable in probability. Moreover, if $\int_0^{r_0} e^{C(x)}\mathrm{d}x = +\infty$, and there exists $p > 0$ such that

$$\int_0^{r_0} \frac{1}{u^p a(u)} e^{-C(u)}\mathrm{d}u < +\infty, \quad \int_0^{r_0} \Big(e^{C(y)} \int_0^y \frac{1}{u^p a(u)} e^{-C(u)}\mathrm{d}u \Big)\mathrm{d}y = +\infty,$$

then $X_t \equiv 0$ is unstable in probability (see Theorem 2.1 below). Obviously, these conditions are easily to be checked. Moreover, these conditions can provide a sharp criterion on the stability of diffusion processes, which can be shown by the following classical example. Consider

$$\mathrm{d}X_t = bX_t\mathrm{d}t + \sigma X_t \mathrm{d}B_t, \quad X_0 = x > 0,$$

and b, σ are constants. Then when $b < \sigma^2/2$, $X_t \equiv 0$ is stable in probability; when $b > \sigma^2/2$, $X_t \equiv 0$ is unstable in probability (see, for example, [11, pp.154, pp.162]). To apply our criterion, we take $r_0 = 1$ and get $C(x) = \int_x^1 \frac{2b}{\sigma^2 u}\mathrm{d}u = -\frac{2b}{\sigma^2}\ln x$. So when $b < \sigma^2/2$, $\int_0^1 e^{C(x)}\mathrm{d}x < +\infty$, and hence $X_t \equiv 0$ is asymptotically stable in probability. When $b > \sigma^2/2$, it is easy to see that there exists $p > 0$ such that

$$\int_0^1 \frac{1}{u^p a(u)} e^{-C(u)}\mathrm{d}u = \frac{2}{\sigma^2} \int_0^1 u^{\frac{2b}{\sigma^2}-2-p}\mathrm{d}u < +\infty,$$
$$\int_0^1 \Big(e^{C(y)} \int_0^y \frac{1}{u^p a(u)} e^{-C(u)}\mathrm{d}u \Big)\mathrm{d}y = \frac{2}{2b-(1+p)\sigma^2} \int_0^1 y^{-1-p}\mathrm{d}y = +\infty.$$

Therefore, when $b > \sigma^2/2$, $X_t \equiv 0$ is unstable in probability. This example is a special case of our Example 2.1:

$$\mathrm{d}X_t = b(X_t^p \wedge X_t)\mathrm{d}t + \sigma(X_t^q \wedge X_t)\mathrm{d}B_t, \tag{1.3}$$

where a complete characterization is given for such class of diffusion processes.

The main aim of this work is to study the stability in probability of regime-switching diffusion processes. This problem has been studied in many works. For instance, [14] and [26] provided some Foster-Lyapunov conditions to study the stability of regime-switching diffusion processes with Markovian switching and state-dependent switching respectively. Moreover, they gave out some explicit criteria for linearized systems. An interesting phenomenon explained in these works is that this process can be stable in some states and unstable in other states, but the random switching can make the process stable or not by changing the switching rate between different states. To solve this problem, in our previous work [20], we choose a priori Lyapunov function $\rho(x)$ independent of jumping states to characterize the stability behavior of this process in each fixed state i in terms of a constant β_i. Then combining these constants (β_i) with the jumping rate matrix (q_{ij}), we provided several criteria to justify the stability in probability of regime-switching diffusion processes. The intuitive reason to use the common function $\rho(x)$ is that only evaluating the stability of the process (X_t) at different states $i \in \mathcal{S}$ using the same ruler $\rho(x)$, we can compare them invoking the switching (q_{ij}). In view of this reason, the previous criteria based on the integrability of coefficients are not applicable for the regime-switching diffusion processes. In this work, based on an idea of variational formula, we develop the results in [20] to provide more explicit formula of the constants β_i, $i \in \mathcal{S}$. Our method is inspired by the idea of Chen and Wang [5], where they provided some sharp estimates for the lower bound of the spectral gap of elliptic operators.

This work is organized as follows. In Section 2, we state our results on stability in probability of diffusion processes. In Section 3, we deal with the regime-switching diffusion processes. Both the state-dependent and state-independent regime-switching diffusion processes are considered in this part.

2 Stability in probability of diffusion processes

In this section we first consider the SDE (1.1) in $\mathbb{R}$. Due to the assumption (H), 0 is inaccessible, so $X_t := X_t^{x_0}$ has the same sign with its initial point $X_0 = x_0$. To simplify the notation, we consider only the case $X_0 = x_0 > 0$.

Theorem 2.1 *Assume (H) holds and $X_0 = x_0 > 0$.*

(i) *If there exists some $r_0 > 0$ such that*

$$\int_0^{r_0} e^{C(x)} \mathrm{d}x < +\infty, \tag{2.1}$$

where $C(x)$ is defined in (1.2) Then $X_t \equiv 0$ of (1.1) is asymptotically stable in probability.

(ii) *If there exists some $r_0 > 0$ and $C(x)$ defined in (1.2) such that*

$$\int_0^{r_0} e^{C(y)} \mathrm{d}y = +\infty, \tag{2.2}$$

and there exists a nonnegative function $f \in C((0, r_0))$ such that

$$\int_0^{r_0} \frac{f(u)}{a(u)} e^{-C(u)} \mathrm{d}u < +\infty, \quad \textit{and} \quad \int_0^{r_0} \Big(e^{C(y)} \int_0^y \frac{f(u)}{a(u)} e^{-C(u)} \mathrm{d}u \Big) \mathrm{d}y = +\infty. \tag{2.3}$$

Then $X_t \equiv 0$ of (1.1) is unstable in probability.

Proof. (i) Let

$$g(x) = \int_0^x (r_0 - y) e^{C(y)} \mathrm{d}y, \quad 0 < x \le r_0. \tag{2.4}$$

By (2.1), it is easy to see g is well-defined on $[0, r_0]$ with $g(0) = 0$. Direct calculation yields that

$$Lg(x) = a(x) g''(x) + b(x) g'(x) = -a(x) e^{C(x)} < 0, \quad \text{for } x \in (0, r_0).$$

Hence, according to the Foster-Lyapunov condition (cf. [11, Theorem 5.5] or [13, Theorem 2.3]), $X_t \equiv 0$ of (1.1) is asymptotically stable in probability.

(ii) According to the assumption,

$$g(x) = \int_x^{r_0} \Big(e^{C(y)} \int_0^y \frac{f(u)}{a(u)} e^{-C(u)} \mathrm{d}u \Big) \mathrm{d}y$$

is well-defined for $x \in (0, r_0]$, and

$$\lim_{x \to 0+} g(x) = \int_0^{r_0} \Big(e^{C(y)} \int_0^y \frac{f(u)}{a(u)} e^{-C(u)} \mathrm{d}u \Big) \mathrm{d}y = +\infty.$$

For $x \in (0, r_0)$,

$$Lg(x) = a(x) g''(x) + b(x) g'(x) = -f(x) \le 0.$$

Therefore, by the Foster-Lyapunov condition (cf. [11, Theorem 5.6]), $X_t \equiv 0$ of (1.1) is unstable in probability. ∎

Note that in Theorem 2.1, condition (2.3) can imply condition (2.2). But compared with condition (2.1), one should check (2.2) first.

Corollary 2.2 *Assume (H) holds and* $X_0 = x_0 > 0$. *Let* $C(x) = \int_x^{r_0} \frac{b(u)}{a(u)} \mathrm{d}u$ *for some* $r_0 > 0$. *Assume*

$$\int_0^{r_0} e^{C(y)} \mathrm{d}y = +\infty,$$

and there exists $p > 0$ *such that*

$$\int_0^{r_0} \frac{1}{u^p a(u)} e^{-C(u)} \mathrm{d}u < +\infty, \quad \text{and} \int_0^{r_0} \Big(e^{C(y)} \int_0^y \frac{1}{u^p a(u)} e^{-C(u)} \mathrm{d}u \Big) \mathrm{d}y = +\infty.$$

Then $X_t \equiv 0$ *of* (1.1) *is unstable in probability.*

Although the proof of Theorem 2.1 is quite easy, this result is very useful. As an application, we give a complete characterization of the stability in probability for the following diffusion process.

