Harnack and Shift Harnack Inequalities for SDEs with Integrable Drifts*

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Abstract

In this paper, the coupling by change of measure is constructed for a class of SDEs with integrable drift and additive noise, from which the Harnack and shift Harnack inequalities are derived. Finally, as applications, the gradient estimate, the regularity of the heat kernel and the distribution properties of the associated transition probability are also obtained. The important tool is Krylov's estimate.

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1 Introduction

Let E be a topological space, P is a Markov operator on $\mathscr{B}_b(E)$ (the bounded measurable functions on E), the dimension-free Harnack inequality with power p, i.e.

$$(Pf)^p(x) \le Pf^p(y)\mathrm{e}^{\Psi(x,y)}, \quad x, y \in E, f \in \mathscr{B}_b^+(E) \tag{1.1}$$

has many applications, for instance, it implies a dimension-free lower bound for logarithmic Sobolev constant on compact manifolds [12]. It also yields strong Feller property, gradient estimate, uniqueness of invariant probability, regularity of the heat kernel with respect to invariant probability, see [14, Chapter 1]. Moreover, it is an important tool in the proof of hypercontractivity of non-symmetric semigroup, [2, 15]. On the other hand, when E is a Banach space, the shift Harnack inequality

$$\Phi(Pf(x)) \le P\{\Phi \circ f(y + \cdot)\}\mathrm{e}^{C_\Phi(x,y)}, \quad x, y \in E, f \in \mathscr{B}_b^+(E) \tag{1.2}$$

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implies the existence and regularity of density of P with respect to the Lebesgue measure. Thus, the Harnack and shift Harnack inequalities attracts much attention and there are many results on this topic, of which [14] gives lots of models satisfying Harnack and shift Harnack inequalities. For simplicity, consider the SDE on $\mathbb{R}^d$ below:

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \mathrm{d}W_t. \tag{1.3}$$

The classical condition for Harnack and shift Harnack inequalities is

$$\langle b_t(x) - b_t(y), x - y\rangle \le C|x-y|^2, \quad t \ge 0, x, y \in \mathbb{R}^d \tag{1.4}$$

for a constant $C > 0$. Recently, Zvonkin type transforms have been used to prove existence and uniqueness of SDEs and SPDEs with singular drift, see e.g. [1, 3, 4, 5, 6, 7, 10, 13, 16, 17, 18, 19, 20]. Following [19], Shao [11] proved the Harnack inequality (1.1) under the condition $|b| + |b|^2 \in L_p^q(T)$ for some $p, q > 1$ satisfying $\frac{d}{p} + \frac{2}{q} < 1$, here $L_p^q(T)$ is defined in (1.6). However, the Harnack inequality in [11] is not precise since $\lim_{y\to x} \mathrm{e}^{\Psi(x,y)} > 1$. In addition, [9] has obtained the precise log-Harnack inequality by gradient-gradient estimate

$$|\nabla P_t f|^2 \le C P_t |\nabla f|^2$$

by approximation method when $|b| \in L_p^q(T)$ for some $p, q > 1$ satisfying $\frac{d}{p} + \frac{2}{q} < 1$, which removes the condition $|b|^2 \in L_p^q(T)$ in [11]. Unfortunately, [9] can not obtain gradient-gradient estimate

$$|\nabla P_t f| \le C P_t |\nabla f|, \tag{1.5}$$

which implies the precise Harnack inequality (1.1) by [14, Theorem 1.3.6 (2)]. To obtain precise Harnack inequality (1.1) in the sense that $\lim_{y\to x} \mathrm{e}^{\Psi(x,y)} = 1$, instead of proving (1.5), we adopt the method of coupling by change of measure. To this end, we introduce an additional condition (1.9) below, which means b satisfying

$$\sup_{y\neq 0} \frac{\|b(\cdot + y) - b\|_p}{|y|} < \infty$$

for some $p > d$ when $q = \infty$, where $\|\cdot\|_p$ is the L^p norm respect to Lebesgue measure, see Remark 1.1 for example and more details.

