The Corners of Core Partitions

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Abstract

In this paper, we are concerned with counting corners of core partitions. We introduce the concepts of stitches and anti-stitches, which are pairs of cells in a quotient space which we call wrap-up space. We prove that the anti-stitches of a rational Dyck path are in bijection with the segments of structure sets of the corresponding core partition, therefore the corners of a core partition can be counted by the number of stitches or anti-stitches. Based on these results, for coprime positive integers a and b, we give two essentially different formulae for the number of corners in all (a, b)-cores. This leads to an unexpected identity, expressing the rational Catalan numbers as weighed sums of binomial numbers. Moreover, we show that for an $(n, n+1)$-core partition λ determined by a certain $(n, n+1)$-Dyck path P, the corners of λ correspond to pairs of consecutive right steps in P. As a consequence, we show that the number of $(n, n+1)$-cores with k corners is the Narayana number $N(n, k+1)$. We also extend these results to multi-cores.

Keywords: core partitions, Dyck paths, cycle lemma, corners of a partition, Narayana number

AMS Subject Classification: 05A17, 05A19

1 Introduction

The objective of this paper is to investigate the number of corners of core partitions. Recall that a partition of a positive integer n is a finite non-increasing sequence of positive integers of sum n. The number of distinct parts of a partition, i.e., the number of corners in its Ferrers diagram, has attracted many researchers. Goh and Schmutz [12] gave a central limit theorem for the number of different parts in a random integer partition. Lovejoy [17] studied the arithmetic properties of partitions with distinct parts. Ono [22] gave weighted recurrence relations for the number of partitions of n with distinct parts.

The corner statistic has been also touched upon in the well-developed theory of overpartitions. Corteel and Lovejoy [18] introduced the overpartitions, which are partitions where each corner has a label chosen from two possible labels. Lovejoy [19] related overpartitions to Andrews' combinatorial generalization of the Gollnitz-Gordon identities and a theorem of Andrews and Santos on partitions with attached odd parts. Lovejoy and Bringmann [20] studied overpartition analogues of Ramanujan's mock theta function. They showed that these functions are related to the generating function of certain Hurwitz class numbers.

In this paper, we will be concerned with the corners of core partitions. Recall that a partition is a t-core partition (or a t-core for short) if none of its cells has hook length divisible by t. The notion of t-core arise from the study of modular representations. Nakayama [21] first conjectured that two characters of S_n are in the same p-block if and only if they are labelled by partitions with the p-core (the p-core of a given partition is obtained by repeatedly deleting border strips of length p). See James and Kerber's book [13] for a detailed and definitive account. Core partitions also play an important role in the emerging theory of k-Schur functions [16].

When $gcd(a, b) = r > 1$, each r-core is both an a-core and a b-core. Since the set of r-cores is infinite for any positive integer r, the set of (a, b)-cores is infinite. Thus, in the remaining of this paper, we shall always assume that a, b are relatively prime integers.

In 2002, Anderson [2] initiated the study on (a, b)-core partitions, namely, partitions that are simultaneously an a-core and a b-core. By giving a bijection which maps (a, b)-cores to a certain class of lattice paths, Anderson proved that the set of (a, b)-cores is counted by the rational Catalan number

$$Cat(a, b) = \frac{1}{a+b}\binom{a+b}{a, b}. \tag{1}$$

The size of a random (a, b)-core partition has been extensively studied. Armstrong, Hanusa and Jones [4] conjectured explicit formulae for the average sizes of (a, b)-cores and self-conjugate (a, b)-cores. Stanley [27], Chen, Huang and Wang [8], Johnson [14], Fayers [11] and Wang [31] obtained many results along this line of research.

Motivated by the above work, we turn to investigate the number of corners of an (a, b)-core partition. In Section 2, we first introduce some new concepts such as the stitch, the anti-stitch and the wrap-up space. Then we prove that the anti-stitches of an (a, b)-Dyck path are in bijection with the segments of structure sets of the corresponding (a, b)-core partition, therefore the number of corners of an (a, b)-core partition can be expressed using the number of stitches or anti-stitches of the corresponding (a, b)-Dyck path. As a consequence, we give two different formulae for the sum of the number of corners over all (a, b)-cores and new expressions for rational Catalan numbers. In Section 3, we give a bijection between corners in $(n, n+1)$-cores and consecutive pairs of right steps in the corresponding $(n, n+1)$-Dyck paths. Consequently, the number of $(n, n+1)$-cores with k corners is the Narayana number $N(n, k+1) = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}$.

2 Corners of (a, b)-cores

In this section, we shall enumerate the corners of all (a, b)-core partitions. First, let us recall Anderson's bijection which will be used in the remainder of this section.

2.1 Anderson's bijection

In this paper all lattice paths consists of North and East steps of length 1. An (a, b)-**Dyck paths** is a lattice path that goes from $(0, 0)$ to (a, b), staying above the diagonal $y = \frac{bx}{a}$. The red line in Figure 2.1 is an example of (3,5)-Dyck path. Note that we draw the x-axis vertically and the y-axis horizontally. It was known to Bizley [6] that the (a, b)-Dyck paths are counted by the rational Catalan number.

A set S is called n-**flush** if and only if for any element $x \in S$ greater than n, $x - n$ is also in S, or equivalently $(S - n) \bigcap \mathbb{N}^+ \subset S$. Denote $h(\lambda)$ by the structure set of a partition λ, which is the set of hooklengths of the cells in the first column of λ. The structure set is also known as β set in [13].

It is well-known that λ is an n-core partition if and only if $h(\lambda)$ is an n-flush (see, for example, Theorem 2.2.10 of [5] or Theorem 2.7.16 of [13]).