Example 2.1 Let

$$\mathrm{d}X_t = b(X_t^p \wedge X_t)\mathrm{d}t + \sigma(X_t^q \wedge X_t)\mathrm{d}B_t, \quad X_0 = x_0 > 0, \tag{2.5}$$

where b, σ, p, q are constants, b, $\sigma \neq 0$, and p, $q \geq 1$. Here, $c \wedge d := \min\{c, d\}$. According to Theorem 2.1 and Corollary 2.2, we obtain the follows:

(i) If $p - 2q = -1$, then when $b < \sigma^2/2$, $X_t \equiv 0$ is asymptotically stable in probability; when $b > \sigma^2/2$, $X_t \equiv 0$ is unstable in probability. These results have been proved in Section 1. When $b = \frac{\sigma^2}{2}$, $X_t \equiv 0$ is unstable in probability. Indeed, set $r_0 = 1/2$, $f(u) = u^{2q-2}(\ln u)^{-2}$, then $C(x) = -\ln 2 - \ln x$ for $x \in (0, \frac{1}{2})$, $\int_0^{1/2} e^{C(x)} \mathrm{d}x = +\infty$,

$$\int_0^{\frac{1}{2}} \frac{f(u)}{a(u)} e^{-C(u)} \mathrm{d}u = \frac{4}{\sigma^2} \int_0^{\frac{1}{2}} \frac{1}{u(\ln u)^2} \mathrm{d}u = \frac{4}{\sigma^2 \ln 2} < +\infty,$$

$$\int_0^{\frac{1}{2}} \Big(e^{C(y)} \int_0^y \frac{f(u)}{a(u)} e^{-C(u)} \mathrm{d}u \Big) \mathrm{d}y = \frac{2}{\sigma^2} \int_0^{\frac{1}{2}} \frac{-1}{y \ln y} \mathrm{d}y = +\infty.$$

(ii) If $p-2q>-1$, then $X_t\equiv 0$ is asymptotically stable in probability.

(iii) If $p-2q<-1$, then when $b<0$, $X_t\equiv 0$ is asymptotically stable in probability; when $b>0$, $X_t\equiv 0$ is unstable in probability. Indeed, when $p-2q<-1$ and $b>0$, setting $r_0=1$, we get $C(x)=2b(1-x^m)/(m\sigma^2)$ for $x\in(0,1)$, where $m=p-2q+1<0$. It is clear that

$$\int_0^1 e^{C(x)}\mathrm{d}x=\int_0^1 e^{\frac{2b}{m\sigma^2}(1-x^m)}\mathrm{d}x=+\infty.$$

Take $f(x)=x^{-\gamma}$ with $\gamma>\max\{1-p,0\}$. Then

$$\int_0^1 \frac{2}{\sigma^2u^{2q+\gamma}}e^{-\frac{2b}{m\sigma^2}(1-u^m)}\mathrm{d}u<+\infty.$$

As

$$\begin{aligned}
&\lim_{y\to 0+} ye^{\frac{2b}{m\sigma^2}(1-y^m)}\int_0^y \frac{2}{\sigma^2u^{2q+\gamma}}e^{-\frac{2b}{m\sigma^2}(1-u^m)}\mathrm{d}u\\
&=\lim_{y\to 0+}\frac{\int_0^y \frac{2}{\sigma^2u^{2q+\gamma}}e^{-\frac{2b}{m\sigma^2}(1-u^m)}\mathrm{d}u+\frac{2}{\sigma^2y^{2q+\gamma-1}}e^{-\frac{2b}{m\sigma^2}(1-y^m)}}{\frac{2b}{\sigma^2}y^{m-1}e^{-\frac{2b}{m\sigma^2}(1-y^m)}}\\
&\geq \lim_{y\to 0+}\frac{1}{by^{2q+\gamma+m-2}}=\lim_{y\to 0+}\frac{1}{by^{\gamma+p-1}}\\
&=+\infty,
\end{aligned}$$

we get

$$\int_0^1\left(e^{\frac{2b}{m\sigma^2}(1-y^m)}\int_0^y \frac{2}{\sigma^2u^{2q+\gamma}}e^{-\frac{2b}{m\sigma^2}(1-u^m)}\mathrm{d}u\right)\mathrm{d}y=+\infty.$$

Then the desired result follows from Corollary 2.2.

Using the results of Example 2.1, we get a criterion on the stability of general diffusion processes by comparing them with the processes in the form (2.5).

Corollary 2.3 *Let (Y_t) be a one-dimensional diffusion process satisfying the SDE*

$$\mathrm{d}Y_t=b(Y_t)\mathrm{d}t+\sigma(Y_t)\mathrm{d}B_t,\quad Y_0=y_0>0,$$

where $b(\cdot)$ and $\sigma(\cdot)$ satisfying (H). Moreover, there exist $p,q>0$ and constants $\hat{b}$, $\hat{\sigma}\neq 0$, $r_0>0$ such that

$$\frac{b(x)}{\sigma(x)^2}\leqslant\frac{\hat{b}}{\hat{\sigma}^2}x^{p-2q},\quad x\in(0,r_0).$$

When one of the following three conditions holds, $Y_t \equiv 0$ is asymptotically stable in probability.

$$(1)\ p-2q=-1 \text{ and } \hat{b}<\hat{\sigma}^2/2;$$
$$(2)\ p-2q>-1;$$
$$(3)\ p-2q<-1 \text{ and } \hat{b}<0.$$

Proof. Since $a(x)=\sigma(x)^2/2$, by simple calculation, for $x\in(0,r_0)$, $r_0>0$, we have

$$C(x)=\int_x^{r_0}\frac{b(u)}{a(u)}\mathrm{d}u\leqslant\frac{2\hat{b}}{\hat{\sigma}^2}\ln\frac{r_0}{x},$$

if $p-2q=-1$ and $C(x)\leqslant C_1+C_2x^{p-2q+1}$ if $p-2q\neq-1$, where

$$C_1=\frac{2\hat{b}}{\hat{\sigma}^2}\frac{r_0^{p-2q+1}}{p-2q+1},\quad C_2=-\frac{2\hat{b}}{\hat{\sigma}^2(p-2q+1)}$$

If $p-2q=-1$ and $\hat{b}<\hat{\sigma}^2/2$, then

$$\int_0^{r_0}\mathrm{e}^{C(x)}\mathrm{d}x\leqslant\int_0^{r_0}r_0^{\frac{2\hat{b}}{\hat{\sigma}^2}}x^{-\frac{2\hat{b}}{\hat{\sigma}^2}}\mathrm{d}x<\infty.$$

By Theorem 2.1, $Y_t\equiv0$ is asymptotically stable in probability.

If $p-2q>-1$, then

$$\sup_{x\in(0,r_0)}C(x)\leqslant C_1+|C_2|r_0^{p-2q+1}<\infty.$$

Moreover,

$$\int_0^{r_0}\mathrm{e}^{C(x)}\mathrm{d}x\leqslant\int_0^{r_0}\mathrm{e}^{C_1+|C_2|r_0^{p-2q+1}}\mathrm{d}x<\infty.$$

By Theorem 2.1, $Y_t\equiv0$ is asymptotically stable in probability.

