Proof of the Andrews-Dyson-Rhoades Conjecture on the spt-Crank

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Abstract. The spt-crank of a vector partition, or an S-partition, was introduced by Andrews, Garvan and Liang. Let $N_S(m,n)$ denote the net number of S-partitions of n with spt-crank m, that is, the number of S-partitions (π_1,π_2,π_3) of n with spt-crank m such that the length of π_1 is odd minus the number of S-partitions (π_1,π_2,π_3) of n with spt-crank m such that the length of π_1 is even. Andrews, Dyson and Rhoades conjectured that $\{N_S(m,n)\}_m$ is unimodal for any n, and they showed that this conjecture is equivalent to an inequality between the rank and crank of ordinary partitions. They obtained an asymptotic formula for the difference between the rank and crank of ordinary partitions, which implies $N_S(m,n) \geq N_S(m+1,n)$ for sufficiently large n and fixed m. In this paper, we introduce a representation of an ordinary partition, called the m-Durfee rectangle symbol, which is a rectangular generalization of the Durfee symbol introduced by Andrews. We give a proof of the conjecture of Andrews, Dyson and Rhoades. For $m \geq 1$, we construct an injection from the set of ordinary partitions of n such that m appears in the rank-set to the set of ordinary partitions of n with rank not less than $-m$. For $m=0$, we need to construct three more injections. We also show that this conjecture implies an inequality between the positive rank and crank moments obtained by Andrews, Chan and Kim.

Keywords: Rank, crank, spt-crank, Andrews' spt-function, rank moment, crank moment.

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1 Introduction

In this paper, we give a proof of a conjecture of Andrews, Dyson and Rhoades on the spt-crank of a vector partition or an S-partition. The spt-function, called the smallest part function, was introduced by Andrews [2]. More precisely, we use $spt(n)$ to denote the total number of smallest parts in all partitions of n. For example, we have $spt(3) = 5$,

$spt(4) = 10$ and $spt(5) = 14$. The smallest part function possesses many arithmetic properties analogous to the ordinary partition function, see, for example, [2, 13, 15, 18].

Andrews [2] showed that the spt-function satisfies the following Ramanujan type congruences:

$$\begin{aligned} spt(5n+4) &\equiv 0 \pmod{5}, &(1.1)\\ spt(7n+5) &\equiv 0 \pmod{7}, &(1.2)\\ spt(13n+6) &\equiv 0 \pmod{13}. &(1.3)\end{aligned}$$

To give combinatorial interpretations of the above congruences, Andrews, Garvan and Liang [6] introduced the spt-crank of an S-partition. Let $\mathcal{D}$ denote the set of partitions into distinct parts and $\mathcal{P}$ denote the set of partitions. For $\pi \in \mathcal{P}$, we use $s(\pi)$ to denote the smallest part of π with the convention that $s(\emptyset) = +\infty$. Let $\ell(\pi)$ denote the number of parts of π and $|\pi|$ denote the sum of parts of π. Define

$$S = \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} \colon \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\}\}.$$

A triple (π_1, π_2, π_3) of partitions in S is called an S-partition, see Andrews, Garvan and Liang [6]. Moreover, if $|\pi_1| + |\pi_2| + |\pi_3| = n$, then (π_1, π_2, π_3) is called an S-partition of n. The spt-crank of an S-partition $\pi = (\pi_1, \pi_2, \pi_3)$, denoted $r(\pi)$, is defined to be the difference between the number of parts of π_2 and π_3, that is,

$$r(\pi) = \ell(\pi_2) - \ell(\pi_3).$$

For an S-partition $\pi = (\pi_1, \pi_2, \pi_3)$, we associate it with a sign $\omega(\pi) = (-1)^{\ell(\pi_1)-1}$ and let $|\pi|$ denote the sum of parts of π_1, π_2 and π_3, that is, $|\pi| = |\pi_1| + |\pi_2| + |\pi_3|$. Let $N_S(m, n)$ denote the net number of S-partitions of n with spt-crank m, that is,

$$N_S(m, n) = \sum_{\substack{|\pi|=n \\ r(\pi)=m}} \omega(\pi) \tag{1.4}$$

and

$$N_S(m, t, n) = \sum_{k \equiv m \pmod{t}} N_S(k, n).$$

Andrews, Garvan and Liang [6] established the following relations:

$$\begin{aligned} N_S(k, 5, 5n+4) &= \frac{spt(5n+4)}{5}, \quad \text{for} \quad 0 \leq k \leq 4,\\ N_S(k, 7, 7n+5) &= \frac{spt(7n+5)}{7}, \quad \text{for} \quad 0 \leq k \leq 6,\end{aligned}$$

which imply the spt-congruences (1.1) and (1.2) respectively.

The following conjecture was posed by Andrews, Dyson and Rhoades [4].

Conjecture 1.1. *For $m \geq 0$ and $n \geq 0$, we have*

$$N_S(m,n) \geq N_S(m+1,n). \tag{1.5}$$

Andrews, Dyson and Rhoades [4] showed that this conjecture is equivalent to an inequality between the rank and crank of ordinary partitions. Recall that the rank of an ordinary partition was introduced by Dyson [10] as the largest part of the partition minus the number of parts. The crank of an ordinary partition was defined by Andrews and Garvan [5] as the largest part if the partition contains no ones, otherwise as the number of parts larger than the number of ones minus the number of ones.

Andrews, Dyson and Rhoades [4] found the following connection between inequality (1.5) on $N_S(m,n)$ and an inequality on the rank and crank for ordinary partitions, as will be stated in (1.9).

Theorem 1.2. *Let $N(m,n)$ denote the number of partitions of n with rank m and $M(m,n)$ denote the number of partitions of n with crank m. Set*

$$M(0,1) = -1, \quad M(-1,1) = M(1,1) = 1, \quad M(m,1) = 0,$$

and define

$$N_{\leq m}(n) = \sum_{|r|\leq m} N(r,n), \tag{1.6}$$

$$M_{\leq m}(n) = \sum_{|r|\leq m} M(r,n). \tag{1.7}$$

Then for $m \geq 0$ and $n > 1$, we have

$$N_S(m,n) - N_S(m+1,n) = \frac{1}{2}\left(N_{\leq m}(n) - M_{\leq m}(n)\right). \tag{1.8}$$

It is clear from (1.8) that Conjecture 1.1 is equivalent to the following conjecture.

Conjecture 1.3. *For $m \geq 0$ and $n \geq 0$, we have*

$$N_{\leq m}(n) \geq M_{\leq m}(n). \tag{1.9}$$

When $m = 0$, inequality (1.9) was conjectured by Kaavya [17]. Andrews, Dyson and Rhoades [4] obtained the following asymptotic formula for $N_{\leq m}(n) - M_{\leq m}(n)$, which implies that Conjecture 1.3 holds for fixed m and sufficiently large n.

Theorem 1.4. *For each $m \geq 0$, we have*

$$\left(N_{\leq m}(n) - M_{\leq m}(n)\right) \sim \frac{(2m+1)\pi^2}{192\sqrt{3}n^2} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \tag{1.10}$$

as $n \to \infty$.

The main objective of this paper is to give a proof of Conjecture 1.3. It is easy to check that Conjecture 1.3 holds for $n = 0$ and $n = 1$. To prove Conjecture 1.3 holds for $n > 1$, we first give a reformulation of Conjecture 1.3 in terms of the rank-set. We then give an injective proof of the equivalent inequality.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an ordinary partition. Recall that the rank-set of λ introduced by Dyson [12] is an infinite sequence

$$[-\lambda_1, 1 - \lambda_2, \ldots, j - \lambda_{j+1}, \ldots, \ell - 1 - \lambda_\ell, \ell, \ell + 1, \ldots].$$

For example, the rank-set of $\lambda = (5, 5, 4, 3, 1)$ is $[-5, -4, -2, 0, 3, 5, 6, 7, 8, \ldots]$.

Dyson [12] also introduced the number of partitions λ of n such that m appears in the rank-set of λ, denoted by $q(m, n)$. For example, there are three partitions of 4 whose rank-set contains the element 1:

$$(4), \quad (2, 1, 1), \quad (1, 1, 1, 1).$$

So we have $q(1, 4) = 3$.

Dyson [12] established a connection between the number $q(m, n)$ and the number of partitions of n with a bounded crank. To be more specific, let $M(\leq m, n)$ denote the number of partitions of n with crank not greater than m. Dyson [12] obtained the following relation for $n > 1$,

$$M(\leq m, n) = q(m, n), \tag{1.11}$$

see also Berkovich and Garvan [8]. Moreover, Dyson [11, 12] proved the following symmetries of $N(m, n)$ and $M(m, n)$:

$$\begin{aligned} N(m, n) &= N(-m, n), & (1.12) \\ M(m, n) &= M(-m, n). & (1.13) \end{aligned}$$

Using relations (1.11), (1.12) and (1.13), we are led to the following connection between $N_{\leq m}(n) - M_{\leq m}(n)$ and $p(-m, n) - q(m, n)$, where $p(-m, n)$ stands for the number of partitions of n with rank not less than $-m$.

Theorem 1.5. *For $m \geq 0$ and $n > 1$, we have*

$$N_{\leq m}(n) - M_{\leq m}(n) = 2(p(-m, n) - q(m, n)). \tag{1.14}$$

It is clear from (1.14) that Conjecture 1.3 is equivalent to the following assertion.

Theorem 1.6. *For $m \geq 0$ and $n \geq 1$, we have*

$$q(m, n) \leq p(-m, n). \tag{1.15}$$

To prove the above theorem, we first introduce a representation of an ordinary partition, called the m-Durfee rectangle symbol, which is a generalization of the Durfee symbol introduced by Andrews [1]. Using this representation, we give characterizations of partitions counted by $q(m,n)$ and $p(-m,n)$. We then construct an injection from the set of partitions of n such that m appears in the rank-set to the set of partitions of n with rank not less than $-m$.

We also note that Conjecture 1.3 implies the following inequality between the positive rank moments $\overline{N}_k(n)$ and the positive crank moments $\overline{M}_k(n)$ obtained by Andrews, Chan and Kim [3], where

$$\overline{N}_k(n) = \sum_{m=1}^{+\infty} m^k N(m,n), \tag{1.16}$$

$$\overline{M}_k(n) = \sum_{m=1}^{+\infty} m^k M(m,n). \tag{1.17}$$

Theorem 1.7. ([3]) *For $k \geq 1$ and $n \geq 1$, we have*

$$\overline{M}_k(n) > \overline{N}_k(n). \tag{1.18}$$

Bringmann and Mahlburg [9] proved that the above inequality (1.18) holds for any fixed positive integer k and sufficiently large n by deriving the following asymptotic formula for $\overline{M}_k(n) - \overline{N}_k(n)$.

Theorem 1.8. *For $k \geq 1$, we have*

$$\overline{M}_k(n) - \overline{N}_k(n) \sim k!\zeta(k-2)(1-2^{3-k}) \frac{6^{\frac{k-1}{2}}}{4\sqrt{3}\pi^{k-1}} n^{\frac{k}{2}-\frac{3}{2}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \tag{1.19}$$

as $n \to \infty$, where $\zeta(s)$ denotes the Riemann ζ-function.

When k is even, inequality (1.18) is equivalent to an inequality of Garvan on the ordinary rank moments $N_k(n)$ and the ordinary crank moments $M_k(n)$ introduced by Atkin and Garvan [7]. For $k \geq 1$ and $n \geq 1$, Garvan [14] proved that

$$M_{2k}(n) > N_{2k}(n), \tag{1.20}$$

where

$$\begin{aligned} N_k(n) &= \sum_{m=-\infty}^{+\infty} m^k N(m,n), \\ M_k(n) &= \sum_{m=-\infty}^{+\infty} m^k M(m,n). \end{aligned}$$

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.5. By Theorem 1.5, we see that Conjecture 1.3 is equivalent to Theorem 1.6. In Section 3, we define m-Durfee rectangle symbols and give characterizations of partitions counted by $q(m,n)$ and $p(-m,n)$. In Section 4, we present an injective proof of Theorem 1.6 for the case $m \geq 1$. To this end, we build an injection from the set of partitions counted by $q(m,n)$ to the set of partitions counted by $p(-m,n)$. We divide the set of partitions counted by $q(m,n)$ into six disjoint subsets $Q_i(m,n)$ $(1 \leq i \leq 6)$ and divide the set of partitions counted by $p(-m,n)$ into eight disjoint subsets $P_i(-m,n)$ $(1 \leq i \leq 8)$. The injection consists of six injections ϕ_i from the set $Q_i(m,n)$ to the set $P_i(-m,n)$, where $1 \leq i \leq 6$. In Section 5, we provide a proof of Theorem 1.6 for the case $m = 0$. It turns out that the case $m = 0$ is not simpler than the general case $m \geq 1$. The injections $\phi_1, \phi_2, \phi_3, \phi_4$ in Section 4 also apply to the sets $Q_i(0,n)$, where $1 \leq i \leq 4$. We further divide $Q_5(0,n) \cup Q_6(0,n)$ into five disjoint subsets $\bar{Q}_i(0,n)$ $(1 \leq i \leq 5)$ and divide $P_5(0,n) \cup P_6(0,n)$ into three disjoint subsets $\bar{P}_i(0,n)$ $(1 \leq i \leq 3)$. In addition to the two injections ϕ_5 and ϕ_6, we need three more injections. In Section 6, we demonstrate that Theorem 1.7 of Andrews, Chan and Kim can be deduced from Conjecture 1.3.