The conventional way to construct core partitions is to use the abacus model. Given an integer $n \geq 2$, list the positive integers in each residue class modulo n increasingly. Then n-flushes can be constructed by choose an arbitrary number of the smallest positive elements from the each residue class independently.

Anderson [2] introduced the following matrix, which is a two-way (vertically and horizontally) abacus. Let A be a matrix of integers, the element on the i-th row and j-th column being

$$A(i, j) = ab - ib - ja.$$

Now put A on the Euclidean plane $\mathbb{R}^2$ so that each element $A_{i,j}$ takes up a unit square. Let $R(A)$ be the rectangle area in $\mathbb{R}^2$ covered by A. Put an (a, b)-Dyck path P in $R(A)$ so that each steps of P is along an edges of a cells of A, and that P runs from the lower-left corner of $R(A)$ to the upper-right corner. Denote those positive elements of A under P by $A(P)$. It has been shown that $A(P)$ is both a-flush and b-flush. Thus the partition λ with structure set $A(P)$ is both an a-core and a b-core. This leads to the following theorem.

Theorem 2.1 (Anderson's bijection) *The (a, b)-cores are in bijection with (a, b)-Dyck paths, and $A(P)$ is the structure set of the corresponding (a, b)-core.*

See Figure 2.1 for an example of the case when $a = 5$ and $b = 8$.

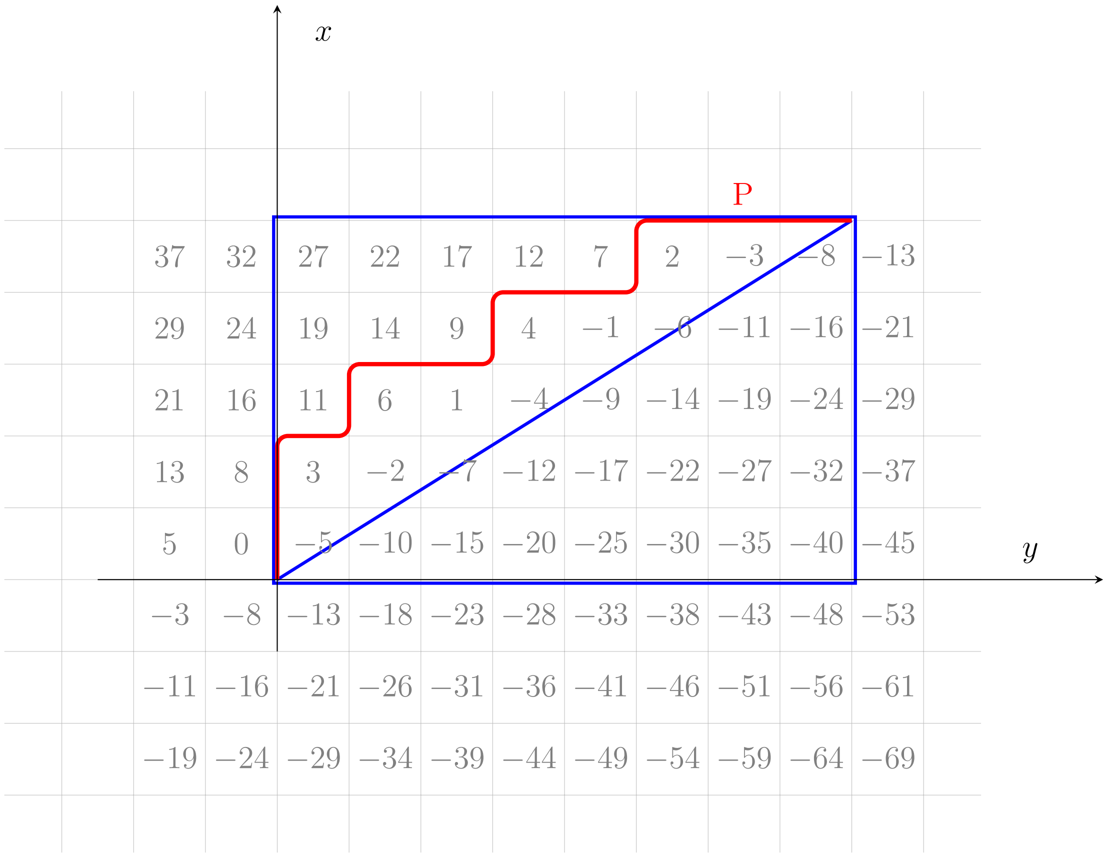

Figure 2.1: The infinite $(5, 8)$-table and a $(5, 8)$-Dyck path. This path corresponds to the $(5, 8)$-core $\lambda = (2, 1, 1, 1, 1)$ with structure set $\{6, 4, 3, 2, 1\}$ under Anderson's bijection. The region inside the rectangle T is the finite $(5, 8)$-table.

2.2 The wrap-up equivalence and stitches

In this subsection, we introduce the new combinatorial objects of stitches and separators, tools we bring in to study the bilinearity of the (a, b)-table.

Definition 2.2 *An **infinite** (a, b)**-table** is an infinite array of cells on the plane that extends infinitely in all directions, with the following labelling*

$$A(i, j) = ab - ib - ja, \tag{1}$$

*where i and j run over all integers. We call the subset of cells with coordinates $1 \leq i \leq a, 1 \leq j \leq b$ the **finite** (a, b)**-table**.*

See Example 3.4 for an example of the (3,10)-table. Note that the above definition is a natural extension of the matrix due to Anderson.

