Ramanujan-type Congruences for Overpartitions Modulo 5

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Abstract. Let $\overline{p}(n)$ denote the number of overpartitions of n. Hirschhorn and Sellers showed that $\overline{p}(4n + 3) \equiv 0 \pmod 8$ for $n \geq 0$. They also conjectured that $\overline{p}(40n + 35) \equiv 0 \pmod{40}$ for $n \geq 0$. Chen and Xia proved this conjecture by using the (p, k)-parametrization of theta functions given by Alaca, Alaca and Williams. In this paper, we show that $\overline{p}(5n) \equiv (-1)^n \overline{p}(4 \cdot 5n) \pmod 5$ for $n \geq 0$ and $\overline{p}(n) \equiv (-1)^n \overline{p}(4n) \pmod 8$ for $n \geq 0$ by using the relation of the generating function of $\overline{p}(5n)$ modulo 5 found by Treneer and the 2-adic expansion of the generating function of $\overline{p}(n)$ due to Mahlburg. As a consequence, we deduce that $\overline{p}(4^k(40n + 35)) \equiv 0 \pmod{40}$ for $n, k \geq 0$. Furthermore, applying the Hecke operator on $\phi(q)^3$ and the fact that $\phi(q)^3$ is a Hecke eigenform, we obtain an infinite family of congruences $\overline{p}(4^k \cdot 5\ell^2 n) \equiv 0 \pmod 5$, where $k \geq 0$ and ℓ is a prime such that $\ell \equiv 3 \pmod 5$ and $\left(\frac{-n}{\ell}\right) = -1$. Moreover, we show that $\overline{p}(5^2 n) \equiv \overline{p}(5^4 n) \pmod 5$ for $n \geq 0$. So we are led to the congruences $\overline{p}\left(4^k 5^{2i+3}(5n \pm 1)\right) \equiv 0 \pmod 5$ for $n, k, i \geq 0$. In this way, we obtain various Ramanujan-type congruences for $\overline{p}(n)$ modulo 5 such as $\overline{p}(45(3n + 1)) \equiv 0 \pmod 5$ and $\overline{p}(125(5n \pm 1)) \equiv 0 \pmod 5$ for $n \geq 0$.

Keywords: overpartition, Ramanujan-type congruence, modular form, Hecke operator, Hecke eigenform

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1   Introduction

The objective of this paper is to use half-integral weight modular forms to derive three infinite families of congruences for overpartitions modulo 5.

Recall that a partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n. An overpartition of n is a partition of n where the first occurrence of each distinct part may be overlined. We denote the number of overpartitions of n by $\overline{p}(n)$. We set $\overline{p}(0) = 1$ and $\overline{p}(n) = 0$ if $n < 0$. For example, there are eight overpartitions of 3

$$3, \ \bar{3}, \ 2+1, \ \bar{2}+1, \ 2+\bar{1}, \ \bar{2}+\bar{1}, \ 1+1+1, \ \bar{1}+1+1.$$

Overpartitions arise in combinatorics [6], q-series [5], symmetric functions [2], representation theory [11], mathematical physics [7, 8] and number theory [15, 16]. They are also called standard MacMahon diagrams, joint partitions, jagged partitions or dotted partitions.

Corteel and Lovejoy [6] showed that the generating function of $\bar{p}(n)$ is given by

$$\sum_{n\geq 0}\bar{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Recall that the generating function of $\bar{p}(n)$ can be expressed as

$$\sum_{n\geq 0}\bar{p}(n)q^n = \frac{1}{\phi(-q)},$$

where $\phi(q)$ is Ramanujan's theta function as defined by

$$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.1}$$

see Berndt [2].

On the other hand, the generating function of $\bar{p}(n)$ has the following 2-adic expansion

$$\sum_{n\geq 0}\bar{p}(n)q^n = 1 + \sum_{k=1}^{\infty} 2^k \sum_{n=1}^{\infty} (-1)^{n+k}c_k(n)q^n, \tag{1.2}$$

where $c_k(n)$ denotes the number of representations of n as a sum of k squares of positive integers. The above 2-adic expansion (1.2) is useful to derive congruences for $\bar{p}(n)$ modulo powers of 2, see, for example [12, 13, 18].

