An Algorithm for Deciding the Summability of Bivariate Rational Functions

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Abstract. Let $\Delta_x f(x,y) = f(x+1,y) - f(x,y)$ and $\Delta_y f(x,y) = f(x,y+1) - f(x,y)$ be the difference operators with respect to x and y. A rational function $f(x,y)$ is called summable if there exist rational functions $g(x,y)$ and $h(x,y)$ such that $f(x,y) = \Delta_x g(x,y) + \Delta_y h(x,y)$. Recently, Chen and Singer presented a method for deciding whether a rational function is summable. To implement their method in the sense of algorithms, we need to solve two problems. The first is to determine the shift equivalence of two bivariate polynomials. We solve this problem by presenting an algorithm for computing the dispersion sets of any two bivariate polynomials. The second is to solve a univariate difference equation in an algebraically closed field. By considering the irreducible factorization of the denominator of $f(x,y)$ in a general field, we present a new criterion which requires only finding a rational solution of a bivariate difference equation. We give a new estimation of the universal denominators based on the m-fold Gosper representation and transform the bivariate difference equation to a system of linear difference equations in one variable. Combining these algorithms, we can decide the summability of a bivariate rational function.

Keywords: summability, bivariate rational function, Gosper's algorithm, dispersion set.

1 Introduction

In 1971, Abramov [1] presented an algorithm to solve the indefinite rational summations. Then in 1978, Gosper [16] presented the celebrated algorithm which solves the problem of determining whether a given hypergeometric term is equal to the difference of another hypergeometric term. Based on Gosper's algorithm, Zeilberger [27,28] gave a fast algorithm for proving terminating hypergeometric identities. Zeilberger's method was further extended to the multivariate case by Wilf and Zeilberger himself in [26]. Paule [22] gave an interpretation of Gosper's algorithm in terms of the greatest factorial factorizations. Chen, Paule and Saad [15] derived an easy understanding version of Gosper's algorithm by considering the convergence of the greatest common divisors of two polynomial sequences.

Other approaches to the summability of rational functions were given by Abramov [2–4], Pirastu and Strehl [24], Ash and Catoiu [10]. The key idea of these methods is to rewrite a rational function α as $\alpha = \Delta(\beta) + \gamma$, where Δ is the difference operator, β and γ are rational functions such that the denominator of γ is shift-free. Then α is summable if and only if γ is zero.

Passing from univariate to multivariate, Zeilberger's algorithm has been discussed by Zeilberger himself [9, 21], Koutschan [20], Schneider [25], Barkatou [11], Chen et.al. [14]. These algorithms are useful in practice. However, they did not provide a complete answer to the summability problem of bivariate hypergeometric terms. Only very recently, Chen and Singer [13] presented criteria for deciding the summability of *bivariate rational functions*.

Let $f(x, y)$ be a rational function over the field $\mathbb{K}$. Chen and Singer considered the partial fraction decomposition of $f(x, y)$ in the field $\overline{\mathbb{K}(x)}(y)$. After merging the summands whose denominators are *shift equivalent*, they showed that $f(x, y)$ is summable if and only if each summand is summable. Moreover, they proved that the ratio $\frac{\alpha(x)}{(y-\beta(x))^j}$ is summable if and only if there exist integers s, t and $c \in \overline{\mathbb{K}}$ such that $\beta(x) = \frac{s}{t}x + c$ and there exists $\gamma(x) \in \overline{\mathbb{K}(x)}$ such that

$$\alpha(x) = \gamma(x + t) - \gamma(x). \tag{1.1}$$

We notice that when applying their criteria, one will encounter two problems. The first one is how to determine whether two bivariate polynomials are shift equivalent. The second one is how to solve the difference equation (1.1) in the field $\overline{\mathbb{K}(x)}$. The main aim of the present paper is to overcome these problems and give an algorithm for deciding the summability of bivariate rational functions. We remark that the general question considered in this paper was raised by Andrews and Paule in [8].

For the first problem, we show that the dispersion set of two bivariate polynomials is computable. Then two polynomials are shift equivalent if and only if the dispersion set is not empty. For the second problem, we present a variation of the criteria by considering the partial fraction decomposition in the field $\mathbb{K}(x)(y)$ instead of the filed $\overline{\mathbb{K}(x)}(y)$. To apply the new criteria, we need only to find rational solutions of a bivariate difference equation. By a discussion similar to Gosper's algorithm, we derive a universal denominator which is a factor of Abramov's universal denominator. Then we reduce the problem of finding rational solutions of the bivariate difference equation to the problem of finding polynomials solutions of a system of linear difference equations in one variable. Abramov and Bronstein have presented an algorithm on solving such systems [6]. Combining these algorithms, we finally obtain an algorithm for deciding the summability of bivariate rational functions.

The paper is organized as follows. In Section 2, we give an algorithm for computing the dispersion set of two bivariate polynomials. In Section 3, we first reduce the summability of a general rational function to that of a rational function whose denominator is a power of an irreducible polynomial. Then we present a criterion on the summability of this special kind of rational functions. This criterion reduces the summability problem to the problem of finding rational solutions of a bivariate difference equation. In Section 4, we give an algorithm for solving the bivariate difference equation. Finally, we present two examples to illustrate the algorithms in Section 5.

Throughout the paper, we take $\mathbb{Q}$, the field of rational numbers, as the ground field. It should be mentioned that the discussions work also for other fields, such as the extension field $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$ where $\alpha_1, \ldots, \alpha_r$ are either algebraic or transcendental over $\mathbb{Q}$.