Note that an infinite (a, b)-table possesses the following anti-diagonal period

$$A(i, j) = A(i + a, j - b). \tag{2}$$

Given an infinite (a, b)-table, there are infinitely many subtables isomorphic to the finite (a, b)-table, and it is impossible to distinguish one from another since one can be obtained from another by sliding in the direction of vector (a, b). This property enables us to define the following equivalence relation.

Definition 2.3 *Two points p_1, p_2 on the plane $\mathbb{R}^2$ with coordinates $(x_1, y_1), (x_2, y_2)$ are **wrap-up equivalent** with respect to the vector (a, b) if and only if there is an integer z satisfying*

$$(x_1 - x_2, y_1 - y_2) = (za, -zb).$$

Two point sets P_1, P_2 are wrap-up equivalent if and only if there is an integer z such that P_1 can be obtained by moving P_2 along the vector $(za, -zb)$.

We use the symbol $\sim_w (a, b)$ to denote wrap-up equivalence, and write $p_1 \sim_w p_2$ for short when the parameters a and b are clear from context.

We call the quotient space $\mathbb{R}^2 \backslash \sim_w$ the **wrap-up space**, because it looks like a carpet rolled up. The interested reader may also wish to compare the wrap-up space to the skyscraper model of Sjöstrand (see [26]).

Lemma 2.4 *Each integer labels exactly one cell in the wrap-up space.*

Proof. Recall that Bézout's Theorem (see pp. 7–11 of [15] for example) states, for co-prime numbers a and b, each integer n can be represented as

$$n = n_1 a + n_2$$

for some integers n_1 and n_2. It follows that n appears in the infinite (a, b)-table, therefore in the wrap-up space.

Now we proceed to prove that only one cell in the wrap-up space is labelled n. If not, assume that two cells $C_1 = (x_1, y_1)$ and $C_2 = (x_2, y_2)$ are both labelled by n. Then

$$b(x_1 - x_2) - a(y_1 - y_2) = 0.$$

Thus the vector from C_1 to C_2 is an integral multiple of $(a, -b)$, so $C_1 \sim_w C_2$. Hence, n appears exactly once in the wrap-up space. This completes the proof. ∎

From the above theorem, one can get that 0 and 1 both exactly appear once in the wrap-up space. Thus 0 and 1 both appear in the infinite (a, b)-table. Now we study the position of 1 relative to 0 in the infinite (a, b)-table.

In the infinite (a, b)-table, we focus on the cell labelled with 0 on the left of the lower left corner of the finite (a, b)-table. From the position of 1 (relative to 0) we extract a quadruple of constants x, y, x' and y', which will play an important role throughout this section.

Suppose the cell labelled 1 in the (a, b)-table is on the x-th row (counting from bottom to top from the row 0 lies in) and the y-th column (counting from left to right from the column 0 lies in), or, x and y satisfies

$$bx - ay = 1.$$

We may think of x and y as the multiplicative inverse of b and a in $\mathbb{Z}_a$ and $\mathbb{Z}_b$, respectively (which are not necessarily fields, and elements are not always invertible). Set $x' = a - x$ and $y' = b - y$. It follows that

$$ay' - bx' = 1.$$

For example, when $(a, b) = (5, 8)$, we have $(x, y) = (2, 3)$ and $(x', y') = (3, 5)$, as illustrated in Figure 2.2.

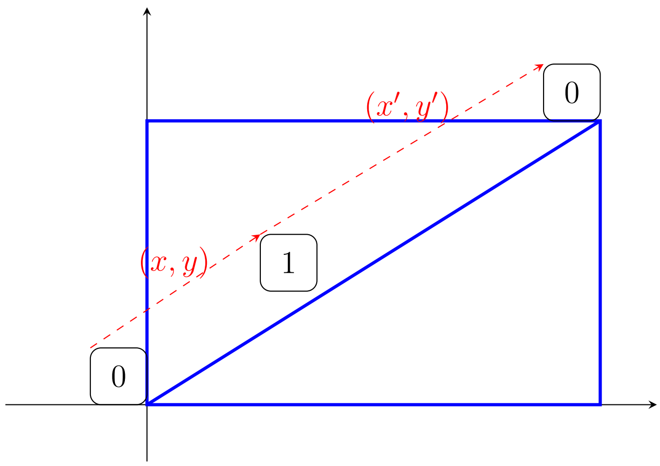

Figure 2.2: Definition of x, y, x' and y'

To track the cells with consecutive labels in a wrap-up space, we introduce the concept of stitches and anti-stitches.

Definition 2.5 *Consider a lattice path P consisting of up steps and right steps extending infinitely on both ends. If it is periodical, and each period consists of a up-steps and b right-steps, then we call it an* ***infinite*** (a, b)***-lattice path****. If the part of P in the finite (a, b)-table is a Dyck path, then P is called an* ***infinite*** (a, b)***-Dyck path****.*

The image of P under the canonical map from $\mathbb{R}^2$ to the wrap-up space $\mathbb{R}^2 \backslash \sim_w$ is denoted by P_w. When P is an infinite (a, b)-lattice (Dyck) path, P_w is said to be a ***cyclic*** (a, b)***-lattice (Dyck) path****.*

Definition 2.6 *Let P be a cyclic (a, b)-Dyck path and (C_0, C_1) is a pair of cells in the wrap-up space, such that P runs below C_1 and above C_0. The label of C_i is l_i, for $i = 0, 1$. We say that (C_0, C_1) is a* ***stitch*** *of P if $l_1 = l_0 + 1$, and that (C_0, C_1) is an* ***anti-stitch*** *of P if $l_1 = l_0 - 1$.*

We usually omit the path P and simply say that (C_0, C_1) is a stitch (or anti-stitch) for short, if no confusion arises.