2 Proof of Theorem 1.5

In this section, we give a proof of relation (1.14) between $N_{\leq m}(n) - M_{\leq m}(n)$ and $p(-m,n) - q(m,n)$.

Proof of Theorem 1.5. Since

$$N_{\leq m}(n) = \sum_{r=-m}^{m} N(r,n)$$

and

$$p(-m,n) = \sum_{r=-m}^{+\infty} N(r,n),$$

we get

$$N_{\leq m}(n) = p(-m,n) - \sum_{r=-\infty}^{+\infty} N(r,n) + \sum_{r=-\infty}^{m} N(r,n).$$

But

$$\sum_{r=-\infty}^{+\infty} N(r,n) = p(n), \tag{2.1}$$

so we have

$$N_{\leq m}(n) = p(-m,n) - p(n) + \sum_{r=-\infty}^{m} N(r,n). \tag{2.2}$$

Replacing r by $-r$ in the summation on the right-hand side of (2.2), and using the symmetry $N(m,n) = N(-m,n)$ in (1.12), we arrive at

$$\sum_{r=-\infty}^{m} N(r,n) = \sum_{r=-m}^{+\infty} N(-r,n) = \sum_{r=-m}^{+\infty} N(r,n) = p(-m,n). \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain

$$N_{\leq m}(n) = 2p(-m,n) - p(n). \tag{2.4}$$

Similarly, for $n > 1$ we get

$$M_{\leq m}(n) = 2q(m,n) - p(n). \tag{2.5}$$

Subtracting (2.5) from (2.4) gives (1.14). This completes the proof. ∎

3 The m-Durfee rectangle symbol

In this section, we define m-Durfee rectangle symbols and give characterizations of partitions counted by $q(m,n)$ and $p(-m,n)$.

Let λ be a partition. The m-Durfee rectangle of λ is defined to be the largest $(m+j)\times j$ rectangle contained in the Ferrers diagram of λ, see Gordon and Houten [16]. An m-Durfee rectangle is referred to as a Durfee square when $m = 0$. The m-Durfee rectangle symbol of λ is defined as

$$(\alpha,\beta)_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}, \tag{3.1}$$

where $(m+j)\times j$ is the m-Durfee rectangle of the Ferrers diagram of λ and α consists of columns to the right of the m-Durfee rectangle and β consists of rows below the m-Durfee rectangle, see Figure 3.1. Clearly, we have $m + j \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s$, $j \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_t$ and

$$|\lambda| = \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{t} \beta_i + j(m+j).$$

For example, the 2-Durfee rectangle symbol of $\lambda = (7,7,6,4,3,3,2,2,2)$ in Figure 3.1 is

$$\begin{pmatrix} 4, & 3, & 3, & 2 \\ 3, & 2, & 2, & 2 \end{pmatrix}_{5\times 3}.$$

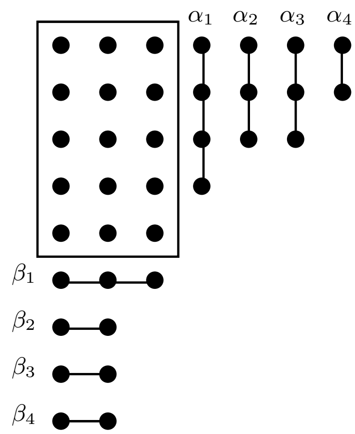

Figure 3.1: The 2-Durfee rectangle symbol of $\lambda = (7,7,6,4,3,3,2,2,2)$.

Notice that for a partition λ with $\ell(\lambda) \leq m$, there is no m-Durfee rectangle. In this case, we adopt a convention that the m-Durfee rectangle of λ is empty, that is, $j = 0$, and so the m-Durfee rectangle symbol is $(\lambda', \emptyset)_{m\times 0}$, where λ' is the conjugate of λ. For example, the 3-Durfee rectangle symbol of $\lambda = (5,5,1)$ is

$$\begin{pmatrix} 3, & 2, & 2, & 2, & 2 \\ & & & & \end{pmatrix}_{3\times 0}.$$

It should be noted that when $m = 0$, a m-Durfee rectangle symbol takes the following form

$$(\alpha,\beta)_{j\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{j\times j}, \tag{3.2}$$

which is a Durfee symbol, see Andrews [1]. In the notation of Andrews, a $D \times D$ Durfee square is simply denoted by D, as shown below

$$(\alpha,\beta)_{D} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{D}. \tag{3.3}$$

For example, the Durfee symbol of $\lambda = (7,7,6,4,3,3,2,2,2)$ in Figure 3.2 is

$$(\alpha,\beta)_{D} = \begin{pmatrix} 3, & 3, & 2 & & \\ 3, & 3, & 2, & 2, & 2 \end{pmatrix}_{4}.$$

The following two properties will be used in the next section to describe partitions counted by $q(m,n)$ and $p(-m,n)$.

Proposition 3.1. *Let λ be a partition and $(\alpha,\beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then m appears in the rank-set of λ if and only if either $j = 0$ or $j \geq 1$ and $\beta_1 = j$.*

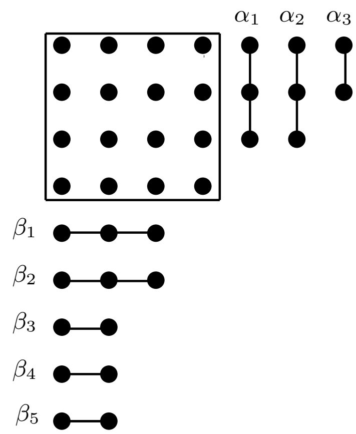

Figure 3.2: The Durfee symbol of $\lambda = (7, 7, 6, 4, 3, 3, 2, 2, 2)$.

Proof. We first show that if m appears in the rank-set of λ, then either $j = 0$ or $j \geq 1$ and $\beta_1 = j$. Assume that m appears in the rank-set of λ. By definition, there exists an integer $k \geq 0$, such that $k - \lambda_{k+1} = m$. Obviously, $k \geq m$. Consider the following two cases.

Case 1: $k = m$. Clearly, λ_{m+1} is equal to zero, which implies that $\ell(\lambda) \leq m$. So we have $j = 0$.

Case 2: $k > m$. We have $\lambda_{k+1} = k - m \geq 1$. Let $\lambda = (\alpha, \beta)_{(m+j)\times j}$. We claim that $j = k - m$. Notice that $\lambda_k \geq \lambda_{k+1} = k - m$ and $\lambda_{k+1} = k - m < k + 1 - m$. By definition, the m-Durfee rectangle of λ is equal to $k \times (k - m)$. This yields $j = k - m \geq 1$, so that the claim is verified. Hence we have $\beta_1 = \lambda_{k+1} = k - m = j$.

We next show that if $j = 0$ or $j \geq 1$ and $\beta_1 = j$, then m appears in the rank-set of λ.

Case 1: $j = 0$. In this case, we have $\ell(\lambda) \leq m$, which implies that $\lambda_{m+1} = 0$. Thus, $m - \lambda_{m+1} = m$. So m appears in the rank-set of λ.

Case 2: $j \geq 1$ and $\beta_1 = j$. By definition, we have $\lambda_{m+j+1} = \beta_1 = j$. Hence $j + m - \lambda_{j+m+1} = j + m - j = m$. In other words, m appears in the rank-set of λ. This completes the proof. ∎

Proposition 3.2. *Let λ be a partition and $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Then the rank of λ is not less than $-m$ if and only if either $j = 0$ or $j \geq 1$ and $\ell(\beta) \leq \ell(\alpha)$.*

Proof. First, we assume that the rank of λ is not less than $-m$, that is, $\lambda_1 - \ell(\lambda) \geq -m$. We aim to show that either $j = 0$ or $j \geq 1$ and $\ell(\beta) \leq \ell(\alpha)$. There are two following cases:

Case 1: $\ell(\lambda) \leq m$. By definition, it is clear that $j = 0$.

Case 2: $\ell(\lambda) \geq m+1$. By definition, we have $j \geq 1$, $\lambda_1 = j+\ell(\alpha)$ and $\ell(\lambda) = j+m+\ell(\beta)$. Hence

$$\lambda_1 - \ell(\lambda) = (j + \ell(\alpha)) - (j + m + \ell(\beta)) = -m + (\ell(\alpha) - \ell(\beta)).$$

Since $\lambda_1 - \ell(\lambda) \geq -m$, we deduce that $\ell(\beta) \leq \ell(\alpha)$.

Conversely, we assume that $j = 0$ or $j \geq 1$ and $\ell(\beta) \leq \ell(\alpha)$. We claim that the rank of λ is not less than $-m$.

Case 1: $j = 0$. Clearly, we have $\ell(\lambda) \leq m$, which implies that the rank of λ is not less than $-m$.

Case 2: $j \geq 1$ and $\ell(\beta) \leq \ell(\alpha)$. By definition, we have $\lambda_1 = j + \ell(\alpha)$ and $\ell(\lambda) = j + m + \ell(\beta)$. Hence

$$\lambda_1 - \ell(\lambda) = (j + \ell(\alpha)) - (j + m + \ell(\beta)) = -m + (\ell(\alpha) - \ell(\beta)). \tag{3.4}$$

Note that $\ell(\alpha) - \ell(\beta) \geq 0$. From (3.4), we deduce that $\lambda_1 - \ell(\lambda) \geq -m$, and so the claim is proved. ∎

4 Proof of Theorem 1.6 for $m \geq 1$

Let $Q(m, n)$ denote the set of partitions λ of n such that m appears in the rank-set of λ and $P(-m, n)$ denote the set of partitions of n with rank not less than $-m$. Theorem 1.6 is equivalent to the following combinatorial statement.

Theorem 4.1. *For $m \geq 0$, there is an injection Φ from the set $Q(m, n)$ to the set $P(-m, n)$.*

In this section, we give a proof of Theorem 4.1 for $m \geq 1$, and hence Theorem 1.6 holds for $m \geq 1$. The proof of Theorem 4.1 for the case $m = 0$ will be given in the next section since it relies on the injections for the case $m \geq 1$.

To establish an injection Φ from the set $Q(m, n)$ to the set $P(-m, n)$, we divide $Q(m, n)$ into six disjoint subsets $Q_i(m, n)$ $(1 \leq i \leq 6)$ and divide $P(-m, n)$ into eight disjoint subsets $P_i(-m, n)$ $(1 \leq i \leq 8)$. We proceed to construct six injections ϕ_i from $Q_i(m, n)$ to $P_i(-m, n)$, where $1 \leq i \leq 6$. Notice that the injections ϕ_1, ϕ_2, ϕ_3 and ϕ_4 hold for $m \geq 0$, and the injections ϕ_5 and ϕ_6 hold only for $m \geq 1$. In fact, the injections ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are needed in the construction of the injection Φ for the case $m = 0$.

To divide $Q(m, n)$ into six classes, let λ be a partition in $Q(m, n)$ and let $(\alpha, \beta)_{(m+j)\times j}$ be the m-Durfee rectangle symbol of λ. Write

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j},$$

that is, we also consider the m-Durfee rectangle symbol as a partition in $Q(m,n)$. By Proposition 3.1, we see that either $j=0$ or $\beta_1=j$ with $j\geq 1$. The subsets $Q_i(m,n)$ can be described by using the m-Durfee rectangle symbol $(\alpha,\beta)_{(m+j)\times j}$.