If $p-2q<-1$ and $\hat{b}<0$, then $C_2<0$ and

$$C(x)\leqslant C_1+C_2x^{p-2q+1}\leqslant C_1$$

for $x\in(0,r_0)$, we have

$$\int_0^{r_0}\mathrm{e}^{C(x)}\mathrm{d}x\leqslant\int_0^{r_0}\mathrm{e}^{C_1}\mathrm{d}x<\infty.$$

By Theorem 2.1, $Y_t \equiv 0$ is asymptotically stable in probability. ∎

Now, we proceed to consider the multidimensional diffusion processes. Firstly, we introduce some notations used below. Let

$$\begin{aligned}
&a(\boldsymbol{x}) = \frac{1}{2}\sigma(x)\sigma(x)^*, \quad \tilde{b}(r) = \sup_{|\boldsymbol{x}|=\sqrt{r}} \big[\langle \boldsymbol{x}, b(\boldsymbol{x})\rangle + \operatorname{trace} a(\boldsymbol{x})\big], \\
&\tilde{a}(r) = 2 \sup_{|\boldsymbol{x}|=\sqrt{r}} \Big[\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) x^i x^j\Big], \\
&\bar{a}(r) = 2 \inf_{|\boldsymbol{x}|=\sqrt{r}} \Big[\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) x^i x^j\Big],
\end{aligned} \tag{2.6}$$

where x^i is the i^{th} component of $\boldsymbol{x} = (x^1, x^2, \cdots, x^n) \in \mathbb{R}^n$ and $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$ denotes the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ with respect to the counting measure.

Theorem 2.4 *Let (X_t) satisfy the SDE (1.1). If there exists some $r_0 > 0$ such that*

$$\limsup_{r\to 0+} \frac{\tilde{b}(r)}{\tilde{a}(r)} r_0 < -1, \tag{2.7}$$

and

$$\int_0^{r_0} e^{\tilde{C}(x)} \mathrm{d}x < \infty, \quad \textit{where} \quad \tilde{C}(x) = \int_x^{r_0} \frac{\tilde{b}(u)}{\tilde{a}(u)} \mathrm{d}u, \ x \in (0, r_0). \tag{2.8}$$

Then $X_t \equiv 0$ of (1.1) is asymptotically stable in probability.

Proof. Let $g(x) = \int_0^x (r_0 - y) e^{\tilde{C}(y)} \mathrm{d}y$ for $x \in [0, r_0)$. By (2.7), there exists $r_1 \in (0, r_0)$ such that for all $x \in (0, r_1)$, $g'(x) = (r_0 - x)e^{\tilde{C}(x)} > 0$ and

$$g''(x) = -\Big[1 + (r_0 - x)\frac{\tilde{b}(x)}{\tilde{a}(x)}\Big] e^{\tilde{C}(x)} \geq 0.$$

Using the Itô formula, we obtain

$$\begin{aligned}
\mathrm{d}g(|X_t|^2) &= 2g'(|X_t|^2)\big[\langle X_t, b(X_t)\rangle + \operatorname{trace} a(X_t)\big]\mathrm{d}t + 4g''(|X_t|^2)\Big(\sum_{i,j=1}^{n} a_{ij}(X_t) X_t^i X_t^j\Big)\mathrm{d}t \\
&\quad + 2g'(|X_t|^2)\langle X_t, \sigma(X_t)\mathrm{d}B_t\rangle \\
&\leq 2g'(|X_t|^2)\tilde{b}(|X_t|^2)\mathrm{d}t + 2g''(|X_t|^2)\tilde{a}(|X_t|^2)\mathrm{d}t + 2g'(|X_t|^2)\langle X_t, \sigma(X_t)\mathrm{d}B_t\rangle.
\end{aligned}$$

Then it follows that

$$\begin{aligned}\mathbb{E}g(|X_{t\wedge\tau}|^2) &\le g(|x|^2) + 2\mathbb{E}\int_0^{t\wedge\tau} \big[g'(|X_s|^2)\tilde{b}(|X_s|^2) + g''(|X_s|^2)\tilde{a}(|X_s|^2)\big]\mathrm{d}s \\ &= g(|x|^2) - 2\mathbb{E}\int_0^{t\wedge\tau} \tilde{a}(|X_s|^2)e^{\bar{C}(|X_s|^2)}\mathrm{d}s,\end{aligned}$$

where $\tau = \inf\{t > 0; |X_t|^2 > r\}$ and r is arbitrarily fixed constant in the interval $(0, r_1)$. Hence,

$$g(r)\mathbb{P}(\tau \le t) \le \mathbb{E}g(|X_{t\wedge\tau}|^2) \le g(|x|^2),$$

and

$$\mathbb{P}(\sup_{0\le s\le t} |X_s|^2 \ge r) = \mathbb{P}(\tau \le t) \le \frac{g(|x|^2)}{g(r)}.$$

Letting $t \to +\infty$, we obtain that $X_t \equiv 0$ is stable in probability by the arbitrariness of $r \in (0, r_1)$. Moreover, we can follow the approach of [26, Lemma 7.6] to prove that $X_t \equiv 0$ is asymptotically stable in probability. We omit the details. ∎

Theorem 2.5 *Let (X_t) satisfy the SDE (1.1). If there exists some $r_0 > 0$ such that*

$$\limsup_{r\to 0+} \frac{\tilde{b}(r)}{\bar{a}(r)} r_0 > -1, \tag{2.9}$$

and

$$\int_0^{r_0} e^{\bar{C}(x)}\mathrm{d}x < \infty, \quad \textit{where } \bar{C}(x) = \int_x^{r_0} \frac{\tilde{b}(u)}{\bar{a}(u)}\mathrm{d}u,\ x \in (0, r_0). \tag{2.10}$$

Then $X_t \equiv 0$ of (1.1) is asymptotically stable in probability.

Proof. Let $g(x) = \int_0^{\sqrt{x}}(r_0 - y)e^{\bar{C}(y)}\mathrm{d}y$ for $x \in [0, r_0)$. Then the proof is similar to that of Theorem 2.4 by noting that it holds $g''(x) < 0$ for x sufficiently near to 0 in present situation. ∎

Remark 2.6 In the previous Theorems 2.4 and 2.5, we reduce the stability problem of a multidimensional diffusion to a one-dimension diffusion process by using the transform $\boldsymbol{x} \mapsto |\boldsymbol{x}|^2$. If one puts some additional conditions on the coefficients $b(\cdot)$ and $\sigma(\cdot)$, the transform $\boldsymbol{x} \mapsto |\boldsymbol{x}|^p$ with $p \ge 1$ can yield different conditions to guarantee $X_t \equiv 0$ to be stable in probability.

3 Stability of regime-switching diffusion processes

In this section we go to study the stability in probability of regime-switching diffusion processes. Let (X_t, Λ_t) satisfy the following SDE:

$$\mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t, \Lambda_t)\mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}^n,\ \Lambda_0 = i \in \mathcal{S}, \tag{3.1}$$

where $\mathcal{S} = \{1, 2, \dots, N\}$ is the state space of the switching process (Λ_t), and (B_t) is a Brownian motion in $\mathbb{R}^n$. Moreover,

$$\mathbb{P}(\Lambda_{t+\delta} = l | \Lambda_t = k, X_t = x) = \begin{cases} q_{kl}(x)\delta + o(\delta), & \text{if } k \neq l, \\ 1 + q_{kk}(x)\delta + o(\delta), & \text{if } k = l, \end{cases} \tag{3.2}$$

for $\delta > 0$. The Q-matrix $(q_{kl}(x))$ is assumed to be irreducible and conservative for each $x \in \mathbb{R}^n$, which means that $q_k(x) = -q_{kk}(x) = \sum_{l \neq k} q_{kl}(x)$, $\forall\, k \in \mathcal{S}$. If $(q_{kl}(x))$ does not depend on x, then (X_t, Λ_t) is called a *state-independent* regime-switching diffusion process; otherwise, it is called a *state-dependent* one. In this work, we focus on the situation that (Λ_t) is in a finite state space $\mathcal{S}$, i.e. $N < \infty$. Using the similar idea of this work, our results can be extended to deal with the case $N = \infty$ by using the finite partition method or the principal eigenvalue method introduced firstly in [20].