Since Lemma 2.4 states that each integer appears in exactly one cell in the wrap-up space, we may call a pair of integers (l_0, l_1) a stitch (or an anti-stitch) if they label two cells that constitute a stitch (or an anti-stitch).

Example 2.7 In Figure 2.3 an infinite $(5, 8)$-Dyck path is drawn in the infinite $(5, 8)$-table. For this (5,8)-Dyck path, the anti-stitches are

$$(0, 1) \text{ and } (5, 6),$$

the stitches are

$$(-1, 0), (4, 5) \text{ and } (6, 7).$$

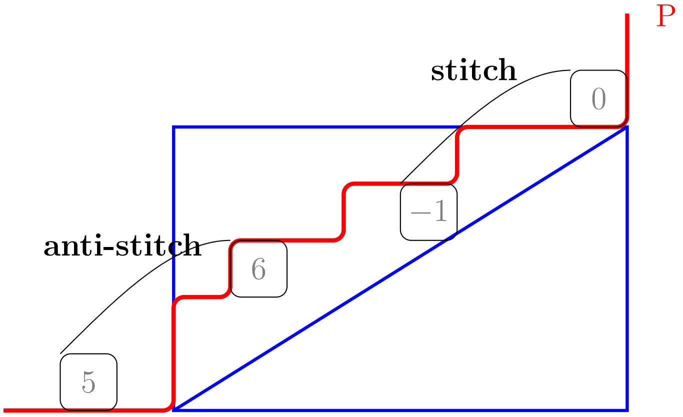

Figure 2.3: An $(5, 8)$-path P, $(6, 5)$ is an anti-stitch and $(-1, 0)$ is a stitch.

A set of integers M can be uniquely decomposed into non-intersecting unions of sets of consecutive numbers

$$M = \bigcup_{i \in I} M_i,$$

where I is the set of indices, each M_i consists of continuous sets of integers, and for distinct integers i and i', M_i and $M_{i'}$ are separated by at least one integer. We call M_i a **segment** of M. Note that a segment M_i can be an infinite set, and the set of segments can be an infinite set. The number of segments of M is denoted by $seg(M)$. The largest element (resp. smallest element) of a segment, if it exists, is called the **end** (resp. **head**) of this segment.

Given a cyclic (a, b)-Dyck path P, denote the set of labels of cells above P (resp. below P) by $\alpha(P)$ (resp. $\beta(P)$). We observe that $\alpha(P)$ and $\beta(P)$ have the following property.

Lemma 2.8 *Let u and v be integers. If (u, v) is a stitch, then u is an end in $\beta(P)$ and v is a head in $\alpha(P)$. If (u, v) is an anti-stitch, then u is a head in $\beta(P)$ and v is an end in $\alpha(P)$.*

By Theorem 2.1 we have the following equivalent definitions for $\alpha(P)$ and $\beta(P)$

$$\beta(P) = \mathbb{Z}^{-} \cup A(P), \tag{3}$$

$$\alpha(P) = (\mathbb{Z}^{+} \cup \{0\}) - A(P). \tag{4}$$

Given a partition λ, we can partition the set of rows by their lengths. The hook-lengths of the leftmost cells in rows with the same length correspond to consecutive integers from a segment in the structure set of λ. This leads to the following lemma.

Lemma 2.9 *For any partition λ, $c(\lambda) = seg(h(\lambda))$.*

Using these properties, we establish the following connection between the number of corners of (a, b)-core partitions and the number of stitches or anti-stitches.

Theorem 2.10 *For a given path P, $c(\lambda(P))$ equals the number of anti-stitches, and $c(\lambda(P)) + 1$ equals the number of stitches.*

Proof. Recall that $A(P)$ is the structure set of λ, that is, $A(P) = h(\lambda)$. Thus, by (3) we have

$$h(\lambda) \cup \mathbb{Z}^{-} = \beta(P).$$

On the other hand, Lemma 2.9 suggests that $c(\lambda(P)) = seg(h(\lambda))$. Thus the number of corners of $\lambda(P)$ is one less than the number of segments of $\beta(P)$. Since the segment $\mathbb{Z}^{-}$ has no head, we get that $c(\lambda(P))$ equals the number of heads of $\beta(P)$.

Given an anti-stitch (u, v), u is a head in $\beta(P)$, and v is an end in $\alpha(P)$. The heads of $\beta(P)$ are as many as the number of corners of $\lambda(P)$. So $c(\lambda(P))$ equals the number of anti-stitches for P.

Similarly, for a stitch (u, v) which is not (-1,0), u is an end in $\beta(P)$ that is not -1, and v is a head in $\alpha(P)$ that is not 0. Either of these two objects is as many as the number of corners of $\lambda(P)$. So $c(\lambda(P)) + 1$ equals the number of stitches.

∎

2.3 Outer-Corners and Stitches

In this subsection we briefly explore the connection between the stitches of P and the outer corners of the (a, b)-core $\lambda(P)$.

An *outer-corner* of a partition λ is a cell C outside λ's Ferrers diagram such that adding C to λ produces the Ferrers diagram of another partition. Outer-corners appear in the theory of representation of symmetric group, especially the branching rule (see [25]) and Jeu de Taquin (see [29]).

Suppose that there is at least one row of λ of length s. Then the lowest row of length s contains a corner, and the highest row of length s is followed by an outer-corner to the right. Since there is an extra outer-corner below the lowest row of λ, we obtain that the number of outer-corners is always one more than the number of corners of λ. Therefore, as a corollary of Theorem 2.10, we have the following result.