We follow the notations used in [13]. Let $f(x, y) \in \mathbb{Q}(x, y)$ be a bivariate rational function. The shift operators σ_x and σ_y are given by

$$\sigma_x f(x, y) = f(x + 1, y) \quad \text{and} \quad \sigma_y f(x, y) = f(x, y + 1).$$

A function $f \in \mathbb{Q}(x, y)$ is said to be (σ_x, σ_y)-*summable* if there exist two rational functions $g, h \in \mathbb{Q}(x, y)$ such that

$$f = \sigma_x g - g + \sigma_y h - h.$$

The summability problem is closely related to the classical problem of the indefinite and definite summation. In practice, when dealing with double summations like $\sum_{n=0}^{x-1} \sum_{k=0}^{y-1} f(n, k)$, we can first test the summability of $f(n, k)$. Suppose $f = \sigma_n g - g + \sigma_k h - h$, then we can reduce the above summation into single summations which can be handled by the Gosper algorithm. If f is not summable, it's easy to prove that the double summation does not have a close form.

2 Dispersion set and shift equivalence

Let $\mathbb{Z}$ denote the set of integers. Recall that given two univariate polynomials, say $f(x)$ and $g(x)$, their *dispersion set* is defined by

$$\mathrm{Disp}_x(f, g) = \{n \in \mathbb{Z} \mid f(x) = g(x + n)\}.$$

It is known that unless f and g are the same constant polynomial, the dispersion set $\mathrm{Disp}_x(f, g)$ is finite and is computable. For the algorithm, see [23, page 79]. We can extend this concept to the bivariate case.

Definition 2.1. *Let f, g be two bivariate polynomials in $\mathbb{Q}[x, y]$ and σ_x, σ_y be the shift operators. The* dispersion set *of f and g is defined by*

$$\mathrm{Disp}(f, g) = \{(m, n) \in \mathbb{Z}^2 \mid f = \sigma_x^m \sigma_y^n g\}.$$

If $\mathrm{Disp}(f, g)$ *is not empty, we say f and g are* shift equivalent*.*

*In particular, when $f = \sigma_x^m g$ (resp. $f = \sigma_y^n g$), we say f, g in the same σ_x-*orbit *(resp. σ_y-*orbit*), denoted by $f \sim_x g$ and $f \sim_y g$ respectively.*

We remark that testing shift equivalence over fields have been considered by Grigoriev and Karpinski [17–19]. More precisely, they gave algorithms to find shifts $(\alpha_1, \ldots, \alpha_r) \in F^r$ such that

$$f(x_1 + \alpha_1, \ldots, x_r + \alpha_r) = g(x_1, \ldots, x_r),$$

where F is a field and $f, g \in F[x_1, \ldots, x_r]$. Instead of considering shifts over a field, we focus on integer shifts, i.e., $m, n \in \mathbb{Z}$.

In the univariate case, the dispersion set of any two polynomials is computable. The following theorem shows that the dispersion set is also computable in the bivariate case.

Theorem 2.2. *Let $f, g \in \mathbb{Q}[x, y]$ be two polynomials. Then we can determine the dispersion set* $\mathrm{Disp}(f, g)$*.*

Proof. Since the shift operators σ_x and σ_y preserve the degree, we get that $\mathrm{Disp}(f,g) = \emptyset$ unless $\deg_x f = \deg_x g$.

When $f = 0$ or $\deg_x(f) = 0$, the computation of $\mathrm{Disp}(f,g)$ reduces to the univariate case. More precisely, we have

$$\mathrm{Disp}(f,g) = \mathbb{Z} \times \mathrm{Disp}_y(f,g).$$

Now assume that $\deg_x f = d > 0$ and write f, g as

$$f = \sum_{k=0}^{d} a_k(y)x^k, \quad g = \sum_{k=0}^{d} b_k(y)x^k.$$

Suppose that $(m,n) \in \mathrm{Disp}(f,g)$. By comparing the leading coefficient with respect to x, we see that n falls in the dispersion set

$$\mathcal{N} = \mathrm{Disp}_y(a_d(y), b_d(y)).$$

If $\mathcal{N}$ is a finite set, we then have

$$\mathrm{Disp}(f,g) = \bigcup_{n_0 \in \mathcal{N}} \mathrm{Disp}_x(f(x,y), g(x, y+n_0)) \times \{n_0\}.$$

Otherwise, we may assume $a_d(y) = b_d(y) = c$, where c is a non-zero constant. By comparing the second leading coefficient with respect to x, we see that

$$a_{d-1}(y) = d \cdot c \cdot m + b_{d-1}(y+n). \tag{2.1}$$

According to the degree of $a_{d-1}(y)$ in variable y, there are three cases.

Case 1. $\deg a_{d-1}(y) > 1$. Then $\mathrm{Disp}(f,g) = \emptyset$ unless the leading term of $a_{d-1}(y)$ and that of $b_{d-1}(y)$ coincide. Assume

$$a_{d-1}(y) = \sum_{j=0}^{h} p_j y^j \quad \text{and} \quad b_{d-1}(y) = \sum_{j=0}^{h} q_j y^j.$$

By comparing the coefficients of y^{h-1} in the expansions of $a_{d-1}(y)$ and $b_{d-1}(y+n)$, we see that n is uniquely determined by

$$hq_h n + q_{h-1} = p_{h-1}. \tag{2.2}$$

Suppose that n_0 is an integer solution of (2.2). We then have

$$\mathrm{Disp}(f,g) = \mathrm{Disp}_x(f(x,y), g(x, y+n_0)) \times \{n_0\}.$$

Case 2. $\deg a_{d-1}(y) = 1$. We also have $\mathrm{Disp}(f,g) = \emptyset$ unless the leading term of $a_{d-1}(y)$ and that of $b_{d-1}(y)$ coincide. Assume

$$a_{d-1}(y) = \hat{p}y + p_0 \quad \text{and} \quad b_{d-1}(y) = \hat{p}y + q_0.$$

Then (2.1) leads to

$$(d \cdot c) \cdot m + \hat{p} \cdot n = p_0 - q_0, \tag{2.3}$$

which is a linear Diophantine equation in unknowns m, n. Either there is no solution, or the solutions are of the form

$$m = ut + v, \quad \text{and} \quad n = u't + v',$$

where u, v, u', v' are explicit integers and t runs over $\mathbb{Z}$. Now by setting all coefficients of x, y in the expansion of $f(x, y) - g(x + ut + v, y + u't + v')$ to be zeros, we arrive at a system of polynomial equations in t. The set of integer solutions of each equation is computable (see, for example [23, page 79]). The final dispersion set of f and g is the intersection of these solution sets.