(1) $Q_1(m,n)$ is the set of m-Durfee rectangle symbols in $Q(m,n)$ for which one of the following conditions holds:
(i) $j=0$;
(ii) $j\geq 1$ and $\ell(\beta)-\ell(\alpha)\leq -1$;
(iii) $j\geq 1$, $\ell(\beta)-\ell(\alpha)=0$ and $\alpha_1=m+j$;

(2) $Q_2(m,n)$ is the set of m-Durfee rectangle symbols in $Q(m,n)$ such that $j\geq 1$, $\ell(\beta)-\ell(\alpha)\geq 0$ and $\alpha_1<m+j$;

(3) $Q_3(m,n)$ is the set of m-Durfee rectangle symbols in $Q(m,n)$ such that $j\geq 1$, $\ell(\beta)-\ell(\alpha)\geq 1$, $\alpha_1=m+j$ and $s(\beta)=1$;

(4) $Q_4(m,n)$ is the set of m-Durfee rectangle symbols in $Q(m,n)$ such that $j\geq 1$, $\ell(\beta)-\ell(\alpha)\geq 1$, $\alpha_1=m+j>\alpha_2$ and $s(\beta)\geq 2$;

(5) $Q_5(m,n)$ is the set of m-Durfee rectangle symbols in $Q(m,n)$ such that $j\geq 1$, $\ell(\beta)-\ell(\alpha)\geq 1$, $\alpha_1=\alpha_2=m+j>\alpha_3$ and $s(\beta)\geq 2$;

(6) $Q_6(m,n)$ is the set of m-Durfee rectangle symbols in $Q(m,n)$ such that $j\geq 1$, $\ell(\beta)-\ell(\alpha)\geq 1$, $\alpha_1=\alpha_2=\alpha_3=m+j$ and $s(\beta)\geq 2$.

To divide the set $P(-m,n)$ into eight classes, we also view $P(-m,n)$ as the set of m-Durfee rectangle symbols of partitions counted by $p(-m,n)$. Let $(\gamma,\delta)_{(m+j')\times j'}$ be the m-Durfee rectangle symbol of a partition μ in $P(-m,n)$. By Proposition 3.2, we have either $j'=0$ or $j'\geq 1$ and $\ell(\delta)-\ell(\gamma)\leq 0$. The subsets $P_i(-m,n)$ can be described as follows.

(1) $P_1(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ for which one of the following conditions holds:
(i) $j'=0$;
(ii) $j'\geq 1$, $\ell(\delta)-\ell(\gamma)\leq -1$ and $\delta_1=j'$;
(iii) $j'\geq 1$, $\ell(\gamma)=\ell(\delta)$, $\gamma_1=m+j'$ and $\delta_1=j'$;

(2) $P_2(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ with $j'\geq 1$ and $\delta_1=j'-1$;

(3) $P_3(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ with $j'\geq 2$ and $\delta_1\leq j'-2$;

(4) $P_4(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ such that $j'\geq 1$, $\ell(\gamma)=\ell(\delta)$, $\gamma_1=m+j'-1$, $\delta_1=j'$ and δ has a part equal to 2;

(5) $P_5(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ such that $j' \geq 1$, $\ell(\gamma)=\ell(\delta)$, $\gamma_1 \leq m+j'-3$ and $\delta_1 = j'$;

(6) $P_6(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ such that $j' \geq 1$, $\ell(\gamma)=\ell(\delta)$, $\gamma_1 = m+j'-2$ and $\delta_1 = j'$;

(7) $P_7(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ such that $j' \geq 1$, $\ell(\gamma)=\ell(\delta)$, $\gamma_1 = m+j'-1 > \gamma_2$, $\delta_1 = j'$ and δ has no parts equal to 2;

(8) $P_8(-m,n)$ is the set of m-Durfee rectangle symbols in $P(-m,n)$ such that $j' \geq 1$, $\ell(\gamma)=\ell(\delta)$, $\gamma_1 = \gamma_2 = m+j'-1$, $\delta_1 = j'$ and δ has no parts equal to 2.

We are now ready to present the six injections ϕ_i from $Q_i(m,n)$ to $P_i(-m,n)$, where $1 \leq i \leq 6$. It is clear that $Q_1(m,n)$ coincides with $P_1(-m,n)$, so that ϕ_1 can be set to the identity map. The following lemma gives an injection from $Q_2(m,n)$ to $P_2(-m,n)$.

Lemma 4.2. *For $m \geq 0$, there is an injection ϕ_2 from $Q_2(m,n)$ to $P_2(-m,n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $Q_2(m,n)$. By definition, we have $\beta_1 = j \geq 1$, $\alpha_1 < m+j$ and $t-s \geq 0$.

Define

$$\phi_2(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \alpha_1+1, & \alpha_2+1, & \ldots, & \alpha_s+1, & 1^{t-s} \\ \beta_1-1, & \beta_2-1, & \ldots, & \beta_t-1 & \end{pmatrix}_{(m+j)\times j}.$$

It is evident that $\ell(\delta) \leq t$ and $\ell(\gamma) = t$, so that $\ell(\delta) - \ell(\gamma) \leq 0$. Moreover it is easy to see that $\delta_1 = j-1$ and $|\phi_2(\lambda)| = |\lambda|$. Hence $\phi_2(\lambda)$ is in $P_2(-m,n)$.

To prove that the map ϕ_2 is an injection, let

$$H(m,n) = \{\phi_2(\lambda) \colon \lambda \in Q_2(m,n)\}.$$

It is easy to check that for $n \neq m+1$, $H(m,n) = P_2(-m,n)$, and for $n = m+1$, we have

$$H(m,n) = P_2(-m,n) \setminus \{(\emptyset,\emptyset)_{(m+1)\times 1}\}.$$

Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{s'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $H(m,n)$. Since $\mu \in P_2(-m,n)$, we have $s' \geq t'$. Define $\sigma(\mu)$ to be

$$\sigma(\mu) = \begin{pmatrix} \gamma_1 - 1, & \gamma_2 - 1, & \ldots, & \gamma_{s'} - 1 & \\ \delta_1 + 1, & \delta_2 + 1, & \ldots, & \delta_{t'} + 1, & 1^{s'-t'} \end{pmatrix}_{(m+j')\times j'}.$$

It can be verified $\sigma(\mu)$ is in $Q_2(m,n)$ and $\sigma(\phi_2(\lambda)) = \lambda$ for any λ in $Q_2(m,n)$. Hence the map ϕ_2 is a bijection between $Q_2(m,n)$ and $H(m,n)$. ∎

For example, for $m = 2$ and $n = 31$, let

$$\lambda = \begin{pmatrix} 4, & 2, & 2 & \\ 3, & 2, & 2, & 1 \end{pmatrix}_{5\times 3}$$

be a 2-Durfee rectangle symbol in $Q_2(2,31)$. Applying the map ϕ_2 to λ, we obtain

$$\phi_2(\lambda) = \begin{pmatrix} 5, & 3, & 3, & 1 \\ 2, & 1, & 1 & \end{pmatrix}_{5\times 3},$$

which is a 2-Durfee rectangle symbol in $P_2(-2,31)$. Applying σ to $\phi_2(\lambda)$, we recover λ, that is, $\sigma(\phi_2(\lambda)) = \lambda$.

Lemma 4.3. *For $m \geq 0$, there is a bijection ϕ_3 between $Q_3(m,n)$ and $P_3(-m,n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $Q_3(m,n)$. By definition, we have $j = \beta_1 \geq \beta_t = 1$, $\alpha_1 = m + j$ and $t - s \geq 1$.

Define

$$\phi_3(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \alpha_2 + 1, & \ldots, & \alpha_s + 1, & 1^{t-s-1} \\ \beta_2 - 1, & \ldots, & \beta_t - 1 & \end{pmatrix}_{(m+j+1)\times(j+1)}.$$

To prove that $\phi_3(\lambda) \in P_3(-m,n)$, we proceed to verify that $\gamma_1 \leq m + j'$, $\delta_1 \leq j' - 2$, $\ell(\delta) - \ell(\gamma) \leq 0$ and $|\lambda| = |\phi_3(\lambda)|$. First, it is easy to see that

$$\gamma_1 = \alpha_2 + 1 \leq m + j + 1 = m + j'$$

and

$$\delta_1 = \beta_2 - 1 \leq j - 1 \leq j' - 2.$$

By definition, $\ell(\gamma) = t - 2$ and $\ell(\delta) \leq t - 2$ for $\beta_t = 1$. Hence $\ell(\delta) - \ell(\gamma) \leq 0$.

Note that

$$|\phi_3(\lambda)| = |\gamma| + |\delta| + (j+1)(m+j+1).$$

But

$$\begin{aligned} |\gamma| + |\delta| &= (|\alpha| - \alpha_1 + t - 2) + (|\beta| - \beta_1 - (t-1)) \\ &= |\alpha| + |\beta| - (m+j) - j - 1, \end{aligned}$$

we find that

$$\begin{aligned} |\phi_3(\lambda)| &= |\alpha| + |\beta| - (m+j) - 1 - j + (j+1)(m+j+1) \\ &= |\alpha| + |\beta| + j(m+j), \end{aligned}$$

which equals $|\lambda|$. Hence $\phi_3(\lambda) \in P_3(-m, n)$. To show that ϕ_3 is a bijection, we construct the inverse map ζ of ϕ_3. Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{s'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $P_3(-m, n)$. Since $\mu \in P_3(-m, n)$, we have $s' \geq t'$, $j' \geq 2$ and $\delta_1 \leq j' - 2$. Define $\zeta(\mu)$ to be

$$\zeta(\mu) = \begin{pmatrix} m + j' - 1, & \gamma_1 - 1, & \gamma_2 - 1, & \ldots, & \gamma_{s'} - 1 & \\ j' - 1, & \delta_1 + 1, & \delta_2 + 1, & \ldots, & \delta_{t'} + 1, & 1^{s'-t'+1} \end{pmatrix}_{(m+j'-1)\times(j'-1)}.$$

It is easy to check that $\zeta(\mu)$ is in $Q_3(m, n)$ and ζ is the inverse map of ϕ_3. So we conclude that ϕ_3 is a bijection. ∎

For example, for $m = 2$ and $n = 34$, let

$$\lambda = \begin{pmatrix} 5, & 4, & 1 & \\ 3, & 3, & 2, & 1 \end{pmatrix}_{5\times 3}$$

be a 2-Durfee rectangle symbol in $Q_3(2, 34)$. Applying the bijection ϕ_3 to λ, we get

$$\phi_3(\lambda) = \begin{pmatrix} 5, & 2 \\ 2, & 1 \end{pmatrix}_{6\times 4},$$

which is in $P_3(-2, 34)$. Applying ζ to $\phi_3(\lambda)$ we recover λ.

The following proposition will be used in the construction of the injection ϕ_4.

