PIECEWISE SMOOTH SEGMENTATION WITH SPARSE PRIOR*

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ABSTRACT

Exploiting sparsity in the image gradient magnitude has proved effective in preserving sharp edges and reducing noises for many image processing tasks. Based on observation, we build up a novel piecewise smooth segmentation model by utilizing a generalized total variation (TV) prior with p-th power for $0 < p < 1$ and a $\ell^1$ data fidelity. We present an efficient algorithm based on the alternating direction method of multipliers (ADMM), where all subproblems can be solved by either one-step Gauss-Seidel iteration or the closed-form solution. Numerical experiments show that the proposed model can achieve more accurate segmentation results than the classical TV based segmentation model.

Index Terms—Image segmentation, intensity inhomogeneity, total variation regularization, $\ell^p$ minimization, ADMM

1. INTRODUCTION

Image segmentation is an important task in image processing, which decomposes the image domain into local regions according to the features such as intensities, edges, colors and so on. Intensity inhomogeneity is a commonly seen artifact in natural image and medical image due to the spatial variation in illuminations and imperfection of imaging devices. The existence of intensity inhomogeneity makes the segmentation algorithms that relying on the assumption of uniform intensity impossible to identify the regions correctly.

Mumford and Shah [1] proposed the most fundamental region-based model for image segmentation, which can deal with intensity inhomogeneous images. Let $\Omega \subset \mathbb{R}^2$ be open and bounded, and $\Gamma$ be a closed subset in $\Omega$. Given an observed image $I : \Omega \to \mathbb{R}$, the MS model finds its piecewise smooth approximation $u$ by solving the following minimization problem

$$\min_{u,\Gamma} \lambda \int_{\Omega} (I - u)^2 dx + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + |\Gamma|,$$

where $\lambda$ and $\beta$ are positive parameters and $|\Gamma|$ denotes the length of $\Gamma$. Because the Mumford-Shah functional is non-convex and the integral regions of last two terms are discontinuous, finding the minimizers is not straightforward and may trap into local minima.

Many variants have been proposed for efficient implementation of the MS model. Chan and Vese [2] sought for a binary approximation of the given image through a level set formulation [3], which has also been extended for multi-phase segmentation in [4]. Lie et al. [5] proposed the piecewise constant level set method to identify curves separating regions into different phases. Chan et al. [6] reformulated the CV model into an equivalent convex minimization, which makes convex optimization techniques be applicable for image segmentation problems. Similar idea has been utilized for multi-phase segmentation in [7, 8]. Because such piecewise constant models approximate the image domain by a set of homogeneous regions, they fail in segmenting images with intensity inhomogeneity.

Piecewise smooth (PS) segmentation models perform better for intensity inhomogeneous images. Le and Vese [9] proposed a piecewise smooth segmentation model by expressing the true intensity as a summation of a piecewise constant component and a smooth component, i.e.,

$$g = \sum_{i=1}^{N} c_i \chi_i + b,$$

where $b : \Omega \to \mathbb{R}$ is a spatially smooth function modeling the intensity inhomogeneity, $\chi_i$ is the characteristic function of set $\Omega_i$, and $\{c_i\}_{i=1}^{N}$ are constant values representing the mean value of intensity inside $\{\Omega_i\}_{i=1}^{N}$. Based on (2), Jung [10] used the $\ell^1$ data fidelity to reformulate the energy functional for better segmentation of images with low contrast or outliers. Indeed, the relationship between the piecewise constant function and the smooth function may be modelled in a multiplicative intrinsic [11, 12].

The aforementioned methods are all based on the total variation (TV) semi-norm, which is defined as the $\ell^1$ norm of image gradient magnitudes. It is well-known the $\ell^1$ norm promotes sparsity in its arguments by preserving the edges of the images as well as eliminating the noises. Indeed, a better choice to enhance the sparsity is to employ the $\ell^p$ quasi-norm for $0 < p < 1$, which has already been used for image reconstruction [13], image reconstruction [14] etc. As only the boundaries between adjacent regions need to be measured for segmentation tasks, we propose a novel PS segmentation model by minimizing the $\ell^p$ norm of image gradient magnitudes, the so-called $\text{TV}_p$ norm. We adopt the $\ell^1$ data fidelity as it was proven efficient in dealing with images with low contrast.
We use the variable splitting technique to separate the $b$ for given $p$ and $c$, and
\[
\min_p \sum_{i=1}^N \left( \lambda \langle |p_i|, R_i \rangle + \frac{\tau}{2} \|p_i - (I - c_i - b) - \frac{\eta_i}{\tau} \|_2^2 \right),
\]
for given $p$ and $c$.