By employing dissection formulas, Fortin, Jacob and Mathieu [7], Hirschhorn and Sellers [9] independently derived various Ramanujan-type congruences for $\bar{p}(n)$, such as

$$\bar{p}(4n + 3) \equiv 0 \pmod{8}. \tag{1.3}$$

Hirschhorn and Sellers [9] proposed the following conjectures

$$\bar{p}(27n + 18) \equiv 0 \pmod{12}, \tag{1.4}$$

$$\bar{p}(40n + 35) \equiv 0 \pmod{40}. \tag{1.5}$$

They also conjectured that if ℓ is prime and r is a quadratic nonresidue modulo ℓ then

$$\bar{p}(\ell n + r) \equiv \begin{cases} 0 \pmod{8} & \text{if } \ell \equiv \pm 1 \pmod{8}, \\ 0 \pmod{4} & \text{if } \ell \equiv \pm 3 \pmod{8}. \end{cases} \tag{1.6}$$

By using the 3-dissection formula for $\phi(-q)$, Hirschhorn and Sellers [10] proved (1.4) and obtained a family of congruences

$$\overline{p}(9^{\alpha}(27n + 18)) \equiv 0 \pmod{12},$$

where $n, \alpha \geq 0$.

Employing the 2-dissection formulas of theta functions due to Ramanujan, Hirschhorn and Sellers [9], Chen and Xia [4] obtained a generating function of $\overline{p}(40n + 35)$ modulo 5. Using the (p, k)-parametrization of theta functions given by Alaca, Alaca and Williams [1], they showed that

$$\overline{p}(40n + 35) \equiv 0 \pmod{5}. \tag{1.7}$$

This proves Hirschhorn and Sellers' conjecture (1.5) by combining congruence (1.3). Applying the 2-adic expansion (1.2), Kim [13] proved (1.6) and obtained congruence properties of $\overline{p}(n)$ modulo 8.

For powers of 2, Mahlburg [18] showed that $\overline{p}(n) \equiv 0 \pmod{64}$ holds for a set of integers of arithmetic density 1. Kim [12] showed that $\overline{p}(n) \equiv 0 \pmod{128}$ holds for a set of integers of arithmetic density 1. For the modulus 3, by using the fact that $\phi(q)^5$ is a Hecke eigenform in the half-integral weight modular form space $M_{\frac{5}{2}}(\tilde{\Gamma}_0(4))$, Lovejoy and Osburn [17] proved that

$$\overline{p}(3\ell^3 n) \equiv 0 \pmod{3},$$

where $\ell \equiv 2 \pmod{3}$ is an odd prime and $\ell \nmid n$. Moreover, by utilizing half-integral weight modular forms, Treneer [21] showed that for a prime ℓ such that $\ell \equiv -1 \pmod{5}$,

$$\overline{p}(5\ell^3 n) \equiv 0 \pmod{5},$$

for all n coprime to ℓ.

In this paper, we establish the following two congruence relations for overpartitions modulo 5 and modulo 8 by using a relation of the generating function of $\overline{p}(5n)$ modulo 5 and applying the 2-adic expansion (1.2).

Theorem 1.1. *For $n \geq 0$, we have*

$$\overline{p}(5n) \equiv (-1)^n \overline{p}(4 \cdot 5n) \pmod{5}. \tag{1.8}$$

Theorem 1.2. *For $n \geq 0$, we have*

$$\overline{p}(n) \equiv (-1)^n \overline{p}(4n) \pmod{8}. \tag{1.9}$$

Combining the above two congruence relations with congruences (1.3) and (1.7), we arrive at a family of congruences modulo 40.

Corollary 1.3. *For $n, k \geq 0$, we have*

$$\overline{p}(4^k(40n + 35)) \equiv 0 \pmod{40}. \tag{1.10}$$

Based on the Hecke operator on $\phi(q)^3$ and the fact that $\phi(q)^3$ is a Hecke eigenform in $M_{\frac{3}{2}}(\tilde{\Gamma}_0(4))$, we obtain a family of congruences for overpartitions modulo 5.