Proposition 4.4. *For $m \geq 0$, let*

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $Q_4(m,n)$. Then there exists an integer $1 \leq k \leq s$ such that

$$\alpha_{k+1} \leq \beta_k - 1 \tag{4.1}$$

and

$$\alpha_k \geq \beta_{k+1} - 1. \tag{4.2}$$

Proof. By the definition of $Q_4(m,n)$, we have $j = \beta_1 \geq \beta_t \geq 2$, $m + j = \alpha_1 > \alpha_2$ and $t - s \geq 1$. When $m = 0$, we may choose $k = 1$, since

$$\alpha_2 \leq j - 1 = \beta_1 - 1$$

and

$$\alpha_1 = j > \beta_2 - 1.$$

When $m \geq 1$, let

$$h = \min\{i \colon 1 \leq i \leq t, \alpha_i \leq \beta_i - 1\}.$$

Setting $k = h - 1$, we proceed to show that $1 \leq k \leq s$ and relations (4.1) and (4.2) hold. Since $\beta_t \geq 2$, $\alpha_{s+1} = 0$ and $t \geq s+1$, we have $\alpha_{s+1} \leq \beta_{s+1} - 1$, which implies that $h \leq s + 1$, that is, $k \leq s$. Observing that $\alpha_1 = j + m > j - 1 = \beta_1 - 1$, we get $h \geq 2$, that is, $k \geq 1$. Thus, we have $1 \leq k \leq s$. By the definition of h, we find that

$$\alpha_h \leq \beta_h - 1$$

and

$$\alpha_{h-1} > \beta_{h-1} - 1.$$

It follows that

$$\alpha_h \leq \beta_h - 1 \leq \beta_{h-1} - 1$$

and

$$\alpha_{h-1} > \beta_{h-1} - 1 \geq \beta_h - 1,$$

which implies that $k = h - 1$. This completes the proof. ∎

Lemma 4.5. *For $m \geq 0$, there is an injection ϕ_4 from $Q_4(m,n)$ to $P_4(-m,n)$.*

Proof. We first construct a map ϕ_4 from $Q_4(m,n)$ to $P_4(-m,n)$, then we show that it is an injection. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $Q_4(m,n)$. By Proposition 4.4, we may choose k to be the minimum integer such that $1 \le k \le s$, $\alpha_{k+1} \le \beta_k - 1$ and $\alpha_k \ge \beta_{k+1} - 1$. By the definition of $Q_4(m,n)$, we have $j = \beta_1 \ge \beta_t \ge 2$, $m + j = \alpha_1 > \alpha_2$ and $t - s \ge 1$. So we may define

$$\begin{aligned}
\phi_4(\lambda) &= \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} \\
&= \begin{pmatrix} \alpha_1 - 1, & \alpha_2, & \ldots, & \alpha_k, & \beta_{k+1} - 1, & \ldots, & \beta_t - 1 & & \\ \beta_1, & \beta_2, & \ldots, & \beta_k, & \alpha_{k+1} + 1, & \ldots, & \alpha_s + 1, & 2, & 1^{t-s-1} \end{pmatrix}_{(m+j)\times j}
\end{aligned} \tag{4.3}$$

Apparently, $\gamma_1 = \alpha_1 - 1 = j' + m - 1$, $\delta_1 = \beta_1 = j' = j$, $\ell(\gamma) = \ell(\delta) = t$ and $\delta_{s+1} = 2$. Furthermore, it can be easily checked that $|\phi_4(\lambda)| = |\lambda|$. This yields that $\phi_4(\lambda) \in P_4(-m,n)$.

To prove that ϕ_4 is an injection, let

$$I(m,n) = \{\phi_4(\lambda)\colon \lambda \in Q_4(m,n)\}$$

be the set of images of ϕ_4, which has been shown to be a subset of $P_4(-m,n)$. We wish to show that the construction of ϕ_4 is reversible, which implies that ϕ_4 is an injection. More precisely, we shall show that there exists a map φ from $I(m,n)$ to $Q_4(m,n)$ such that for any λ in $Q_4(m,n)$ we have

$$\varphi(\phi_4(\lambda)) = \lambda.$$

We now describe the map φ. Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{t'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{(m+j')\times j'} \tag{4.4}$$

be an m-Durfee rectangle symbol in $I(m,n)$. The following procedure generates an m-Durfee rectangle symbol $\varphi(\mu)$ in $Q_4(m,n)$.

We claim that for $\mu \in I(m,n)$ given by (4.4), there exists an integer k' such that $1 \le k' \le \ell(\gamma) - 1$ and

$$\delta_{k'} - 1 \ge \gamma_{k'+1}, \quad \gamma_{k'} \ge \delta_{k'+1} - 1 \ge 1. \tag{4.5}$$

Since $\mu \in I(m,n)$, there exists $\lambda \in Q_4(m,n)$ such that $\phi_4(\lambda) = \mu$. By the choice of k in the construction $\phi_4(\lambda)$, we see that

$$1 \le k \le s \le t - 1 = \ell(\gamma) - 1.$$

Again, from the construction (4.3) of $\phi_4(\lambda)$, we find that

$$\delta_k \ge \gamma_{k+1} + 1$$

and

$$\gamma_k \geq \delta_{k+1} - 1 \geq 1.$$

So k satisfies the conditions in (4.5). Thus the claim is verified.

Now, we may choose k' to be the minimum integer such that $1 \leq k' \leq \ell(\gamma)-1$, $\delta_{k'}-1 \geq \gamma_{k'+1}$ and $\gamma_{k'} \geq \delta_{k'+1} - 1 \geq 1$. Since μ is in $P_4(-m,n)$, the partition δ in the m-Durfee rectangle symbol of μ has a part equal to 2. Assume that $\delta_{s'} = 2 > \delta_{s'+1}$. Then we may define

$$\begin{aligned}
\varphi(\mu) &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} \\
&= \begin{pmatrix} \gamma_1+1, & \gamma_2, & \ldots, & \gamma_{k'}, & \delta_{k'+1}-1, & \ldots, & \delta_{s'-1}-1 \\ \delta_1, & \delta_2, & \ldots, & \delta_{k'}, & \gamma_{k'+1}+1, & \ldots, & \gamma_{t'}+1 \end{pmatrix}_{(m+j')\times j'}. \qquad (4.6)
\end{aligned}$$

Evidently, $\beta_1 = \delta_1 = j$, $\alpha_1 = \gamma_1 + 1 = m + j > \alpha_2$, $\beta_{t'} = \gamma_{t'} + 1 \geq 2$ and $t' > s' - 1$. Moreover, it is easy to check that $|\varphi(\mu)| = |\mu|$. So we deduce that $\varphi(\mu) \in Q_4(m,n)$.

It remains to verify that $\varphi(\phi_4(\lambda)) = \lambda$. By the constructions (4.3) and (4.6) of $\phi_4(\lambda)$ and $\varphi(\mu)$, it suffices to show that the integer k appearing in the representation of $\phi_4(\lambda)$ coincides with the integer k' appearing in the representation of $\varphi(\phi_4(\lambda))$.

Recall that k is the minimum integer determined by λ subject to the conditions

$$1 \leq k \leq s, \quad \alpha_k \geq \beta_{k+1} - 1, \quad \text{and} \quad \alpha_{k+1} \leq \beta_k - 1. \qquad (4.7)$$

On the other hand, it can be shown that k is also the minimum integer k' depending on $\phi_4(\lambda)$ such that

$$1 \leq k' \leq \ell(\gamma) - 1, \quad \delta_{k'} - 1 \geq \gamma_{k'+1} \quad \text{and} \quad \gamma_{k'} \geq \delta_{k'+1} - 1 \geq 1. \qquad (4.8)$$

From the definitions of k and s, we find that $s \leq t - 1 = \ell(\gamma) - 1$, which implies $k \leq \ell(\gamma) - 1$. By the construction (4.3) in $\phi_4(\lambda)$, we have $\gamma_{k+1} = \beta_{k+1} - 1$ and $\delta_k = \beta_k$. Furthermore, we have $\gamma_1 = \alpha_1 - 1$, and $\gamma_k = \alpha_k$ for $k \geq 2$. It can also be seen that $\delta_{s+1} = 2$ and $\delta_{k+1} = \alpha_{k+1} + 1$ for $1 \leq k \leq s - 1$. Hence we deduce that $\delta_k - 1 \geq \gamma_{k+1}$ and $\gamma_k \geq \delta_{k+1} - 1 \geq 1$ for $1 \leq k \leq s$, that is, k satisfies the conditions in (4.8).

Finally, we need to show that k is the minimum integer satisfying conditions in (4.8). Assume to the contrary that there is an integer $1 \leq p \leq k-1$ for which the conditions in (4.8) are satisfied, that is,

$$\delta_p - 1 \geq \gamma_{p+1} \quad \text{and} \quad \gamma_p \geq \delta_{p+1} - 1 \geq 1.$$

From the construction (4.3) of $\phi_4(\lambda)$ and the assumption $1 \leq p \leq k-1$, we find that

$$\alpha_{p+1} = \gamma_{p+1}, \quad \beta_p = \delta_p, \quad \beta_{p+1} = \delta_{p+1}.$$

Moreover, by (4.3) we see that $\alpha_p = \gamma_p + 1$ if $p = 1$ and $\alpha_p = \gamma_p$ if $p \geq 2$. In either case, we have

$$\alpha_p \geq \beta_{p+1} - 1 \quad \text{and} \quad \alpha_{p+1} \leq \beta_p - 1.$$

This means that p also satisfies the conditions in (4.7), contradicting the choice of k. So we conclude that k is the minimum integer satisfying conditions in (4.8), which implies that $\varphi(\phi_4(\lambda)) = \lambda$. This completes the proof. ∎

For example, for $m = 2$ and $n = 41$, consider the following 2-Durfee rectangle symbol in $Q_4(2, 41)$:

$$\lambda = \begin{pmatrix} 5, & 4, & 2, & 1 & & \\ 3, & 3, & 2, & 2, & 2, & 2 \end{pmatrix}_{5\times 3}.$$

It can be checked that $k = 2$. Applying the injection ϕ_4 to λ, we get

$$\mu = \phi_4(\lambda) = \begin{pmatrix} 4, & 4, & 1, & 1, & 1, & 1 \\ 3, & 3, & 3, & 2, & 2, & 1 \end{pmatrix}_{5\times 3},$$

which is in $P_4(-2, 41)$. Applying φ to μ, we obtain that $k' = 2$ and $\varphi(\mu) = \lambda$.

We next describe the injection ϕ_5 from $Q_5(m, n)$ to $P_5(-m, n)$.

Lemma 4.6. *For $m \geq 1$, there is an injection ϕ_5 from $Q_5(m, n)$ to $P_5(-m, n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $Q_5(m, n)$. By definition, we have $j = \beta_1 \geq \beta_t \geq 2$, $\alpha_1 = \alpha_2 = m + j > \alpha_3$ and $t - s \geq 1$.

Since $\alpha_2 - m + 2 = j + 2 > \beta_3 - 1$, we may choose the maximum number k such that $1 \leq k \leq t - 1$ and $\alpha_k - m + 2 \geq \beta_{k+1} - 1$. To define $\phi_5(\lambda)$, we construct two partitions γ and δ. It is clear that $k \geq 2$. So we may define

$$\gamma = (\beta_2 + m - 2,\ \ldots,\ \beta_k + m - 2,\ \alpha_{k+1} + 1,\ \ldots,\ \alpha_t + 1) \tag{4.9}$$

and

$$\delta = (\alpha_2 + 1 - m,\ \alpha_3 + 2 - m,\ \ldots,\ \alpha_k + 2 - m,\ \beta_{k+1} - 1,\ \ldots,\ \beta_t - 1). \tag{4.10}$$

Notice that when $k = 2$ the above definition (4.10) may be ambiguous. In this case, (4.10) is interpreted as

$$\delta = (\alpha_2 + 1 - m,\ \beta_3 - 1,\ \ldots,\ \beta_t - 1).$$

We now define

$$\phi_5(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j+1)\times(j+1)}. \tag{4.11}$$

We first prove that $(\gamma,\delta)_{(m+j+1)\times(j+1)}$ is an m-Durfee rectangle symbol. To this end, we need to show that γ and δ are partitions with $\gamma_1 \le m+j+1$ and $\delta_1 \le j+1$. We then verify that $(\gamma,\delta)_{(m+j+1)\times(j+1)}$ satisfies the conditions for $P_5(-m,n)$.

To prove that δ is a partition, it suffices to show that when $k=2$, we have

$$\alpha_2+1-m \ge \beta_3-1, \tag{4.12}$$

and when $k\ge 3$, we have

$$\alpha_2+1-m \ge \alpha_3+2-m \tag{4.13}$$

and

$$\alpha_k+2-m \ge \beta_{k+1}-1. \tag{4.14}$$

When $k=2$, since $\alpha_2-m+1=j+1$ and $\beta_3\le\beta_1=j$, we see that (4.12) holds, and so δ is a partition. When $k\ge 3$, since $\alpha_2>\alpha_3$, we get (4.13). On the other hand, (4.14) follows from the choice of k. Hence δ forms a partition when $k\ge 3$. Furthermore, it is clear from (4.10) that $\delta_1=\alpha_2+1-m=j+1$.

We now verify that γ is a partition. From the definition (4.9) of γ, it suffices to show that

$$\beta_k+m-2 \ge \alpha_{k+1}+1. \tag{4.15}$$

Keep in mind that k is in the range from 2 to $t-1$. When $k=t-1$, (4.15) becomes $\beta_{t-1}+m-2\ge\alpha_t+1$, which is valid since $\beta_{t-1}\ge 2$ and $\alpha_t=0$. When $2\le k\le t-2$, since k is the maximum integer such that $\alpha_k-m+2\ge\beta_{k+1}-1$, we have $\alpha_{k+1}-m+2<\beta_{k+2}-1$, which implies (4.15). This proves that γ is a partition. It is clear from (4.9) that $\gamma_1=\beta_2+m-2\le j+m-2$.