Compared with the existed precise Harnack inequalities, the drift in this paper is allowed to be integrable and not continuous. As to the shift Harnack inequality, it is very new since there is few result for SDE with integrable drift on this topic.

Throughout the paper, the letter C or c will denote a positive constant, and $C(\theta)$ or $c(\theta)$ stands for a constant depending on θ. The value of the constants may change from one appearance to another.

For a measurable function f defined on $[0, T] \times \mathbb{R}^d$, let

$$\|f\|_{L_p^q(s,t)} = \left(\int_s^t \left(\int_{\mathbb{R}^d} |f_r(x)|^p \mathrm{d}x\right)^{\frac{q}{p}} \mathrm{d}r\right)^{\frac{1}{q}}, \quad p, q \ge 1, 0 \le s \le t \le T. \tag{1.6}$$

When $s=0$, we simply denote $\|f\|_{L^q_p(0,t)}=\|f\|_{L^q_p(t)}$. Let W_t be an m-dimensional Brownian motion on a complete filtration probability space $(\Omega,\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P})$. Consider the following SDEs on $\mathbb{R}^d$:

$$\mathrm{d}X_t=b_t(X_t)\mathrm{d}t+\sigma_t\mathrm{d}W_t, \tag{1.7}$$

where

$$b:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d;\quad \sigma:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^m$$

are measurable. Throughout this paper, we make the following assumptions:

(H1) There exists constants $p,q>1$ with $\frac{d}{p}+\frac{2}{q}<1$ such that

$$\|b\|_{L^q_p(T)}<\infty,\quad T\geq 0. \tag{1.8}$$

Moreover, there exists an nonnegative function $K\in L^q_{loc}([0,\infty))$ such that

$$\left(\int_{\mathbb{R}^d}|b_t(x+y)-b_t(x)|^p\mathrm{d}x\right)^{\frac{1}{p}}\leq K(t)|y|,\quad t\geq 0, y\in\mathbb{R}^d. \tag{1.9}$$

(H2) There exists a constant $\delta\in(1,\infty)$ such that for any $t\in[0,\infty)$,

$$\delta^{-1}I_{d\times d}\leq\sigma_t\sigma_t^*\leq\delta I_{d\times d}.$$

According to [19, Theorem 1.1], under (1.8) and **(H2)**, the equation (1.7) has a unique non-explosive strong solution X^x_t with $X_0=x\in\mathbb{R}^d$. Let P_t be the associated Markov semigroup, i.e.

$$P_tf(x)=\mathbb{E}f(X^x_t),\quad f\in\mathscr{B}_b(\mathbb{R}^d).$$

Remark 1.1. *To obtain precise Harnack inequality, we introduce* (1.9) *instead of the Lipschitzian continuity for b, i.e.*

$$\|b_t(\cdot+y)-b_t(\cdot)\|_\infty<C(t)|y|. \tag{1.10}$$

To see the difference between (1.9) *and* (1.10)*, we give an example as follows. Let $b=a1_{[c_1,c_2]}$ for $a,c_1,c_2\in\mathbb{R}^d$ with $a\neq 0$ and $c_1\leq c_2$. Obviously, b does not satisfy* (1.10) *(in fact, b does not satisfy* (1.4) *either), but by a simple calculus,* (1.9) *holds. From this example, we see that b may be not continuous if* (1.9) *holds. On the other hand, it is well known that*

$$\|f(\cdot+y)-f\|_p\leq\|\nabla f\|_p|y|,\quad p>0,$$

where ∇ is the weak gradient. This means that if $\|\nabla b_t\|_p<K(t)$ for some $K\in L^q_{loc}([0,\infty))$, then (1.9) *holds.*

Let

$$\mathscr{K}:=\left\{(p,q)\in(1,\infty)\times(1,\infty):\ \frac{d}{p}+\frac{2}{q}<2\right\}.$$

We firstly give an important lemma which will be used in the sequence.