If we use $\mathcal{S}$ to denote the state space of the environment, for example, $\mathcal{S} = \{1, 2\}$, where "1" denotes the hot environment, and "2" denotes the cold environment. Then the process (Λ_t) describes the random changing of the environment between states "1" and "2". (X_t) can be looked on as a diffusion process in a random environment. An interesting phenomenon of (X_t, Λ_t) is that $X_t \equiv 0$ could be stable in probability when the environment is at state "1", but is unstable in probability when the environment is at state "2". When the environment changes randomly between states "1" and "2", $X_t \equiv 0$ could be stable in probability or not depending on the changing rate of the environment. But it is difficult to get an explicit criterion to justify whether $X_t \equiv 0$ is stable or not. In [14] and [26], some explicit criteria are provided for regime-switching diffusion processes with linear coefficients. In [20], we provided some criteria based on the existence of some Lyapunov function $\rho(x)$ independent of environment i. In this work, we shall give out more explicit criteria depending explicitly on the coefficients of (X_t, Λ_t). Now, we collect some basic assumptions on the process (X_t, Λ_t) used below.

(H.1) $(q_{ij}(x))$ is conservative and irreducible for each $x \in \mathbb{R}^n$, and $q_{ij}(\cdot)$ is a bounded continuous function for each pair of i, $j \in \mathcal{S}$.

(H.2) $b(0,i)=0$ and $\sigma(0,i)=0$ for every $i\in\mathcal{S}$. Moreover, for any sufficiently small $0<\varepsilon<r_0$, there exist $l\in\{1,\dots,n\}$ and $\kappa(\varepsilon)>0$ such that $a_{ll}(x,i)>\kappa(\varepsilon)$ for all $(x,i)\in\{x;\varepsilon<|x|<r_0\}\times\mathcal{S}$, where $a(x,i)=\frac{1}{2}\sigma(x,i)\sigma(x,i)^*$.

(H.3) There exists a constant $\bar K>0$ so that

$$|b(x,i)-b(y,i)|+\|\sigma(x,i)-\sigma(y,i)\|\le \bar K|x-y|,\quad \forall\, x,y\in\mathbb{R}^n,\ i\in\mathcal{S}.$$

Conditions (H.1)–(H.3) can guarantee the existence of the weak solution of SDE (3.1), (3.2). We refer the reader to [18] for more discussion on the existence and uniqueness of strong solutions for state-dependent regime-switching diffusion processes under non-Lipschitz conditions.

We first introduce a class of functions on $[0,+\infty)$. For $r>0$, let

$$\begin{aligned}\mathscr{D}(r)=\Big\{f\in C^2((0,r));\, f(x)>0, \int_0^x f(u)\mathrm{d}u<+\infty, \forall\, x\in(0,r),\\ \lim_{x\to 0+} a(x,i)f'(x)+b(x,i)f(x)=0, \forall\, i\in\mathcal{S}\Big\}.\end{aligned}\tag{3.3}$$

According to [21], the generator $\mathscr{A}$ of the regime-switching diffusion process (X_t,Λ_t) can be written as

$$\begin{aligned}\mathscr{A}V(x,i)&=L^{(i)}V(\cdot,i)(x)+QV(x,\cdot)(i)\\ &=\sum_{k=1}^n b_k(x,i)\frac{\partial V}{\partial x_k}(x,i)+\sum_{k,l=1}^n a_{kl}(x,i)\frac{\partial^2 V}{\partial x_k\partial x_l}(x,i)\\ &\quad+\sum_{j\neq i} q_{ij}(x)(V(x,j)-V(x,i)),\quad V\in C^2(\mathbb{R}^n\times\mathcal{S}),\end{aligned}$$

where $L^{(i)}$ is the infinitesimal generator of X_t in fixed environment $\Lambda_t\equiv i$.

Theorem 3.1 *Let (X_t,Λ_t) be a one-dimensional state-independent regime-switching process satisfying* (3.1), (3.2) *with $X_0=x_0>0$. Assume (H.2), (H.3) hold. Let (π_i) denote the invariant probability measure of (Λ_t). Assume that there exist some $r_0>0$, $f\in\mathscr{D}(r_0)$ such that $b(\cdot,i)$ and $a(\cdot,i)$ are differentiable on $(0,r_0)$ for each $i\in\mathcal{S}$, and*

$$\sum_{i=1}^N \pi_i\beta_f(i)<0,\tag{3.4}$$

where

$$\beta_f(i) = \limsup_{x\to 0+} b'(x,i) + \frac{a(x,i)f''(x) + (a'(x,i)+b(x,i))f'(x)}{f(x)}.$$

Then $X_t \equiv 0$ *of* (3.1), (3.2) *is asymptotically stable in probability.*

Proof. Since $\sum_{i=1}^N \pi_i\beta_f(i) < 0$, there exist $0 < r_1 < r_0$, $\varepsilon > 0$ such that

$$b'(x,i) + \frac{a(x,i)f''(x) + (a'(x,i)+b(x,i))f'(x)}{f(x)} < \beta_f(i) + \varepsilon \tag{3.5}$$

for all $x \in (0, r_1)$, and $\sum_{i=1}^N \pi_i(\beta_f(i)+\varepsilon) < 0$.

Set $\tilde\beta(i) = \beta_f(i)+\varepsilon$, and $Q_p = Q + p\,\mathrm{diag}(\tilde\beta(1),\dots,\tilde\beta(N))$, $p > 0$, where $\mathrm{diag}(\tilde\beta(1),\dots,\tilde\beta(N))$ denotes the diagonal matrix generated by vector $(\tilde\beta(1),\dots,\tilde\beta(N))$ as usual. Let

$$\eta_p = -\max\{Re(\gamma);\ \gamma \in \text{ the spectrum of } Q_p\}.$$

Set $Q_{p,t} = e^{tQ_p}$. Since all coefficients of $Q_{p,t}$ are positive (see [1, Proposition 4.1]), the Perron-Frobenius theorem [3, Chapter 2] yields that $-\eta_p$ is a single eigenvalue of Q_p. Moreover, the eigenvector of $Q_{p,t}$ corresponding to $e^{-\eta_p t}$ is also an eigenvector of Q_p corresponding to $-\eta_p$. Then the Perron-Fronenius theorem ensures that there exists an eigenvector $\boldsymbol{\xi}$ of Q_p associated with $-\eta_p$ so that $\boldsymbol{\xi} \gg 0$, which means that all elements of $\boldsymbol{\xi}$ are positive. By [1, Proposition 4.2], there exists some $p_0 > 0$ so that $\eta_p > 0$ for all $0 < p < p_0$ since $\sum_{i=1}^N \pi_i\tilde\beta(i) < 0$. Fix a p with $0 < p < \min\{1, p_0\}$ and an eigenvector $\boldsymbol{\xi}$ with $\boldsymbol{\xi} \gg 0$. Then one gets

$$Q_p\boldsymbol{\xi}(i) = \big(Q + p\mathrm{diag}(\tilde\beta)\big)\boldsymbol{\xi}(i) = -\eta_p\xi_i < 0, \quad \forall\, i = 1,\dots,N.$$

Set $V(x,i) = g(x)^p\xi_i$ with $g(x) = \int_0^x f(u)\mathrm{d}u$ for $x \ge 0$, $i \in \mathcal{S}$. Then

$$\begin{aligned} \mathscr{A}V(x,i) &= Q\boldsymbol{\xi}(i)g(x)^p + \xi_i L^{(i)}g(x)^p \\ &\le Q\boldsymbol{\xi}(i)g(x)^p + p\xi_i g(x)^{p-1}L^{(i)}g(x), \quad 0 < x < r_1. \end{aligned} \tag{3.6}$$

Note that for each $i \in \mathcal{S}$,

$$\begin{aligned} L^{(i)}g(x) &= a(x,i)g''(x) + b(x,i)g'(x) \\ &= g(x)\Big(\frac{a(x,i)f'(x) + b(x,i)f(x)}{g(x)}\Big). \end{aligned}$$

Let $h(x,i) = a(x,i)f'(x)+b(x,i)f(x)$. As $f \in \mathscr{D}(r_0)$, we get $h(0,i) := \lim_{x\to 0+} a(x,i)f'(x)+ b(x,i)f(x) = 0$. So $h(\cdot,i) \in C([0,r_1]) \cap C^1((0,r))$. By the mean value theorem,

$$\begin{aligned}L^{(i)}g(x) &= g(x)\frac{h(x,i)-h(0,i)}{g(x)-g(0)}\\ &\le g(x)\sup_{x\in(0,r_1)}\frac{\big(a(x,i)f'(x)+b(x,i)f(x)\big)'}{f(x)}\\ &\le \tilde{\beta}(i)g(x), \quad \forall\, x\in(0,r_1).\end{aligned}$$