Next we demonstrate that $(\gamma,\delta)_{(m+j+1)\times(j+1)}$ is an m-Durfee rectangle symbol in $P_5(-m,n)$. It is clear from (4.9) and (4.10) that $\delta_1=\alpha_2+1-m=j+1$, $\gamma_1=\beta_2+m-2\le j+m-2$, and $\ell(\gamma)=\ell(\delta)=t-1$. It remains to check that $|(\gamma,\delta)_{(m+j+1)\times(j+1)}|=|\lambda|$. Note that

$$\begin{aligned} |\gamma|+|\delta| &= |\alpha|-\alpha_1+(2-m)(k-2)+1-m+(t-k) \\ &\quad +|\beta|-\beta_1+(m-2)(k-1)-(t-k) \\ &= |\alpha|-\alpha_1+|\beta|-\beta_1-1. \end{aligned}$$

Since $\beta_1=j$ and $\alpha_1=m+j$, we get

$$|\gamma|+|\delta|=|\alpha|+|\beta|-(2j+m+1).$$

Hence

$$\begin{aligned} |(\gamma,\delta)_{(m+j+1)\times(j+1)}| &= |\gamma|+|\delta|+(m+j+1)(j+1) \\ &= |\alpha|+|\beta|+j(j+m), \end{aligned}$$

which equals $|\lambda|$. So we arrive at the conclusion $(\gamma,\delta)_{(m+j+1)\times(j+1)} \in P_5(-m,n)$.

We are now in a position to prove that ϕ_5 is an injection. Let

$$J(m,n)=\{\phi_5(\lambda)\colon \lambda\in Q_5(m,n)\}$$

be the set of images of ϕ_5. It has been shown that $J(m,n)$ is a subset of $P_5(-m,n)$. We wish to construct a map τ from $J(m,n)$ to $Q_5(m,n)$ such that for any λ in $Q_5(m,n)$, we have

$$\tau(\phi_5(\lambda))=\lambda.$$

To describe the map τ, let

$$\mu=\begin{pmatrix}\gamma\\ \delta\end{pmatrix}_{(m+j')\times j'}=\begin{pmatrix}\gamma_1, & \gamma_2, & \ldots, & \gamma_{t'}\\ \delta_1, & \delta_2, & \ldots, & \delta_{t'}\end{pmatrix}_{(m+j')\times j'}$$

be an m-Durfee rectangle symbol in $J(m,n)$, that is, there is an m-Durfee rectangle symbol $\lambda=(\alpha,\beta)_{(m+j)\times j}$ in $Q_5(m,n)$ such that $\phi_5(\lambda)=\mu$. We claim that $\gamma_{t'}=1$ and there exists an integer k' such that

$$1\le k'\le t'-1 \quad\text{and}\quad \gamma_{k'}-m+1\ge\delta_{k'+1}. \tag{4.16}$$

From the constructions (4.9) and (4.10) of ϕ_5, we see that $\gamma_{t'}=\alpha_t+1=1$, $\gamma_{k-1}=\beta_k+m-2$ and $\delta_k=\beta_{k+1}-1$. It follows that $\gamma_{k-1}-m+1\ge\delta_k$. Since $1\le k-1\le t-2=t'-1$, we reach the conclusion that $k-1$ satisfies the conditions in (4.16). This proves the claim.

By the above claim, we may choose k' to be the maximum integer such that $1\le k'\le t'-1$ and

$$\gamma_{k'}-m+1\ge\delta_{k'+1}. \tag{4.17}$$

The choice of k' yields that $\gamma_{k'+1}-m+1<\delta_{k'+2}$ when $1\le k'\le t'-2$, which implies $\gamma_{k'+1}-1<\delta_{k'}-2+m$. When $k'=t'-1$, we also have $\gamma_{k'+1}-1\le\delta_{k'}-2+m$ since $\gamma_{t'}=1$. Combining the above two cases for k', we obtain that

$$\gamma_{k'+1}-1\le\delta_{k'}-2+m. \tag{4.18}$$

In view of (4.17) and (4.18), we may define

$$\tau(\mu)=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}_{(m+j'-1)\times(j'-1)},$$

where

$$\alpha=(j'+m-1,\,\delta_1-1+m,\,\delta_2-2+m,\,\ldots,\,\delta_{k'}-2+m,\,\gamma_{k'+1}-1,\,\ldots,\,\gamma_{t'}-1) \tag{4.19}$$

and

$$\beta=(j'-1,\,\gamma_1+2-m,\,\ldots,\,\gamma_{k'}+2-m,\,\delta_{k'+1}+1,\,\ldots,\,\delta_{t'}+1). \tag{4.20}$$

It is easily checked that $\tau(\mu) \in Q_5(m, n)$.

Finally, we verify that $\tau(\phi_5(\lambda)) = \lambda$. By the constructions of $\phi_5(\lambda)$ and $\tau(\mu)$, it suffices to show that the integer k appearing in the representation of $\phi_5(\lambda)$ is equal to the integer k' appearing in the representation of $\tau(\phi_5(\lambda))$ plus 1, namely, $k' = k - 1$.

Recall that k is the maximum integer determined by λ subject to the conditions

$$1 \leq k \leq t - 1 \quad \text{and} \quad \alpha_k - m + 2 \geq \beta_{k+1} - 1. \tag{4.21}$$

On the other hand, it can be shown that $k - 1$ is the maximum integer k' determined by $\phi_5(\lambda)$ subject to the conditions

$$1 \leq k' \leq t' - 1 \quad \text{and} \quad \gamma_{k'} - m + 1 \geq \delta_{k'+1}. \tag{4.22}$$

Using (4.9) and (4.10), we find that $1 \leq k-1 \leq t-2 = t'-1$ and $\gamma_{k-1} - m + 1 = \beta_k - 1 \geq \beta_{k+1} - 1 = \delta_k$, that is, the conditions in (4.22) hold with k' replaced by $k - 1$. It remains to show that $k - 1$ is the maximum integer satisfying the conditions in (4.22). Assume to the contrary that there is an integer $p \geq k$ for which the conditions in (4.22) are satisfied, that is, $k \leq p \leq t' - 1$ and

$$\gamma_p - m + 1 \geq \delta_{p+1}. \tag{4.23}$$

Since $t' = t - 1$, we have

$$k \leq p \leq t - 2. \tag{4.24}$$

From the constructions (4.9) and (4.10) of $\phi_5(\lambda)$, we find that $\gamma_p = \alpha_{p+1} + 1$ and $\delta_{p+1} = \beta_{p+2} - 1$. By (4.23), we deduce that $\alpha_{p+1} - m + 2 \geq \beta_{p+2} - 1$. Moreover, it follows from (4.24) that $k + 1 \leq p + 1 \leq t - 1$. Thus, (4.21) is valid with k replaced by $p + 1$, which contradicts the choice of k. So we conclude that $k - 1$ is the maximum integer satisfying conditions in (4.22). This implies that $\tau(\phi_5(\lambda)) = \lambda$, and hence the proof is complete. ∎

For example, for $m = 1$ and $n = 34$, consider the following 1-Durfee rectangle symbol in $Q_5(1, 34)$:

$$\lambda = \begin{pmatrix} 4, & 4, & 2 & & \\ 3, & 3, & 2, & 2, & 2 \end{pmatrix}_{4\times 3}.$$

It can be checked that $k = 4$. Applying the injection ϕ_5 to λ, we get

$$\mu = \phi_5(\lambda) = \begin{pmatrix} 2, & 1, & 1, & 1 \\ 4, & 3, & 1, & 1 \end{pmatrix}_{5\times 4},$$

which is in $P_5(-1, 34)$. Applying τ to μ, we obtain that $k' = 3$ and $\tau(\mu) = \lambda$.

It should be remarked that the injection ϕ_5 is not valid for $m = 0$. More precisely, ϕ_5 does not apply to Durfee symbols $\lambda = (\alpha, \beta)_j$ in $Q_5(0, n)$ with $\beta_{t-1} = 2$, where $\ell(\beta) = t$ and $\ell(\alpha) = s < t$. Assume that $\beta_{t-1} = 2$. Then we have $\alpha_{t-1} + 2 \geq 2 > \beta_t - 1$, so that $k = t - 1$. Applying ϕ_5 to $(\alpha, \beta)_j$, we get

$$\gamma = (\beta_2 - 2, \, \ldots, \, \beta_{t-1} - 2, \, \alpha_t + 1),$$

which is not a partition, since $\gamma_{t-2} = \beta_{t-1} - 2 = 0$ and $\gamma_{t-1} = \alpha_t + 1 = 1$.

In the following lemma, we give an injection ϕ_6 from $Q_6(m,n)$ to $P_6(-m,n)$.

Lemma 4.7. *For $m \geq 1$, there is an injection ϕ_6 from $Q_6(m,n)$ to $P_6(-m,n)$.*

Proof. To define the map ϕ_6, let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{(m+j)\times j} = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_{(m+j)\times j}$$

be an m-Durfee rectangle symbol in $Q_6(m,n)$. By definition, we have $j = \beta_1 \geq \beta_t \geq 2$, $\alpha_1 = \alpha_2 = \alpha_3 = m+j$ and $t - s \geq 1$.

Since $\alpha_3 - m + 1 = j + 1 > \beta_3 - 1$, there exists a maximum integer k such that $k \leq s$ and $\alpha_k - m + 1 \geq \beta_k - 1$. We aim to construct two partitions γ and δ from λ. It is clear that $k \geq 3$. So we may define

$$\gamma = (\beta_1 + m - 1,\, \ldots,\, \beta_{k-1} + m - 1,\, \alpha_{k+1} + 1,\, \ldots,\, \alpha_s + 1,\, 2,\, 1^{t-s-1}) \tag{4.25}$$

and

$$\delta = (\alpha_3 + 1 - m,\, \ldots,\, \alpha_k + 1 - m,\, \beta_k - 1,\, \ldots,\, \beta_t - 1). \tag{4.26}$$

To avoid ambiguity, when $k = s$, we set

$$\gamma = (\beta_1 + m - 1,\, \ldots,\, \beta_{s-1} + m - 1,\, 2,\, 1^{t-s-1}).$$

Using the argument in the proof of Lemma 4.6, it can be shown that $(\gamma,\delta)_{(m+j+1)\times(j+1)}$ is an m-Durfee rectangle symbol. Define

$$\phi_6(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j+1)\times(j+1)}.$$

We claim that $\phi_6(\lambda)$ is an m-Durfee rectangle symbol in $P_6(-m,n)$. It is clear from (4.25) and (4.26) that $\gamma_1 = j + m - 1$, $\delta_1 = j + 1$ and $\ell(\gamma) = \ell(\delta) = t - 1$. It remains to check that $|\phi_6(\lambda)| = |\lambda|$. Observe that

$$\begin{aligned} |\gamma| + |\delta| &= |\alpha| - \alpha_1 - \alpha_2 + (1-m)(k-2) + (s-k) + 2 + (t-s-1) \\ &\quad + |\beta| + (m-1)(k-1) - (t-k+1) \\ &= |\alpha| + |\beta| - \alpha_1 - \alpha_2 + m - 1. \end{aligned} \tag{4.27}$$

By the definition of $Q_6(m,n)$, we have $\alpha_1 = \alpha_2 = j + m$. Thus, it follows from (4.27) that

$$|\gamma| + |\delta| = |\alpha| + |\beta| - (2j + m + 1).$$

Hence,

$$\begin{aligned}|\phi_6(\lambda)| &= |\gamma| + |\delta| + (j+1)(j+m+1)\\ &= |\alpha| + |\beta| + j(j+m),\end{aligned}$$

which equals to $|\lambda|$. This proves that $\phi_6(\lambda) \in P_6(-m,n)$.