Lemma 1.2. *Let $T>0$. Assume* (1.8) *and* **(H2)**. *Then for any $(\alpha,\beta)\in\mathscr{K}$, there exists a constant $\kappa=\kappa(T,\delta,\alpha,\beta,\|b\|_{L^q_p(T)})>0$ such that for any $s_0\in[0,T)$ and any solution $(X_{s_0,t})_{t\in[s_0,T]}$ of* (1.7) *from time s_0,*

$$\mathbb{E}\left[\int_s^t |f|(r,X_{s_0,r})\mathrm{d}r\Big|\mathscr{F}_s\right]\le\kappa\|f\|_{L^\beta_\alpha(T)},\ \ s_0\le s<t\le T, f\in L^\beta_\alpha(T). \tag{1.11}$$

Then for any $\lambda>0$, there exists a constant $\gamma=\gamma(\lambda,\kappa,\|f\|_{L^\beta_\alpha(T)})>0$ such that

$$\mathbb{E}\big(\mathrm{e}^{\lambda\int_s^T|f|(r,X_{s_0,r})\mathrm{d}r}\big|\mathscr{F}_s\big)\le\gamma,\quad s_0\le s\le T. \tag{1.12}$$

Moreover,

$$\mathbb{E}\big(\mathrm{e}^{\lambda\int_s^T|f|(r,X_{s_0,r})\mathrm{d}r}\big|\mathscr{F}_s\big)\le\frac{1}{1-\lambda\kappa\|f\|_{L^\beta_\alpha(T)}},\quad s_0\le s\le T \tag{1.13}$$

when $\|f\|_{L^\beta_\alpha(T)}<\frac{1}{\lambda\kappa}$.

Proof. (1.11), which is called Krylov's estimate, was proved in [8, Lemma 3.3], see also [7, Lemma 3.1] for the multiplicative noise case. (1.12) follows from (1.11) and Khasminskii's estimate. We only need to prove (1.13). Since (1.11) implies that for any $n\ge1$, $\lambda>0$,

$$\lambda^n\mathbb{E}\left[\left(\int_s^T|f|(r,X_{s_0,r})\mathrm{d}r\right)^n\Big|\mathscr{F}_s\right]\le n!\left(\lambda\kappa\|f\|_{L^\beta_\alpha(T)}\right)^n,\quad s_0\le s\le T, f\in L^\beta_\alpha(T). \tag{1.14}$$

Thus, if $\|f\|_{L^\beta_\alpha(T)}<\frac{1}{\lambda\kappa}$, we have

$$\mathbb{E}\big(\mathrm{e}^{\lambda\int_s^T|f|(r,X_{s_0,r})\mathrm{d}r}\big|\mathscr{F}_s\big)\le\sum_{n=0}^\infty\left(\lambda\kappa\|f\|_{L^\beta_\alpha(T)}\right)^n=\frac{1}{1-\lambda\kappa\|f\|_{L^\beta_\alpha(T)}},\quad s\in[s_0,T]. \tag{1.15}$$

□

The paper is organized as follows: In Section 2, we give main results on Harnack and shift Harnack inequality and their applications respectively; In Section 3, we prove Harnack inequality; In Section 4, we prove shift Harnack inequality.

2 Main Results

2.1 Harnack Inequality and Its Applications

Theorem 2.1. *Assume* **(H1)**-**(H2)**. *Let $T>0$ and $\beta(T,K,\delta,\kappa)=\delta\left(\frac{1}{T}+\kappa\|K\|^2_{L^q([0,T])}\right)$. Then for any nonnegative $f\in\mathscr{B}_b(\mathbb{R}^d)$ and any $p>1$,*

$$(P_Tf)^p(y)\le P_Tf^p(x)\left(1-\frac{(2p+2)\beta(T,K,\delta,\kappa)|x-y|^2}{(p-1)^2}\right)^{-\frac{p-1}{2}}$$

holds for any $x,y\in\mathbb{R}^d$ with $|x-y|^2<\left(\frac{(2p+2)\beta(T,K,\delta,\kappa)}{(p-1)^2}\right)^{-1}$.