Inserting previous inequality into (3.6), we get

$$\mathscr{A}V(x,i) \le Q\boldsymbol{\xi}(i)g(x)^p + p\tilde{\beta}(i)\xi_i g(x)^p = -\eta_p\xi_i g(x)^p = -\eta_p V(x,i).$$

Therefore, according to the Foster-Lyapunov criteria ([26, Lemma 7.6]), we obtain that $X_t \equiv 0$ is asymptotically stable in probability. ∎

Example 3.1 Consider the following regime-switching diffusion process

$$\mathrm{d}X_t = b_{\Lambda_t}X_t\mathrm{d}t + \sigma_{\Lambda_t}(X_t^q \wedge X_t)\mathrm{d}B_t,$$

where Λ_t is a Markov chain on $S = \{1,2,\cdots,N\}$ with invariant probability measure (π_i). X_t is asymptotically stable in probability provided one of the following conditions holds: (1) $q > 1$ and $\sum_{i=1}^n \pi_i b_i < 0$; (2) $q = 1$ and $\sum_{i=1}^n \pi_i(2b_i + \sigma_i^2\gamma) < 0$.

Proof. Let $f(x) = x^\gamma$, $\gamma > 0$. Then $f \in C^2(0,1)$, $f > 0$ on $(0,1)$, $\int_0^1 f(u)\mathrm{d}u < \infty$. Since

$$b(x,i) = b_i x, \quad a(x,i) = \frac{1}{2}\sigma_i^2 x^{2q}, \quad x \in (0,1),$$

we have

$$\lim_{x\to 0^+}\big(af'(x)+bf(x)\big) = \lim_{x\to 0^+}\left(\frac{1}{2}\gamma\sigma_i^2 x^{2q+\gamma-1} + b_i x^{1+\gamma}\right) = 0$$

since $q \geqslant 1$. That is $f \in \mathscr{D}(1)$. By simple calculation, we get

$$\begin{aligned}&a(x,i)f''(x) + (a'(x,i)+b(x,i))f'(x)\\ &= \frac{1}{2}\sigma_i^2\gamma(\gamma-1)x^{2q+\gamma-2} + q\gamma\sigma_i^2 x^{2q+\gamma-2} + b_i\gamma x^\gamma,\end{aligned}$$

Therefore,

$$\begin{aligned}&b'(x,i) + \frac{a(x,i)f''(x) + (a'(x,i)+b(x,i))f'(x)}{f(x)}\\ &= b_i + \frac{1}{2}\sigma_i^2\gamma(\gamma-1)x^{2q-2} + q\gamma\sigma_i^2 x^{2q-2} + b_i\gamma.\end{aligned}$$

If $q > 1$,

$$\beta_f(i) = \lim_{x\to 0^+} b'(x,i) + \frac{a(x,i)f''(x) + (a'(x,i)+b(x,i))f'(x)}{f(x)} = (1+\gamma)b_i.$$

If $q = 1$,

$$\beta_f(i) = (1+\gamma)\big(b_i + \frac{1}{2}\gamma\sigma_i^2\big).$$

The required assertions follows by Theorem 3.1. ∎

Next, we shall use the M-matrix theory, which was applied to diffusion processes for the first time in [24], to study the stability in probability of regime-switching diffusion processes. From the perspective of M-matrix, we present a new comparable theorem between the regime-switching diffusion processes with N switching states and that with two switching states (see Corollary 3.6 and the remark following it). We recall that a square matrix $A = (a_{ij})_{m\times m}$ is called an M-matrix if A can be expressed in the form $A = sI - B$ with some $B \geq 0$ and $s \geq \mathrm{Ria}(B)$, where I is the $m \times m$ identity matrix, and $\mathrm{Ria}(B)$ denotes the spectral radius of B. Here $B \geq 0$ denotes all elements of B are non-negative. If further $s > \mathrm{Ria}(B)$ then A is called a nonsingular M-matrix. There are many equivalent conditions to justify whether a matrix is a nonsingular M-matrix. We introduce several conditions here and refer to [3] for more details.

Proposition 3.2 ([3]) *The following statements are equivalent.*

1. *A is a nonsingular $n \times n$ M-matrix.*

2. *All of the principal minors of A are positive, that is,*

$$\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{1k} & \dots & a_{kk} \end{vmatrix} > 0 \quad \textit{for every } k = 1, 2, \dots, n.$$

3. *Every real eigenvalue of A is positive.*

4. *A is semipositive, that is, there exists $x \gg 0$ in $\mathbb{R}^n$ such that $Ax \gg 0$.*

We first use the M-matrix theory to yield a criterion on the stability in probability for state-independent regime-switching diffusion processes, then go to deal with state-dependent processes. For a vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_N)^* \in \mathbb{R}^N$, $\mathrm{diag}(\beta_1, \dots, \beta_N)$ denotes the diagonal matrix generated by the vector $\boldsymbol{\beta}$ as usual.

Theorem 3.3 *Let (X_t, Λ_t) be a state-independent regime-switching diffusion process satisfying* (3.1) *and* (3.2)*. Assume (H.2), (H.3) hold. Suppose that there exist $r_0 > 0$, $f \in \mathscr{D}(r_0)$ so that $b(\cdot, i)$ and $a(\cdot, i)$ are differentiable on $(0, r_0)$ for each $i \in \mathcal{S}$ and*

$$-\big(\mathrm{diag}(\beta_f(1), \ldots, \beta_f(N)) + Q\big) \tag{3.7}$$

is a nonsingular M-matrix, where

$$\beta_f(i) = \limsup_{x \to 0+} b'(x, i) + \frac{a(x, i)f''(x) + (a'(x, i) + b(x, i))f'(x)}{f(x)}.$$

Then $X_t \equiv 0$ of (3.1)*,* (3.2) *is asymptotically stable in probability.*

Proof. Because $-(\mathrm{diag}(\beta_f(1), \ldots, \beta_f(N)) + Q)$ is a nonsingular M-matrix, by Proposition 3.2, there exists a vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)^* \gg 0$ such that

$$\boldsymbol{\lambda} = -(\mathrm{diag}(\beta_f(1), \ldots, \beta_f(N)) + Q)\boldsymbol{\xi} \gg 0.$$

Set $g(x) = \int_0^x f(u)\mathrm{d}u$ for $x \in (0, r_0)$. By the definition of $\beta_f(i)$, $i \in \mathcal{S}$, for any $\varepsilon > 0$, there exists $r_1 \in (0, r_0)$ such that

$$\begin{aligned}
L^{(i)}g(x) &= g(x)\frac{a(x, i)g''(x) + b(x, i)g'(x)}{g(x) - g(0)}\\
&\le g(x) \sup_{x \in (0, r_1)} \frac{\big(a(x, i)f'(x) + b(x, i)f(x)\big)'}{f(x)}\\
&\le (\beta_f(i) + \varepsilon)g(x), \quad \forall\, x \in (0, r_1),\ i \in \mathcal{S},
\end{aligned}$$

where in the second step the mean value theorem has been used. Setting $V(x, i) = g(x)\xi_i$, we get

$$\begin{aligned}
\mathscr{A}V(x, i) &= L^{(i)}g(x)\xi_i + g(x)Q\boldsymbol{\xi}(i)\\
&\le g(x)Q\boldsymbol{\xi}(i) + (\beta_f(i) + \varepsilon)\xi_i g(x)\\
&= (-\lambda_i + \varepsilon\xi_i)g(x), \quad \forall\, x \in (0, r_1),\ i \in \mathcal{S}.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary and $(\xi_i; 1 \le i \le N)$ is bounded, we can take $\varepsilon > 0$ sufficiently small so that $-\lambda_i + \varepsilon\xi_i < 0$ for every $i \in \mathcal{S}$. Then we can choose constant $r_1 \in (0, r_0)$ such that $\mathscr{A}V(x, i) \le 0$ holds for all $x \in (0, r_1)$, $i \in \mathcal{S}$. By the Foster-Lyapunov condition (cf. [26, Lemma 7.6]), we conclude that $X_t \equiv 0$ is asymptotically stable in probability. ∎