Case 3. $\deg a_{d-1}(y) = 0$ or $a_{d-1}(y) = 0$. If $\deg_y b_{d-1}(y) > 0$, we then have $\mathrm{Disp}(f, g) = \emptyset$. Otherwise, m is uniquely determined by (2.1). Suppose m_0 is an integer solution of (2.1), we have

$$\mathrm{Disp}(f, g) = \{m_0\} \times \mathrm{Disp}_y(f(x, y), g(x + m_0, y)).$$

This completes the proof. ∎

Based on the proof as above, we can describe an algorithm for computing the dispersion set of two polynomials in $\mathbb{Q}[x, y]$.

Algorithm DispSet

Input: Two polynomials $f = \sum_{k=0}^{d_1} a_k(y)x^k$ and $g = \sum_{k=0}^{d_2} b_k(y)x^k$.

Output: The dispersion set $\mathrm{Disp}(f, g)$.

1. If $d_1 \neq d_2$, return $\emptyset$. Else set $d = d_1 = d_2$.

2. If $d \leq 0$, return the set $\mathbb{Z} \times \mathrm{Disp}_y(f, g)$. Else continue the following steps.

3. If $\deg a_d(y) > 0$, compute $\mathcal{N} = \{n \mid a_d(y) = b_d(y + n)\}$ and for each $n_0 \in \mathcal{N}$, compute the set S_{n_0} of integers m such that $f = \sigma_x^m \sigma_y^{n_0} g$. Return the set

$$\bigcup_{n_0 \in \mathcal{N}} S_{n_0} \times \{n_0\}.$$

   Else set $c := a_d(y)$ and continue the following steps.

4. If $\deg_y a_{d-1}(y) > 1$, compute the unique n_0 according to (2.2). If n_0 is an integer, then return $\mathrm{Disp}_x(f(x, y), g(x, y + n_0)) \times \{n_0\}$. Else return $\emptyset$.

5. If $\deg_y a_{d-1}(y) = 1$. If the leading terms of $a_{d-1}(y)$ and $b_{d-1}(y)$ are different, then return $\emptyset$. Else solve the linear Diophantine equation (2.3). Suppose that the solutions are of the form

$$m = ut + v \quad \text{and} \quad n = u't + v'.$$

   Substituting m by $ut+v$ and n by $u't+v'$ in $f = \sigma_x^m \sigma_y^n g$ and comparing the coefficients of each power of x and y to get a system of polynomial equations in t. Return all integer solutions if there are. Else return $\emptyset$.

6. If $\deg_y a_{d-1}(y) = 0$ or $a_{d-1}(y) = 0$. If $\deg_y b_{d-1}(y) > 0$ then return $\emptyset$. Else compute the unique m_0 satisfying (2.1). If m_0 is not an integer, then return $\emptyset$. Else return the set
$$\{m_0\} \times \mathrm{Disp}_y(f(x,y), g(x+m_0,y)).$$

The following is an example which shows how to determine the shift equivalence of any two given bivariate polynomials.

Example 2.3. *Let*
$$f = 2x^2 + 2xy + y^2 + y + 1 \quad \textit{and} \quad g = 2x^2 + 2xy + y^2 + 2x + y + 1.$$
We try to determine whether f and g are shift equivalent according to the proof of Theorem 2.2. Rewrite f, g as
$$f = 2x^2 + (2y)x + (y^2 + y + 1), \; \textit{and} \quad g = 2x^2 + (2y+2)x + (y^2 + y + 1).$$
It's easy to check that this meets Case 2 in the proof. Thus m, n satisfy the linear equation $2m + n = -1$ whose solutions are
$$m = t \text{ and } n = -2t - 1, \quad t \in \mathbb{Z}.$$
Now by setting all coefficients of x, y in the expansion of $f(x,y) - g(x+t, y-2t-1)$ to be zeros, we obtain an integer solution $t = -1$. It means that $f(x,y) = g(x-1, y+1)$ and thus f, g are shift equivalent.

3 Summability criterion

As stated in the introduction, one can decompose a univariate rational function α into the form $\alpha = \Delta\beta + \gamma$. The goal of this section is to introduce a bivariate variant of such additive decomposition and thus reduce the bivariate summability problem of a general rational function to that of a rational function whose denominator is a power of an irreducible polynomial. We then present a criterion for the summability of this kind of special rational functions.

Let $f \in \mathbb{Q}(x,y)$ be a bivariate rational function. Assume that the irreducible factorization of the denominator $D(x,y)$ of $f(x,y)$ is
$$D(x,y) = \prod_{i=1}^{m} d_i^{n_i}(x,y),$$
where $d_i(x,y)$ are irreducible polynomials and n_i are positive integers. Viewing f as a rational function of y over the field $\mathbb{Q}(x)$, we have the partial fraction decomposition
$$f = P + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{a_{i,j}}{d_i^j}, \tag{3.1}$$
where $P \in \mathbb{Q}(x)[y]$, $a_{i,j} \in \mathbb{Q}(x)[y]$ and $\deg_y(a_{i,j}) < \deg_y(d_i)$. It is well known that the polynomial P is the difference of a polynomial w.r.t. y.