The next corollary following from Theorem 2.1 describes the property of the transition probability, see [14, Theorem 1.4.2 (1)] for the proof.

Corollary 2.2. *Let the assumption in Theorem 2.1 hold. Let $T > 0$, $p > 1$, $x, y \in \mathbb{R}^d$ with $|x-y|^2 < \left(\frac{(2p+2)\beta(T,K,\delta,\kappa)}{(p-1)^2}\right)^{-1}$. Then $P_T(x,\cdot)$ is equivalent to $P_T(y,\cdot)$ and*

$$P_T\left\{\left(\frac{\mathrm{d}P_T(x,\cdot)}{\mathrm{d}P_T(y,\cdot)}\right)^{\frac{1}{p-1}}\right\}(x) \le \left(1 - \frac{(2p+2)\beta(T,K,\delta,\kappa)|x-y|^2}{(p-1)^2}\right)^{-\frac{1}{2}}.$$

2.2 Shift Harnack Inequality and Its Applications

The following theorem gives the result on the shift Harnack inequality.

Theorem 2.3. *Let $T > 0$. Assume (1.8) and* **(H2)**. *Then the following assertions hold.*

(i) For any $x, y \in \mathbb{R}^d$ and positive $f \in \mathscr{B}_b(\mathbb{R}^d)$, the shift log-Harnack inequality holds, i.e.

$$P_T \log f(x) \le \log P_T f(y+\cdot)(x) + \delta\left(\frac{|y|^2}{T} + 4\kappa\|b\|^2_{L^q_p(T)}\right).$$

Moreover, for any $p > 1$, and any nonnegative $f \in \mathscr{B}_b(\mathbb{R}^d)$, it holds that

$$(P_T f)^p(x) \le P_T f^p(y+\cdot)(x)\gamma \mathrm{e}^{\frac{\delta(p+1)|y|^2}{2(p-1)T}}$$

with some constant $\gamma > 0$ depending on $p, \kappa, \delta, \|b\|^2_{L^q_p(T)}$.

(ii) If in addition (1.9) holds, then for any $x, y \in \mathbb{R}^d$ and positive $f \in \mathscr{B}_b(\mathbb{R}^d)$, the shift log-Harnack inequality holds, i.e.

$$P_T \log f(x) \le \log P_T f(y+\cdot)(x) + \beta(T,K,\delta,\kappa)|y|^2.$$

Moreover, for any $p > 1$, and any nonnegative $f \in \mathscr{B}_b(\mathbb{R}^d)$,

$$(P_T f)^p(x) \le P_T f^p(y+\cdot)(x)\left(1 - \frac{(2p+2)\beta(T,K,\delta,\kappa)|y|^2}{(p-1)^2}\right)^{-\frac{p-1}{2}}$$

holds for any $x, y \in \mathbb{R}^d$ with $|y|^2 < \left(\frac{(2p+2)\beta(T,K,\delta,\kappa)}{(p-1)^2}\right)^{-1}$. Here, $\beta(T,K,\delta,\kappa)$ is defined in Theorem 2.1.

According to [14, Theorem 1.4.3, Proposition 1.3.9 (2)], we have the following corollary from Theorem 2.3.

Corollary 2.4. *Let $T > 0$. Assume (1.8) and* **(H2)**.

(i) For any $x, y \in \mathbb{R}^d$, P_T has transition density $p_T(x,y)$ with respect to the Lebesgue measure such that

$$\int_{\mathbb{R}^d} p_T(x,y)^{\frac{p}{p-1}}\mathrm{d}y \le \frac{1}{\left(\gamma^{-1}\int_{\mathbb{R}^d} \mathrm{e}^{-\frac{\delta(p+1)|y|^2}{2(p-1)T}}\mathrm{d}y\right)^{\frac{1}{p-1}}}$$

for any $p>1$ and some constant $\gamma>0$ depending on $p, \kappa, \delta, \|b\|^2_{L^q_p(T)}$.