For a state-dependent regime-switching diffusion process (X_t, Λ_t), we should use some suitable transform to change it into a new state-independent one. Then we apply the criterion obtained above based on the M-matrix theory to get a criterion for the original process. Note that

$$\beta_f(i) = \limsup_{x\to 0+} b'(x,i) + \frac{a(x,i)f''(x) + (a'(x,i) + b(x,i))f'(x)}{f(x)},$$

Assume that $M_1 = \inf_{i\in \boldsymbol{S}} \beta_f(i) > -\infty$, $M_2 = \sup_{i\in \boldsymbol{S}} \beta_f(i) < \infty$. We first divide the space $\boldsymbol{S}$ into $m(1 < m \leqslant N)$ subsets according to the stability of (X_t) in each fixed environment as follows. Let $k_i \in [M_1, M_2], i = 0, 1, \cdots, m$, satisfy

$$M_1 = k_0 < k_1 < \cdots < k_{m-1} < k_m = M_2,$$

and

$$F_i = \big\{ j \in \boldsymbol{S} : \beta_f(j) \in (k_{i-1}, k_i] \big\}, \quad i = 1, \cdots, m,$$

is nonempty. Let

$$q_{ik}^F = \begin{cases} \sup_{x\in\mathbb{R}} \sup_{r\in F_i} \sum_{j\in F_k} q_{rj}(x), & k < i, \\ \inf_{x\in\mathbb{R}} \inf_{r\in F_i} \sum_{j\in F_k} q_{rj}(x), & k > i, \end{cases} \tag{3.8}$$

for i, $k \in \{1, \cdots, m\}$ and $i \neq k$. Set $q_i^F = -q_{ii}^F = \sum_{k\neq i} q_{ik}^F < \infty$. Put $Q^F = (q_{ij}^F)$. Define

$$\beta_f^F(i) = \sup_{j\in F_i} \beta_f(j), \quad i \in \{1, 2, \cdots m\}.$$

Then

$$\beta_f^F(i) \leqslant \beta_f^F(i+1).$$

Theorem 3.4 *Let (X_t, Λ_t) be a state-dependent regime-switching diffusion process satisfying* (3.1), (3.2)*. Assume (H.1), (H.2), (H.3) hold. Suppose that there exist $r_0 > 0$, $f \in \mathscr{D}(r_0)$ such that $b(\cdot, i)$, $a(\cdot, i)$ are differentiable on $(0, r_0)$ for each $i \in \mathcal{S}$ and*

$$-\big(\mathrm{diag}(\beta_f^F(1), \ldots, \beta_f^F(m)) + Q^F\big) H_m$$

is a nonsingular M-matrix, where

$$H_m = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m\times m}. \tag{3.9}$$

Then $X_t \equiv 0$ of (3.1), (3.2) *is asymptotically stable in probability if $f \in \mathscr{D}(r_0)$.*

Proof. As $-\big(Q^F+\mathrm{diag}(\beta_f^F(1),\dots,\beta_f^F(m))\big)H_m$ is a nonsingular M-matrix, by Proposition 3.2, there exists a vector $\boldsymbol{\eta}=(\eta_1,\dots,\eta_m)^*\gg 0$ such that

$$\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_m)^*:=-\big(Q^F+\mathrm{diag}(\beta_f^F(1),\dots,\beta_f^F(m))H_m\boldsymbol{\eta}\gg 0.$$

Set $\boldsymbol{\xi}^F=H_m\boldsymbol{\eta}$. Then

$$\xi_i^F=\eta_i+\eta_{i+1}+\ldots+\eta_m,\quad i=1,\dots,m,$$

which yields that $\xi_{i+1}^F<\xi_i^F$ for $i=1,\dots,m-1$ and $\boldsymbol{\xi}^F\gg 0$. Extend $\boldsymbol{\xi}$ to a vector $\boldsymbol{\xi}$ on $\boldsymbol{S}$ by setting $\xi_j=\xi_i^F$ if $j\in F_i$. By the definition of Q^F, $\boldsymbol{\beta}_f^F$ and the decreasing property of ξ_i^F, for $i\in F_r$, we have

$$\begin{aligned}
Q_x\boldsymbol{\xi}(i)&:=\sum_{k\neq i}q_{ik}(x)(\xi_k-\xi_i)=\sum_{k\notin F_r}q_{ik}(x)(\xi_k-\xi_i)\\
&=\sum_{j<r}\Big(\sum_{k\in F_j}q_{ik}(x)\Big)(\xi_j^F-\xi_r^F)+\sum_{j>r}\Big(\sum_{k\in F_j}q_{ik}(x)\Big)(\xi_j^F-\xi_r^F)\quad(\text{since }\xi_i=\xi_r^F,i\in F_r)\\
&\le\sum_{j<r}q_{rj}^F(\xi_j^F-\xi_r^F)+\sum_{j>r}q_{rj}^F(\xi_j^F-\xi_r^F)\quad(\text{since }i\in F_r)\\
&=Q^F\boldsymbol{\xi}^F(\phi(i)),
\end{aligned}\tag{3.10}$$

where ϕ: $\boldsymbol{S}\to\{1,\cdots,m\}$ is a map defined by $\phi(i)=j$ if $i\in F_j$.

Let $g(x)=\int_0^x f(u)\mathrm{d}u$ for $x\ge 0$, where f is given in the assumption. By definition of $\beta_f(i)$, for any $\varepsilon>0$, there exists $r_1\in(0,r_0)$ such that

$$\sup_{x\in(0,r_1)}\frac{\big(a(x,i)f'(x)+b(x,i)f(x)\big)'}{f(x)}\leqslant\beta_f(i)+\varepsilon.$$

Therefore,

$$\begin{aligned}
L^{(i)}g(x)&=g(x)\Big(\frac{a(x,i)g''(x)+b(x,i)g'(x)}{g(x)}\Big)\\
&\le g(x)\sup_{x\in(0,r_1)}\frac{\big(a(x,i)f'(x)+b(x,i)f(x)\big)'}{f(x)}\\
&\le(\beta_f(i)+\varepsilon)g(x)\le(\beta_f^F(\phi(i))+\varepsilon)g(x),\quad\forall\,x\in(0,r_1),\ i\in\mathcal{S},
\end{aligned}\tag{3.11}$$

where in the second step we have used the mean value theorem and the fact $f\in\mathscr{D}(r_0)$ and $g(0)=0$, in the last step, we have used the definition of $\beta_f^F(\phi(i))$.

Letting $V(x,i)=g(x)\xi_i$, it follows from (3.10) and (3.11) that

$$\begin{aligned}\mathscr{A}V(x,i)&=Q_x\boldsymbol{\xi}(i)g(x)+\xi_iL^{(i)}g(x)\\&\leq Q^F\xi^F(\phi(i))g(x)+(\beta_f^F(\phi(i))+\varepsilon)\xi^F_{\phi(i)}g(x)\quad(\text{since } \xi_i=\xi^F_{\phi(i)})\\&=(-\lambda_{\phi(i)}+\varepsilon\xi^F_{\phi(i)})g(x),\quad\forall\, x\in(0,r_1),\ i\in\mathcal{S}.\end{aligned}$$

As $\varepsilon>0$ is arbitrary, we get that there exists $r_1>0$ such that for every $i\in\mathcal{S}$, $-\lambda_{\phi(i)}+\varepsilon\xi^F_{\phi(i)}<0$. Hence, we get $\mathscr{A}V(x,i)\leq 0$. By the Foster-Lyapunov condition (cf. [26, Lemma 7.6]), we get $X_t\equiv 0$ is asymptotically stable in probability. ∎

Remark 3.5 In Theorem 3.4, m takes integer value in $\{1,\dots,N\}$. If we take $m=2$ in (3.8), then we obtain Corollary 3.6 below, which is of important use in application in spite of the loss of certain precision. Corollary 3.7 is obtained by taking $m=N$ in (3.8), which preserves the precision of this type of criterion.