The optimality condition of (6) gives us the following closed-form solution
\[
c_i = \int_{\Omega} I - b - p_i + \frac{\eta_i}{\tau} dx, \quad \text{for } i = 1, \ldots, N.
\]

For (7), the Euler-Lagrange equation gives a linear equation
\[
(\tau N - 2\mu \Delta) b = \tau \sum_{i=1}^N (I - c_i - p_i + \frac{\eta_i}{\tau}),
\]
which can be solved by the discrete Fast Fourier transform (FFT) under the periodic boundary condition, i.e.,
\[
b = F^{-1} \left( \frac{F(\tau \sum_{i=1}^N (I - c_i - p_i + \frac{\eta_i}{\tau}))}{(\tau N - 2\mu \Delta)} \right).
\]

For (8), after the elementary calculations of the optimality condition, we can obtain a closed-form solution as
\[
p_i = \text{shrink}(I - c_i - b + \frac{\eta_i}{\tau}, \frac{\lambda R_i}{\tau}),
\]
where $\text{shrink}(s, t) = \frac{s}{|s|} \max(|s| - t, 0)$.

### 2.2. Sub-minimization problem w.r.t. $u$

The subproblem of $u$ is a constrained minimization problem as follows
\[
\min_u \sum_{i=1}^N \langle f_i, R_i \rangle + \sum_{l=1}^m \|\nabla u_l\|_2^2, \quad u_l \in [0, 1],
\]
where $f_i = |f - c_i - b|$. The energy functional in (13) has multi-variables $u_1, \ldots, u_m$, which can be solved individually. Thus, we compute each $u_l$ by minimizing the functional with respect to $u_l$ while the others are fixed
\[
\min_{u_l \in [0, 1]} \lambda \langle r_l, u_l \rangle + \|\nabla u_l\|_2^2, \quad \text{for } l = 1, \ldots, m,
\]
where $r_1 = r_2$ for $N = 2$ and $r_1 = (f_1 - f_3)u_2 + (f_2 - f_4)(1 - u_2)$, $r_2 = (f_1 - f_2)u_1 + (f_3 - f_4)(1 - u_1)$ for $N = 4$, and so forth.

For simplicity, we denote $r = r_l$ and $u = u_l$. By introducing a new variable $q$, the subproblem (14) becomes
\[
\min_{q \in [0, 1]} \lambda \langle r, u \rangle + \|q\|_2^2, \quad \text{s.t. } q = \nabla u.
\]
Based on the augmented Lagrangian method, we can reformulate the constrained optimization problem (15) into an equivalent saddle-point problem as follows
\[
\mathcal{L}_u(u, q; \xi) = \lambda \langle r, u \rangle + \|q\|_2^2 - \langle \xi, q - \nabla u \rangle + \gamma \|q - \nabla u\|_2^2,
\]
where $\xi$ is the Lagrange multiplier and $\gamma$ is a positive parameter. Similarly, two subproblems are solved instead of the saddle-point problem
\begin{equation}
\min_u \lambda(r,u) + \frac{\gamma}{2} \| q - \nabla u - \frac{\xi}{\gamma} \|^2,
\end{equation}
for a given $q$ and $\xi$, and
\begin{equation}
\min_q \| q \|^2 + \frac{\gamma}{2} \| q - \nabla u - \frac{\xi}{\gamma} \|^2,
\end{equation}
for a given $u$.