(ii) If in addition (1.9) holds, then

$$|P_T(\nabla_y f)(x)|^2 \le 2\beta(T,K,\delta,\kappa)\{P_T f^2(x) - (P_T f)^2(x)\}, \quad x, y \in \mathbb{R}^d, f \in C^1_b(\mathbb{R}^d).$$

Moreover, for any $p>1$, $x,y\in\mathbb{R}^d$ with $|y|^2 < \left(\frac{(2p+2)\beta(T,K,\delta,\kappa)}{(p-1)^2}\right)^{-1}$, $P_T(x,\cdot)$ is equivalent to $P_T(x,\cdot - y)$ and

$$P_T\left\{\left(\frac{\mathrm{d}P_T(x,\cdot)}{\mathrm{d}P_T(x,\cdot-y)}\right)^{\frac{1}{p-1}}\right\}(x) \le \left(1-\frac{(2p+2)\beta(T,K,\delta,\kappa)|y|^2}{(p-1)^2}\right)^{-\frac{1}{2}}.$$

3 Proof of Theorem 2.1

We use the coupling by change of measure to derive the Harnack inequality.

Proof of Theorem 2.1. For any $x\in\mathbb{R}^d$, let X_t^x solve (1.7) with $X_0 = x$, and Y_t solve the equation

$$\mathrm{d}Y_t = b_t(X_t^x)\mathrm{d}t + \sigma_t \mathrm{d}W_t + \frac{x-y}{T}\mathrm{d}t \tag{3.1}$$

with $Y_0 = y$. Then we have

$$Y_s = X_s^x + \frac{(s-T)(x-y)}{T}, \quad s\in[0,T]. \tag{3.2}$$

In particular, $X_T^x = Y_T$. Set

$$R(s) = \exp\left[-\int_0^s \langle \sigma_u^*(\sigma_u\sigma_u^*)^{-1}\Phi(u), \mathrm{d}W_u\rangle - \frac{1}{2}\int_0^s |\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\Phi(u)|^2\mathrm{d}u\right],$$

and

$$\bar{W}_s = W_s + \int_0^s \sigma_u^*(\sigma_u\sigma_u^*)^{-1}\Phi(u)\mathrm{d}u,$$

where

$$\Phi(s) = b_s(X_s^x) - b_s(Y_s) + \frac{x-y}{T}.$$

By Lemma 1.2 for X_t^x and $\alpha = p/2$, $\beta = q/2$, (1.8) and (3.2) imply that

$$\begin{aligned}\mathbb{E}\int_0^T |b_s(X_s^x) - b_s(Y_s)|^2 \,\mathrm{d}s &= \mathbb{E}\int_0^T \left|b_s(X_s^x) - b_s\left(X_s^x + \frac{(s-T)(x-y)}{T}\right)\right|^2 \mathrm{d}s \\ &\le \kappa \left\{\int_0^T \left(\int_{\mathbb{R}^d} \left|b_s(z + \frac{(s-T)(x-y)}{T}) - b_s(z)\right|^p \mathrm{d}z\right)^{\frac{q}{p}} \mathrm{d}s\right\}^{\frac{2}{q}} \\ &\le 4\kappa \|b\|^2_{L^q_p(T)}.\end{aligned}$$

Then by (1.12) and **(H2)**, we have

$$\mathbb{E}\exp\left\{\int_0^T |\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\Phi(u)|^2 \mathrm{d}u\right\} < \infty.$$

By Girsanov's theorem, $\{\bar{W}_s\}_{s\in[0,T]}$ is a Brownian motion under $\mathbb{Q}_T = R(T)\mathbb{P}$. Then (3.1) reduces to

$$\mathrm{d}Y_t = b_t(Y_t)\mathrm{d}t + \sigma_t \mathrm{d}\bar{W}_t, \tag{3.3}$$

which together with the weak uniqueness of (1.7) implies the distribution of Y_T under $\mathbb{Q}_T$ coincides with the one of X_T^y under $\mathbb{P}$. Thus, from (3.2), (1.9), and (1.11) with $\alpha = p/2$, $\beta = q/2$, it holds that