Corollary 3.6 *Let (X_t,Λ_t) be a state-dependent regime-switching diffusion process satisfying* (3.1), (3.2)*. Assume (H.1), (H.2), (H.3) hold. Suppose that there exist $r_0>0$, $f\in\mathscr{D}(r_0)$ such that $b(\cdot,i)$, $a(\cdot,i)$ are differentiable on $(0,r_0)$ for each $i\in\mathcal{S}$ and*

$$-\big(\mathrm{diag}(\beta_f^F(1),\beta_f^F(2))+Q^F\big)H_2$$

is a nonsingular M-matrix, where

$$H_2=\begin{pmatrix}1&1\\0&1\end{pmatrix}_{2\times 2}.$$

Then $X_t\equiv 0$ of (3.1), (3.2) *is asymptotically stable in probability.*

Let $m=N$ in (3.8). Then $\tilde q_{ik}:=q^F_{ik}$ becomes

$$\tilde q_{ik}=\begin{cases}\sup_{x\in\mathbb{R}}q_{ik}(x), & k<i,\\ \inf_{x\in\mathbb{R}}q_{ik}(x), & k>i,\end{cases}$$

for i, $k\in\mathcal{S}$ and $i\neq k$. Set $\tilde q_i=-\tilde q_{ii}=\sum_{k\neq i}\tilde q_{ik}$. Put $\tilde Q=(\tilde q_{ij})$. The following result holds immediately by Theorem 3.4.

Corollary 3.7 *Let (X_t,Λ_t) be a state-dependent regime-switching diffusion process satisfying* (3.1), (3.2)*. Assume (H.1), (H.2), (H.3) hold. Suppose that there exist $r_0>0$, $f\in\mathscr{D}(r_0)$ such that $b(\cdot,i)$, $a(\cdot,i)$ are differentiable on $(0,r_0)$ for each $i\in\mathcal{S}$ and*

$$-\big(\mathrm{diag}(\beta_f(1),\dots,\beta_f(N))+\tilde Q\big)H_N$$

is a nonsingular M-matrix, where H_N is a $N \times N$ matrix with $m = N$ in (3.9)*. Then* $X_t \equiv 0$ *of* (3.1), (3.2) *is asymptotically stable in probability.*

Remark 3.8 Replace q_{ik}^F in (3.8) with

$$q_{ik}^F = \begin{cases} \sup_{r\in F_i} \sum_{j\in F_k} q_{rj}, & k < i, \\ \inf_{r\in F_i} \sum_{j\in F_k} q_{rj}, & k > i, \end{cases}$$

Theorem 3.4, Corollaries 3.6 and 3.7 still hold for state independent regime-switching diffusion processes.

Example 3.2 Put $q > 1$. Consider the following regime-switching diffusion process

$$\mathrm{d}X_t = b_{\Lambda_t} X_t \mathrm{d}t + \sigma_{\Lambda_t}(X_t^q \wedge X_t)\mathrm{d}B_t,$$

where Λ_t is a birth-death process on $S = \{1, 2, \cdots, N\}$ with $q_{ii+1}(x) = c_i + (i-c)\cos x$ for $i \geqslant 1$, $q_{ii-1}(x) = a_i + (i-a)\cos x$ for $i \geqslant 2$ and $q_{ij}(x) = 0$ for $j \notin \{i-1, i, i+1\}$, where a_i, c_i, a, c are all positive constants. Assume that $\sup_{i\geqslant 2} b_i = \kappa > b_1$. If $\kappa(1-c-c_1+b_1+\gamma b_1) > b_1(a_2 + a - 2)$ for some $\gamma > 0$, then $X_t \equiv 0$ is asymptotically stable in probability.

Proof. Let $f(x) = x^\gamma$, $\gamma > 0$. Then $f \in \mathscr{D}(1)$. Since $a(x,i) = \sigma_i^2 x^{2q}/2$, $b(x,i) = b_i x$, we get

$$\begin{aligned} \beta_f(i) &= \lim_{x\to 0} b'(x,i) + \frac{a(x,i)f''(x) + (a'(x,i) + b(x,i))f'(x)}{f(x)} \\ &= \lim_{x\to 0}(1+\gamma)b_i + \left(\frac{\gamma-1}{2} + q\right)\gamma\sigma_i^2 x^{2q-2} = (1+\gamma)b_i. \end{aligned}$$

Let $k \in (b_1, \kappa)$. Then $F_1 = \{j : \beta_f(j) \leqslant k\} = \{1\}$, $F_2 = \{j : \beta_f(j) > k\} = \{2, 3, \cdots\}$ and

$$q_{21}^F = \sup_{x\in\mathbb{R}}(a_2 + (2-a)\cos x) = a_2 + (2-a),$$

$$q_{12}^F = \inf_{x\in\mathbb{R}}(c_1 + (1-c)\cos x) = c_1 - (1-c).$$

$q_1^F = -q_{11}^F = \sum_{k\neq 1} q_{1k}^F = (1-c) - c_1 < \infty$, $q_2^F = -q_{22}^F = \sum_{k\neq 2} q_{2k}^F = (a-2) - a_2 < \infty$.

$$\beta_f^F(1) = \sup_{j\in F_1} \beta_f(j) = (b_1(1+\gamma)),$$

$$\beta_f^F(2) = \sup_{j\in F_2} \beta_f(j) = \kappa(1+\gamma).$$

Put $Q^F = (q^F_{ij})$, by calculation,

$$-(Q^F + \mathrm{diag}(\beta^F_f(1), \beta^F_f(2)))H_2 = -\begin{pmatrix} 1-c-c_1+(1+\gamma)b_1 & b_1(1+\gamma) \\ a_2-a+2 & \kappa(1+\gamma) \end{pmatrix}.$$

Since

$$\big| -(Q^F + \mathrm{diag}(\beta^F_f(1), \beta^F_f(2)))H_2 \big| = (1+\gamma)\big[\kappa(1-c-c_1+b_1+\gamma b_1) - b_1(a_2+a-2)\big],$$

we have $-(Q^F + \mathrm{diag}(\beta^F_f(1), \beta^F_f(2)))H_2$ is a nonsingular M-matrix if and only if $\kappa(1-c-c_1+b_1+\gamma b_1) > b_1(a_2+a-2)$. By Corollary 3.6, when $\kappa(1-c-c_1+b_1+\gamma b_1) > b_1(a_2+a-2)$, $X_t = 0$ is asymptotically stable in probability. ∎

Remark 3.9 For fixed environment i, let $(X^{(i)}_t)$ be the diffusion process associated with (X_t) in environment i in Example 3.2, i.e.

$$\mathrm{d}X^{(i)}_t = b_i X^{(i)}_t \mathrm{d}t + \sigma_i((X^{(i)}_t)^q \wedge X^{(i)}_t)\mathrm{d}B_t.$$

By Example 2.1, it is easy to see that when $b_i < 0$, $X_t \equiv 0$ is asymptotically stable in probability; when $b_i > 0$, $X_t \equiv 0$ is unstable in probability.

Let $b_1 = -1$, $\kappa = 1$. Then $X^{(i)}_t \equiv 0$ is unstable in probability for some $i \geqslant 2$ and $X^{(1)}_t \equiv 0$ is asymptotically stable in probability. However, $X_t \equiv 0$ is asymptotically stable in probability once $c + c_1 + \gamma < a_2 + a - 2$ for some $\gamma > 0$. Therefore, the diffusion process with switching may be stable even if for some fixed environment it is unstable.