Now suppose that $d_i(x,y) = d_k(x+m, y+n)$ for some index $i \neq k$. Then we have

$$\frac{a_{i,j}}{d_i^j} = \sigma_x(g) - g + \sigma_y(h) - h + \frac{\sigma_x^{-m}\sigma_y^{-n}(a_{i,j})}{d_k^j},$$

where

$$g = \begin{cases} \sum_{\ell=0}^{m-1} \frac{\sigma_x^{\ell-m}(a_{i,j})}{\sigma_x^{\ell}\sigma_y^{n}(d_k^j)}, & \text{if } m \geq 0, \\ -\sum_{\ell=0}^{-m-1} \frac{\sigma_x^{\ell}(a_{i,j})}{\sigma_x^{m+\ell}\sigma_y^{n}(d_k^j)}, & \text{if } m < 0, \end{cases}$$

and

$$h = \begin{cases} \sum_{\ell=0}^{n-1} \frac{\sigma_y^{\ell-n}\sigma_x^{-m}(a_{i,j})}{\sigma_y^{\ell}(d_k^j)}, & \text{if } n \geq 0, \\ -\sum_{\ell=0}^{-n-1} \frac{\sigma_y^{\ell}\sigma_x^{-m}(a_{i,j})}{\sigma_y^{n+\ell}(d_k^j)}, & \text{if } n < 0. \end{cases}$$

Repeating the above transformation, we arrive at the following decomposition.

Lemma 3.1. *For a rational function $f \in \mathbb{Q}(x,y)$, we can decompose it into the form*

$$f = \Delta_x(g) + \Delta_y(h) + r,$$

where $g, h \in \mathbb{Q}(x,y)$ and r is of the form

$$r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{a_{i,j}(x,y)}{d_i^j(x,y)}, \tag{3.2}$$

with $a_{i,j} \in \mathbb{Q}(x)[y]$, $\deg_y(a_{i,j}) < \deg_y(d_i)$, $d_i \in \mathbb{Q}[x,y]$ are irreducible polynomials, and d_i and $d_{i'}$ are not shift equivalent for any $1 \leq i \neq i' \leq m$.

From Lemma 3.1, we see that f is (σ_x, σ_y)-summable if and only if r is (σ_x, σ_y)-summable. The following lemma shows that the summability of r is equivalent to the summability of each summand of r.

Lemma 3.2. *Let $r \in \mathbb{Q}(x,y)$ be of the form (3.2). Then r is (σ_x, σ_y)-summable if and only if $\frac{a_{i,j}(x,y)}{d_i^j(x,y)}$ is (σ_x, σ_y)-summable for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$.*

Proof. The sufficiency follows from the linearity of the difference operators Δ_x and Δ_y. It suffices to prove the necessity. Assume that r is (σ_x, σ_y)-summable, then there exist $g, h \in \mathbb{Q}(x,y)$ such that $r = \sigma_x(g) - g + \sigma_y(h) - h$. We can always decompose g, h as

$$g = \frac{A_1}{D_1} + \frac{A_2}{D_2} \quad \text{and} \quad h = \frac{B_1}{C_1} + \frac{B_2}{C_2},$$

where $A_i, B_i, C_i, D_i (i = 1, 2)$ are polynomials in y over $\mathbb{Q}(x)$, $\deg_y(A_1) < \deg_y(D_1)$, $\deg_y(B_1) < \deg_y(C_1)$, D_1 (resp. C_1) contains only irreducible factors that are shift equivalent to d_i, while D_2 (resp. C_2) contains no such factors. Let $r_i = \sum_{j=1}^{n_i} \frac{a_{i,j}(x,y)}{d_i^j(x,y)}$. We then have

$$r_i - \left(\sigma_x \frac{A_1}{D_1} - \frac{A_1}{D_1} + \sigma_y \frac{B_1}{C_1} - \frac{B_1}{C_1}\right) = \sigma_x \frac{A_2}{D_2} - \frac{A_2}{D_2} + \sigma_y \frac{B_2}{C_2} - \frac{B_2}{C_2} - \sum_{j \neq i} r_j.$$

Note that σ_x, σ_y preserve the (σ_x, σ_y)-equivalence. Therefore, we have

$$r_i = \sigma_x \frac{A_1}{D_1} - \frac{A_1}{D_1} + \sigma_y \frac{B_1}{C_1} - \frac{B_1}{C_1},$$

which means r_i is (σ_x, σ_y)-summable.

By the same observation as in [13, Page 330], we see that σ_x and σ_y preserve the multiplicities of the fractions $a_{i,j}/d_i^j$. This implies that r_i is (σ_x, σ_y)-summable if and only if each summand $a_{i,j}/d_i^j$ is (σ_x, σ_y)-summable. This concludes the proof. ∎

Now we only need to study the summability problem of rational functions of the form a/d^j, where $d \in \mathbb{Q}[x, y]$ is irreducible, $a \in \mathbb{Q}(x)[y]$, and $\deg_y(a) < \deg_y(d)$. For this kind of rational functions, we have the following criterion for their summability.

Theorem 3.3. *Let $f = \frac{a(x,y)}{d^j(x,y)}$, where $d(x, y) \in \mathbb{Q}[x, y]$ is an irreducible polynomial, $a \in \mathbb{Q}(x)[y]$ is non-zero and $\deg_y(a) < \deg_y(d)$. Then f is (σ_x, σ_y)-summable if and only if*

(1) *there exist integers t, ℓ with $t \neq 0$ such that*

$$\sigma_x^t d(x, y) = \sigma_y^\ell d(x, y), \tag{3.3}$$

(2) *for the smallest positive integer t such that (3.3) holds, we have*

$$a = \sigma_x^t \sigma_y^{-\ell} p - p, \tag{3.4}$$

for some $p \in \mathbb{Q}(x)[y]$ with $\deg_y(p) < \deg_y(d)$.