Theorem 2.11 *the number of stitches of P equals the number of outer-corners of the (a, b)-core $\lambda(P)$.*

Here we also give a direct combinatorial proof of this property. Assume that a cell C lies in the i-th row. The left-most cell of the i-th row has hooklength $m = \lambda_i + (l - i)$. It is easily seen that C is an outer-corner if and only if $\lambda_{i-1} > \lambda_i$. Since the left-most cell of the $(i-1)$-th row has hooklength $m' = \lambda_{i-1} + (l - i) + 1$, this inequality holds if and only if $m' > m + 1$, which is equivalent to $m + 1 \notin H(\lambda)$. Thus, C is an outer-corner if and only if $m \in \beta(P)$, and $m + 1 \notin \beta(P)$. Since $\alpha(P) = \mathbb{Z} - \beta(P)$, we have C is an outer-corner if and only if the pair $(l, l + 1)$ labels a stitch.

2.4 Cyclic paths and Patterns

First we define a shifting action on lattice paths (see also [29], Section 5.3).

Definition 2.12 *The action of deleting the first step of an (a, b)-lattice path and attaching the step to the end of the path generates a cyclic group C_{a+b}. We call this action* ***rotation***.

The Cycle Lemma is a powerful tool in enumerative combinatorics concerning the number of desired objects in each orbit under certain cyclic group action (for example, the rotation action defined above). Given a positive integer k, a word P of letters U and D is called **k-dominating** if in any prefix of P, the number of Us is more than k times the number of Ds. The conventional Cycle Lemma gives the number of k-dominating words:

Lemma 2.13 (Cycle Lemma, [10]) *Let $P = (p_1, p_2, \ldots, p_{m+n})$ be a sequence that consists of m copies of U and n copies of D, where $m \geq nk$. Of the $m + n$ sequences of the form $(p_i, p_{i+1}, \ldots, p_{m+n}, p_1, \ldots, p_{i-1})$ obtained from P by the rotation action, exactly $m - kn$ are k-dominating.*

See [9] for more details.

In this paper we will use the following rational form of Cycle Lemma. It is also known as Spitzer's Lemma (Lemma 10.4.3 of [7]).

Lemma 2.14 (Rational Cycle Lemma) *Let a and b be coprime positive integers. Given a finite lattice paths L with steps $x_1 x_2 \ldots x_{a+b}$, where $x_i \in \{R, U\}$. Then the orbit of L under the action of cyclic group C_{a+b} consists of $a + b$ elements. In this orbit, there is exactly one lattice that is a rational Dyck path.*

By a **pattern** in a lattice path we mean a certain sequence of steps. Note that a lattice path is always viewed as a cyclic lattice path, i.e., the last step is again followed by the first step and a pattern can be a suffix of the path followed by a prefix of the path. For example, the pattern $Q = RRUUU$ appears once in the finite $(3,4)$-Dyck path $UUURRRR$. Now let us concern with the enumeration of any given pattern Q.

Theorem 2.15 *For any given pattern Q, assume that Q contains m up steps and n right steps, we have that Q appears*

$$(a+b)\binom{a-m+b-n}{b-n}$$

times in all (a,b)-lattice paths and

$$\binom{a-m+b-n}{b-n}$$

times in all (a,b)-Dyck paths.

Proof. A cyclic (a,b)-lattice (or Dyck) paths with a highlighted segment Q is a cyclic (a,b)-lattice (or Dyck) paths with some steps drawn in a highlighted color. These steps are continuous and form a pattern Q.

To enumerate the appearances of Q in all (a,b)-lattice (or Dyck) paths, we count cyclic (a,b)-lattice (or Dyck) paths with exactly one highlighted segment Q.

Consider (a,b)-lattice paths which starts with Q. By Definition 2.12 of C_{a+b}, rotating these paths under the action of C_{a+b} produces all the highlighted (a,b)-cyclic lattice paths. It implies that the number of the highlighted (a,b)-cyclic lattice paths can be counted by the product of the number of (a,b)-lattice paths which starts with Q and the cardinality of the group C_{a+b}.

It is easily seen that the number of (a,b)-lattice paths that begins with Q is $\binom{a-m+b-n}{b-n}$. Since C_{a+b} is the cyclic group of order $a+b$, we have that the number of highlighted (a,b)-lattice paths is

$$(a+b)\binom{a-m+b-n}{b-n}.$$

By Lemma 2.14, each orbit of highlighted (a,b)-lattice paths contains exactly one highlighted (a,b)-Dyck path. Since the size of an orbit is $a+b$, we get that the number of highlighted (a,b)-Dyck path is $\binom{a-m+b-n}{b-n}$, or equivalently, Q appears in all (a,b)-Dyck paths

$$\binom{a-m+b-n}{b-n}$$

times. This completes the proof. ∎

2.5 The sum of corners of (a,b)-cores

To enumerate the corners of all (a,b)-cores, we introduce a special type of pattern. Given a stitch(or an anti-stitch) with cells C_0 and C_1, there is a minimal lattice rectangle containing C_0 and C_1. Denote the height and width of the rectangle by h and w, and call (h,w) the **type** of the stitch (or anti-stitch) (C_0, C_1), written as (h,w)-stitch (or (h,w)-anti-stitch). Note that since the labels of C_0 and C_1 are two consecutive integers, the type (h,w) is either $(x,y)+n(a,b)$ or$(x',y')+n(a,b)$.