Theorem 1.4. *Let $\left(\frac{\cdot}{\ell}\right)$ denote the Legendre symbol. Assume that k is a nonnegative integer and ℓ is a prime with $\ell \equiv 3 \pmod{5}$. Then we have*

$$\overline{p}(4^k \cdot 5\ell^2 n) \equiv 0 \pmod{5},$$

where n is a nonnegative integer such that $\left(\frac{-n}{\ell}\right) = -1$.

Using the properties of the Hecke operator $T_{\frac{3}{2},16}(\ell^2)$ and the Hecke eigenform $\phi(q)^3$, we are led to another congruence relation for overpartitions modulo 5.

Theorem 1.5. *For $n \geq 0$, we have*

$$\overline{p}(5^2 n) \equiv \overline{p}(5^4 n) \pmod{5}. \tag{1.11}$$

Combining (1.8) and (1.11), we find the following family of congruences modulo 5.

Corollary 1.6. *For $n, k, i \geq 0$, we have*

$$\overline{p}\big(4^k 5^{2i+3}(5n \pm 1)\big) \equiv 0 \pmod{5}. \tag{1.12}$$

2 Preliminaries

To make this paper self-contained, we recall some definitions and notation on half-integral weight modular forms. For more details, see [3, 14, 19–21].

Let k be an odd positive integer and N be a positive integer with $4|N$. We use $M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$ to denote the space of holomorphic modular forms on $\Gamma_0(N)$ of weight $\frac{k}{2}$.

Definition 2.1. *Let*

$$f(z) = \sum_{n \geq 0} a(n)q^n$$

be a modular form in $M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$. For any odd prime $\ell \nmid N$, the action of the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$ on $f(z) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$ is given by

$$f(z) \mid T_{\frac{k}{2},N}(\ell^2) = \sum_{n \geq 0} \left(a(\ell^2 n) + \left(\frac{(-1)^{\frac{k-1}{2}}n}{\ell} \right) \ell^{\frac{k-3}{2}} a(n) + \ell^{k-2} a\left(\frac{n}{\ell^2}\right) \right) q^n, \tag{2.1}$$

where $a(\frac{n}{\ell^2}) = 0$ if n is not divisible by ℓ^2.

The following proposition says that the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$ maps the modular form space $M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$ into itself.

Proposition 2.2. *Let ℓ be an odd prime and $f(z) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(N))$, then*

$$f(z) \mid T_{\frac{k}{2},N}(\ell^2) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(N)).$$

A Hecke eigenform associated with the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$ is defined as follows.

Definition 2.3. *A half-integral weight modular form $f(z) \in M_{\frac{k}{2}}(\tilde{\Gamma}_0(4N))$ is called a Hecke eigenform for the Hecke operator $T_{\frac{k}{2},N}(\ell^2)$, if for every prime $\ell \nmid 4N$ there exists a complex number $\lambda(\ell)$ for which*

$$f(z) \mid T_{\frac{k}{2},N}(\ell^2) = \lambda(\ell)f(z).$$

For the space of half-integral weight modular forms on $\Gamma_0(4)$, we have the following dimension formula.

Proposition 2.4. *We have*

$$\dim M_{\frac{k}{2}}(\tilde{\Gamma}_0(4)) = 1 + \left\lfloor \frac{k}{4} \right\rfloor.$$

By the above dimension formula, we see that $\dim M_{\frac{3}{2}}(\tilde{\Gamma}_0(4)) = 1$. From the fact that $\phi(q)^3 \in M_{\frac{3}{2}}(\tilde{\Gamma}_0(4))$, it is easy to deduce that

$$\phi(q)^3 \mid T_{\frac{3}{2},4}(\ell^2) = (\ell+1)\phi(q)^3, \tag{2.2}$$

see, for example [21, P. 18].

3   Proofs of Theorem 1.1 and Theorem 1.2

In this section, we give proofs of Theorem 1.1 and Theorem 1.2 by using a relation of the generating function of $\bar{p}(5n)$ modulo 5 and the 2-adic expansion (1.2) of $\bar{p}(n)$.