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}_T}\int_0^T |\Phi(s)|^2 \mathrm{d}s \\ &\le 2\mathbb{E}^{\mathbb{Q}_T}\int_0^T \left(\frac{|x-y|^2}{T^2} + |b_s(X_s^x) - b_s(Y_s)|^2\right) \mathrm{d}s \\ &= 2\frac{|x-y|^2}{T} + 2\mathbb{E}^{\mathbb{Q}_T}\int_0^T |b_s(X_s^x) - b_s(Y_s)|^2 \,\mathrm{d}s \\ &= 2\frac{|x-y|^2}{T} + 2\mathbb{E}^{\mathbb{Q}_T}\int_0^T \left|b_s(Y_s) - b_s\left(Y_s + \frac{(T-s)(x-y)}{T}\right)\right|^2 \mathrm{d}s \\ &\le 2\frac{|x-y|^2}{T} + 2\kappa\left\{\int_0^T \left(\int_{\mathbb{R}^d}\left|b_s(z + \frac{(T-s)(x-y)}{T}) - b_s(z)\right|^p \mathrm{d}z\right)^{\frac{q}{p}} \mathrm{d}s\right\}^{\frac{2}{q}} \\ &\le 2\frac{|x-y|^2}{T} + 2\kappa\left\{\int_0^T K(s)^q \mathrm{d}s\right\}^{\frac{2}{q}} |x-y|^2 \\ &\le 2\left(\frac{1}{T} + \kappa\|K\|^2_{L^q([0,T])}\right)|x-y|^2.\end{aligned} \tag{3.4}$$

By Hölder inequality, we have

$$P_T f(y) = \mathbb{E}^{\mathbb{Q}_T} f(Y_T) = \mathbb{E}^{\mathbb{Q}_T} f(X_T^x) \le (P_T f^p(x))^{\frac{1}{p}} \{\mathbb{E}R(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}}. \tag{3.5}$$

Combining (3.4) and **(H2)**, it follows from Hölder inequality and (1.13) that

$$\left(\mathbb{E}R(T)^{\frac{p}{p-1}}\right)^2 \le \mathbb{E}^{\mathbb{Q}_T}\exp\left\{\frac{p+1}{(p-1)^2}\int_0^T |(\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\Phi(u)|^2\mathrm{d}u\right\}$$
$$\le \frac{1}{1-\frac{(2p+2)\beta(T,K,\delta,\kappa)|x-y|^2}{(p-1)^2}}$$

if $|x-y|^2 < \left(\frac{(2p+2)\beta(T,K,\delta,\kappa)}{(p-1)^2}\right)^{-1}$. Substituting this into (3.5), we complete the proof. □

4 Proof of Theorem 2.3

Proof of Theorem 2.3. For any $x \in \mathbb{R}^d$, let X_t^x solve (1.7) with $X_0 = x$, and $\tilde{Y}_t$ solve the equation

$$\mathrm{d}\tilde{Y}_t = b_t(X_t^x)\mathrm{d}t + \sigma_t\mathrm{d}W_t + \frac{y}{T}\mathrm{d}t \tag{4.1}$$

with $\tilde{Y}_0 = x$. Then we have

$$\tilde{Y}_s = X_s^x + \frac{s}{T}y, \quad s \in [0,T]. \tag{4.2}$$

In particular, $X_T^x + y = \tilde{Y}_T$. Let

$$\tilde{R}(s) = \exp\left[-\int_0^s \langle\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u), \mathrm{d}W_u\rangle - \frac{1}{2}\int_0^s |\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)|^2\mathrm{d}u\right],$$

and

$$\tilde{W}_s = W_s + \int_0^s \sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)\mathrm{d}u,$$

where

$$\tilde{\Phi}(s) = b_s(X_s^x) - b_s(\tilde{Y}_s) + \frac{y}{T}.$$

Again by Lemma 1.2 for X_t^x and $\alpha = p/2$, $\beta = q/2$, it follows from (1.8) and (4.2) that