A **separator** is a section of the finite (a,b)-path P that lies in the interior of the minimal rectangle that contains the two cells C_0 and C_1 of a stitch (or an anti-stitch). See Fig. 2.5 for an example. Note that steps of P on the boundary of the rectangular area are excluded from the separator. For an (x,y)-type stitch, the corresponding separator can be written as a sequence of right steps R and up steps U

$$(s_0, s_1, \ldots, s_{m-1}, s_m),$$

where $y+1$ of the steps are R (including s_0 and s_m), and the rest of the steps are U. Since the separator lies in the interior of a rectangle of size x by y, the number of up steps is equal or lower than $x-1$. For example, the “UURU” part of P in the minimal rectangle of the anti-stitch $(6,5)$ is a separator of the anti-stitch, and “RURRR” is a separator of the stitch $(-1,0)$.

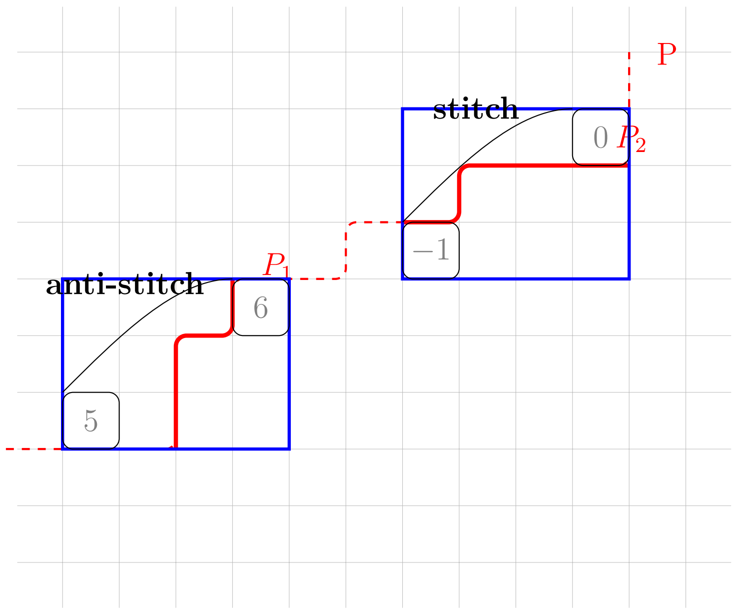

Figure 2.4: The blue rectangles are minimal rectangles for the anti-stitch $(6,5)$ and the stitich $(-1,0)$. The sections of P that lie in the minimal rectangles are the corresponding separators.

Now we are in a position to apply Theorem 2.15 to the enumeration of corners of all (a,b)-cores. Recall that $Cat(a,b)$ denotes the rational Catalan number $\frac{1}{a+b}\binom{a+b}{a,b}$.

Theorem 2.16 *The sum of number of corners over all (a,b)-cores is represented by any of the following four formulae*

$$\begin{aligned}
&\sum_{0\le t\le x}(x-t)\binom{a-t+b-y-1}{b-y-1}\binom{t+y-1}{y-1}-Cat(a,b) &(5)\\
=&\sum_{x\le t\le a}(t-x)\binom{a-t+b-y-1}{b-y-1}\binom{t+y-1}{y-1} &(6)\\
=&\sum_{0\le s\le y}(y-s)\binom{b-s+a-x-1}{a-x-1}\binom{s+x-1}{x-1}-Cat(a,b) &(7)\\
=&\sum_{y\le s\le b}(s-y)\binom{b-s+a-x-1}{a-x-1}\binom{s+x-1}{x-1}. &(8)
\end{aligned}$$

Proof. We count the corners of all (a,b)-cores by finding stitches for all paths P. Given a stitch (C_0,C_1), consider the two columns in which the C_0 and C_1 lie. These two columns cut out a segment of a given path P, which is an (x,y)-separator.

Assume this (x,y)-separator consists of $x-t$ up steps and $y+1$ right steps. Then there are t stitches in total (associated with one particular path P), each of which cut out the same (x,y)-separator.

This separator may appear in $\binom{a-(x-t)+b-y-1}{b-y-1}$ paths P. On the other hand, such separators are as many as

$$\binom{x-t+y-1}{y-1}.$$

So the total number of (x,y)-stitches is

$$\sum_{1\le t\le x} t\binom{a-(x-t)+b-y-1}{b-y-1}\binom{x-t+y-1}{y-1}. \tag{9}$$

Substituting t with $x-t$ in the above formula, we get (8).

Similarly, the total number of (x',y')-stitches is

$$\sum_{0\le t\le x}(x-t)\binom{a-t+b-y-1}{b-y-1}\binom{t+y-1}{y-1}. \tag{10}$$

Note that (the equivalence class of) any segment-end of $b(P)$ corresponds to a stitch. So we have (5). The other two formulae are similarly obtained. ∎

Theorem 5 leads to some interesting identities involving the rational Catalan numbers. For instance, combining (5)-(8), we have the following identities.

Corollary 2.17 *For coprime positive integers a and b, we have*

$$Cat(a,b)=\sum_{0\le t\le a}(x-t)\binom{a-t+b-y-1}{b-y-1}\binom{t+y-1}{y-1}$$

and

$$Cat(a,b) = \sum_{0\le s\le b} (y-s)\binom{b-s+a-x-1}{a-x-1}\binom{s+x-1}{x-1}.$$

In the Catalan case, we have $n\cdot n-(n-1)\cdot(n+1)=1$, so $x=n, y=n-1$. Hence, Corollary 2.17 reduces to the following result.

Corollary 2.18 *For $n\ge 1$,*

$$Cat(n,n+1) = \sum_{0\le t\le n} (1-t)\binom{n-t+n-1}{n-1}.$$

When $a=n, b=nk+1$ for a fixed positive integer k, we have that $x=1$ and $y=k$. Thus Corollary 2.17 reduces to the following equality.