For (16), the optimality condition gives the following linear equation
\begin{equation}
\gamma \Delta u = \lambda r + \gamma \text{div}(q - \frac{\xi}{\gamma}),
\end{equation}
which can be efficiently solved by one-step Gauss-Seidel iteration. Denote $T_{i,j} = q_{i,j}^x + q_{i-1,j}^x + q_{i,j}^y + q_{i,j-1}^y - \frac{1}{\gamma} (s_{i,j}^x + s_{i-1,j}^x + s_{i,j}^y + s_{i,j-1}^y)$. The solution of $u$ is achieved by
\begin{equation}
u^{k+1}_{i,j} = \frac{1}{4} ((u^{k+1}_{i+1,j} + u^{k+1}_{i-1,j} + u^{k+1}_{i,j+1} + u^{k+1}_{i,j-1}) - T_{i,j} - \lambda r_{i,j}),
\end{equation}
with a projection step on $u^{k+1}$ to restrict its value to $[0,1]$.

In the end, we calculate (17) explicitly according to the following proposition.

**Proposition 2.1** ([13]) Let $0 < p < 1$. The solution to minimization problem (17) is given by
\begin{equation}
q = \eta^*(\nabla u + \frac{\xi}{\gamma}), \quad \text{where} \quad \eta^* \in [0,1],
\end{equation}
with
\begin{align*}
\eta^* = \begin{cases} 
0, & \text{if } \omega \leq \pi, \\
p(p-2) + \omega, & \text{if } \omega > \pi,
\end{cases}
\end{align*}
where we set
\begin{equation}
\omega := \gamma \| \nabla u + \frac{\xi}{\gamma} \|_2^{-p}, \quad \pi := \frac{2-p}{2(2-2p)}.
\end{equation}

We conclude this section with the alternating minimization algorithm for solving (4); see Algorithm I.

### Algorithm I: Alternating minimization algorithm for (4)

**Initialize:** choose $\lambda, \mu, \tau, \gamma > 0$, and let $p_i^0 = \eta_i^0 = 0$ for $i = 1, \ldots, N$; $q_i^0 = \xi_i^0 = 0$, for $l = 1, \ldots, m$; $u_l^0 = 1$ in some reign, $u_l^0 = 0$ otherwise.

**Iterate for** $k = 0, 1, 2, \ldots$.
- With $p^k$, $b^{k+1}$ and $\eta^k$, solve $c^{k+1}$ from (9);
- With $p^k$, $b^{k+1}$ and $\eta^k$, solve $b^{k+1}$ from (11);
- With $u^k$, $c^{k+1}$, $b^{k+1}$ and $\eta^k$, solve $p^{k+1}$ from (12);
- With $\eta^k$ and $\xi^k$, solve $u^{k+1}$ from (19);
- With $u^{k+1}$ and $\xi^k$, solve $q^{k+1}$ from (20);
- Update Lagrange multiplier $\eta$ and $\xi$ from
\begin{align*}
\eta^{k+1} &= \eta^k - \tau (p^{k+1} - (I - c^{k+1} - b^{k+1})); \\
\xi^{k+1} &= \xi^k - \gamma (q^{k+1} - \nabla u^{k+1}).
\end{align*}

### 3. NUMERICAL EXPERIMENTS

This section presents the numerical results of the proposed PS model of TV-$p$ regularization and compare the results with the PS model of TV regularization [10]1, which are denoted as TV$p$ model and TV model, respectively. We give the following two remarks before illustrating our results:

1. We use the Jaccard Similarity (JS) to quantitatively evaluate the segmentation accuracy, which is defined as
\begin{equation}
JS(S_1, S_2) = \frac{|S_1 \cap S_2|}{|S_1 \cup S_2|},
\end{equation}
where $S_1$ is the region segmented by the algorithm and $S_2$ is the corresponding region in the ground truth.

2. Both algorithms are terminated when either the maximum iteration number exceeds $K = 1000$, or the relative error is reached as
\begin{equation}
\frac{\| e^{k+1} - e^k \|_2}{\| e^{k+1} \|_2} < \epsilon \quad \text{with} \quad \epsilon = 10^{-6}.
\end{equation}

#### 3.1. Two-phase segmentation

Firstly, we test our model on two grayscale images. As shown in Fig. 1 and Fig. 2, different intensity inhomogeneities from small to large are introduced into the test images. We set $p$ as $p = 1/3, p = 1/2, p = 2/3$ in the TV$p$ model. We can observe that both TV$p$ model and TV model can well segment the images when the intensity inhomogeneity is small. As the degree of nonuniformity increases, the superiority of the TV$p$ model becomes more pronounced, which can preserve corners and curves of the images. Meanwhile, we tabulate the JS values of both TV$p$ model and TV model in Table 1, the values of which correspond to the visual results. Better JS values can be achieved by TV$p$ model and JS values are slightly improved as the sparsity increases. Thus, we can select $p$ as $p = 1/3$ in the applications.