$$\mathbb{E}\int_0^T \left|b_s(X_s^x) - b_s(\tilde{Y}_s)\right|^2\mathrm{d}s$$
$$= \mathbb{E}\int_0^T \left|b_s(X_s^x) - b_s\left(X_s^x + \frac{s}{T}y\right)\right|^2\mathrm{d}s$$
$$\le \kappa\left\{\int_0^T\left(\int_{\mathbb{R}^d}\left|b_s(z+\frac{s}{T}y) - b_s(z)\right|^p\mathrm{d}z\right)^{\frac{q}{p}}\mathrm{d}s\right\}^{\frac{2}{q}}$$
$$\le \kappa\left\{\int_0^T\left(\left(\int_{\mathbb{R}^d}\left|b_s(z+\frac{s}{T}y)\right|^p\mathrm{d}z\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^d}|b_s(z)|^p\mathrm{d}z\right)^{\frac{1}{p}}\right)^q\mathrm{d}s\right\}^{\frac{2}{q}}$$

$$\leq 4\kappa\|b\|^2_{L^q_p(T)}.$$

Then by (1.12) and **(H2)**, we have

$$\mathbb{E}\exp\left\{\int_0^T |\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)|^2 \mathrm{d}u\right\} < \infty.$$

Applying Girsanov's theorem, we obtain that $\{\tilde{W}_s\}_{s\in[0,T]}$ is a Brownian motion under $\tilde{\mathbb{Q}}_T = \tilde{R}(T)\mathbb{P}$. Then (4.1) reduces to

$$\mathrm{d}\tilde{Y}_t = b_t(\tilde{Y}_t)\mathrm{d}t + \sigma_t \mathrm{d}\tilde{W}_t, \tag{4.3}$$

and this together with the weak uniqueness of (1.7) yields the distribution of $\tilde{Y}_T$ under $\tilde{\mathbb{Q}}_T$ coincides with the one of X_T^y under $\mathbb{P}$. By Young's inequality,

$$P_T \log f(x) = \mathbb{E}^{\tilde{\mathbb{Q}}_T} \log f(\tilde{Y}_T) = \mathbb{E}^{\tilde{\mathbb{Q}}_T} \log f(X_T^x + y) \leq \log P_T f(y+\cdot)(x) + \mathbb{E}\tilde{R}(T)\log\tilde{R}(T),$$

and by Hölder inequality,

$$P_T f(x) = \mathbb{E}^{\tilde{\mathbb{Q}}_T} f(\tilde{Y}_T) = \mathbb{E}^{\tilde{\mathbb{Q}}_T} f(X_T^x + y) \leq (P_T f^p(y+\cdot))^{\frac{1}{p}}(x)\{\mathbb{E}\tilde{R}(T)^{\frac{p}{p-1}}\}^{\frac{p-1}{p}}.$$

(i) (1.8), (1.11) yield that

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T |\tilde{\Phi}(s)|^2\mathrm{d}s &\leq 2\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T \left(\frac{|y|^2}{T^2} + \left|b_s(X_s^x) - b_s(\tilde{Y}_s)\right|^2\right)\mathrm{d}s \\
&= 2\frac{|y|^2}{T} + 2\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T \left(\left|b_s(X_s^x) - b_s(\tilde{Y}_s)\right|^2\right)\mathrm{d}s \\
&= 2\frac{|y|^2}{T} + 2\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T \left|b_s(\tilde{Y}_s) - b_s\left(\tilde{Y}_s + \frac{s}{T}y\right)\right|^2\mathrm{d}s \\
&\leq 2\frac{|y|^2}{T} + 8\kappa\|b\|^2_{L^q_p(T)}.
\end{aligned} \tag{4.4}$$

Combining (4.4) and **(H2)**, we arrive at

$$\begin{aligned}
\mathbb{E}\tilde{R}(T)\log\tilde{R}(T) = \mathbb{E}^{\tilde{\mathbb{Q}}_T}\log\tilde{R}(T) =& \frac{1}{2}\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T |(\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)|^2\mathrm{d}u \\
&\leq \delta\left(\frac{|y|^2}{T} + 4\kappa\|b\|^2_{L^q_p(T)}\right).
\end{aligned}$$