Corollary 2.19 *For $n\ge 1$ and $k\ge 1$,*

$$Cat(n,kn{+}1) = \frac{1}{nk+n+1}\binom{nk+n+1}{n} = \sum_{0\le t\le n}(1{-}t)\binom{n-t+nk-k}{nk-k}\binom{t+k-1}{k-1}$$

and

$$Cat(n,kn+1) = \sum_{0\le s\le kn+1}(k-s)\binom{kn-s+n-1}{n-2}$$

3 (a,b)-cores with specified number of corners

3.1 Catalan case

In this subsection we focus on the Catalan case when $a=n$ and $b=n+1$. Under this assumption the separators are reduced to simpler forms. This allows us to enumerate explicitly $(n,n+1)$-cores with specified number of corners.

Theorem 3.1 *The set of $(n,n{+}1)$-cores with k corners is counted by Narayana number*

$$N(n,k+1) = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}.$$

Proof. Given an $(n,n+1)$-Dyck path L, each corner of $\lambda(L)$ corresponds to two consecutive up steps in L. Note that an $(n,n+1)$-Dyck path L always ends with a down step. So each up step is either followed by a right step and forming a peak, or followed by another up step and forming a pair of consecutive up steps. Therefore, an $(n,n+1)$-Dyck path has k pairs of consecutive up steps if and only if it has $n-k$ peaks.

Since an $(n, n+1)$-Dyck path is essentially juxtaposition of a Dyck path with n up steps and n right steps and a final right step, the number of $(n, n+1)$-Dyck paths with $n-k$ peaks is counted by Narayana number $N(n, n-k)$. Note that the sequence of Narayana number is palindromic, i.e., $N(n, n-k) = N(n, k+1)$. The result is immediate. ∎

Example 3.2 Here is a list of $(n, n+1)$-Dyck paths and $(n, n+1)$-cores for $n = 3$.

path	*URURUR*	*UURRUR*	*URUURR*	*UURURR*	*UUURRR*
number of peaks	*3*	*2*	*2*	*2*	*1*
$(n, n+1)$*-core*	∅	1	2 1	2 1	5 2 1 2 1
number of corners	*0*	*1*	*1*	*1*	*2*

Since the number of $(n, n+1)$-cores with a specified number of corners is counted by Narayana number, we want to find out the number of (a, b)-cores with a specified number of corners, which will be a generalization of Narayana number. Note that there exists another rational Narayana number $\frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}$ in the literature (see, for example, [24]).

Open Problem 3.3 *Enumerate (a, b)-cores with k corners.*

Note that A. Reifegerste pointed out in Remark 2.6(c) of [23] that the diagrams of certain permutations fitting in the shape $(n-1, n-2, ..., 1)$ and having k corners are enumerated by the Narayana number. It would be interesting to know if this result is connected to Theorem 3.1.

3.2 Fuß-Catalan case

In combinatorics and statistics, the Fuß-Catalan numbers are numbers of the form

$$C_m(p, r) = \frac{r}{mp+r}\binom{mp+r}{m}.$$

They are named after N.I. Fuß and E.C. Catalan. This notion appeared in Fuß's work on dissection of a convex $(kn+2)$-gon into $(k+2)$-gons in the 18th century. See Armstrong's thesis [3] for more details.

It can be readily checked that

$$C_m(p, r) = r\ Cat(mp + r - m, m).$$

In this subsection we study corners in $(n, kn+1)$-cores. These core partitions are in bijection with $(n, kn+1)$ rational Dyck paths.

Recall that in an infinite $(n, kn+1)$-table, the cell (i, j) is labelled $A_{i,j} = n(kn+1) - i(kn+1) - jn$. So given cell (i, j) labelled m, cell $(i-1, j+k)$ is labelled $m+1$.

17	14	11	8	5	2	−1	−4	−7	−10
7	4	1	−2	−5	−8	−11	−14	−17	−20
−3	−6	−9	−12	−15	−18	−21	−24	−27	−30

Figure 3.5: The (3,10)-table.

Example 3.4 In the case of $n = k = 3$*, we have the following* $(3, 10)$*-table*

Thus we get that in the $(n, kn+1)$-case, a corner corresponds to k consecutive right steps, except the last k steps in the $(n, kn + 1)$-path, which corresponds to the stitch $(-1, 0)$. Similarly, in the $(n, kn-1)$ case, each $k+1$ consecutive right steps corresponds to an anti-stitch, therefore a corner.

Open Problem 3.5 *Find an analog of Narayana number to enumerate* $(n, kn + 1)$*-cores (or* $(n, kn - 1)$*-cores) with specified number of corners.*

3.3 Multi-Catalan Case

Fix positive integer n and $k \geq 2$, Amdeberhan and Leven [1] gave a bijection between $(n, n+1, \ldots, n+k)$-cores and a certain family of paths called **(n, k)-generalized Dyck paths**. Recall that an (n, k)-generalized Dyck path is a path staying above the line $y = x$ and consists of

- vertical steps of length k (which we call a k-up step),
- horizontal steps of length k (which we call a k-down step),
- diagonal steps (i, i) for $1 \leq i \leq k - 1$ (which we call an (i, i)-diagonal step).

Figure 3.6 (which is excerpt from [1]) shows the correspondence between $(10, 3)$-generalized Dyck paths and $(10, 11, 12, 13)$-cores. These numbers $10, 11, 12, 13$ can be easily read off the figure as those numbers missing between the tail of the first diagonal and the head of the second diagonal.

Note that in the case $k = 1$, we can still generate $(n, n + 1)$ cores from $(n, 1)$-Dyck paths, which are the usual Dyck paths of length $2n$, but the following arguments fall apart.