Next, we go to study the stability of multidimensional regime-switching diffusion processes by using an analogous method for multidimensional diffusion processes in Section 2. We first introduce some useful notations. Let

$$\mathscr{D}_+(r) = \{f \in \mathscr{D}(r); f'(x) > 0, \forall\, x \in (0,r)\},\ \mathscr{D}_-(r) = \{f \in \mathscr{D}(r); f'(x) < 0, \forall\, x \in (0,r)\}. \tag{3.12}$$

Set

$$a(x,i) = \frac{1}{2}\sigma(x,i)\sigma(x,i)^*, \quad \tilde{b}(r,i) = 2\sup_{|x|=\sqrt{r}}\big[\langle x, b(x,i)\rangle + \mathrm{trace}\, a(x,i)\big],$$

$$\tilde{a}(r,i) = 4\sup_{|x|=\sqrt{r}}\big[\sum_{k,l=1}^n a_{kl}(x,i)x^k x^l\big],$$

$$\bar{a}(r,i) = 4\inf_{|x|=\sqrt{r}}\big[\sum_{k,l=1}^n a_{kl}(x,i)x^k x^l\big]$$

for $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, $r \geq 0$, $i \in \mathcal{S}$.

Theorem 3.10 *Let (X_t, Λ_t) be a state-independent regime-switching diffusion process satisfying* (3.1) *and* (3.2)*. Suppose (H.2), (H.3) hold. Assume there exist $r_0 > 0$ and $f \in \mathscr{D}_+(r_0)$ such that $\tilde b(\cdot, i)$, $\tilde a(\cdot, i)$ are differentiable on $(0, r_0)$ for each $i \in \mathcal{S}$ and*

$$-(\mathrm{diag}(\beta_f(1), \ldots, \beta_f(N)) + Q)$$

is a nonsingular M-matrix, where

$$\beta_f(i) = \limsup_{r\to 0+} \tilde b'(r, i) + \frac{\tilde a(r, i) f''(r) + (\tilde a'(r, i) + \tilde b(r, i)) f'(r)}{f(r)}. \tag{3.13}$$

Then $X_t \equiv 0$ of (3.1), (3.2) *is asymptotically stable in probability.*

Proof. As $-(\mathrm{diag}(\beta_f(1), \ldots, \beta_f(N)) + Q)$ is a nonsingular M-matrix, it is easy to see from Proposition 3.2 that there exists $\varepsilon > 0$ such that

$$-\big(\mathrm{diag}(\tilde\beta_f(1), \ldots, \tilde\beta_f(N)) + Q)$$

is a nonsingular M-matrix, where $\tilde\beta_f(i) = \beta_f(i) + \varepsilon$ for $i \in \mathcal{S}$. Consequently, using Proposition 3.2 again, there exists a vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)^* \gg 0$ so that

$$\boldsymbol{\lambda} = -(\mathrm{diag}(\tilde\beta_f(1), \ldots, \tilde\beta_f(N)) + Q)\boldsymbol{\xi} \gg 0.$$

Let $g(x) = \int_0^x f(u)\mathrm{d}u$, $x \in (0, r_0)$. Let $\tau = \inf\{t > 0; |X_t|^2 \geq r_1\}$ for $r_1 \in (0, r_0)$ to be determined later. By Itô's formula (see [21]),

$$\begin{aligned}
&\mathbb{E}[g(|X_{t\wedge\tau}|^2)\xi_{\Lambda_{t\wedge\tau}}] \\
&= g(|x_0|^2)\xi_i + \mathbb{E}\int_0^{t\wedge\tau} \xi_{\Lambda_s}\big\{2g'(|X_s|^2)\big[\langle X_s, b(X_s, \Lambda_s)\rangle + \mathrm{trace}\, a(X_s, \Lambda_s)\big] \\
&\qquad + 4g''(|X_s|^2)\big(\sum_{k,l=1}^n a_{kl}(X_s, \Lambda_s)X_s^k X_s^l\big)\big\} + g(|X_s|^2)Q\boldsymbol{\xi}(\Lambda_s)\mathrm{d}s \\
&\leq g(|x_0|^2)\xi_i + \mathbb{E}\int_0^{t\wedge\tau} \xi_{\Lambda_s}\big[g'(|X_s|^2)\tilde b(|X_s|^2, \Lambda_s) + g''(|X_s|^2)\tilde a(|X_s|^2, \Lambda_s)\big] \\
&\qquad + g(|X_s|^2)Q\boldsymbol{\xi}(\Lambda_s)\mathrm{d}s.
\end{aligned}$$

By the definition of $\beta_f(i)$, there exists $r_1 \in (0, r_0)$ such that

$$g'(r)\tilde b(r, i) + g''(r)\tilde a(r, i)$$

$$
\begin{aligned}
&= g(r)\frac{f(r)\tilde{b}(r,i)+f'(r)\tilde{a}(r,i)}{g(r)-g(0)}\\
&= g(r)\frac{\big(f(r)\tilde{b}(r,i)+f'(r)\tilde{a}(r,i)\big)'}{f(r)}\\
&\le g(r)(\beta_f(i)+\varepsilon)=g(r)\tilde{\beta}_f(i),\quad \forall\, r\in(0,r_1).
\end{aligned}
$$

Combining with the fact $\boldsymbol{\xi}\gg 0$, we get

$$
\begin{aligned}
&\mathbb{E}[g(|X_{t\wedge\tau}|^2)\xi_{\Lambda_{t\wedge\tau}}]\\
&\le g(|x_0|^2)\xi_i+\mathbb{E}\int_0^{t\wedge\tau}\tilde{\beta}_f(\Lambda_s)\xi_{\Lambda_s}g(|X_s|^2)+Q\boldsymbol{\xi}(\Lambda_s)g(|X_s|^2)\mathrm{d}s\\
&= g(|x_0|^2)\xi_i-\mathbb{E}\int_0^{t\wedge\tau}\lambda_{\Lambda_s}g(|X_s|^2)\mathrm{d}s\\
&\le g(|x_0|^2)\xi_i.
\end{aligned}
$$

This further yields that

$$
g(r_1)\min_{1\le j\le N}\xi_j\mathbb{P}(\tau\le t)\le\mathbb{E}[g(|X_{t\wedge\tau}|^2)\xi_{\Lambda_{t\wedge\tau}}]\le g(|x_0|^2)\xi_i.
$$

Therefore,

$$
\mathbb{P}(\sup_{0\le s\le t}|X_s|^2\ge r_1)=\mathbb{P}(\tau\le t)\le\frac{g(|x_0|^2)\xi_i}{g(r_1)\min_{1\le j\le N}\xi_j}.
$$

Letting $t\to+\infty$, it follows that

$$
\mathbb{P}(\sup_{s\ge 0}|X_s|^2\ge r_1)\le\frac{g(|x_0|^2)\xi_i}{g(r_1)\min_{1\le j\le N}\xi_j},
$$

which yields that $X_t\equiv 0$ is stable in probability. Similarly, the asymptotic stability in probability follows from the proof of [26, Lemma 7.6, Remark 7.8]. ∎

Theorem 3.11 *Let (X_t,Λ_t) be a state-independent regime-switching diffusion process satisfying* (3.1), (3.2). *Suppose (H.2) (H.3) hold. Assume that there exist $r_0>0$, $f\in\mathscr{D}_-(r_0)$ such that $\tilde{b}(\cdot,i)$, $\bar{a}(\cdot,i)$ are differentiable on $(0,r_0)$ for each $i\in\mathcal{S}$ and*

$$
-\big(\mathrm{diag}(\beta_f(1),\dots,\beta_f(N))+Q\big)
$$

is a nonsingular M-matrix, where

$$
\beta_f(i)=\limsup_{r\to 0+}\tilde{b}'(r,i)+\frac{\bar{a}(r,i)f''(r)+(\bar{a}'(r,i)+\tilde{b}(r,i))f'(r)}{f(r)}. \tag{3.14}
$$

Then $X_t\equiv 0$ of (3.1), (3.2) *is asymptotically stable in probability.*

Proof. This theorem can be proved in the same way as Theorem 3.10 by noting that for $f \in \mathscr{D}_-(r_0)$ we should use $\bar{a}(x,i)$ to replace $\tilde{a}(x,i)$ to get the desired inequality. The details is omitted. ∎

Remark 3.12 Applying the same method as Theorems 3.10 and 3.11 which extend Theorem 3.3 to multidimensional case, Theorem 3.7 can be extended to deal with the state-dependent multidimensional regime-switching diffusion processes, which is omitted to save space.

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