Next we show that ϕ_6 is an injection. Let

$$K(m,n) = \{\phi_6(\lambda)\colon \lambda \in Q_6(m,n)\}$$

be the set of images of ϕ_6, which has been shown to be a subset of $P_6(-m,n)$. It suffices to construct a map χ from $K(m,n)$ to $Q_6(m,n)$ such that for any λ in $Q_6(m,n)$,

$$\chi(\phi_6(\lambda)) = \lambda.$$

To describe the map χ, let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{t'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{(m+j')\times j'} \tag{4.28}$$

be an m-Durfee rectangle symbol in $K(m,n)$, that is, there is an m-Durfee rectangle symbol $\lambda = (\alpha,\beta)_{(m+j)\times j}$ in $Q_6(m,n)$ such that $\phi_6(\lambda) = \mu$. From the defining relation (4.25) of ϕ_6, we see that γ has a part equal to 2. Moreover, s is the maximum number such that $\gamma_s = 2$. This property enables us to determine s from γ. We claim that there exists an integer k' such that

$$1 \le k' \le s-1 \quad \text{and} \quad \gamma_{k'} - m \ge \delta_{k'}. \tag{4.29}$$

By (4.25) and (4.26), we have

$$2 \le k-1 \le s-1, \quad \gamma_{k-1} = \beta_{k-1} + m - 1, \quad \delta_{k-1} = \beta_k - 1$$

which implies that

$$1 \le k-1 \le s-1 \quad \text{and} \quad \gamma_{k-1} - m \ge \delta_{k-1}.$$

Hence the conditions in (4.29) are satisfied with k' replaced by $k-1$. So the claim is proved.

Now we may choose k' to be the maximum integer for which (4.29) holds. This choice of k' implies that $\gamma_{k'+1} - m < \delta_{k'+1}$ when $1 \le k' \le s-2$. It follows that $\delta_{k'-1} + m > \gamma_{k'+1}$ when $1 \le k' \le s-2$. When $k' = s-1$, since $\gamma_s = 2$, we have $\delta_{s-2} + m \ge \gamma_s$. Combining the above two cases for k', we deduce that

$$\delta_{k'-1} + m \ge \gamma_{k'+1}. \tag{4.30}$$

By (4.29) and (4.30), we may define

$$\chi(\mu)=\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)_{(m+j'-1)\times(j'-1)}.$$

where

$$\alpha=(j'+m-1,\ j'+m-1,\ \delta_1-1+m,\ \ldots,\ \delta_{k'-1}-1+m,\ \gamma_{k'+1}-1,\ \ldots,\ \gamma_{s-1}-1) \tag{4.31}$$

and

$$\beta=(\gamma_1+1-m,\ \ldots,\ \gamma_{k'}+1-m,\ \delta_{k'}+1,\ \ldots,\ \delta_{t'}+1). \tag{4.32}$$

It can be easily checked that $\chi(\mu)\in Q_6(m,n)$.

Finally, we verify that $\chi(\phi_6(\lambda))=\lambda$. By the constructions of $\phi_6(\lambda)$ and $\chi(\mu)$, it suffices to show that the integer k appearing in the representation of $\phi_6(\lambda)$ is equal to the integer k' appearing in the representation of $\chi(\phi_6(\lambda))$ plus 1, that is, $k'=k-1$. This assertion can be justified by using the same argument as in the proof of Lemma 4.6. For completeness, we include a proof.

Recall that k is the maximum integer determined by λ subject to the conditions

$$3\le k\le s,\quad \alpha_k-m+1\ge\beta_k-1. \tag{4.33}$$

We proceed to show that $k-1$ is the maximum integer k' determined by $\phi_6(\lambda)$ such that

$$1\le k'\le s-1,\quad \gamma_{k'}-m\ge\delta_{k'}. \tag{4.34}$$

From the constructions (4.25) and (4.26) of ϕ_6, it can be checked that (4.34) is valid with k' replaced by $k-1$. So it suffices to show that $k-1$ is the maximum integer satisfying conditions in (4.34). Assume to the contrary that there is an integer $k\le p\le s-1$ for which the conditions in (4.34) are satisfied, that is,

$$\gamma_p-m\ge\delta_p. \tag{4.35}$$

In view of the constructions (4.25) and (4.26) of ϕ_6, and noting that $k\le p\le s-1$, we find that

$$\gamma_p=\alpha_{p+1}+1\quad\text{and}\quad\delta_p=\beta_{p+1}-1. \tag{4.36}$$

Substituting (4.36) into (4.35), we arrive at

$$\alpha_{p+1}-m+1\ge\beta_{p+1}-1.$$

This means that (4.33) holds with k being replaced by $p+1$. But this contradicts the maximality of k. So we conclude that $k-1$ is the maximum integer satisfying conditions in (4.34), which implies that $\chi(\phi_6(\lambda))=\lambda$. This completes the proof. ∎

For example, for $m = 2$ and $n = 60$, let

$$\lambda = \begin{pmatrix} 5, & 5, & 5, & 5, & 3, & 2 & & \\ 3, & 3, & 3, & 3, & 2, & 2, & 2, & 2 \end{pmatrix}_{5\times 3}$$

be a 2-Durfee rectangle symbol in $Q_6(2, 60)$. It can be checked that $k = 6$. Applying ϕ_6 to λ, we get

$$\mu = \phi_6(\lambda) = \begin{pmatrix} 4, & 4, & 4, & 4, & 3, & 2, & 1 \\ 4, & 4, & 2, & 1, & 1, & 1, & 1 \end{pmatrix}_{6\times 4},$$

which in $P_6(-2, 60)$. Applying χ to μ, we obtain that $s = 6$, $k' = 5$ and $\chi(\mu) = \lambda$.

It should be noted that the injection ϕ_6 is not valid for $m = 0$. To be more specific, ϕ_6 does not apply to Durfee symbols $\lambda = (\alpha, \beta)_j$ in $Q_6(0, n)$ with $\beta_{s-1} = 2$, where $\ell(\alpha) = s$ and $\ell(\beta) = t$. Assume that $(\gamma, \delta)_{j'} = \phi_6(\lambda)$. Since $\beta_{s-1} = 2$, we have $\alpha_s + 1 \geq 2 > \beta_s - 1$, which implies that $k = s$. Thus

$$\gamma = (\beta_1 - 1, \ldots, \beta_{s-1} - 1, 2, 1^{t-s-1}),$$

which is not a partition, since $\gamma_{s-1} = \beta_{s-1} - 1 = 1$ and $\gamma_s = 2$.

Combining the above injections ϕ_i $(1 \leq i \leq 6)$, we are led to an injection from $Q(m, n)$ to $P(-m, n)$ for the case $m \geq 1$.

Proof of Theorem 4.1 for $m \geq 1$. Assume that $m \geq 1$. Let λ be a partition in $Q(m, n)$, define

$$\Phi(\lambda) = \begin{cases} \phi_1(\lambda), & \text{if} \quad \lambda \in Q_1(m, n); \\ \phi_2(\lambda), & \text{if} \quad \lambda \in Q_2(m, n); \\ \phi_3(\lambda), & \text{if} \quad \lambda \in Q_3(m, n); \\ \phi_4(\lambda), & \text{if} \quad \lambda \in Q_4(m, n); \\ \phi_5(\lambda), & \text{if} \quad \lambda \in Q_5(m, n); \\ \phi_6(\lambda), & \text{if} \quad \lambda \in Q_6(m, n). \end{cases}$$

Using the divisions of $Q(m, n)$ and $P(-m, n)$ and combining Lemmas 4.2–4.7, we conclude that Φ is an injection from $Q(m, n)$ to $P(-m, n)$. ∎

5 Proof of Theorem 1.6 for $m = 0$

In this section, we give a proof of Theorem 4.1 for $m = 0$, and so Theorem 1.6 holds for $m = 0$. In addition to the injections ϕ_1, ϕ_2, ϕ_3 and ϕ_4 for $m \geq 0$ and restrictions of ϕ_5 and ϕ_6, this seemingly special case requires three more injections.

Recall that $Q(0,n)$ denotes the set of Durfee symbols $(\alpha,\beta)_j$ of n such that $\beta_1 = j$ and $P(0,n)$ denotes the set of Durfee symbols $(\gamma,\delta)_j$ of n such that $\ell(\delta) - \ell(\gamma) \leq 0$. From the definitions of $Q_i(m,n)$ and $P_i(-m,n)$ given in Section 4, it can be seen that

$$Q(0,n) = \bigcup_{i=1}^{6} Q_i(0,n)$$

and

$$P(0,n) = \bigcup_{i=1}^{8} P_i(0,n).$$

It is known that $Q_1(0,n) = P_1(0,n)$. By Lemmas 4.2, 4.3 and 4.5, we see that the injections ϕ_2, ϕ_3, ϕ_4 can be applied to $Q_2(0,n)$, $Q_3(0,n)$ and $Q_4(0,n)$, so that we get three injections from $Q_i(0,n)$ to $P_i(0,n)$, where $2 \leq i \leq 4$.

As mentioned in the previous section, the injections ϕ_5 and ϕ_6 do not apply to $Q_5(0,n)$ and $Q_6(0,n)$. We need to construct an injection from $Q_5(0,n) \cup Q_6(0,n)$ to $P_5(0,n) \cup P_6(0,n) \cup P_7(0,n) \cup P_8(0,n)$. To this end, we shall divide the set $Q_5(0,n) \cup Q_6(0,n)$ into five disjoint subsets $\bar{Q}_1(0,n)$, $\bar{Q}_2(0,n)$, $\bar{Q}_3(0,n)$, $\bar{Q}_4(0,n)$ and $\bar{Q}_5(0,n)$:

(1) $\bar{Q}_1(0,n)$ is the set of Durfee symbols $(\alpha,\beta)_j \in Q_5(0,n)$ with $s(\beta) \geq 3$;

(2) $\bar{Q}_2(0,n)$ is the set of Durfee symbols $(\alpha,\beta)_j \in Q_6(0,n)$ with $s(\beta) \geq 3$;

(3) $\bar{Q}_3(0,n)$ is the set of Durfee symbols $(\alpha,\beta)_j \in Q_5(0,n) \cup Q_6(0,n)$ with $s(\alpha) = 1$ and $s(\beta) = 2$;

(4) $\bar{Q}_4(0,n)$ is the set of Durfee symbols $(\alpha,\beta)_j \in Q_5(0,n) \cup Q_6(0,n)$ with $s(\alpha) \geq 2$, $\beta_1 = \beta_2$ and $s(\beta) = 2$;

(5) $\bar{Q}_5(0,n)$ is the set of Durfee symbols $(\alpha,\beta)_j \in Q_5(0,n) \cup Q_6(0,n)$ with $s(\alpha) \geq 2$, $\beta_1 > \beta_2$ and $s(\beta) = 2$.

On the other hand, we divide the set $P_5(0,n) \cup P_6(0,n)$ into three disjoint subsets $\bar{P}_1(0,n)$, $\bar{P}_2(0,n)$ and $\bar{P}_3(0,n)$:

(1) $\bar{P}_1(0,n)$ is the set of Durfee symbols $(\gamma,\delta)_{j'} \in P_5(0,n)$ with $s(\delta) \geq 2$;

(2) $\bar{P}_2(0,n)$ is the set of Durfee symbols $(\gamma,\delta)_{j'} \in P_6(0,n)$ with $s(\delta) \geq 2$;

(3) $\bar{P}_3(0,n)$ is the set of Durfee symbols $(\gamma,\delta)_{j'} \in P_5(0,n) \cup P_6(0,n)$ with $s(\delta) = 1$.

In the following lemmas, we shall show that there exist an injection ψ_1 from $\bar{Q}_1(0,n)$ to $\bar{P}_1(0,n)$, an injection ψ_2 from $\bar{Q}_2(0,n)$ to $\bar{P}_2(0,n)$, an injection ψ_3 from $\bar{Q}_3(0,n)$ to $\bar{P}_3(0,n)$, an injection ψ_4 from $\bar{Q}_4(0,n)$ to $P_7(0,n)$ and an injection ψ_5 from $\bar{Q}_5(0,n)$ to $P_8(0,n)$. It should be noted that ψ_1 is a restriction of ϕ_5 to $\bar{Q}_1(0,n)$ and ψ_2 is a restriction of ϕ_6 to $\bar{Q}_2(0,n)$. Then the injection Φ for $m = 0$ consists of injections ϕ_i $(1 \leq i \leq 4)$ and injections ψ_i $(1 \leq i \leq 5)$.