It follows from Hölder inequality and (1.12) that

$$\begin{aligned}
\left(\mathbb{E}\tilde{R}(T)^{\frac{p}{p-1}}\right)^2 &\leq \mathbb{E}^{\tilde{\mathbb{Q}}_T}\exp\left\{\frac{p+1}{(p-1)^2}\int_0^T |(\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)|^2\mathrm{d}u\right\} \\
&\leq \gamma \mathrm{e}^{\frac{\delta(p+1)|y|^2}{T(p-1)^2}}
\end{aligned}$$

with γ depending on $\kappa, \delta, \|b\|^2_{L^q_p(T)}$. Thus, we finish the proof of (i).

(ii) If moreover (1.9) holds, then (1.9) and (1.11) yield that

$$
\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T |\tilde{\Phi}(s)|^2 \mathrm{d}s &\le 2\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T \left(\frac{|y|^2}{T^2} + \left|b_s(X_s^x) - b_s(\tilde{Y}_s)\right|^2\right)\mathrm{d}s \\
&= 2\frac{|y|^2}{T} + 2\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T \left(\left|b_s(X_s^x) - b_s(\tilde{Y}_s)\right|^2\right)\mathrm{d}s \\
&= 2\frac{|y|^2}{T} + 2\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T \left|b_s(\tilde{Y}_s) - b_s\left(\tilde{Y}_s + \frac{s}{T}y\right)\right|^2 \mathrm{d}s \\
&\le 2\frac{|y|^2}{T} + 2\kappa\left\{\int_0^T\left(\int_{\mathbb{R}^d}\left|b_s(z+\frac{s}{T}y) - b_s(z)\right|^p \mathrm{d}z\right)^{\frac{q}{p}}\mathrm{d}s\right\}^{\frac{2}{q}} \\
&\le 2\frac{|y|^2}{T} + 2\kappa\left\{\int_0^T K(s)^{2q}\mathrm{d}s\right\}^{\frac{1}{q}}|y|^2 \\
&\le 2\left(\frac{1}{T} + \kappa\|K\|^2_{L^q([0,T])}\right)|y|^2.
\end{aligned}
\tag{4.5}
$$

Combining (4.5) and **(H2)**, we arrive at

$$
\begin{aligned}
\mathbb{E}\tilde{R}(T)\log\tilde{R}(T) = \mathbb{E}^{\tilde{\mathbb{Q}}_T}\log\tilde{R}(T) =& \frac{1}{2}\mathbb{E}^{\tilde{\mathbb{Q}}_T}\int_0^T |(\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)|^2\mathrm{d}u \\
&\le \delta\left(\frac{1}{T} + \kappa\|K\|^2_{L^q([0,T])}\right)|y|^2.
\end{aligned}
$$

It follows from Hölder inequality and (1.13) that

$$
\begin{aligned}
\left(\mathbb{E}\tilde{R}(T)^{\frac{p}{p-1}}\right)^2 &\le \mathbb{E}^{\tilde{\mathbb{Q}}_T}\exp\left\{\frac{p+1}{(p-1)^2}\int_0^T |(\sigma_u^*(\sigma_u\sigma_u^*)^{-1}\tilde{\Phi}(u)|^2\mathrm{d}u\right\} \\
&\le \frac{1}{1 - \frac{(2p+2)\beta(T,K,\delta,\kappa)|y|^2}{(p-1)^2}}
\end{aligned}
$$

if $|y|^2 < \left(\frac{(2p+2)\beta(T,K,\delta,\kappa)}{(p-1)^2}\right)^{-1}$. Thus, the proof is completed. □

Remark 4.1. *In fact, from the construction of the coupling by change of measure, we only use the weak existence and uniqueness of* (1.7)*. Thus, we may replace* (1.8) *by some weaker integrable condition that ensures the weak existence and uniqueness of* (1.7)*.*

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