The rest part of this section is devoted to proving this Theorem.

Firstly, we need some preparations. Analogue to the discrete residue given by Chen and Singer [13], we introduce the concept of polynomial residues. Let $\mathbb{K}$ be a field and $f \in \mathbb{K}(x)$. By partial fraction decomposition, f can be written as

$$f = p(x) + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{k_{i,j}} \frac{a_{i,j,\ell}(x)}{\sigma_x^\ell d_i^j(x)}, \tag{3.5}$$

where $p(x) \in \mathbb{K}[x]$, $m, n_i, k_{i,j} \in \mathbb{N}$, $\deg_x(a_{i,j,\ell}) < \deg_x(d_i)$, and $d_i(x)\,(i = 1, \ldots, m)$ are irreducible polynomials that in distinct σ_x-orbits. The summation

$$\sum_{\ell=0}^{k_{i,j}} \sigma_x^{-\ell}(a_{i,j,\ell})$$

is called the *polynomial residue* of f at the σ_x-orbit of $d_i(x)$ of multiplicity j, denoted by $\text{res}_x(f(x), d_i(x), j)$.

The necessary and sufficient conditions on the summability of rational functions in $\mathbb{K}(x)$ can be given in terms of polynomial residues. The proof is similar to the case of discrete residue [12, 13] and is omitted.

Proposition 3.4. *Let $f(x) \in \mathbb{K}(x)$ be of the form (3.5). Then $f(x)$ is σ_x-summable in $\mathbb{K}(x)$ if and only if the polynomial residue $\operatorname{res}_x(f(x), d_i(x), j)$ is zero for any polynomial $d_i(x)$ and any multiplicity j.*

Now we are ready to prove Theorem 3.3. For the sufficiency, let $g = \sum_{k=0}^{t-1} \frac{\sigma_x^k(p)}{\sigma_x^k(d^j)}$, then we will get

$$\frac{a}{d^j} - (\sigma_x g - g) = \frac{a}{d^j} - \frac{\sigma_x^t p}{\sigma_x^t d^j} + \frac{p}{d^j} = \frac{a+p}{d^j} - \frac{\sigma_x^t p}{\sigma_y^\ell d^j} = -\sigma_y^\ell \left(\frac{\sigma_x^t \sigma_y^{-\ell} p}{d^j} \right) + \frac{\sigma_x^t \sigma_y^{-\ell} p}{d^j}, \tag{3.6}$$

which means $\frac{a}{d^j}$ is (σ_x, σ_y)-summable.

For the necessity, suppose $f = a/d^j$ is (σ_x, σ_y)-summable and assume that

$$f = \sigma_x g - g + \sigma_y h - h, \tag{3.7}$$

where $g, h \in \mathbb{Q}(x, y)$. As a univariate analogue to Lemma 3.1, we can decompose g into the form

$$g = \sigma_y g_1 - g_1 + g_2 + \frac{\lambda_1}{\sigma_x^{\mu_1} d^j} + \cdots + \frac{\lambda_s}{\sigma_x^{\mu_s} d^j},$$

where $g_1, g_2 \in \mathbb{Q}(x, y)$ with g_2 containing no term of the form $\frac{\lambda}{\sigma_x^u d^j}$ in its partial fraction decomposition with respect to y, $\mu_\ell \in \mathbb{Z}$, $\lambda_\ell \in \mathbb{Q}(x)[y]$, and $\sigma_x^{\mu_\ell} d \, (\ell = 1, \ldots, s)$ are irreducible polynomials in distinct σ_y-orbits.

Claim 1. Let

$$\Lambda := \{\sigma_x^{\mu_1} d, \ldots, \sigma_x^{\mu_s} d, \sigma_x^{\mu_1+1} d, \ldots, \sigma_x^{\mu_s+1} d\}.$$

Then

(a) At least one element of Λ is in the same σ_y-orbit as d.

(b) For each element $\eta \in \Lambda$, there is one element of $\Lambda \backslash \{\eta\} \bigcup \{d\}$ that is in the same σ_y-orbit as η.

Proof of Claim 1. (a) Suppose there is no element of Λ that is in the same σ_y-orbit as d. Since $f = a/d^j$, we have $\operatorname{res}_y(f, d, j) = a \neq 0$. While by (3.7) and Proposition 3.4, we deduce that

$$\operatorname{res}_y(f, d, j) = \operatorname{res}_y(\sigma_x g - g, d, j) = 0,$$

which is a contradiction.

(b) The assertion follows from the same argument when considering the polynomial residues of η on both sides of (3.7). ∎

Claim 1 implies that either $d \sim_y \sigma_x^{\mu_1'} d$ or $d \sim_y \sigma_x^{\mu_1'+1} d$ for some $\mu_1' \in \{\mu_1, \ldots, \mu_s\}$. We will only consider the first case. The second case can be treated similarly.

Claim 2. Assume $d \sim_y \sigma_x^{\mu_1'} d$. We have the following assertions.

(a) Suppose $k \geq 2$ be an integer such that $\sigma_x^l d \nsim_y d$ for $1 \leq l \leq k-1$. Then there exist $\mu'_1, \ldots, \mu'_k \in \{\mu_1, \ldots, \mu_s\}$ such that

$$\sigma_x^{\mu'_1+1} d \sim_y \sigma_x^{\mu'_2} d, \quad \sigma_x^{\mu'_2+1} d \sim_y \sigma_x^{\mu'_3} d, \quad \ldots, \quad \sigma_x^{\mu'_{k-1}+1} d \sim_y \sigma_x^{\mu'_k} d,$$

and

$$\sigma_x^{k-1} d \sim_y \sigma_x^{\mu'_k} d.$$

(b) There exists a positive integer $t \leq s$ such that $\sigma_x^t d \sim_y d$.