Lemma 5.1. *There exists an injection ψ_1 from $\bar{Q}_1(0,n)$ to $\bar{P}_1(0,n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_j = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_j$$

be a Durfee symbol in $\bar{Q}_1(0,n)$. By definition, we have $j = \beta_1 \geq \beta_t \geq 3$, $j = \alpha_1 = \alpha_2 > \alpha_3$ and $t - s \geq 1$. Consequently, we have $\alpha_2 + 2 = j + 2 > \beta_3 - 1$. Hence there exists a maximum number k such that $1 \leq k \leq t-1$ and $\alpha_k + 2 \geq \beta_{k+1} - 1$. So we may define

$$\psi_1(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j+1}, \tag{5.1}$$

where

$$\gamma = (\beta_2 - 2,\, \ldots,\, \beta_k - 2,\, \alpha_{k+1} + 1,\, \ldots,\, \alpha_t + 1)$$

and

$$\delta = (\alpha_2 + 1,\, \alpha_3 + 2,\, \ldots,\, \alpha_k + 2,\, \beta_{k+1} - 1,\, \ldots,\, \beta_t - 1).$$

Using the same argument as in the proof of Lemma 4.6, we deduce that $\psi_1(\lambda)$ is a Durfee symbol in $\bar{P}_1(0,n)$ and the construction of ψ_1 is reversible. Thus ψ_1 is an injection from $\bar{Q}_1(0,n)$ to $\bar{P}_1(0,n)$. This completes the proof. ∎

Lemma 5.2. *There exists an injection ψ_2 from $\bar{Q}_2(0,n)$ to $\bar{P}_2(0,n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_j = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_j$$

be a Durfee symbol in $\bar{Q}_2(0,n)$. In this case, we have $j = \beta_1 \geq \beta_t \geq 3$, $j = \alpha_1 = \alpha_2 = \alpha_3$ and $t - s \geq 1$. Thus, $\alpha_3 + 1 = j + 1 > \beta_3 - 1$. So we may assume that k is the maximum integer such that $k \leq s$ and $\alpha_k + 1 \geq \beta_k - 1$. Define

$$\psi_2(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j+1}, \tag{5.2}$$

where

$$\gamma = (\beta_1 - 1,\, \ldots,\, \beta_{k-1} - 1,\, \alpha_{k+1} + 1,\, \ldots,\, \alpha_s + 1,\, 2,\, 1^{t-s-1})$$

and

$$\delta = (\alpha_3 + 1,\, \ldots,\, \alpha_k + 1,\, \beta_k - 1,\, \ldots,\, \beta_t - 1).$$

It can be checked that $\psi_2(\lambda)$ is a Durfee symbol in $\bar{P}_2(0,n)$. Moreover, it can be shown that ψ_2 is reversible by using the same reasoning as in the proof of Lemma 4.7. Hence ψ_2 is an injection, and the proof is complete. ∎

Lemma 5.3. *There is an injection ψ_3 from $\bar{Q}_3(0,n)$ to $\bar{P}_3(0,n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_j = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_j$$

be a Durfee symbol in $\bar{Q}_3(0,n)$. So we have $\alpha_1 = \alpha_2 = j$, $\alpha_s = 1$, $\beta_1 = j$, $\beta_t = 2$ and $t - s \geq 1$. It follows that $\beta_2 \leq j$ and $\alpha_2 = j$. This enables us to define

$$\psi_3(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j'} = \begin{pmatrix} \beta_2 - 1, & \ldots, & \beta_t - 1 & \\ \alpha_2 + 1, & \ldots, & \alpha_{s-1} + 1, & 1^{t-s+1} \end{pmatrix}_{j+1}. \tag{5.3}$$

Note that $\gamma_1 = \beta_2 - 1 \leq j - 1 = j' - 2$, $\delta_1 = \alpha_2 + 1 = j + 1 = j'$ and $\ell(\gamma) = \ell(\delta)$. Since $t - s \geq 1$, we see that $s(\delta) = 1$. Since $\alpha_s = 1$, it is easily checked that $|\psi_3(\lambda)| = |\lambda|$. This proves that $\psi_3(\lambda)$ is in $\bar{P}_3(0,n)$.

To show that ψ_3 is an injection, let

$$L(m,n) = \{\psi_3(\lambda) \colon \lambda \in \bar{Q}_3(0,n)\}$$

be the set of images of ψ_3, which has been shown to be a subset of $\bar{P}_3(0,n)$. It suffices to construct a map ϑ from $L(m,n)$ to $\bar{Q}_3(0,n)$ such that for any λ in $\bar{Q}_3(0,n)$,

$$\vartheta(\psi_3(\lambda)) = \lambda. \tag{5.4}$$

Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{t'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{j'}$$

be a Durfee symbol in $L(m,n)$. We claim that $\gamma_{t'} = 1$ and $\delta_{t'-1} = 1$. By the definition of $L(m,n)$, there exists a Durfee symbol $\lambda = (\alpha, \beta)_j$ in $\bar{Q}_3(0,n)$ such that $\psi_3(\lambda) = \mu$. Since $t - s + 1 \geq 2$ and $\beta_t = 2$, from the definition (5.3) of $\psi_3(\lambda)$, we get

$$\gamma_{t'} = \beta_t - 1 = 1 \quad \text{and} \quad \delta_{t'-1} = 1. \tag{5.5}$$

So the claim holds.

We next define the map ϑ. Let h' be the largest index such that $\delta_{h'} > 1$. By the above claim, we have $\delta_{t'-1} = 1$, and so $h' \leq t' - 2$. Define

$$\vartheta(\mu) = \begin{pmatrix} j' - 1, & \delta_1 - 1, & \ldots, & \delta_{h'} - 1, & 1 \\ j' - 1, & \gamma_1 + 1, & \ldots, & \gamma_{t'} + 1 & \end{pmatrix}_{j'-1}.$$

It is not difficult to check that $\vartheta(\mu) \in \bar{Q}_3(0,n)$ and $\vartheta(\psi_3(\lambda)) = \lambda$ for $\lambda \in \bar{Q}_3(0,n)$. Therefore, ψ_3 is an injection from $\bar{Q}_3(0,n)$ to $\bar{P}_3(0,n)$. This completes the proof. ∎

For example, for $n = 35$, let

$$\lambda = \begin{pmatrix} 3, & 3, & 2, & 2, & 1 & \\ 3, & 3, & 3, & 2, & 2, & 2 \end{pmatrix}_3$$

be a Durfee symbol in $\bar{Q}_3(0, 35)$. Applying the injection ψ_3 to λ, we get

$$\psi_3(\lambda) = \begin{pmatrix} 2, & 2, & 1, & 1, & 1 \\ 4, & 3, & 3, & 1, & 1 \end{pmatrix}_4,$$

which is in $\bar{P}_3(0, 35)$. Applying ϑ to μ, we find that $h' = 3$ and $\vartheta(\mu) = \lambda$.

Lemma 5.4. *There is a bijection ψ_4 between $\bar{Q}_4(0, n)$ and $P_7(0, n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_j = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_j$$

be a Durfee symbol in $\bar{Q}_4(0, n)$. By definition, $\alpha_1 = \alpha_2 = j$, $\alpha_s \geq 2$, $\beta_1 = \beta_2 = j$, $\beta_t = 2$ and $t - s \geq 1$. Thus, $\alpha_2 = j > \beta_3 - 1$ and $\beta_2 = j \geq \alpha_3$. So we may define

$$\psi_4(\lambda) = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j'} = \begin{pmatrix} \alpha_2, & \beta_3 - 1, & \ldots, & \beta_{t-1} - 1 & \\ \beta_2 + 1, & \alpha_3 + 1, & \ldots, & \alpha_s + 1, & 1^{t-s-1} \end{pmatrix}_{j+1}.$$

Note that $\delta_{s-1} = \alpha_s + 1 \geq 3$ and $\delta_i = 1$ for $s \leq i \leq t - 2$. It is clear that δ has no parts equal to 2. Since $\beta_t = 2$, we find that $|\psi_4(\lambda)| = |\lambda|$. Moreover, we have $\ell(\gamma) = \ell(\delta) = t - 2$, $\delta_1 = \beta_2 + 1 = j + 1 = j'$ and $\gamma_1 = \alpha_2 = j = j' - 1 > \beta_3 - 1 = \gamma_2$. So $\psi_4(\lambda)$ is in $P_7(0, n)$.

To show that ψ_4 is a bijection, we construct the inverse map ξ of ψ_4. Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{(m+j')\times j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{t'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{j'}$$

be a Durfee symbol in $P_7(0, n)$. By the definition of $P_7(0, n)$, we have $\delta_1 = j'$, $\gamma_1 = j' - 1 > \gamma_2$ and δ has no part equal to 2. We define $\xi(\mu)$ as follows:

$$\xi(\mu) = \begin{pmatrix} j' - 1, & \gamma_1, & \delta_2 - 1, & \ldots, & \delta_{t'} - 1 & \\ j' - 1, & \delta_1 - 1, & \gamma_2 + 1, & \ldots, & \gamma_{t'} + 1, & 2 \end{pmatrix}_{j'-1}.$$

It can be checked that $\xi(\mu) \in \bar{Q}_4(0, n)$ and ξ is the inverse map of ψ_4. Thus ψ_4 is a bijection. ∎

For example, for $n = 40$, consider the following Durfee symbol in $\bar{Q}_4(0, 40)$:

$$\lambda = \begin{pmatrix} 3, & 3, & 3, & 2, & 2 & & \\ 3, & 3, & 3, & 3, & 2, & 2, & 2 \end{pmatrix}_3 .$$

Applying the bijection ψ_4, we get

$$\psi_4(\lambda) = \begin{pmatrix} 3, & 2, & 2, & 1, & 1 \\ 4, & 4, & 3, & 3, & 1 \end{pmatrix}_4 ,$$

which is in $P_7(0, 40)$. Applying ξ to $\psi_4(\lambda)$, we recover λ.

Lemma 5.5. *There is an injection ψ_5 from $\bar{Q}_5(0, n)$ to $P_8(0, n)$.*

Proof. Let

$$\lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_j = \begin{pmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_s \\ \beta_1, & \beta_2, & \ldots, & \beta_t \end{pmatrix}_j$$

be a Durfee symbol in $\bar{Q}_5(0, n)$. In this case, we have $\alpha_1 = \alpha_2 = j$, $\alpha_s \geq 2$, $j = \beta_1 > \beta_2$, $\beta_t = 2$ and $t - s \geq 1$. Observe that $j \geq 3$, since $j = \beta_1 > \beta_2 \geq \beta_t = 2$.

We next define the map ψ_5. Given $\alpha_1 = \alpha_2 = j$, we may choose k to be the maximum integer such that $\alpha_k = j$. Clearly, $k \geq 2$. Using $\beta_t = 2$, we may choose h to be the minimum integer such that $\beta_h = 2$. Since $\beta_1 = j > 2$, we get $2 \leq h \leq t$.

By the choice of k, we see that $\alpha_k = j$ and $\alpha_{k+1} < j$. Combining $\beta_1 = j$ and $\beta_2 < j$, we see that $\alpha_k > \beta_2$ and $\beta_1 > \alpha_{k+1}$. On the other hand, by the choice of h, we see that $\beta_{h-1} > \beta_h$. So we may define

$$\begin{aligned} \psi_5(\lambda) &= \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j'} \\ &= \begin{pmatrix} \alpha_1 - 1, & \ldots, & \alpha_k - 1, & \beta_2 - 1, & \ldots, & \beta_{h-1} - 1, & \beta_h, & 1^{t-h} \\ \beta_1, & \alpha_{k+1} + 1, & \ldots, & \alpha_s + 1, & 1^{2k-2+t-s} & & & \end{pmatrix}_j . \end{aligned} \tag{5.6}$$

Since $\delta_{s-k+1} = \alpha_s + 1 \geq 3$ and $\delta_i = 1$ for $s - k + 2 \leq i \leq t - 1 + k$, we deduce that δ has no parts equal to 2. Furthermore, it is easily checked that $\ell(\gamma) = \ell(\delta) = t + k - 1$, $\delta_1 = j'$, $\gamma_1 = \gamma_2 = j' - 1$ and $|\psi_5(\lambda)| = |\lambda|$. So $\psi_5(\lambda)$ is in $P_8(0, n)$.

To prove that ψ_5 is an injection, let

$$R(0, n) = \{\psi_5(\lambda) \colon \lambda \in \bar{Q}_5(0, n)\}$$

be the set of images of ψ_5, which has been shown to be a subset of $P_8(0, n)$. We shall construct a map θ from $R(0, n)$ to $\bar{Q}_5(0, n)$ such that for any λ in $\bar{Q}_5(0, n)$,

$$\theta(\psi_5(\lambda)) = \lambda.$$

Let

$$\mu = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}_{j'} = \begin{pmatrix} \gamma_1, & \gamma_2, & \ldots, & \gamma_{t'} \\ \delta_1, & \delta_2, & \ldots, & \delta_{t'} \end{pmatrix}_{j'}$$

be a Durfee symbol in $R(0,n)$. Let k' denote the number of occurrences of $j'-1$ in γ and let $n_1(\delta)$ denote the number of occurrences of 1 in δ. We claim that for $j' \geq 4$, we have $k' \geq 2$ and $n_1(\delta) \geq 2k'-1$, and for $j'=3$, we have $k' \geq 3$ and $n_1(\delta) \geq 2k'-3$.