Proof of Theorem 1.1. Recall the following 2-dissection formula for $\phi(q)$,

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{3.1}$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n^2+n}{2}},$$

see, for example, Hirschhorn and Sellers [9]. Replacing q by $-q$, (3.1) becomes

$$\phi(-q) = \phi(q^4) - 2q\psi(q^8). \tag{3.2}$$

We now consider the generating function of $\bar{p}(5n)$ modulo 5. The following relation is due to Treneer [21, p. 18],

$$\sum_{n \geq 0} \bar{p}(5n)q^n \equiv \phi(-q)^3 \pmod{5}. \tag{3.3}$$

Plugging (3.2) into (3.3) yields that

$$\sum_{n \geq 0} \bar{p}(5n)q^n \equiv \phi(q^4)^3 - q\phi(q^4)^2\psi(q^8) + 2q^2\phi(q^4)\psi(q^8)^2 - 3q^3\psi(q^8)^3 \pmod{5}. \tag{3.4}$$

Extracting the terms of q^{4n+i} for $i = 0, 1, 2, 3$ on both sides of (3.4) and setting q^4 to q, we obtain

$$\sum_{n \geq 0} \bar{p}(20n)q^n \equiv \phi(q)^3 \pmod{5}, \tag{3.5}$$

$$\sum_{n \geq 0} \bar{p}(20n+5)q^n \equiv -\phi(q)^2\psi(q^2) \pmod{5}, \tag{3.6}$$

$$\sum_{n \geq 0} \bar{p}(20n+10)q^n \equiv 2\phi(q)\psi(q^2)^2 \pmod{5}, \tag{3.7}$$

$$\sum_{n \geq 0} \bar{p}(20n+15)q^n \equiv -3\psi(q^2)^3 \pmod{5}. \tag{3.8}$$

Substituting the 2-dissection formula (3.1) into (3.5), we find that

$$\sum_{n \geq 0} \bar{p}(20n)q^n \equiv \phi(q^4)^3 + q\phi(q^4)^2\psi(q^8) + 2q^2\phi(q^4)\psi(q^8)^2 + 3q^3\psi(q^8)^3 \pmod{5}. \tag{3.9}$$

Extracting the terms of q^{4n+i} for $i = 0, 1, 2, 3$ on both sides of (3.9) and setting q^4 to q, we obtain

$$\sum_{n \geq 0} \bar{p}(4 \cdot 20n) \equiv \phi(q)^3 \pmod{5}, \tag{3.10}$$

$$\sum_{n \geq 0} \bar{p}(4 \cdot (20n+5)) \equiv \phi(q)^2\psi(q^2) \pmod{5}, \tag{3.11}$$

$$\sum_{n \geq 0} \bar{p}(4 \cdot (20n+10)) \equiv 2\phi(q)\psi(q^2)^2 \pmod{5}, \tag{3.12}$$

$$\sum_{n \geq 0} \bar{p}(4 \cdot (20n+15)) \equiv 3\psi(q^2)^3 \pmod{5}. \tag{3.13}$$

Comparing the equations (3.5)–(3.8) with (3.10)–(3.13), we deduce that

$$\overline{p}(5 \cdot (4n)) \equiv \overline{p}(4 \cdot 5 \cdot 4n) \pmod 5,$$

$$\overline{p}(5 \cdot (4n+1)) \equiv -\overline{p}(4 \cdot 5 \cdot (4n+1)) \pmod 5,$$

$$\overline{p}(5 \cdot (4n+2)) \equiv \overline{p}(4 \cdot 5 \cdot (4n+2)) \pmod 5,$$

$$\overline{p}(5 \cdot (4n+3)) \equiv -\overline{p}(4 \cdot 5 \cdot (4n+3)) \pmod 5.$$

So we conclude that

$$\overline{p}(5n) \equiv (-1)^n \overline{p}(4 \cdot 5n) \pmod 5.$$

This completes the proof. ∎

We note that extracting the terms of odd powers of q on both sides of (3.8) leads to the congruence $\overline{p}(40n + 35) \equiv 0 \pmod 5$ due to Chen and Xia [4].