Proof of Claim 2. (a) From Claim 1(b), we derive that $\sigma_x^{\mu'_1+1} d$ is σ_y-equivalent to an element of $\Lambda \backslash \{\sigma_x^{\mu'_1+1} d\} \bigcup \{d\}$. If $\sigma_x^{\mu'_1+1} d \sim_y d$, then $\sigma_x^{\mu'_1+1} d \sim_y \sigma_x^{\mu'_1} d$ and thus $\sigma_x d \sim_y d$, which contradicts to the hypothese on k. If $\sigma_x^{\mu'_1+1} d \sim_y \sigma_x^{\mu'_l+1} d$, then $\sigma_x^{\mu'_1} d \sim_y \sigma_x^{\mu'_l} d$ for some l, which contradicts to the assumption that $\sigma_x^{\mu_\ell}$ are in distinct σ_y-orbits. Therefore we are left with the only possibility that $\sigma_x^{\mu'_1+1} d \sim_y \sigma_x^{\mu'_2} d$ for some $\mu'_2 \in \{\mu_1, \ldots, \mu_s\} \setminus \{\mu'_1\}$. Continue this process, we will find $\mu'_3, \ldots, \mu'_k$ such that

$$\sigma_x^{\mu'_2+1} d \sim_y \sigma_x^{\mu'_3} d, \quad \ldots, \quad \sigma_x^{\mu'_{k-1}+1} d \sim_y \sigma_x^{\mu'_k} d.$$

Finally, we have

$$\sigma_x^{\mu'_k} d \sim_y \sigma_x^{\mu'_1+k-1} d \sim_y \sigma_x^{k-1} d.$$

(b) If such t does not exist, then one could find $\{\mu'_1, \ldots, \mu'_{s+1}\}$ satisfying the constraints in (a). Thus, it holds that $\mu'_r = \mu'_t$ for some $r > t$. Hence $\sigma_x^{\mu'_1+r} d \sim_y \sigma_x^{\mu'_1+t} d$, which leads to $\sigma_x^{r-t} d \sim_y d$, a contradiction. ∎

Suppose t is the smallest integer such that $\sigma_x^t d \sim_y d$. Then taking $k = t$ in Claim 2(a), we derive that there exist $\mu'_1, \ldots, \mu'_t \in \{\mu_1, \ldots, \mu_s\}$ such that

$$\sigma_x^{\mu'_1+1} d \sim_y \sigma_x^{\mu'_2} d, \quad \sigma_x^{\mu'_2+1} d \sim_y \sigma_x^{\mu'_3} d, \quad \ldots, \quad \sigma_x^{\mu'_{t-1}+1} d \sim_y \sigma_x^{\mu'_t} d,$$

and

$$\sigma_x^{\mu'_t+1} d \sim_y \sigma_x^t d \sim_y d.$$

Recall that $\sigma_x^{\mu'_1} d \sim_y d$. By the definition of $\sim_y$, there exist integers $s_0, s_1, \ldots, s_t$ such that

$$\sigma_x^{\mu'_k+1} d = \sigma_x^{\mu'_{k+1}} \sigma_y^{s_{k+1}} d, \ 1 \leq k \leq t-1, \quad \sigma_x^{\mu'_t+1} d = \sigma_y^{s_1} d, \quad \text{and} \quad \sigma_x^{\mu'_1} d = \sigma_y^{s_0} d.$$

Hence,

$$\sigma_y^{s_0} d = \sigma_x^{\mu'_1} d = \sigma_x^{\mu'_2-1} \sigma_y^{s_2} d = \sigma_x^{\mu'_3-2} \sigma_y^{s_2+s_3} d = \cdots = \sigma_x^{\mu'_t-t+1} \sigma_y^{s_2+s_3+\cdots+s_t} d = \sigma_y^{s_1+\cdots+s_t} \sigma_x^{-t} d.$$

Setting $\ell = s_1 + \cdots + s_t - s_0$, we then have $\sigma_x^t d = \sigma_y^\ell d$.

Now we compare the polynomial residues on both sides of (3.7). We list the residues in Table 1, where the first column consists of the σ_y-orbits of elements in Λ and the second column consists of the equations obtained by equating the corresponding polynomial residues

σ_y-orbit	Comparison of two sides of (3.7)
$d,\ \sigma_x^{\mu'_t+1}d$	$a = \sigma_x\sigma_y^{-s_0}\lambda'_t - \sigma_y^{-s_1}\lambda'_1$
$\sigma_x^{\mu'_{t-1}+1}d,\ \sigma_x^{\mu'_t}d$	$0 = \sigma_x\sigma_y^{-s_t}\lambda'_{t-1} - \lambda'_t$
$\sigma_x^{\mu'_{t-2}+1}d,\ \sigma_x^{\mu'_{t-1}}d$	$0 = \sigma_x\sigma_y^{-s_{t-1}}\lambda'_{t-2} - \lambda'_{t-1}$
$\vdots$	$\vdots$
$\sigma_x^{\mu'_2+1}d,\ \sigma_x^{\mu'_3}d$	$0 = \sigma_x\sigma_y^{-s_3}\lambda'_2 - \lambda'_3$
$\sigma_x^{\mu'_1+1}d,\ \sigma_x^{\mu'_2}d$	$0 = \sigma_x\sigma_y^{-s_2}\lambda'_1 - \lambda'_2$

Table 1: Orbits and their corresponding polynomial residues.

on both sides of (3.7). By investigating the equations in Table 1 from bottom to top, we find that

$$a = \sigma_x^t\sigma_y^{-\ell}p - p,$$

where $p = \sigma_y^{-s_1}\lambda'_1(x, y)$. Since $\deg_y \lambda'_1 < \deg_y d$, we have $\deg_y p < \deg_y d$. This completes the proof of Theorem 3.3. ∎

The criterion (3.3) can be tested by computing the dispersion set $\mathrm{Disp}(d, d)$. In the next section, we will give an algorithm for solving the equation (3.4). Then combining Lemma 3.1, Lemma 3.2 and Theorem 3.3, we will obtain an algorithm for determining whether a bivariate rational function is summable.