#### Table 1. JS of synthetic images.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Fig. 1</th>
<th>Fig. 2</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>1st row</td>
<td>2nd row</td>
</tr>
<tr>
<td>$p = 1/3$</td>
<td>100</td>
<td>99.98</td>
</tr>
<tr>
<td>$p = 1/2$</td>
<td>100</td>
<td>99.98</td>
</tr>
<tr>
<td>$p = 2/3$</td>
<td>100</td>
<td>99.98</td>
</tr>
<tr>
<td>TV model</td>
<td>100</td>
<td>99.98</td>
</tr>
</tbody>
</table>

Secondly, we present an example of our TV$p$ model on noisy image. Fig. 3 illustrates that the proposed model can achieve better or comparable segmentation results as TV model when the original images are corrupted by both noises and intensity inhomogeneity.

#### 3.2. Multi-phase segmentation

In this subsection, we test our model on two brain images, both of which have slowly varying intensities inside the brain. As there mainly exists three different tissues, i.e., white matter (WM), gray matter (GM) and cerebrospinal fluid (CSF), we implement a four phase segmentation model. In Fig. 4 and Fig. 5, we present the original image, the initialization of

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1 The authors would like to thank Prof. Miyoun Jung from Hankuk University of Foreign Studies for providing us with MATLAB code of L1 model [10].
Fig. 1. From left to right: the original image, the segmentation results of TVp model with $p=1/3$, $1/2$, $2/3$ and TV model.

Fig. 2. From left to right: the original image, the segmentation results of TVp model with $p=1/3$, $1/2$, $2/3$ and TV model.

Fig. 3. From left to right: the original image, the segmentation results of TVp model with $p=1/3$, $1/2$, $2/3$ and TV model.

level set functions, the piecewise smooth image $g$, the final membership functions $u_1$, $u_2$, and the segmentation results of WM, GM, CSF generated by $u_1$ and $u_2$.

As shown by 4, the TV model struggles to segment the left part of the brain, while our model can achieve a better segmentation of WM. Simultaneously, the JS values in Table 2 demonstrate that the segmentation result of our TVp model is more accurate than TV model. In Fig.5, obvious inhomogeneous intensities can be observed in the top part of the brain. The comparison shows that TV model classified the top part as GM while our TVp model correctly identified the boundary between WM and GM. Compared to the TV model, much better JS values are obtained by our model. Both examples demonstrate that TV$_p$ regularization functions better than TV regularization for segmentation of image corrupted by intensity inhomogeneity.

Table 2. JS of brain MR images.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Fig. 4 WM</th>
<th>Fig. 4 GM</th>
<th>Fig. 4 CSF</th>
<th>Fig. 5 WM</th>
<th>Fig. 5 GM</th>
<th>Fig. 5 CSF</th>
</tr>
</thead>
<tbody>
<tr>
<td>TVp model</td>
<td>97.40</td>
<td>91.00</td>
<td>99.61</td>
<td>96.38</td>
<td>89.58</td>
<td>89.12</td>
</tr>
<tr>
<td>TV model</td>
<td>91.00</td>
<td>83.53</td>
<td>90.86</td>
<td>78.82</td>
<td>66.79</td>
<td>81.71</td>
</tr>
</tbody>
</table>

4. CONCLUSION

In this paper, we presented a novel piecewise smooth segmentation approach by modeling the true image as a sum of a piecewise constant function and a smooth function. The sparsity was enforced on both image gradient and data fidelity. An efficient minimization algorithm based on the ADMM strategy has been developed to solve the proposed model. Experiments have demonstrated its good performance.
5. REFERENCES