Next, we prove Theorem 1.2 by using the 2-adic expansion (1.2). Recall that $c_k(n)$ in (1.2) denotes the number of representations of n as a sum of k squares of positive integers. In particular, $c_1(n) = 1$ if n is a square; otherwise, $c_1(n) = 0$.

Proof of Theorem 1.2. It follows from (1.2) that

$$\overline{p}(n) \equiv (-1)^n \left(-2c_1(n) + 4c_2(n)\right) \pmod 8, \tag{3.14}$$

where $n \geq 1$. Replacing n by $4n$ in (3.14), we get

$$\overline{p}(4n) \equiv -2c_1(4n) + 4c_2(4n) \pmod 8. \tag{3.15}$$

Since $c_1(n) = c_1(4n)$ and $c_2(n) = c_2(4n)$, (3.15) can be rewritten as

$$\overline{p}(4n) \equiv -2c_1(n) + 4c_2(n) \pmod 8. \tag{3.16}$$

Substituting (3.16) into (3.14), we arrive at

$$\overline{p}(n) \equiv (-1)^n \overline{p}(4n) \pmod 8,$$

as claimed. ∎

It is easy to see that Corollary 1.3 can be obtained by iteratively applying Theorem 1.1 and Theorem 1.2 to the congruences $\overline{p}(40n + 35) \equiv 0 \pmod 5$ and $\overline{p}(40n + 35) \equiv 0 \pmod 8$ that can be deduced from congruence (1.3) by replacing n with $10n + 8$.

4   Proof of Theorem 1.4

In this section, we prove Theorem 1.4 by using the Hecke operator on $\phi(q)^3$ along with the fact that $\phi(q)^3$ is a Hecke eigenform in $M_{\frac{3}{2}}(\tilde{\Gamma}_0(4))$.

In view of Theorem 1.1, to prove Theorem 1.4, it suffices to consider the special case $k = 0$ that takes the following form.

Theorem 4.1. *Let ℓ be a prime with $\ell \equiv 3 \pmod 5$. Then*

$$\overline{p}(5\ell^2 n) \equiv 0 \pmod 5 \tag{4.1}$$

holds for any nonnegative integer n with $\left(\frac{-n}{\ell}\right) = -1$.

Proof. Recall that $\phi(-q)^3$ is a modular form in $M_{\frac{3}{2}}(\tilde{\Gamma}_0(16))$. Suppose that

$$\phi(-q)^3 = \sum_{n \geq 0} a(n)q^n \tag{4.2}$$

is the Fourier expansion of $\phi(-q)^3$.

Applying the Hecke operator $T_{\frac{3}{2},16}(\ell^2)$ to $\phi(-q)^3$ and using (2.1), we find that

$$\phi(-q)^3 \mid T_{\frac{3}{2},16}(\ell^2) = \sum_{n=0}^{\infty} \left(a(\ell^2 n) + \left(\frac{-n}{\ell}\right)a(n) + \ell a\left(\frac{n}{\ell^2}\right) \right) q^n, \tag{4.3}$$

where ℓ is an odd prime. Replacing q by $-q$ in (2.2), we see that $\phi(-q)^3$ is a Hecke eigenform in the space $M_{\frac{3}{2}}(\tilde{\Gamma}_0(16))$, and hence

$$\phi(-q)^3 \mid T_{\frac{3}{2},16}(\ell^2) = (\ell+1)\phi(-q)^3. \tag{4.4}$$

Comparing the coefficients of q^n in (4.3) and (4.4), we deduce that

$$a(\ell^2 n) + \left(\frac{-n}{\ell}\right)a(n) + \ell a\left(\frac{n}{\ell^2}\right) = (\ell+1)a(n). \tag{4.5}$$