Now we briefly describe how to obtain the corresponding $(n, \ldots, n + k)$-core from an (n, k)-generalized Dyck paths. Given an (n, k)-generalize Dyck path P, change each of the (i, i) diagonal paths into i up steps followed by i right steps and obtain a new path Q. Call Q a **peaked (n, k)-generalize Dyck path**, or the peaked form of P.

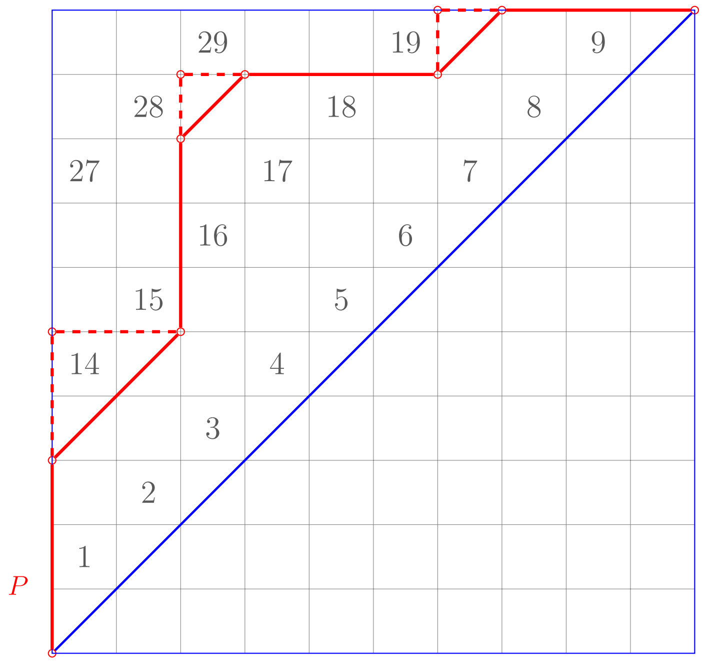

Figure 3.6: This is Fig. 8 of [1]. The thick red path is a $(10, 3)$-generalized Dyck paths P, and the dashed line (partly under P) is its peaked form.

The numbers in cells below the peaked path Q are the structure numbers of the corresponding core partition.

To reverse this process, find the the cells labelled with the structure numbers $H(\lambda)$ of the given core partition λ, and find the lattice path that covers most cells but not the cells with labels outside $H(\lambda)$. This lattice path is the peaked path of the desired (n, k)-generalized Dyck path. Then flatten the peaks at height $km + j$, where m is an integer and $1 \leq j \leq k - 1$, to a plateau at height km. (The height of a lattice point on a Dyck paths is understood to be the number of up steps to the left of the point minus the number of right steps to its right.) The resulting path is the (n, k)-generalized Dyck path.

Note that if we add the (n, k)-generalized Dyck paths with the missing peaks, which are shown as dotted lines in the above figure, then we obtain exactly those $2n$-Dyck paths whose valleys have height divisible by k. This can be verified by checking that end points of k-right steps or i-diagonal paths of (n, k)-generalized Dyck paths end at a height divisible by k.

We have the following partial result concerning the number of corners of $(n, \ldots, n+k)$-core partitions.

Theorem 3.6 *Given an (n, k)-generalized Dyck paths and the corresponding $(n, \ldots, n+k)$-core partition, there is a bijection that sends each corner of the core partition to either an (i, i) diagonal steps with $i \geq 2$ or a k-down step.*

Proof. Consider an (n,k)-generalized Dyck paths P and its peaked form Q. Let λ be the partition that corresponds to P under Amdeberhan and Leven's bijection. Recall from Lemma 2.9 that the corners of λ are in bijection with the segments of the structure numbers of λ. Note that the structure numbers of λ are the labels of the cells under Q, and the segments can be divided into the following two categories.

Some segments consist of labels of cells under P (e.g., the diagonal 16,17,18 in the figure). The head of such a segment labels a cell that contains the starting point of a k-up step, and the end of the segment labels a cell that contains the ending point of a k-down step. Therefore such a segment corresponds to a k-down step. Conversely, we obtain a segment from a k-down step by taking the labels of the consecutive cells to the south-west of the end point of the k-down step.

The other segments consists of labels of cells between P and Q (e.g. the single 14 in the left figure). Each of these segments lie in a triangular area between P and Q, and the edge that lies in P a vertical (i,i)-diagonal step with $i \geq 2$. Conversely, given an (i,i)-diagonal step, we may find the triangle bounded by this diagonal step and P, and the desired segment consists of the labels of the cells that lie in this triangle.

Thus we have proved that each corner of λ is in bijection with either a k-down step or an (i,i)-diagonal step. ∎

The above theorem is only a first step towards understanding the distribution of the number of corners of multi-cores. Therefore we raise the following open problem.

Open Problem 3.7 *Enumerate $(n,\ldots,n+k)$-cores with j corners. Equivalently, count (n,k)-generalized Dyck paths without $(1,1)$-diagonals by the number of steps.*

The following theorem gives the distribution of the number of corners for $(n,n+1,n+2)$-cores.

Theorem 3.8 *The number of $(n,n+1,n+2)$-cores with j corners is*

$$\binom{n}{2j} C_j, \tag{1}$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Proof. Suppose there are $n-2j$ $(1,1)$-diagonal steps in the $(n,2)$-generalized Dyck paths. Each of these paths corresponds to a core partition with j corners. Then there are j 2-up steps and j 2-down steps. So the number of such $(n,2)$-generalized Dyck paths is $\binom{n}{2j}C_j$. ∎

Corollary 3.9 *The total number of corners of all $(n,n+1,n+2)$-cores is*

$$\sum_{j=0}^{n} j\binom{n}{2j} C_j. \tag{2}$$

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