By the definition of $R(0,n)$, there exists a Durfee symbol $\lambda = (\alpha,\beta)_j$ in $\bar{Q}_5(0,n)$ such that $\psi_5(\lambda) = \mu$. From the construction (5.6) of ψ_5, we find that $j' = j$ and $n_1(\delta) = 2k-2+t-s$. Since $t-s \geq 1$, we get $n_1(\delta) \geq 2k-1$. Moreover, since $k \geq 2$, it suffices to show that $k' = k$ if $j \geq 4$ and $k' = k+1$ if $j = 3$. From the construction (5.6) of ψ_5, we get $\gamma_i = \alpha_i - 1$ for $1 \leq i \leq k$. Since $\alpha_i = j$ for $1 \leq i \leq k$, we deduce that $\gamma_i = j-1$ for $1 \leq i \leq k$, which implies $k' \geq k$.

It remains to show that $\gamma_{k+1} < j-1$ for $j \geq 4$ and $\gamma_{k+1} = j-1 > \gamma_{k+2}$ for $j = 3$. By (5.6), we have either $\gamma_{k+1} = \beta_2 - 1$ or $\gamma_{k+1} = \beta_h$. For $j \geq 4$, in either case, we have $\gamma_{k+1} < j-1$ since $\beta_2 < j$ and $\beta_h = 2$. For $j = 3$, we have $\beta_1 = 3$ and $\beta_2 = 2$, so that $h = 2$, where h is the minimum integer such that $\beta_h = 2$. This implies that $\gamma_{k+1} = \beta_2 = 2 = j-1$. Since $\gamma_{k+2} \leq 1$, we find that $\gamma_{k+2} < j-1$. Thus, we arrive at the conclusion that $k' = k+1$ for $j = 3$. This proves the claim.

From the construction (5.6) of ψ_5, it can be seen that $\gamma_{k+h-1} = \beta_h = 2$. So we may choose h' to be the maximum integer such that $\gamma_{h'} = 2$. Recall that k' denotes the number of occurrences of $j'-1$ in γ. We consider the following two cases:

Case 1: $j' \geq 4$. Define

$$\theta(\mu) = \begin{pmatrix} \gamma_1+1, & \ldots, & \gamma_{k'}+1, & \delta_2-1, & \ldots, & \delta_{t'}-1 & & & & \\ \delta_1, & \gamma_{k'+1}+1, & \ldots, & \gamma_{h'-1}+1, & \gamma_{h'}, & \gamma_{h'+1}+1, & \ldots, & \gamma_{t'}+1 \end{pmatrix}_{j'}.$$

By the above claim, we have $k' \geq 2$ and $n_1(\delta) \geq 2k'-1$. Now it is easy to check that $\theta(\mu) \in \bar{Q}_5(0,n)$.

Case 2: $j' = 3$. By the definitions of k' and h', we have $k' = h'$. Let $r' = n_1(\delta)$ and define

$$\theta(\mu) = \begin{pmatrix} 3^{k'-1}, & 2^{t'-r'-1} \\ 3, & 2^{t'-k'+1} \end{pmatrix}_3.$$

From the above claim, we deduce that $r' \geq 2k'-3$ and $k' \geq 3$. Then it is easily checked that $\theta(\mu) \in \bar{Q}_5(0,n)$.

Finally, from the constructions of ψ_5 and θ together with the above claim, it is straightforward to verify that $\theta(\psi_5(\lambda)) = \lambda$ for any $\lambda \in \bar{Q}_5(0,n)$. This completes the proof. ∎

For example, for $n = 51$, consider the following Durfee symbol in $\bar{Q}_5(0, 51)$:

$$\lambda = \begin{pmatrix} 4, & 4, & 4, & 2, & 2 & & \\ 4, & 3, & 3, & 3, & 2, & 2, & 2 \end{pmatrix}_4 .$$

Applying the injection ψ_5, we obtain that $k = 3$, $h = 5$, and

$$\mu = \psi_5(\lambda) = \begin{pmatrix} 3, & 3, & 3, & 2, & 2, & 2, & 2, & 1, & 1 \\ 4, & 3, & 3, & 1, & 1, & 1, & 1, & 1, & 1 \end{pmatrix}_4 ,$$

which is in $P_8(0, 51)$. Applying θ to μ, we find that $k' = 3$, $h' = 7$ and $\theta(\mu) = \lambda$.

We are now ready to complete the proof of Theorem 4.1 for the case $m = 0$.

Proof of Theorem 4.1 for $m = 0$. From the definitions of $Q_i(0, n)$ $(1 \le i \le 6)$ and $\bar{Q}_i(0, n)$ $(1 \le i \le 5)$, we have

$$\begin{aligned} Q(0, n) \;=\;& Q_1(0, n) \cup Q_2(0, n) \cup Q_3(0, n) \cup Q_4(0, n) \cup \bar{Q}_1(0, n) \cup \bar{Q}_2(0, n) \\ & \cup \bar{Q}_3(0, n) \cup \bar{Q}_4(0, n) \cup \bar{Q}_5(0, n). \end{aligned}$$

By the definitions of $P_i(0, n)$ $(1 \le i \le 8)$ and $\bar{P}_i(0, n)$ $(1 \le i \le 3)$, we have

$$\begin{aligned} P(0, n) \;=\;& P_1(0, n) \cup P_2(0, n) \cup P_3(0, n) \cup P_4(0, n) \cup \bar{P}_1(0, n) \cup \bar{P}_2(0, n) \\ & \cup \bar{P}_3(0, n) \cup P_7(0, n) \cup P_8(0, n). \end{aligned}$$

Let $\lambda \in Q(0, n)$, define

$$\Phi(\lambda) = \begin{cases} \phi_1(\lambda), & \text{if} \quad \lambda \in Q_1(0, n); \\ \phi_2(\lambda), & \text{if} \quad \lambda \in Q_2(0, n); \\ \phi_3(\lambda), & \text{if} \quad \lambda \in Q_3(0, n); \\ \phi_4(\lambda), & \text{if} \quad \lambda \in Q_4(0, n); \\ \psi_1(\lambda), & \text{if} \quad \lambda \in \bar{Q}_1(0, n); \\ \psi_2(\lambda), & \text{if} \quad \lambda \in \bar{Q}_2(0, n); \\ \psi_3(\lambda), & \text{if} \quad \lambda \in \bar{Q}_3(0, n); \\ \psi_4(\lambda), & \text{if} \quad \lambda \in \bar{Q}_4(0, n); \\ \psi_5(\lambda), & \text{if} \quad \lambda \in \bar{Q}_5(0, n). \end{cases}$$

From Lemmas 4.2 to 4.5 and Lemmas 5.1 to 5.5, it immediately follows that Φ is an injection from $Q(0, n)$ to $P(0, n)$. This completes the proof. ∎

By a closer examination of the injections in the proof of Theorem 4.1, we can characterize the numbers n and m for which $N_{\le m}(n) = M_{\le m}(n)$. The details are omitted.

6 Connection to Theorem 1.7

In this section, we establish a connection between Conjecture 1.3 and Theorem 1.7 of Andrews, Chan and Kim. More precisely, we relate the positive rank (crank) moments $\overline{N}_k(n)$ ($\overline{M}_k(n)$) to the functions $N_{\leq m}(n)$ ($M_{\leq m}(n)$) defined by Andrews, Dyson and Rhoades. Based on this connection, it can be seen that Theorem 1.7 of Andrews, Chan and Kim on the positive rank and crank moments can be deduced from Conjecture 1.3. This leads to an alternative proof of the theorem of Andrews, Chan and Kim.

Theorem 6.1. *For $k \geq 1$ and $n \geq 1$, we have*

$$\overline{N}_k(n) = \frac{1}{2}\sum_{m=1}^{+\infty}(m^k-(m-1)^k)\left(p(n)-N_{\leq m-1}(n)\right), \tag{6.1}$$

$$\overline{M}_k(n) = \frac{1}{2}\sum_{m=1}^{+\infty}(m^k-(m-1)^k)\left(p(n)-M_{\leq m-1}(n)\right). \tag{6.2}$$

Proof. We only give a proof of (6.1) since (6.2) can be justified in the same vain. Recall that

$$\overline{N}_k(n) = \sum_{j=1}^{+\infty} j^k N(j,n). \tag{6.3}$$

Express (6.3) in the following form:

$$\begin{aligned}\overline{N}_k(n) &= \sum_{j=1}^{+\infty} N(j,n)\left(\sum_{m=1}^{j} m^k - \sum_{m=1}^{j}(m-1)^k\right)\\ &= \sum_{j=1}^{+\infty}\sum_{m=1}^{j}(m^k-(m-1)^k)N(j,n).\end{aligned}$$

Changing the order of summations, we find that

$$\overline{N}_k(n) = \sum_{m=1}^{+\infty}(m^k-(m-1)^k)\sum_{j=m}^{+\infty}N(j,n). \tag{6.4}$$

Writing the second sum in (6.4) as

$$\sum_{j=m}^{+\infty}N(j,n) = \sum_{j=-\infty}^{+\infty}N(j,n) - \sum_{j=-\infty}^{m-1}N(j,n), \tag{6.5}$$

and substituting the relations

$$\sum_{r=-\infty}^{\infty}N(r,n) = p(n)$$

and

$$\sum_{j=-\infty}^{m-1} N(j,n) = p(-m+1,n)$$

as given by (2.1) and (2.3) into (6.5), we deduce that

$$\sum_{j=m}^{+\infty} N(j,n) = p(n) - p(-m+1,n). \tag{6.6}$$

Replacing m by $m-1$ in (2.4) yields

$$p(-m+1,n) = \frac{p(n) + N_{\leq m-1}(n)}{2}. \tag{6.7}$$

Substituting (6.7) into (6.6), we obtain

$$\sum_{j=m}^{+\infty} N(j,n) = \frac{p(n) - N_{\leq m-1}(n)}{2}. \tag{6.8}$$

Combining (6.4) and (6.8), we arrive at relation (6.1). This completes the proof. ∎

In view of Theorem 6.1, it can be seen that Theorem 1.7 follows from Conjecture 1.3.

Proof of Theorem 1.7. Subtracting (6.1) from (6.2) in Theorem 6.1, we obtain

$$\overline{M}_k(n) - \overline{N}_k(n) = \frac{1}{2}\sum_{m=1}^{+\infty}(m^k - (m-1)^k)(N_{\leq m-1}(n) - M_{\leq m-1}(n)). \tag{6.9}$$

From the definitions of the rank and crank, we have for $m \geq n+1$,

$$\begin{aligned} N_{\leq m-1}(n) &= p(n), \\ M_{\leq m-1}(n) &= p(n). \end{aligned}$$

It follows that for $m \geq n+1$

$$N_{\leq m-1}(n) - M_{\leq m-1}(n) = 0. \tag{6.10}$$

For $m = n$, from the definitions of the rank and crank, we find that

$$\begin{aligned} N_{\leq n-1}(n) &= p(n), \\ M_{\leq n-1}(n) &= p(n) - 2. \end{aligned}$$

Consequently,

$$N_{\leq n-1}(n) - M_{\leq n-1}(n) = 2. \tag{6.11}$$

Substituting (6.10) and (6.11) into (6.9), we obtain

$$\overline{M}_k(n) - \overline{N}_k(n) = \frac{1}{2}\sum_{m=1}^{n-1}(m^k - (m-1)^k)(N_{\leq m-1}(n) - M_{\leq m-1}(n)) \\ + n^k - (n-1)^k. \tag{6.12}$$

Since $m^k - (m-1)^k > 0$ for $m \geq 1$ and $k \geq 1$, by Conjecture 1.3, that is, $N_{\leq m-1}(n) - M_{\leq m-1}(n) \geq 0$, we reach the assertion that $\overline{M}_k(n) - \overline{N}_k(n) > 0$ for $n \geq 1$ and $k \geq 1$. This completes the proof. ∎

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