Revoking the congruence (3.3), that is,

$$\phi(-q)^3 \equiv \sum_{n \geq 0} \overline{p}(5n)q^n \pmod 5, \tag{4.6}$$

and comparing (4.2) with (4.6), we get

$$a(n) \equiv \overline{p}(5n) \pmod 5. \tag{4.7}$$

Plugging (4.7) into (4.5), we deduce that

$$\overline{p}(5\ell^2 n) + \left(\frac{-n}{\ell}\right)\overline{p}(5n) + \ell\overline{p}\left(\frac{5n}{\ell^2}\right) \equiv (\ell+1)\overline{p}(5n) \pmod 5. \tag{4.8}$$

Since $\ell \equiv 3 \pmod 5$ and $\left(\frac{-n}{\ell}\right) = -1$, we see that $\ell \nmid 5$ and $\ell \nmid n$, so that $\ell^2 \nmid 5n$ and $\overline{p}\left(\frac{5n}{\ell^2}\right) = 0$. Moreover, we have $\left(\frac{-n}{\ell}\right) \equiv (\ell+1) \equiv -1 \pmod 5$. Hence congruence (4.8) becomes

$$\overline{p}(5\ell^2 n) \equiv 0 \pmod 5.$$

This completes the proof. ∎

8

We now give some special cases of Theorem 1.4. Setting $\ell = 3$ and $k = 0, 1$ in Theorem 1.4, respectively, we obtain the following congruences for $n \geq 0$,

$$\overline{p}\big(45(3n + 1)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(180(3n + 1)\big) \equiv 0 \pmod{5}.$$

Setting $\ell = 13$, $k = 0$ in Theorem 1.4, we obtain the following congruences for $n \geq 0$,

$$\overline{p}\big(845(13n + 2)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(845(13n + 5)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(845(13n + 6)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(845(13n + 7)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(845(13n + 8)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(845(13n + 11)\big) \equiv 0 \pmod{5}.$$

5 Proof of Theorem 1.5

In this section, we complete the proof of Theorem 1.5 by using the Hecke operator $T_{\frac{3}{2},16}(\ell^2)$ and the Hecke eigenform $\phi(-q)^3$.

Proof of Theorem 1.5. Setting $\ell = 5$ in the congruence relation (4.8), we find that

$$\overline{p}(5n) \equiv \overline{p}(5^3 n) + \left(\frac{n}{5}\right) \overline{p}(5n) \pmod{5}. \tag{5.1}$$

By the definition of the Legendre symbol, we see that if $n \equiv 0 \pmod{5}$, then $\left(\frac{n}{5}\right) = 0$. Hence, by replacing n with $5n$ in congruence (5.1), we obtain that

$$\overline{p}(5^2 n) \equiv \overline{p}(5^4 n) \pmod{5}, \tag{5.2}$$

as claimed. ∎

Furthermore, we note that if $n \equiv \pm 1 \pmod{5}$, then $\left(\frac{n}{5}\right) = 1$. Hence by setting n to $5n \pm 1$ in (5.1), we deduce that

$$\overline{p}(5^3(5n \pm 1)) \equiv 0 \pmod{5}. \tag{5.3}$$

By iteratively applying the congruence $\overline{p}(5n) \equiv (-1)^n \overline{p}(4 \cdot 5n) \pmod{5}$ given in Theorem 1.1 and congruence (5.2) to (5.3), we obtain that

$$\overline{p}(4^k 5^{2i+3}(5n \pm 1)) \equiv 0 \pmod{5}, \tag{5.4}$$

where $n, k, i \geq 0$. This proves Corollary 1.6.

For $n \geq 0$, setting $i = 0$ and $k = 0, 1$ in (5.4), we obtain the following special cases

$$\overline{p}\big(125(5n \pm 1)\big) \equiv 0 \pmod{5},$$

$$\overline{p}\big(500(5n \pm 1)\big) \equiv 0 \pmod{5}.$$

By replacing n by $5n \pm 2$ in (5.1) and iteratively using the congruence relation (5.2), we obtain the following relation.

Corollary 5.1. *For $n, i \geq 0$, we have*

$$\overline{p}\big(5(5n \pm 2)\big) \equiv 3\,\overline{p}\big(5^{2i+3}(5n \pm 2)\big) \pmod{5}.$$

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