4 Rational solutions of bivariate difference equations

Let d_0 be a positive integer and u be a polynomial in y over $\mathbb{Q}(x)$ with $\deg_y(u) < d_0$. In this section, we present a method of finding solutions $p \in \mathbb{Q}(x)[y]$ with $\deg_y(p) < d_0$ to the following difference equation

$$u = \sigma_x^m\sigma_y^{-n}p - p, \tag{4.1}$$

where m, n are given integers and $m > 0$.

Noting that $\deg_y(p) < d_0$, we may assume

$$p = p_0(x) + \hat{p}(x)y + \cdots + p_{d_0-1}(x)y^{d_0-1},\ p_i(x) \in \mathbb{Q}(x).$$

Comparing the coefficients of each power of y on both sides of (4.1), we obtain a system of linear difference equations on $p_i(x)$. Abramov and coauthors have presented algorithms for finding a universal denominator for the system (see, for example [5, 7]). That is, we can compute a polynomial $d(x)$ such that $\hat{p}_i(x) = p_i(x)d(x)$ is a polynomial in x for each i. The universal denominator can also be obtained by using the convergence argument introduced by Chen, Paule and Saad [15].

Assume that $u(x, y) = a(x, y)/b(x)$, where a, b are polynomials in x and y. Abramov's universal denominator $d(x)$ can be computed as follows. Let

$$\mathcal{N} = \mathrm{Disp}_x(b, b) \bigcap \{m, 2m, 3m, \ldots\} = \{s_1 > s_2 > \cdots > s_r\}. \tag{4.2}$$

Initially, let $f_1 = g_1 = b$. For $i = 1, 2, \ldots, r$, set

$$\begin{aligned} &h_i(x) = \gcd(f_i(x), g_i(x + s_i)), \\ &f_{i+1}(x) = f_i(x)/h_i(x), \quad g_{i+1}(x) = g_i(x)/h_i(x - s_i). \end{aligned}$$

Then

$$d(x) = \prod_{i=1}^{r} \prod_{j=1}^{s_i/m} h_i(x - mj). \tag{4.3}$$

We will show that Abramov's universal denominator can be reduced for the special equation (4.1). Recall that an m-fold Gosper representation [23, page 80] of a rational function $r(x)$ of x is

$$r(x) = \frac{A(x)}{B(x)} \frac{C(x+m)}{C(x)},$$

where $A(x), B(x), C(x)$ are polynomials in x and

$$\gcd(A(x), B(x + mh)) = 1, \quad \forall\, h = 1, 2, \ldots.$$

The following theorem says that a universal denominator can be deduced from an m-fold Gosper representation of $b(x)/b(x + m)$.

Theorem 4.1. *Let $(A(x), B(x), C(x))$ be an m-fold Gosper representation of $\frac{b(x)}{b(x+m)}$. Then each solution $p(x, y) \in \mathbb{Q}(x)[y]$ to Equation (4.1) is of the form*

$$p(x, y) = \frac{B(x - m)\hat{p}(x, y)}{b(x)C(x)},$$

where $\hat{p}(x, y)$ is a polynomial in both x and y.

Proof. Rewrite Equation (4.1) as

$$a(x, y) = \frac{b(x)}{b(x + m)} \sigma_x^m \sigma_y^{-n} (b(x)p(x, y)) - b(x)p(x, y). \tag{4.4}$$

Assume that

$$b(x)p(x, y) = \frac{g(x, y)}{q(x)C(x)}, \tag{4.5}$$

where $g(x, y) \in \mathbb{Q}[x, y]$, $q(x) \in \mathbb{Q}[x]$ is a monic polynomial and $\gcd(q(x), g(x, y)) = 1$. By the definition of m-fold Gosper representations, we have

$$\frac{b(x)}{b(x + m)} = \frac{A(x)}{B(x)} \frac{C(x + m)}{C(x)}. \tag{4.6}$$

Substituting (4.5) and (4.6) into Equation (4.4), we deduce that

$$a(x,y)B(x)C(x)q(x)q(x+m) = A(x)g(x+m,y-n)q(x) - B(x)g(x,y)q(x+m). \tag{4.7}$$

It's easy to check that

$$q(x) \mid g(x,y)B(x)q(x+m).$$

Since $\gcd(q(x), g(x,y)) = 1$, we obtain

$$q(x) \mid B(x)q(x+m).$$

Using this divisibility repeatedly, we get

$$q(x) \mid B(x)B(x+m)\cdots B(x+(r-1)m)q(x+rm).$$

When $r > \max \mathrm{Disp}_x(q(x), q(x))$, we have $\gcd(q(x), q(x+rm)) = 1$, and thus

$$q(x) \mid B(x)B(x+m)\cdots B(x+(r-1)m).$$

From Equation (4.7), we also derive that

$$q(x+m) \mid g(x+m,y-n)A(x)q(x).$$

By a similar discussion, we arrive at

$$q(x) \mid A(x-m)A(x-2m)\cdots A(x-rm).$$

By the definition of Gosper representation, we know that $\gcd(A(x), B(x+hm)) = 1$ for any $h \in \mathbb{N}$. Thus the only opportunity for $q(x)$ is $q(x) = 1$.

When $q(x) = 1$, Equation (4.7) will be reduced to

$$a(x,y)B(x)C(x) = A(x)g(x+m,y-n) - B(x)g(x,y).$$

It's easy to see that

$$B(x) \mid A(x)g(x+m,y-n),$$

and hence $B(x) \mid g(x+m,y-n)$. Setting $g(x,y) = B(x-m)\hat{p}(x,y)$ concludes the proof. ∎

Let $\mathcal{N}$ be the dispersion set of $b(x)$ itself given by (4.2). A minimal m-fold Gosper representation of the ratio $b(x)/b(x+m)$ can be computed as follow. Initially, let $f_r = g_r = b$. For $i = r, r-1, \ldots, 1$, set

$$\tilde{h}_i(x) = \gcd(f_i(x), g_i(x+s_i)),$$
$$f_{i-1}(x) = f_i(x)/\tilde{h}_i(x), \quad g_{i-1}(x) = g_i(x)/\tilde{h}_i(x-s_i).$$

Then

$$A(x) = f_0(x), \quad B(x) = g_0(x+m), \quad C(x) = \prod_{i=1}^{r} \prod_{j=1}^{s_i/m-1} \tilde{h}_i(x-mj).$$

Hence the universal denominator given by Theorem 4.1 is

$$\tilde{d}(x) = \frac{b(x)C(x)}{B(x-m)} = \prod_{i=1}^{r} \prod_{j=1}^{s_i/m} \tilde{h}_i(x-mj). \tag{4.8}$$

The difference between the new universal denominator $\tilde{d}(x)$ given by (4.8) and Abramov's universal denominator $d(x)$ given by (4.3) lies in the order of elements in $\mathcal{N}$. By the properties of Gosper representations, $\tilde{d}(x)$ is a factor of $d(x)$.

Example 4.2. *Let m and ℓ be two positive integers and let $b(x) = x^2(x+m\ell)(x+m\ell+m)$. Abramov's universal denominator is*

$$(x+m\ell)\prod_{i=0}^{\ell-1}(x+mi)^2$$

and the universal denominator given by (4.8) is

$$\prod_{i=0}^{\ell}(x+mi).$$

Substituting $p(x,y) = \frac{B(x-m)\hat{p}(x,y)}{b(x)C(x)}$ into (4.4), we obtain

$$a(x,y)C(x) = A(x)\hat{p}(x+m, y-n) - B(x-m)\hat{p}(x,y). \tag{4.9}$$

Assume that

$$\hat{p}(x,y) = \hat{p}_0(x) + \hat{p}_1(x)y + \cdots + \hat{p}_{d_0-1}(x)y^{d_0-1}.$$

Equation (4.9) is equivalent to a linear system of difference equations on $\hat{p}_0(x), \ldots, \hat{p}_{d_0-1}(x)$. Abramov and Bronstein have presented an algorithm on solving such systems [6].

5 Examples

In this section, we give two examples to illustrate how to use our criterion for deciding the summability of some rational functions.

Example 5.1. *Decide whether*

$$f(x,y) = -\frac{(x+y+4)}{(x^2+2x+2xy-1+2y+y^2)(x^2+2xy+y^2-2)}.$$

is summable.

The first step is to find the partial fraction decomposition of $f(x,y)$ in variable y. By standard `Maple` command, we find that

$$f(x,y) = \frac{y+x+2}{x^2+2\,xy+y^2+2\,x+2\,y-1} + \frac{-y-x}{x^2+2\,xy+y^2-2}$$

Denote

$$d(x,y) = x^2+2xy+y^2-2 \quad \text{and} \quad d'(x,y) = x^2+2\,xy+y^2+2\,x+2\,y-1.$$

By computing the dispersion set of d' and d, we find that $d'(x,y) = d(x+1,y)$. Therefore, by applying the (σ_x, σ_y)-reduction, we derive that

$$f(x,y) = \Delta_x(g_1) + r(x,y), \tag{5.1}$$

where

$$g_1(x,y) = \frac{x+y+1}{d(x,y)} \quad \text{and} \quad r(x,y) = \frac{1}{d(x,y)}.$$

It remains to decide whether $r(x,y)$ is summable. By computing the dispersion of d itself, we find that $\sigma_x d(x,y) = \sigma_y d(x,y)$. Thus we need to find $p(x,y) \in \mathbb{Q}(x)[y]$ such that

$$p(x+1, y-1) - p(x,y) = 1 \quad \text{and} \quad \deg_y p < \deg_y d = 2.$$

The universal denominator is 1 and we may assume

$$p(x,y) = p_0(x) + p_1(x)y,$$

where $p_0(x)$ and $p_1(x)$ are polynomials in x. By the `Maple` package `LinearFunctionalSystems`, we find that

$$p_0(x) = p_1(x) = -1.$$

By Equation (3.6), we deduce that

$$r(x,y) = \sigma_x g_2(x,y) - g_2(x,y) + \sigma_y h(x,y) - h(x,y),$$

where

$$g_2(x,y) = \frac{-y-1}{d(x,y)} \quad \text{and} \quad h(x,y) = \frac{y}{d(x,y)}.$$

Substituting into (5.1), we finally derive that

$$f(x,y) = \sigma_x g(x,y) - g(x,y) + \sigma_y h(x,y) - h(x,y),$$

where

$$g(x,y) = \frac{x}{(x+y)^2-2}, \quad \text{and} \quad h(x,y) = \frac{y}{(x+y)^2-2}.$$

Example 5.2. *Decide whether*

$$f(x,y) = \frac{1}{(x+y)x}$$

is summable.

According to Theorem 3.3, it's easy to see that we only need to check whether

$$1/x = \sigma_x \sigma_y^{-1} p - p \tag{5.2}$$

is satisfied for some $p \in \mathbb{Q}(x)$. However Theorem 4.1 implies that $p(x)$ satisfying Equation 5.2 must be a polynomial in x which is impossible. Thus $\frac{1}{(x+y)x}$ is not (σ_x, σ_y)-summable.

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